ADE Exam, Spring 2023
Department of Mathematics, UCLA

1. [10 points]

   (a) Consider the dynamical system

   $$\frac{dx}{dt} = y\,,$$
   $$\frac{dy}{dt} = -x - \epsilon x^2 y\,, \quad (x, y) \in \mathbb{R}^2\,, \tag{1}$$

   where $\epsilon \geq 0$ is a parameter.

   Determine the stability of the equilibrium point at $(0, 0)$.

   (b) Consider the dynamical system

   $$\frac{dx}{dt} = y\,,$$
   $$\frac{dy}{dt} = x - x^3 - \delta y + x^2 y\,, \quad (x, y) \in \mathbb{R}^2\,, \tag{2}$$

   where $\delta > 0$ is a constant.

   Determine the equilibrium points of (2), and use linear stability analysis to classify their type and (when possible) their stability. Show that the two vertical lines $x = \pm\sqrt{\delta}$ divide the phase plane into three regions such that a periodic orbit cannot exist entirely in one of these regions.

2. [10 points] Consider the Legendre equation

   $$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \ell(\ell + 1)y = 0\,, \quad -1 \leq x \leq 1\,, \tag{3}$$

   where $\ell \geq 0$ is an integer. Let P_ℓ denote the solution of (3) that satisfies $P_\ell(1) = 1$.

   (a) Show that $x = 1$ is a regular singular point. Find the indicial equation and indicial exponents, and find the leading terms of the series expansion at $x = 1$ for two linearly independent solutions. Use them to explain why the condition $P_\ell(1) = 1$ is sufficient to uniquely determine P_ℓ.

   (b) Derive a recursion relation for the coefficients of the series expansion $y(x) = \sum_{k=0}^{\infty} a_k x^k$ for solutions of (3). Using this relation, show that P_ℓ is a polynomial that (i) consists only of even powers when ℓ is even and (ii) consists only of odd powers when ℓ is odd.

(c) Using the Rodrigues formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} \left[\left(x^2 - 1\right)^\ell \right] , \tag{4}$$

or otherwise, show that P_ℓ satisfies the orthogonality relation

$$\int_{-1}^{1} P_\ell(x) P_m(x) \, dx = 0 \, , \tag{5}$$

and determine the value of the integral (5) when $\ell = m$.

3. [10 points] Consider the energy functional $E[u]$, which is defined for $u \in \mathcal{C}^2(D)$ by

$$E[u] = \frac{1}{2} \int_D \left(|\nabla u|^2 + u^2 \right) \, d^n x \, ,$$

where $D \subset \mathbb{R}^n$ is a bounded and open set. Assume that $u = g(x)$ is known on ∂D.

(a) Derive the partial differential equation that is satisfied by the minimizer of E. Starting from the minimization principle, prove that solutions of this PDE are unique.

(b) Suppose that $n = 1$, $D = (-1, 1)$, and $u(-1) = u(1) = 1$. Find an approximate solution of your PDE from (a) that takes the form $u = 1 + A(1 - x^2)$. That is, find the value of A that minimizes the energy functional.

4. [10 points] Consider the nonlinear partial differential equation

$$u_t = \Delta u - u^3 \ , \quad x \in D \ , \quad 0 < t < T \, ,$$

where $D \subset \mathbb{R}^n$ is a bounded and open set. You may assume that solutions exist and are $\mathcal{C}^{2,1}(D \times (0, T)) \cap \mathcal{C}(\bar{D} \times [0, T])$. Show that the solutions of the PDE are unique.

5. [10 points] Consider the one-dimensional partial differential equation

$$u_t + \frac{1}{2} u_x^2 = 0 \ , \quad -\infty < x < \infty \, ,$$

with initial condition

$$u(x, 0) = 0 \, , \quad x < 0 \, ; \quad u(x, 0) = 1 \, , \quad x > 0$$

and boundary conditions

$$u \to 0 \ \text{ as } \ x \to -\infty \quad \text{and} \quad u \to 1 \ \text{ as } \ x \to \infty \ .$$

(a) Show that the PDE does *not* have a traveling-wave solution that is compatible with these boundary conditions, even when the derivatives are interpreted in the sense of distributions.

(b) Derive the weak solution of the PDE.

[Note: You do not need to derive the Hopf–Lax formula, but you should state the formula carefully if you use it.]

6. [10 points] Consider the partial differential equation

$$u_t + uu_x = -u\,,$$

with initial condition

$$u(x,0) = 1\,,\ x < 0\,; \quad u(x,0) = 0\,,\ x > 0\,.$$

(a) Show that for a smooth solution, the PDE can be written in terms of the "characteristic" variable $x = \xi(t)$ as

$$\frac{d}{dt}u(\xi(t)) = -u(\xi(t))\,; \quad \frac{d\xi}{dt} = u\,.$$

(b) Suppose that $u(0,0)$ takes values $\alpha \in (0,1)$. Solve for the characteristics $\xi_\alpha(t)$ starting from $x = 0$ with the initial value $u = \alpha$.

(c) Using the result from (b), solve the Riemann problem with the initial condition above. Write your answer in terms of the Eulerian variables x and t.

7. [10 points] Consider the Korteweg–de Vries (KdV) equation

$$u_t + u_{xxx} + 6uu_x = 0\,, \quad x \in \mathbb{R}\,, t > 0\,, \quad u(x,0) = f(x)\,.$$

Assume that the function $u(x,t)$ and all of its derivatives vanish as $|x| \to \infty$.

(a) Show that the following are conserved quantities in time:

$$\int_{-\infty}^{\infty} u^2(x,t)\,dx\,; \quad \int_{-\infty}^{\infty} \left[\frac{1}{2}u_x^2(x,t) - u^3(x,t]\right)\right]\,dx\,.$$

(b) Show that the KdV equation does not preserve positivity of the solution by constructing an initial condition f that is nonnegative for which the solution becomes negative at a later time.

[Hint: Consider a local minimum for which the third derivative in space is nonzero.]

8. [10 points] Solve the initial-value problem

$$u_{tt} - 2u_{xt} - 15u_{xx} = 0\,,$$

with $u(x,0) = g(x)$ and $u_t(x,0) = h(x)$.

[Hint: Consider factoring the differential operator.]