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WEAK LIMITS OF MULTI-SCALE HOMOGENIZATION: A PHENOMENON OF CONVERGENCE TO AVERAGES ON SUBMANIFOLDS OF THE TORUS†

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Abstract. This paper studies weak limits of oscillatory functions where the oscillations are introduced through a multiple number of vanishing scales – $f(x, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n})$, $\varepsilon_i \downarrow 0$. It is shown that the weak limit equals the average of the function over an affine submanifold of the torus T^n. The submanifold and its dimension are determined by the limit ratios between the scales, $\alpha_i = \lim \frac{\varepsilon_n}{\varepsilon_i}$, their linear dependence over the integers and also on the rate in which the ratios between the scales approach their limit. This unexpected phenomenon, of unstable dependence of the weak limit on the small scales, is also demonstrated graphically. Applications to a dynamical system and to homogenization of convection-diffusion equations are given.

To Judith, Dukshi and Dafna with love.

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Contents.

1. Introduction. This study revolves around the following question: Let $f(x,y)$ be a function of a real variable x and the periodic variables $y = (y_1, ..., y_n)$ on the unit n-dimensional torus, $T^n = [0,1]^n$, and let $f_\varepsilon(x)$ be the oscillatory function

$$f_\varepsilon(x) = f(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n}) \ , \tag{1.1}$$

where $\varepsilon_i > 0$, $1 \le i \le n$, are small scale parameters. Then what is the weak limit of $f_\varepsilon(x)$ when $\varepsilon_i \downarrow 0$?

This question is motivated by one of the most important problems in nonlinear partial differential equations – translating a microscopic level model to an effective macroscopic one. One of the forms that this problem may take is the following: given oscillatory solutions of a small scale dependent problem, what are the corresponding homogenized equation and data which determine the weak limit of those solutions when the scales tend to zero? (see [3], [5]).

In [10] we have addressed this question in the case where only one vanishing scale exists. We have studied oscillatory solutions to convection-diffusion problems which are subject to initial and forcing data with modulated 1-scale-oscillations, i.e., functions of the form (1.1) with $n = 1$. As a first step, we determined the weak limit of such functions [10, Lemma 2.1]:

LEMMA 1.1. *Assume that $f = f(x,y) \in BV_x(\Omega \times T^1)$ and let $f_\varepsilon(x) := f(x, \frac{x}{\varepsilon})$. Then*

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le C\varepsilon \quad , \quad C \sim \|f\|_{L^1(T^1;BV(\Omega))} \ ,$$

where $\bar{f}(x) = \int_{T^1} f(x,y)dy$.

Here, and henceforth, $\Omega = [a,b]$ denotes a bounded interval in $\mathbb{R}_x$, $\|\cdot\|_{W^{-1,\infty}(\Omega)}$ stands for the $W^{-1,\infty}$-norm in Ω,

$$\|g(x)\|_{W^{-1,\infty}(\Omega)} = \| \int_a^x g(\xi)d\xi \|_{L^\infty(\Omega)} \ ,$$

and $BV_x(\Omega \times T^n)$ is the space of all bounded functions $f = f(x,y)$, $x \in \Omega$, $y = (y_1, y_2, ..., y_n) \in T^n$, which have a bounded variation in x.

In this paper we are concerned with the multiscale case. To this end, we assume henceforth that $f = f(x,y) \in BV_x(\Omega \times T^n)$ and view all scales as continuous functions of a common parameter, $\varepsilon_i = \varepsilon_i(\varepsilon) > 0$, such that $\lim_{\varepsilon \to 0^+} \varepsilon_i = 0$, $1 \le i \le n$. Hence, we seek $\bar{f}(x)$ – the weak limit of $f_\varepsilon(x)$, (1.1), when $\varepsilon \downarrow 0$.

This question turns out to be more complex than the analogous question in the simpler 1-scale case. The answer, or, better yet, the array of answers which we reveal here is quite interesting, sometimes even surprising. T. Hou has discovered in [5] a part of the picture. He dealt with the 2-scale case and found that $\bar{f}(x)$ depends on the limit ratio $\alpha = \lim_{\varepsilon \to 0} \frac{\varepsilon_1}{\varepsilon_2}$ in the following unstable manner: If α is 0 (or, equivalently, infinite) or an irrational number, the weak limit is the average of $f(x,y)$ over the 2-dimensional torus,

$$\bar{f}(x) = \int_{T^2} f(x,y)dy \ ; \tag{1.2}$$

in case α is a nonzero rational number, $\frac{m}{n}$, the weak limit is the average of $f(x,y)$ over the projection of the straight line $Span_{\mathbb{R}}\{(n,m)\}$ on T^2,

$$\bar{f}(x) = \int_{T^1} f(x, ny_1, my_1)dy_1 \ . \tag{1.3}$$

The assumption under which these limits where obtained, was that $r := \frac{\varepsilon_1}{\varepsilon_2} - \alpha$ tends to zero **faster** than ε_1 and ε_2.

In §2 we complete the task and unveil the entire picture in the 2-scale case. If α is zero or irrational, we prove that the weak limit is as in (1.2), **regardless** of the rate in which r vanishes (Theorems 2.1 and 2.7). If, however, α is a nonzero rational number, the weak limit depends on the value of α and, in addition, on the **rate** in which α is approached by $\frac{\varepsilon_1}{\varepsilon_2}$, namely – the order of magnitude of r. In Theorem 2.4 we show that (1.3) holds only when $|r| \ll \mathcal{O}(\varepsilon_1, \varepsilon_2)$; if $|r| = \mathcal{O}(\varepsilon_1, \varepsilon_2)$, $\bar{f}(x)$ takes a similar form of an f-average over an affine curve on T^2 which is parallel to the linear curve along which the integral in (1.3) is taken; however, if $|r| \gg \mathcal{O}(\varepsilon_1, \varepsilon_2)$, the weak limit switches unexpectedly from a one-dimensional integral to the double integral in (1.2). Our convergence proofs, in the cases where α is rational, are accompanied by convergence rate estimates.

