ADE Exam, Fall 2023
Department of Mathematics, UCLA

1. [10 points]

   (a) Find all equilibrium solutions of the dynamical system

   $$\frac{dx}{dt} = 1 - xy\,,$$
   $$\frac{dy}{dt} = x - y^3 \tag{1}$$

   and determine (if possible) whether they are stable or unstable.

   (b) Consider the dynamical system

   $$\frac{dx}{dt} = ax - bxy - ex^2\,,$$
   $$\frac{dy}{dt} = -cy + dxy - fy^2\,, \tag{2}$$

   where $a, b, c, d, e, f > 0$ are constants. Suppose that $c/d > a/e$. Prove that every solution of (2) with $x(0), y(0) > 0$ approaches the equilibrium $(x, y) = (a/e, 0)$ as $t \to \infty$.

2. [10 points] Consider the equation

   $$5x^2y'' + x(1+x)y' - y = 0\,. \tag{3}$$

   Find two independent series solutions about $x = 0$, and find a closed-form expression for one of these solutions.

3. [10 points] Suppose that $u(x)$ is a harmonic function on $\mathbb{R}^n$.

   (a) Prove that $u(x)$ has the mean value property; that is, prove that if $B(x, R)$ is a ball of radius R centered on x, then

   $$u(x) = \frac{1}{\mathrm{vol}(B(x,R))} \int_{y \in B(x,R)} u(y)\, d^n y\ . \tag{4}$$

   (b) Suppose additionally that $u(x) > 1$ for all x. Show, by applying the above result to the balls $B(x, R)$ and $B(y, \|x - y\| + R)$, that $u(x)$ is a constant function.

4. [10 points] Let $u(x,t)$ be a $\mathcal{C}^2 \times \mathcal{C}^1$ solution of the PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u\ , \quad 0 < x < 1\ , \quad t > 0\ , \tag{5}$$

with $u(0,t) = 0$ and $u(1,t) = 1$. For any initial condition $u(x,0)$, show that $u(x,t) \to U(x)$ as $t \to \infty$, where $U(x)$ is independent of the initial conditions and should be calculated as part of your solution.

5. [10 points] Show that $\mathcal{C}^2$ solutions of the PDE

$$\Delta u - u^3 = 0 \quad , \quad x \in \Omega\, , \tag{6}$$

where Ω is a bounded and open domain with boundary $\partial\Omega$, depend continuously on its boundary conditions. Specifically, given any $\epsilon > 0$, show that there exists $\delta > 0$ such that if u_1 and u_2 are both $\mathcal{C}^2$ solutions of (6) and $\|u_1 - u_2\|_\infty < \delta$ on $\partial\Omega$, then $\|u_1 - u_2\|_\infty < \epsilon$ on Ω.

6. [10 points] Consider the Euler equations in two dimensions for an incompressible, inviscid flow:

$$v_t + v \cdot \nabla v = -\nabla p\, , \ \nabla \cdot v = 0\, ,$$

where $\vec{v}(x,y,t) = (v^1, v^2)$ is the two-dimensional velocity field and $p(x,t)$ is a scalar pressure.

(a) Show that the scalar vorticity $\omega(x,y,t) = v^2_x - v^1_y$ satisfies the "vorticity stream" form of the Euler equation:

$$\omega_t + v \cdot \nabla \omega = 0\, .$$

(b) Prove that ω is conserved along particle paths that move with the flow according to $x_t = v^1(x,y,t)\, ,\ y_t = v^2(x,y,t)$.

(c) Define a scalar stream function Ψ according to

$$\Delta \Psi = \omega$$

in all of $\mathbb{R}^2$ and solve for Ψ in terms of ω.

(d) The stream function defines a velocity field by $v^1 = -\Psi_y\, ,\ v^2 = \Psi_x$.

Show that v is divergence-free and use the formula from (c) to derive the "Biot–Savart" law for the velocity of the fluid in terms of the vorticity.

7. [10 points] A ferry of length L pulls up to a terminal with cars filled to capacity at the maximum density $\rho = 1$. When the ferry docks, the gate opens and the cars leave the ferry. Assume that the car density ρ evolves according to the evolution equation

$$\rho_t + (v\rho)_x = 0\,, \tag{7}$$

where

$$\begin{aligned} v &= 1 - \rho\,, \quad 0 < \rho < 1 \\ v &= 0\,, \qquad \text{otherwise}\,. \end{aligned} \tag{8}$$

At the "maximum packing density" $\rho = 1$, we assume that cars are parked so close together that they cannot move. The initial condition is then $\rho = 1$ for $-L < x < 0$ and $\rho = 0$ otherwise.

(a) Solve the Riemann problem for equation (7) with initial condition $\rho = 1$ for $x < 0$ and $\rho = 0$ for $x > 0$ and v defined by equation (8).

[You may use the fact that for $F(\rho) = \rho(1-\rho)$, we have $F'^{-1}(x/t) = 0.5(1-x/t)$.]

(b) Use the solution to part (a) to solve for $\rho(x,t)$ on the interval $-L < x < \infty$ for all $t > 0$ with the ferry initial condition $\rho = 1$ for $-L < x < 0$ and $\rho = 0$ otherwise.

The solution $\rho(x,t)$ should be broken down into two parts: (i) the solution during which some of the cars are still densely packed and cannot move; and (ii) the solution after all of the cars have started moving.

(c) If you are in a car at the back of the ferry (i.e., at $x = -L$ at time $t = 0$), what is the trajectory of your car for all $t > 0$?

[You can break the trajectory down into the time it takes for the cars in front of you to start moving (during which time your car cannot move) and then the trajectory that you follow once your car starts moving. Note that the car will travel with speed $v(x,t)$, rather than the characteristic speed that is associated with the conservation law.]

8. [10 points] Consider the PDE

$$u_t + u \cdot \nabla u = -u$$

in $\mathbb{R}^2$, where u is a two-dimensional vector field with initial data $u = \vec{x}$.

Using the method of characteristics, solve for u for all $t > 0$ and $\vec{x} \in \mathbb{R}^2$.