ALGEBRA QUALIFYING EXAM

2023 MARCH 28

Problem 1. Let $F, F' \colon \mathcal{C} \to \mathcal{D}$ and $G, G' \colon \mathcal{D} \to \mathcal{C}$ be four functors such that F is left adjoint to G, and F' is left adjoint to G'. Establish a bijection between the natural transformations $\alpha\colon F \Rightarrow F'$ and the natural transformations $\beta\colon G' \Rightarrow G$. [Hint: Use $G\alpha G' \colon GFG' \Rightarrow GF'G'$.]

Problem 2. Let p, q be distinct prime numbers and consider the number field $K = \mathbb{Q}(\sqrt{p} + \sqrt{q})$. Describe all subfields of K and inclusions between them.

Problem 3. Give an example of an infinite field extension $K \subset L$ such that L has only finitely many field automorphisms fixing K.

Problem 4. Let $\mathrm{M}_n(K)$ be the ring of $n \times n$-matrices with coefficients in a field K, for $n \geq 1$. Describe all possible ring homomorphisms $\mathrm{M}_n(K) \to K$.

Problem 5. Let A be a local commutative noetherian ring and M a finitely generated A-module such that every exact sequence $0 \to M'' \to M' \to M \to 0$ remains exact after tensoring with the residue field k of A. Show that M is free.

Problem 6. Let A be a commutative ring and let $s \in A$. Let $S = \{1, s, s^2, \ldots\}$. Show that the following assertions are equivalent:
(a) The canonical morphism $A \to S^{-1}A$ is surjective.
(b) There is $N > 0$ such that $s^n A = s^N A$ for all $n \geq N$.
(c) For n large enough, the ideal $s^n A$ is generated by an element e with $e^2 = e$.

Problem 7. Let k be a field and let $A = k[X, Y]/(X^2, XY, Y^2)$.
(a) Determine the invertible elements of A.
(b) Determine the ideals of A.
(c) Determine the principal ideals of A.

Problem 8. Let G be a finite group and let p be the smallest prime dividing the order of G. Show that a subgroup $H \leq G$ of index p must be normal.

Problem 9. Let G be a non-abelian finite group of order pq where p and q are prime numbers with $q > p$. Determine the degrees of the irreducible characters of G, and determine the number of irreducible characters of a given degree.

Problem 10. Let A be an artinian ring and let M be an A-module. Let $B = \mathrm{End}_A(M)$. Let $f \in B$ such that $f(M) \subset \mathrm{Rad}(A) \cdot M$, where $\mathrm{Rad}(A) = J(A)$ is the Jacobson radical. Show that $f \in \mathrm{Rad}(B)$.