In §3 we deal again with the case where the limit of the ratio $\frac{\varepsilon_1}{\varepsilon_2}$ is a nonzero irrational number. We offer alternative convergence proofs which – apart from being interesting and different from the one in §2 – provide, in some cases, convergence rate estimates as well. These proofs are based on some results from number theory and the theory of quasi-Monte Carlo integration methods which we review briefly in the Appendix.

In §4 we extend our discussion to the case of a multiple number of scales. First, we introduce an equivalence relation, $\sim$, on the set of scales, $\mathcal{S} = \{\varepsilon_i\}_{1 \le i \le n}$. This relation enables us to reduce the problem of homogenization of f with respect to $\mathcal{S}$ to a problem of homogenization of another function with respect to the smaller set of scales, $\mathcal{S}/\sim$. After that we show that the weak limit of $f_\varepsilon(x)$ is an average of $f(x,\cdot)$ over a submanifold of T^n. This submanifold is determined by the ratios between the scales and, like in the 2-scale case, takes the form of a projection of an $\mathbb{R}^n$-affine subspace onto the torus T^n. In case that all scales are proportional, i.e., $\frac{\varepsilon_n}{\varepsilon_i} = \alpha_i > 0$ for all $1 \le i \le n$, the dimension of this submanifold is determined by the degree of linear dependence between the α_i's over $\mathbb{Z}$: if they are linearly independent, the submanifold is the entire n-dimensional torus; in the other extreme case where they are all rationally-proportional, the submanifold is 1-dimensional. In the general case where the scales are not necessarily proportional, the limit-submanifold depends also on the number of different orders of magnitude among the scales, as well as on the rate in which the ratio between two scales of the same order of magnitude tends to its limit, in resemblance to the 2-scale case.

§5 is devoted to applications of our analysis. In §5.1 we study the motion of harmonic and quasi-harmonic oscillators in several dimensions; in §5.2 we apply our results to the homogenization of nonlinear convection-diffusion equations. Finally, in §6 we provide convincing visual illustrations of our weak convergence results.

We conclude the Introduction with some notation remarks. Throughout this paper, $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ denote, respectively, the sets of integer, rational and real numbers; $\mathbb{Z}^*, \mathbb{Q}^*$ and $\mathbb{R}^*$ stand for the same sets with the exclusion of 0. We also define the following smoothness classes:

DEFINITION 1.2. *$BV(y_i)$, $1 \leq i \leq n$, is the class of all functions $f \in BV_x(\Omega \times T^n)$ which have a uniformly bounded variation with respect to y_i in $\Omega \times T^n$.*

$Lip(y_i)$ (or $Lip(x)$) is the class of all functions $f \in BV_x(\Omega \times T^n)$ which are uniformly Lipschitz continuous with respect to y_i (respectively, x) in $\Omega \times T^n$.

2. $W^{-1,\infty}$-Convergence Analysis with Two Scales.

Throughout this section and the following one, §3, $y = (y_1, y_2) \in T^2$, $f = f(x,y) \in BV_x(\Omega \times T^2)$ and $f_\varepsilon(x) = f(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2})$. We identify the common parameter ε with ε_2, i.e.,

$$\varepsilon_1 = \varepsilon_1(\varepsilon) \quad , \quad \varepsilon_2 = \varepsilon \ . \tag{2.1}$$

Denoting, as before,

$$r = \frac{\varepsilon_1}{\varepsilon_2} - \alpha \to 0 \ , \tag{2.2}$$

we get that

$$\varepsilon_1 = \alpha\varepsilon + \delta \quad \text{where} \quad \delta = r\varepsilon = o(\varepsilon) \ . \tag{2.3}$$

2.1. Case 1: Zero limit.

Here we deal with the case where the ratio $\frac{\varepsilon_1}{\varepsilon_2}$ tends to zero. In this simple case, the following holds:

THEOREM 2.1. *Assume that $\frac{\varepsilon_1}{\varepsilon_2} \to 0$ and that $f \in Lip(y_2)$ or $f \in Lip(x) \cap BV(y_2)$. Then*

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq Const \cdot \left(\varepsilon_2 + \frac{\varepsilon_1}{\varepsilon_2}\right) \ , \tag{2.4}$$

where

$$\bar{f}(x) = \int_{T^2} f(x,y)dy \ . \tag{2.5}$$

Proof. Defining

$$g(x, y_1) := f(x, y_1, \frac{x}{\varepsilon_2}) \ , \qquad \bar{g}(x) := \int_{T^1} g(x, y_1)dy_1 \ ,$$

and

$$h(x, y_2) = \int_{T^1} f(x, y_1, y_2)dy_1 \ , \qquad \bar{h}(x) = \int_{T^1} h(x, y_2)dy_2 \ ,$$

the difference in (2.4) may be decomposed as follows:

$$(2.6)\ \|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le \|g(x, \frac{x}{\varepsilon_1}) - \bar{g}(x)\|_{W^{-1,\infty}(\Omega)} + \|h(x, \frac{x}{\varepsilon_2}) - \bar{h}(x)\|_{W^{-1,\infty}(\Omega)} \ .$$

Using Lemma 1.1 for the two terms on the right hand side of (2.6), we get that

$$(2.7)\quad \|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le Const \cdot \left(\|g\|_{L^1(T^1;BV(\Omega))} \cdot \varepsilon_1 + \|h\|_{L^1(T^1;BV(\Omega))} \cdot \varepsilon_2 \right) \ .$$

One can verify that either of the smoothness assumptions that we made on f implies that

$$(2.8)\qquad \|g\|_{L^1(T^1;BV(\Omega))} \le Const \cdot \varepsilon_2^{-1} \ .$$

Therefore, (2.4) follows from (2.7) and (2.8). □

2.2. Case 2: A nonzero rational limit. Here we deal with the case where the ratio $\frac{\varepsilon_1}{\varepsilon_2}$ tends to a nonzero rational limit. We first assume that this limit equals 1, namely, in view of (2.3), $\varepsilon_1 = \varepsilon + \delta$ where $\delta = o(\varepsilon)$.

