BASIC EXAM: FALL 2023

Test instructions:

Write your UCLA ID number on the upper right corner of *each* sheet of paper you use. Do not write your name anywhere on the exam. All answers must be justified. If you wish to use a known theorem, make sure to give a precise statement.

Important: No books, notes, calculators, computers or other printed or electronic materials can be used on the exam.

The final score will be the sum of the best **FIVE** analysis problems (Problems 1–6) and the best **FIVE** linear algebra problems (Problems 7–12).

On the front of your paper indicate which 10 problems you wish to have graded otherwise the first 10 problems you attempted will be graded. Please be reminded that to pass the exam you need to show mastery of both subjects.

Please staple your problems in the order they are listed in the exam.

1	2	3	4
5	6	7	8
9	10	11	12

Problem 1. Let $\{f_n\}$ be a sequence of differentiable functions on $[0, 1]$ with the property that $f_n(0) = f_n'(0)$ for all $n \in \mathbb{N}$. Furthermore, assume that $|f_n'(x)| \leq 3$ for all $n \in \mathbb{N}$ and for all $x \in [0, 1]$. Prove that there is a subsequence of $\{f_n\}$ that converges uniformly on $[0, 1]$.

Problem 2. Given some $\alpha > 0$, show that the sequence:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$$

for $n \geq 0$, converges to the unique positive fixed-point for any $x_0 > 0$.

Problem 3. Let (X, dist) be a complete metric space. Suppose that a function $f : X \to X$ has the property that

$$\theta_n = \sup_{x,y \in X, x \neq y} \frac{\mathrm{dist}(f^n(x), f^n(y))}{\mathrm{dist}(x, y)}$$

with $\sum_{n=1}^{\infty} \theta_n < \infty$, where f^n denotes the composition of f n-times. Prove that f has a unique fixed point in X.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function.

(a) For $\delta > 0$ we define the points $x_n = n\delta$ for all $n \in \mathbb{Z}$. Prove that there is a $\delta > 0$ such that for all $x_n = n\delta$,
$$|f(x_n)| \leq |f(0)| + |n|.$$
(b) Prove that for all $x \in \mathbb{R}$,
$$|f(x)| \leq a|x| + b,$$
where a, b are non-negative constants.

Problem 5. Consider the power series:

$$\sum_{n=0}^{\infty} n^2 x^n.$$

(a) Find the radius of convergence. Justify your solution.
(b) Compute the value of the series at $x = \frac{1}{2}$.

Problem 6. Let Y be a closed subset of a metric space (X, dist). Prove that if $S \subset X$ is compact, then $S \cap Y = \emptyset$ if and only if

$$\inf_{x \in S, y \in Y} \mathrm{dist}(x, y) > 0.$$

Problem 7. Solve the following two problems regarding linear transformations over real vector spaces.

(i) Find a linear transformation $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ for which

$$\mathrm{Ker}(f) = \langle (1,0,1,0)\rangle \text{ and } \mathrm{Im}(f^2) = \langle (1,0,1,0),(1,1,1,1)\rangle.$$

(iI) Find all the possible values of $\dim(\mathrm{Ker}(f) \cap \mathrm{Im}(f^2))$ where $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ is a linear transformation.

Problem 8. Consider the following matrix:

$$A := \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Compute A^{2023}.

Problem 9. Show that two diagonalizable matrices are simultaneously diagonalizable if and only if they commute.

Problem 10. Let V be a finite-dimensional vector space. Let $f_1, \dots, f_k$ be linear functionals on V with nullspaces $N_1, \dots, N_k$, respectively. Show that a linear functional g on V is a linear combination of $f_1, \dots, f_k$ if an only if the nullspace of g contains $N_1 \cap \cdots \cap N_k$.

Problem 11. Let A be the 100×100 matrix defined by

$$A_{i,j} = \begin{cases} 1 & \text{if } [i+j]_{50} = 1 \\ 2 & \text{if } [i-j]_{150} \in \{0, 50\} \\ 1 & \text{if } [i-j]_{150} = 100 \\ 0 & \text{otherwise} \end{cases}$$

Find the determinant of A.

Problem 12. Let A be a non-singular $n \times n$ complex matrix. Show that there exists a non-singular $n \times n$ complex matrix B for which $B^2 = A$.