Algebra Qualifying Exam
Fall 2023
Complete 8 of the following 10 problems.
(If attempting more than 8, indicate which 8 you wish to have graded.)

1. Let G be a group, let $H \subset G$ be a subgroup of finite index $n \geq 2$, and let $x \in G$. Prove that $[H : H \cap xHx^{-1}] \leq n - 1$.

2. Let A be a commutative Noetherian ring. Prove that every nonzero ideal I of A contains a finite product of nonzero prime ideals.

3. Show that there is an isomorphism of $\mathbb{Q}$-algebras $\mathbb{Q}[t] \otimes_{\mathbb{Q}[t^2]} \mathbb{Q}[t] \cong \mathbb{Q}[x,y]/(x^2 - y^2)$.

4. Let K/F be a (finite) Galois extension of fields, and let $\alpha \in K \setminus F$. Let E be a subfield of K containing F of largest degree over F such that $\alpha \notin E$. Prove that $E(\alpha)/E$ is a Galois extension of prime degree.

5. Let F be a field, and let $f(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial of degree $n \geq 1$ with coefficients $a_i \in F$. Show that the splitting field of $f(x^2)$ over F contains a square root of $(-1)^n a_0 a_n^{-1}$.

6. For a positive integer n, let C_n be the category with objects $[1,n] := \{1, 2, \ldots, n\}$ and morphisms $\mathrm{Mor}(i,j)$ an empty set if $i > j$ and a singleton otherwise. For positive integers m and n, a nonstrictly increasing function $f \colon [1,n] \to [1,m]$ can be viewed as a functor $C_n \to C_m$. Prove that this functor f has right adjoint if and only if $f(1) = 1$.

7. Let R be a PID and $n \geq 1$. Let M be a finitely generated R^n-module, where R^n is the product of n copies of R. Show that there exists an exact sequence

$$0 \to P \to Q \to M \to 0$$

with P and Q finitely generated projective R^n-modules.

8. Let A be a domain that is normal (i.e., integrally closed in its quotient field), and let $\mathfrak{p}$ be a prime ideal of A.

a. Show that the localization $A_{\mathfrak{p}}$ is a normal domain.

b. Suppose that A is Noetherian and that $\mathfrak{p}$ is a minimal nonzero prime ideal of A. Show that $A_{\mathfrak{p}}$ is a DVR.

9. Find the dimensions and characters of all irreducible $\mathbb{Q}$-representations of the cyclic group of order a prime p.

10. Let $\rho \colon G \to \mathrm{GL}(V)$ be a finite dimensional irreducible representation of a finite group G over the field of complex numbers. Prove that for every central element $g \in G$, the operator $\rho(g)$ is multiplication by a scalar.