In Lemmas 2.2 and 2.3 below we prove that the weak limit depends on the ratio between δ and ε^2 in the following way:

$$(2.9)\qquad f(x, \frac{x}{\varepsilon+\delta}, \frac{x}{\varepsilon}) \longrightarrow \begin{cases} \int_{T^1} f(x, y_1 - cx, y_1)dy_1 & \text{if } \frac{\delta}{\varepsilon^2} \to c \\ \int\int_{T^2} f(x, y_1, y_2)dy_1 dy_2 & \text{if } \frac{\varepsilon^2}{\delta} \to 0 \end{cases} \ .$$

LEMMA 2.2. *Assume that* $\frac{\delta}{\varepsilon^2} \to c$ *and* $f \in Lip(y_1)$. *Then*

$$(2.10)\qquad \|f(x, \frac{x}{\varepsilon+\delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le Const \cdot \left(\varepsilon + \left| \frac{\delta}{\varepsilon^2} - c \right| \right) \ ,$$

where

$$(2.11)\qquad \bar{f}(x) = \int_{T^1} f(x, y_1 - cx, y_1)dy_1 \ .$$

Proof. We decompose the error in (2.10) as follows:

$$(2.12)\quad \|f(x, \frac{x}{\varepsilon+\delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le$$

$$\|f(x, \frac{x}{\varepsilon+\delta}, \frac{x}{\varepsilon}) - f(x, \frac{x}{\varepsilon} - cx, \frac{x}{\varepsilon})\|_{W^{-1,\infty}(\Omega)} + \|f(x, \frac{x}{\varepsilon} - cx, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \ .$$

The first term on the right hand side of (2.12) may be upper bounded in L^∞, using the Lipschitz continuity of $f(x, \cdot, y_2)$:

$$(2.13)\quad \|f(x, \frac{x}{\varepsilon+\delta}, \frac{x}{\varepsilon}) - f(x, \frac{x}{\varepsilon} - cx, \frac{x}{\varepsilon})\|_{W^{-1,\infty}(\Omega)} \le$$

$$|\Omega| \cdot \|f(x, \frac{x}{\varepsilon+\delta}, \frac{x}{\varepsilon}) - f(x, \frac{x}{\varepsilon} - cx, \frac{x}{\varepsilon})\|_{L^\infty(\Omega)} \leq$$

$$Const \cdot \left| \frac{x}{\varepsilon+\delta} - \left(\frac{x}{\varepsilon} - cx\right) \right| \leq Const \cdot \left| c - \frac{\delta}{\varepsilon^2 + \varepsilon\delta} \right| .$$

Finally, one can easily verify that the upper bound on the right of (2.13) may be written as

$$\|f(x, \frac{x}{\varepsilon+\delta}, \frac{x}{\varepsilon}) - f(x, \frac{x}{\varepsilon} - cx, \frac{x}{\varepsilon})\|_{W^{-1,\infty}(\Omega)} \leq Const \cdot \left(\varepsilon + \left| \frac{\delta}{\varepsilon^2} - c \right| \right) . \tag{2.14}$$

In order to upper bound the second term on the right hand side of (2.12), we define $g(x, y_1) = f(x, y_1 - cx, y_1)$ and $\bar{g}(x) = \int_{T^1} g(x, y_1) dy_1$. Clearly, $g \in BV_x(\Omega \times T^1)$ and $\bar{g}(x) = \bar{f}(x)$. Hence, by Lemma 1.1,

$$\|f(x, \frac{x}{\varepsilon} - cx, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} = \|g(x, \frac{x}{\varepsilon}) - \bar{g}(x)\|_{W^{-1,\infty}(\Omega)} \leq Const \cdot \varepsilon . \tag{2.15}$$

Error estimate (2.10) now follows from (2.12), (2.14) and (2.15). □

LEMMA 2.3. *Assume that $\frac{\varepsilon^2}{\delta} \to 0$ and that $f \in Lip(y_2)$ or $f \in Lip(x) \cap BV(y_2)$. Then*

$$\|f(x, \frac{x}{\varepsilon+\delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq Const \cdot \left(\frac{|\delta|}{\varepsilon} + \frac{\varepsilon^2}{|\delta|} \right) , \tag{2.16}$$

where

$$\bar{f}(x) = \int_{T^2} f(x, y) dy . \tag{2.17}$$

Proof. We consider the function $g(x, y_1, y_2) = f(x, y_1, y_1 + y_2)$, which is 1-periodic with respect to y_1, y_2. Clearly, g is as smooth as f with respect to x and y_2 and, therefore, satisfies the assumptions of Theorem 2.1. With this, we rewrite our function as follows:

$$f(x, \frac{x}{\varepsilon+\delta}, \frac{x}{\varepsilon}) = g(x, \frac{x}{\eta_1}, \frac{x}{\eta_2}) \quad \text{where} \quad \eta_1 = \varepsilon + \delta \; , \; \eta_2 = \frac{\varepsilon^2 + \varepsilon\delta}{\delta} . \tag{2.18}$$

Since $\varepsilon \gg |\delta| \gg \varepsilon^2$, we have that

$$\eta_1 \sim \varepsilon \to 0 \quad \text{and} \quad \eta_2 \sim \frac{\varepsilon^2}{\delta} \to 0 . \tag{2.19}$$

