Basic Exam: Spring 2023

March 27,2023

Test instructions:

Write your UCLA ID number on the upper right corner of each sheet of paper you use. Do not write your name anywhere on the exam! All answers must be justified. If you wish to use a known theorem, make sure to give a precise statement. Work out 10 problems and clearly indicate, by circling the number of the problem on the front page, which 10 of the 12 problems you want us to grade. Graders will grade only 10 problems per exam!

Important: No books, notes, calculators, computers or other printed or electronic materials can be used on the exam.

Problem Scores:

1.

2.

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12.

Total Score:

Problem 1:

Consider a matrix with complex entries of the form

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{pmatrix}$$

Give a basis of $\mathbb{C}^3$ which consists of simultaneous eigenvectors for all such M. Give also a formula for the associated eigenvalues as functions of the entries of M.

Problem 2:

Let V be a real inner product space and P be a linear map on V such that $P^2 = P$. Prove that the operator $I - 2P$ is an isometry if and only if P is self-adjoint.

Problem 3:

Let V be a real inner product space and let $u \in V$ be a unit vector. Let R be the linear transformation defined by the formula, for $v \in V$,

$$R(v) = v - 2 < v, u > u.$$

Such linear transformations are called orthogonal reflections.

(a) If $V = \mathbb{R}^2$, show that every orthogonal matrix of determinant 1 is the product of two orthogonal reflections.

(b) If $V = \mathbb{R}^2$, show that every orthogonal matrix of determinant -1 is an orthogonal reflection.

(c) For $V = \mathbb{R}^n$, $n > 1$, show that any orthogonal matrix is a product of at most n orthogonal reflections

Problem 4: Consider a normal matrix $A \in \mathbb{C}^{d \times d}$ with the property that $A^4 = A^3$.

(a) Prove or disprove that A is a Hermitian matrix.

(b) Prove or disprove that $A^2 = A$ (i.e. A is a projection).

(c) Prove or disprove that Part (a) holds if A is not necessarily normal.

Problem 5:

A 4 by 4 matrix A satisfies $P(A) = 0$ where $P(X) = X^4 - X^3$. The rank of A is 2 and the rank of $A - Id_4$ is 3. Give a representative of each possible similarity class of A in Jordan canonical form.

Problem 6:

Determine the QR decomposition of the matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

Problem 7:

(a) Recall that a function $f(x)$ is **Lipschitz** in x on an interval $[a, b]$ if there exists a constant $L > 0$, called the Lipschitz constant, with

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2|,$$

for all $x_1, x_2 \in [a, b]$.

Assume that f is Lipschitz in $[a, b]$ with Lipschitz constant $L \leq 1 - \epsilon$ for some $\epsilon > 0$ and that $f : [a, b] \to [a, b]$. Prove that f has a unique fixed-point in $[a, b]$.

(b) Suppose that $f \in C^1([a, b])$, $f'(x) \leq 1 - \epsilon$ for some $\epsilon > 0$, and $f : [a, b] \to [a, b]$. Prove that f has a unique fixed-point in $[a, b]$.

Problem 8:

Let f_n be a sequence of Riemann integrable functions defined on $[0, 1]$ such that $|f_{n+1}(x)| \leq |f_n(x)|$ for all $n \geq 1$ and for all $x \in [0, 1]$, and consider the sequence g_n defined by

$$g_n(x) = \int_0^x f_n(t)\, dt.$$

Show that there is a subsequence $n_k \to \infty$ so that g_{n_k} converges uniformly on $[0, 1]$.

Problem 9:

Let $\epsilon > 0$, and $f(x)$ be a continuous function on $[0, \pi/2]$ such that $f(\pi/2) = 0$, and

$$\text{for all } 0 \le x \le \pi/2, \quad \text{we have} \quad |f(x)^2 + \sin(x) f(x)| \le \epsilon^2.$$

Show that for small enough $\epsilon > 0$,

$$\max_{0 \le x \le \pi/2} |f(x)| \le 3\epsilon.$$

Problem 10:

Let $f(x)$ be a non-increasing differentiable function on $[0, +\infty)$ such that

$$f(0) = 1, \quad \lim_{x \to +\infty} f(x) = 0$$

and

$$\int_0^\infty f'(x) x^4 dx < \infty.$$

Show that

$$\lim_{x \to \infty} x^4 f(x) = 0.$$

Problem 11:

Give an example of a sequence of continuous functions on $[0, 1]$ that converges pointwise to a nonintegrable function.

Problem 12:

Let $\mathbb{H} = \big\{(x, y) \in \mathbb{R}^2 : x, y \ge 0, x^2 + y^2 \le 1\big\}$.

(a) Prove that for any $\epsilon > 0$ and any continuous function $f : \mathbb{H} \to \mathbb{R}$ there exists a function $g(x, y)$ of the form

$$g(x, y) = \sum_{m=0}^{N} \sum_{n=0}^{N} a_{mn} x^{2m} y^{2n}, \quad N = 0, 1, 2, \cdots, \quad a_{mn} \in \mathbb{R}$$

such that

$$\sup_{(x,y) \in \mathbb{H}} |f(x, y) - g(x, y)| < \epsilon.$$

(b) Does the result in (a) hold if $\mathbb{H}$ is replaced by the disk $\mathbb{D} = \big\{(x, y) : x^2 + y^2 \le 1\big\}$?