Moreover,

$$\frac{\eta_1}{\eta_2} \sim \frac{\delta}{\varepsilon} \to 0 . \tag{2.20}$$

Hence, in light of (2.19)–(2.20), we may apply Theorem 2.1 to g and conclude that

$$\|g(x,\frac{x}{\eta_1},\frac{x}{\eta_2})-\bar{g}(x)\|_{W^{-1,\infty}(\Omega)}\leq Const\cdot\left(\frac{|\delta|}{\varepsilon}+\frac{\varepsilon^2}{|\delta|}\right)\ , \tag{2.21}$$

where

$$\bar{g}(x)=\int_0^1\int_0^1 g(x,y_1,y_2)dy_1dy_2\ . \tag{2.22}$$

Using the definition of g and a change of variables in (2.22), we obtain that

$$\bar{g}(x)=\iint_D f(x,y_1,y_2)dy_1dy_2\ , \tag{2.23}$$

where D is the parallelogram with the vertices (0,0), (0,1), (1,1) and (1,2). Finally, using the 1-periodicity of f with respect to y_2, we get that

$$\bar{g}(x)=\bar{f}(x)=\int_{T^2} f(x,y)dy \tag{2.24}$$

and (2.16) follows from (2.18), (2.21) and (2.24). □

Remark. In Lemma 2.2 we assumed that $f\in Lip(y_1)$. In case this assumption does not hold but $f\in Lip(y_2)$, we may define a new scale, $\tilde{\varepsilon}:=\varepsilon+\delta$, with respect to which $f(x,\frac{x}{\varepsilon+\delta},\frac{x}{\varepsilon})=f(x,\frac{x}{\tilde{\varepsilon}},\frac{x}{\tilde{\varepsilon}-\delta})$ and proceed as before. Also in Lemma 2.3, where we assumed smoothness with respect to y_2, we may interchange the roles of y_1 and y_2 by applying the above rescaling.

The same principle holds for all our statements henceforth where the smoothness assumptions with respect to y_1 and y_2 are different: when needed, one may always apply an appropriate rescaling in order to interchange between those two variables.

Finally, we deal with the general case of a nonzero rational limit:

THEOREM 2.4. *Assume that* $\frac{\varepsilon_1}{\varepsilon_2}\to\frac{m}{n}$ *where* $m,n\in\mathbb{Z}^*$ *and* $r=\frac{\varepsilon_1}{\varepsilon_2}-\frac{m}{n}$.
(1) If $\frac{r}{\varepsilon_2}\to c$ *then*

$$\|f_\varepsilon(x)-\bar{f}(x)\|_{W^{-1,\infty}(\Omega)}\leq Const\cdot\left(\varepsilon_2+\left|\frac{r}{\varepsilon_2}-c\right|\right)\ , \tag{2.25}$$

where

$$\bar{f}(x)=\int_{T^1} f(x,ny_1-\frac{n^2cx}{m^2},my_1)dy_1\ , \tag{2.26}$$

provided that $f\in Lip(y_1)$.
(2) If $\frac{\varepsilon_2}{r}\to 0$ *then*

$$\|f_\varepsilon(x)-\bar{f}(x)\|_{W^{-1,\infty}(\Omega)}\leq Const\cdot\left(|r|+\frac{\varepsilon_2}{|r|}\right)\ , \tag{2.27}$$

where

$$\bar{f}(x) = \int_{T^2} f(x,y)dy \ , \tag{2.28}$$

provided that $f \in Lip(y_2)$ or $f \in Lip(x) \cap BV(y_2)$.

Proof. Introducing the notations $\tilde{\varepsilon} := m\varepsilon_2$ and $\tilde{\delta} := nr\varepsilon_2$, we get that $n\varepsilon_1 = \tilde{\varepsilon} + \tilde{\delta}$ and

$$f_\varepsilon(x) = f(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}) = f(x, \frac{nx}{n\varepsilon_1}, \frac{mx}{m\varepsilon_2}) = g(x, \frac{x}{\tilde{\varepsilon}+\tilde{\delta}}, \frac{x}{\tilde{\varepsilon}}) \ , \tag{2.29}$$

where $g(x, y_1, y_2) = f(x, ny_1, my_2)$. Since $\frac{\tilde{\delta}}{\tilde{\varepsilon}} = \frac{n}{m} \cdot r \to 0$, we may apply Lemmas 2.2 and 2.3 in order to obtain the $W^{-1,\infty}$-limit of $g(x, \frac{x}{\tilde{\varepsilon}+\tilde{\delta}}, \frac{x}{\tilde{\varepsilon}})$ and, consequently, that of $f_\varepsilon(x)$.

In the first case, $\frac{r}{\varepsilon_2} \to c$, we have that

$$\frac{\tilde{\delta}}{\tilde{\varepsilon}^2} = \frac{nr}{m^2\varepsilon_2} \to \frac{nc}{m^2} \ .$$

Hence, since the assumption on f implies that also $g \in Lip(y_1)$, we conclude by Lemma 2.2 that

$$\|g(x, \frac{x}{\tilde{\varepsilon}+\tilde{\delta}}, \frac{x}{\tilde{\varepsilon}}) - \bar{g}(x)\|_{W^{-1,\infty}(\Omega)} \le Const \cdot \left(\tilde{\varepsilon} + \left|\frac{\tilde{\delta}}{\tilde{\varepsilon}^2} - \frac{nc}{m^2}\right|\right) \ , \tag{2.30}$$

where

$$\bar{g}(x) = \int_{T^1} g(x, y_1 - \frac{nc}{m^2}x, y_1)dy_1 \ . \tag{2.31}$$

As the definition of g implies that

$$\bar{g}(x) = \int_{T^1} f(x, ny_1 - \frac{n^2c}{m^2}x, my_1)dy_1 \ , \tag{2.32}$$

(2.25)–(2.26) follow from (2.30) and (2.32).

In the second case we have that $\frac{\tilde{\varepsilon}^2}{\tilde{\delta}} \to 0$. We, therefore, apply Lemma 2.3 to g and conclude that

$$\|g(x, \frac{x}{\tilde{\varepsilon}+\tilde{\delta}}, \frac{x}{\tilde{\varepsilon}}) - \bar{g}(x)\|_{W^{-1,\infty}(\Omega)} \le Const \cdot \left(\frac{|\tilde{\delta}|}{\tilde{\varepsilon}} + \frac{\tilde{\varepsilon}^2}{|\tilde{\delta}|}\right) \ , \tag{2.33}$$

where

$$\bar{g}(x) = \int_{T^2} g(x,y)dy \ . \tag{2.34}$$

Since the 1-periodicity of f implies that $\int_{T^2} g(x,y)dy = \int_{T^2} f(x,y)dy$, we arrive at (2.27)–(2.28). □

Remark. Consider the first case where $\frac{r}{\varepsilon_2} \to c$, $r = \frac{\varepsilon_1}{\varepsilon_2} - \frac{m}{n}$. It may be verified that in this case $\frac{r'}{\varepsilon_1} \to -\frac{n^3 c}{m^3}$ where $r' = \frac{\varepsilon_2}{\varepsilon_1} - \frac{n}{m}$ (consult Proposition 4.2 in §4). Hence, the weak limit $\bar{f}(x)$ in (2.26) takes also the alternative form

$$\bar{f}(x) = \int_{T^1} f(x, ny_2, my_2 + \frac{ncx}{m}) dy_2 \ . \tag{2.35}$$

Indeed, replacing the integration variable in (2.26) with $y_2 = y_1 - \frac{ncx}{m^2}$, we see that the two forms of the weak limit in this case, (2.26) and (2.35), agree.

Example. Consider the basic functions which span $BV_x(\Omega \times T^2)$,

$$E_{m,n}(y_1, y_2) = e^{2\pi i(my_1 + ny_2)} \ , \quad m, n \in \mathbb{Z} \ , \ y_1, y_2 \in T^1 \ . \tag{2.36}$$

According to Theorem 2.4, if $|\delta| \ll \varepsilon \downarrow 0$, the following hold on any bounded interval Ω:

(1) If $\frac{\delta}{\varepsilon^2} \to c$,

$$\|E_{m,n}(\frac{x}{\varepsilon + \delta}, \frac{x}{\varepsilon}) - \bar{E}^1_{m,n}(cx)\|_{W^{-1,\infty}(\Omega)} \to 0 \ , \tag{2.37}$$

where

$$\bar{E}^1_{m,n}(cx) = \int_{T^1} E_{m,n}(y_1 - cx, y_1) dy_1 = \begin{cases} e^{2\pi i ncx} & \text{if } \ m + n = 0 \\ 0 & \text{otherwise} \end{cases} \ ; \tag{2.38}$$

(2) If $\frac{\varepsilon^2}{\delta} \to 0$,

$$\|E_{m,n}(\frac{x}{\varepsilon + \delta}, \frac{x}{\varepsilon}) - \bar{E}^2_{m,n}\|_{W^{-1,\infty}(\Omega)} \to 0 \ , \tag{2.39}$$

where

$$\bar{E}^2_{m,n} = \int\int_{T^2} E_{m,n}(y_1, y_2) dy_1 dy_2 = \begin{cases} 1 & \text{if } \ m = n = 0 \\ 0 & \text{otherwise} \end{cases} \ . \tag{2.40}$$

Indeed, denoting $E(x) = e^{2\pi i x}$,

$$E_{m,n}(\frac{x}{\varepsilon + \delta}, \frac{x}{\varepsilon}) = E(\frac{x}{\eta}) \quad \text{where} \quad \eta = \frac{\varepsilon^2 + \varepsilon\delta}{(m+n)\varepsilon + n\delta} \ .$$

When $\frac{\varepsilon^2}{\delta} \to 0$, $\eta \to 0$ for all $(m, n) \neq (0, 0)$. Therefore, by Lemma 1.1, $\|E(\frac{x}{\eta})\|_{W^{-1,\infty}} \leq Const \cdot |\eta| \to 0$. Since the case $m = n = 0$ is straightforward ($E_{0,0} \equiv 1$), we get the weak limit in (2.39)–(2.40).

When $\frac{\delta}{\varepsilon^2} \to c$, we consider two cases: if $m + n \neq 0$ then $\eta \sim \varepsilon \to 0$ and, therefore, as before, the weak limit is zero. However, if $m + n = 0$, $E_{m,n}(\frac{x}{\varepsilon+\delta}, \frac{x}{\varepsilon}) = E(nx \cdot \frac{\delta}{\varepsilon^2 + \varepsilon\delta})$. Since $\frac{\delta}{\varepsilon^2 + \varepsilon\delta} \to c$, we obtain, owing to the Lipschitz continuity of E, the strong convergence in L^∞ $E_{m,n}(\frac{x}{\varepsilon+\delta}, \frac{x}{\varepsilon}) \to E(ncx) = e^{2\pi i ncx}$, in agreement with (2.37)–(2.38). ♠

The integral in (2.26), or (2.35), is taken along a closed spiral curve in T^2. The larger are m and n – the longer is the curve. Let α be an irrational number and let $\{\frac{m_k}{n_k}\}_{k \in \mathbb{N}}$ be a sequence of rational numbers which converges to α as $k \to \infty$. Let $\mathcal{L}_k = \{(n_k y_1, m_k y_1) : y_1 \in T^1\}$ be a typical curve in T^2 associated with $\frac{m_k}{n_k}$ by (2.26). Then, since $m_k, n_k \to \infty$, the length of $\mathcal{L}_k$ tends to infinity as $\frac{m_k}{n_k} \to \alpha$ and the "limit-curve", so to speak, covers the entire torus T^2. Hence, it is natural to expect that when $\frac{\varepsilon_1}{\varepsilon_2} \to \alpha$, α irrational, the corresponding weak limit of $f_\varepsilon(x)$ will take the form of a two-dimensional integral over T^2, like in (2.28), rather than a line integral as in (2.26). This is the subject of our discussion in the following subsection.

2.3. Case 3: A nonzero irrational limit. Here, $\frac{\varepsilon_1}{\varepsilon_2} \to \alpha \in \mathbb{R} \setminus \mathbb{Q}$. We start with the following straightforward lemmas:

LEMMA 2.5. *Let $E_{m,n}(y_1, y_2)$ be as in (2.36), $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $|\delta| \ll \varepsilon \downarrow 0$. Then there exists a constant $C > 0$, such that for every fixed $(m,n) \neq (0,0)$,*

$$(2.41) \qquad \|E_{m,n}(\frac{x}{\alpha\varepsilon + \delta}, \frac{x}{\varepsilon})\|_{W^{-1,\infty}(\Omega)} \leq C \cdot |\eta| \quad \textit{where} \quad \eta = \frac{\alpha\varepsilon^2 + \varepsilon\delta}{(m + n\alpha)\varepsilon + n\delta} \to 0 \ .$$

Proof. Denoting $E(x) = e^{2\pi i x}$, $E_{m,n}(\frac{x}{\alpha\varepsilon+\delta}, \frac{x}{\varepsilon}) = E(\frac{x}{\eta})$, with η as in (2.41). Since the irrationality of α implies that $m + n\alpha \neq 0$, we conclude that $\eta \sim \varepsilon \to 0$. Hence, applying Lemma 1.1 to the real and imaginary parts of $E(\frac{x}{\eta})$, both of which have a zero average, we obtain the weak convergence rate in (2.41). □

LEMMA 2.6. *Let $g \in BV(\Omega)$ and $f \in W^{-1,\infty}(\Omega)$. Then $g \cdot f \in W^{-1,\infty}(\Omega)$ and*

$$(2.42) \qquad \|gf\|_{W^{-1,\infty}(\Omega)} \leq (\|g\|_{L^\infty(\Omega)} + \|g\|_{BV(\Omega)}) \cdot \|f\|_{W^{-1,\infty}(\Omega)} \ .$$

Proof. Let $F(x)$ denote the primitive of $f(x)$, $F(x) = \int_a^x f(\xi) d\xi$. Then

$$(2.43) \qquad \int_a^x gf = g(x)F(x) - \int_a^x g'F \ .$$

Taking the supremum in absolute value over Ω on both sides of (2.43) we arrive at (2.42). □

We may now proceed to prove the main theorem of this subsection:

THEOREM 2.7. *Assume that $\frac{\varepsilon_1}{\varepsilon_2} \to \alpha \in \mathbb{R} \setminus \mathbb{Q}$ and that $f \in L^\infty(\Omega, H^s(T^2))$, $s > 1$. Then*

$$(2.44) \qquad \|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \to 0 \ ,$$

where

$$(2.45) \qquad \bar{f}(x) = \int_{T^2} f(x,y) dy \ .$$

Proof. Using the notations (2.1)–(2.3), we shall show that for any $\mu > 0$

$$\|f(x, \frac{x}{\alpha\varepsilon + \delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq \mu \tag{2.46}$$

for sufficiently small ε.

Let f_N denote the Nth order Fourier approximation of f,

$$f_N(x, y) = f_N(x, y_1, y_2) = \sum_{-N \leq m,n \leq N} \hat{f}_{m,n}(x) E_{m,n}(y_1, y_2) \ , \tag{2.47}$$

where $\hat{f}_{m,n}(x)$ are the corresponding Fourier coefficients. Then, for any value of r, $1 < r < s$, it holds that

$$\|f(x, \cdot) - f_N(x, \cdot)\|_{H^r(T^2)} \leq Const \cdot \frac{\|f(x, \cdot)\|_{H^s(T^2)}}{N^{s-r}} \qquad \forall x \in \Omega \tag{2.48}$$

(consult [9]). Hence, since the L^∞-norm in $\mathbb{R}^2$ is dominated by the H^r-norm for $r > 1$, we conclude that

$$\|f - f_N\|_{L^\infty(\Omega \times T^2)} = \sup_{x \in \Omega} \|f(x, \cdot) - f_N(x, \cdot)\|_{L^\infty(T^2)} \leq Const \cdot \frac{\|f\|_{L^\infty(\Omega, H^s(T^2))}}{N^{s-r}} \underset{N \to \infty}{\longrightarrow} 0 \ . \tag{2.49}$$

We may now proceed to prove (2.46). By (2.49), there exists N such that

$$\|f(x, y) - f_N(x, y)\|_{L^\infty(\Omega \times T^2)} \leq \frac{\mu}{2|\Omega|} \ . \tag{2.50}$$

Therefore, for this value of N,

$$\|f(x, \frac{x}{\alpha\varepsilon + \delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq \frac{\mu}{2} + \|f_N(x, \frac{x}{\alpha\varepsilon + \delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \ . \tag{2.51}$$

Since $\bar{f}(x) = \hat{f}_{0,0}(x)$, we may upper bound the second term on the right of (2.51), using Lemma 2.6, as follows:

$$\|f_N(x, \frac{x}{\alpha\varepsilon + \delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq \sum |\hat{f}_{m,n}(x)| \cdot \|E_{m,n}(\frac{x}{\alpha\varepsilon + \delta}, \frac{x}{\varepsilon})\|_{W^{-1,\infty}(\Omega)} \ , \tag{2.52}$$

where the sum is taken over $-N \leq m, n \leq N$, $(m, n) \neq (0, 0)$ and $|\cdot| := \|\cdot\|_{L^\infty(\Omega)} + \|\cdot\|_{BV(\Omega)}$. Since, by Lemma 2.5, each of the terms on the right of (2.52) tends to zero when $\varepsilon \downarrow 0$,

$$\|f_N(x, \frac{x}{\alpha\varepsilon + \delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq \frac{\mu}{2} \ , \tag{2.53}$$

for sufficiently small ε. Therefore, (2.46) follows from (2.51) and (2.53). □

<u>*Remarks.*</u>

1. The assumption $f \in L^\infty(\Omega, H^s(T^2))$, $s > 1$, could have been replaced by the weaker assumption that f has a uniformly convergent Fourier series; namely, $\|f(x, y) - f_N(x, y)\|_{L^\infty(\Omega \times T^2)} \to 0$ as $N \to \infty$.
2. It is possible to quantify the convergence rate in (2.44), by assuming a rate of decay of $|\hat{f}_{m,n}(x)|$ and a lower bound for $C_N := \min |m + n\alpha|$ where the minimum is taken over all $-N \leq m, n \leq N$, $(m, n) \neq (0, 0)$.

3. The case of a nonzero irrational limit revisited. In this section we deal once again with the case where the ratio $\frac{\varepsilon_1}{\varepsilon_2}$ tends to a nonzero irrational limit. In §2.3 we have obtained the weak limit of $f_\varepsilon(x)$, without a convergence rate estimate. Here, we attack the same problem with entirely different techniques, which – apart from being interesting for their own sake – render convergence rate estimates in some subcases.

We first handle the relatively easy case where the ratio between the scales is kept fixed, $\frac{\varepsilon_1}{\varepsilon_2} = \alpha \in \mathbb{R} \setminus \mathbb{Q}$, and then we proceed to the closely related case where $r = \frac{\varepsilon_1}{\varepsilon_2} - \alpha$ is of order of magnitude no larger than $\mathcal{O}(\varepsilon_1, \varepsilon_2)$, i.e., $\frac{r}{\varepsilon_2} \to c \in \mathbb{R}$. We establish the weak convergence of $f_\varepsilon(x)$ to the average of $f(x, \cdot)$ on T^2, by classical arguments of ergodic theory; moreover, for some values of α we obtain convergence rate estimates as well (see Theorem 3.2 below).

The case where $|r| \gg \mathcal{O}(\varepsilon_1, \varepsilon_2)$ is much more intricate. Here, we are able to obtain convergence rate estimates only for a subsequence of $f_\varepsilon(x)$ (Theorem 3.3) or for the entire sequence, whenever α is an algebraic number (Theorem 3.4).

The analysis presented here involves some terminology and results from number theory and the theory of quasi-Monte Carlo integration methods. The reader is referred to §7 where a brief review of these terms and results is provided.

We start with the simplest case where the ratio between the two scales remains fixed. The following lemma is a modification of [5, Lemma 2.2]:

LEMMA 3.1. *Let $f \in BV_x(\Omega \times T^2)$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Assume that f is differentiable with respect to x and is in class $BV(y_2)$. Then*

$$\|f(x, \frac{x}{\alpha\varepsilon}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq C(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0 , \tag{3.1}$$

where

$$\bar{f}(x) = \int_{T^2} f(x, y) dy . \tag{3.2}$$

Furthermore:

(1) If α is proper, then $C(\varepsilon) = \mathcal{O}(\varepsilon |\log \varepsilon|)$;

(2) If α is of type η and $f(x, y_1, \cdot)$ is in class $\mathcal{E}^k$ for $k > \eta$, then $C(\varepsilon) = \mathcal{O}(\varepsilon)$.

Proof. By normalizing f, we may assume that $\bar{f}(x) \equiv 0$. For the sake of conveniency, we shift the x-domain, Ω, so that $\Omega = [0, b]$. We therefore have to show that for any $x_0 \in [0, b]$,

$$\left| \int_0^{x_0} f(x, \frac{x}{\alpha\varepsilon}, \frac{x}{\varepsilon}) dx \right| \underset{\varepsilon \to 0}{\longrightarrow} 0 . \tag{3.3}$$

We first prove the assertion for functions f which do not depend on their first variable, $f(x, y_1, y_2) = f(y_1, y_2)$. By a change of variable in (3.3) we get that

$$\left| \int_0^{x_0} f(\frac{x}{\alpha\varepsilon}, \frac{x}{\varepsilon}) dx \right| \leq b \cdot \left| \frac{1}{M} \int_0^M f(y_1, \alpha y_1) dy_1 \right| , \tag{3.4}$$

where $M = \frac{x_0}{\alpha\varepsilon}$. We need to show that the right hand side of (3.4) tends to zero as $M \to \infty$. It suffices to show that only for integer values of M. Using the 1-periodicity of f with respect to y_1, we get that

$$\frac{1}{M}\int_0^M f(y_1, \alpha y_1)dy_1 = \frac{1}{M}\sum_{n=0}^{M-1}\int_0^1 f(y_1, \alpha y_1 + \alpha n)dy_1 = \frac{1}{M}\sum_{n=0}^{M-1} F(n\alpha) \ , \tag{3.5}$$

where $F(z) := \int_0^1 f(y_1, \alpha y_1 + z)dy_1$. Since F is 1-periodic, we may apply the ergodic theorem of equipartition modulo 1, (7.4), to conclude that

$$\frac{1}{M}\sum_{n=0}^{M-1} F(n\alpha) \underset{M\to\infty}{\longrightarrow} \int_0^1 F(z)dz = \int_0^1\int_0^1 f(y_1, y_2)dy_1 dy_2 = 0 \ . \tag{3.6}$$

Therefore, (3.3) follows in this case from (3.4)–(3.6). The convergence rate estimates (1) and (2) are consequences of Propositions 7.2 and 7.5 in §7.

Next, we deal with the general case where f depends on its first variable, $f = f(x, y_1, y_2)$. Using the identity

$$f(x, \frac{x}{\alpha\varepsilon}, \frac{x}{\varepsilon}) = \frac{d}{dx}\int_0^x f(x, \frac{s}{\alpha\varepsilon}, \frac{s}{\varepsilon})ds - \int_0^x f_x(x, \frac{s}{\alpha\varepsilon}, \frac{s}{\varepsilon})ds \ ,$$

we get that

$$\int_0^{x_0} f(x, \frac{x}{\alpha\varepsilon}, \frac{x}{\varepsilon})dx = \int_0^{x_0} f(x_0, \frac{s}{\alpha\varepsilon}, \frac{s}{\varepsilon})ds - \int_0^{x_0}\int_0^x f_x(x, \frac{s}{\alpha\varepsilon}, \frac{s}{\varepsilon})ds dx \ . \tag{3.7}$$

Since the function $f_x(x, y_1, y_2)$ is 1-periodic with respect to y_1, y_2 and has a zero average, $\int_{T^2} f_x(x, y)dy = \frac{d}{dx}\bar{f}(x) = 0$, we may apply our previous arguments to the integrals with respect to ds in (3.7) and thus conclude the proof. □

THEOREM 3.2. *Assume that $\frac{\varepsilon_1}{\varepsilon_2} \to \alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $r = \frac{\varepsilon_1}{\varepsilon_2} - \alpha$. Let f be differentiable with respect to x and in class $Lip(y_1) \cap BV(y_2)$. Then if*

$$\frac{r}{\varepsilon_2} \to c \ , \tag{3.8}$$

it holds that

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le C(\varepsilon_1, \varepsilon_2) \underset{\varepsilon_1,\varepsilon_2\to 0}{\longrightarrow} 0 \ , \tag{3.9}$$

where

$$\bar{f}(x) = \int_{T^2} f(x, y)dy \ . \tag{3.10}$$

Furthermore:

(1) If α is proper, then $C(\varepsilon_1, \varepsilon_2) = Const \cdot \left(\varepsilon_2|\log\varepsilon_2|) + \left|\frac{r}{\varepsilon_2} - c\right|\right)$;

(2) If α is of type η and $f(x, y_1, \cdot)$ is in class $\mathcal{E}^k$, $k > \eta$, then $C(\varepsilon_1, \varepsilon_2) = Const \cdot \left(\varepsilon_2 + \left|\frac{r}{\varepsilon_2} - c\right|\right)$.

Proof. The proof of this theorem is similar to the one of Lemma 2.2 and, therefore, is outlined shortly:

$$(3.11) \qquad \|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq$$

$$\|f(x, \frac{x}{\alpha\varepsilon + \delta}, \frac{x}{\varepsilon}) - f(x, \frac{x}{\alpha\varepsilon} - \frac{cx}{\alpha^2}, \frac{x}{\varepsilon})\|_{W^{-1,\infty}(\Omega)} + \|f(x, \frac{x}{\alpha\varepsilon} - \frac{cx}{\alpha^2}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \cdot$$

Since

$$\left|\frac{1}{\alpha\varepsilon + \delta} - \left(\frac{1}{\alpha\varepsilon} - \frac{c}{\alpha^2}\right)\right| = \left|\frac{1}{\alpha^2}\left(c - \frac{\delta}{\varepsilon^2}\right) + \frac{\delta^2}{\alpha^3\varepsilon^3 + \delta\alpha^2\varepsilon^2}\right| \leq \mathcal{O}\left(\left|c - \frac{r}{\varepsilon_2}\right| + \varepsilon_2\right) ,$$

we conclude, using the Lipschitz continuity of $f(x, \cdot, y_2)$, that

$$(3.12) \ \|f(x, \frac{x}{\alpha\varepsilon + \delta}, \frac{x}{\varepsilon}) - f(x, \frac{x}{\alpha\varepsilon} - \frac{cx}{\alpha^2}, \frac{x}{\varepsilon})\|_{W^{-1,\infty}(\Omega)} \leq Const \cdot \left(\left|c - \frac{r}{\varepsilon_2}\right| + \varepsilon_2\right) \underset{\varepsilon_1, \varepsilon_2 \to 0}{\longrightarrow} 0 .$$

Applying Lemma 3.1 to the function $g(x, y_1, y_2) = f(x, y_1 - \frac{cx}{\alpha^2}, y_2)$ (which, like f, is in class $BV(y_2)$) and observing that $\int_{T^2} g(x, y)dy = \int_{T^2} f(x, y)dy$, we conclude that

$$(3.13) \qquad \|f(x, \frac{x}{\alpha\varepsilon} - \frac{cx}{\alpha^2}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \underset{\varepsilon_1, \varepsilon_2 \to 0}{\longrightarrow} 0 .$$

This concludes the proof of (3.9)–(3.10). As for the convergence rate estimates (1) and (2), they follow from the analogous estimates in Lemma 3.1 and from (3.12). □

Now, we are concerned with the case where $|r| \gg \varepsilon_2$. The approach here is different from the one employed above for the case where $|r| \sim \varepsilon_2$ or $|r| \ll \varepsilon_2$. Our two convergence rate results in this case are as follows:

THEOREM 3.3. *Let $\varepsilon_i = \varepsilon_i(\nu) > 0$, $i = 1, 2$, be two vanishing scales as $\nu \to 0^+$. Assume that $r := \frac{\varepsilon_1}{\varepsilon_2} - \alpha \to 0$, where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and that $f \in Lip(y_1)$. Then if*

$$(3.14) \qquad \frac{\varepsilon_2}{r} \to 0 ,$$

there exists a subsequence of scales, $\varepsilon_i(\nu_k)$, $\nu_k \to 0^+$, $i = 1, 2$, such that the corresponding sequence of functions, $f_{\nu_k}(x)$, satisfies:

$$(3.15) \ \ \|f_{\nu_k}(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \leq Const \cdot \left(\frac{\varepsilon_2}{|r|} + \sqrt{|r|}\right) \quad , \quad \bar{f}(x) = \int_{T^2} f(x, y)dy .$$

Proof. By Proposition 7.7, there exists an infinite sequence of pairs of nonzero integers, (m_k, n_k), such that

$$|n_k\alpha - m_k| \le \frac{1}{n_k} \ . \tag{3.16}$$

For a given pair in this sequence, (m_k, n_k), we define ν_k to be the scaling parameter for which

$$\frac{|\delta(\nu_k)|}{\varepsilon(\nu_k)} = |r(\nu_k)| = \frac{2}{n_k^2} \ , \tag{3.17}$$

(δ, ε and r are as in (2.1)–(2.3)); if the convergence of $|r| = |r(\nu)|$ to zero is not monotonic in ν, we choose ν_k to be the first value of ν for which (3.17) holds. For the sake of conveniency, we use henceforth the notations $m, n, \varepsilon, \delta$ instead of $m_k, n_k, \varepsilon(\nu_k), \delta(\nu_k)$. We shall show below that for these values of ε and δ, it holds that

$$\|f(x, \frac{x}{\alpha\varepsilon+\delta}, \frac{x}{\varepsilon}) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le Const \cdot \left(\frac{\varepsilon^2}{|\delta|} + \sqrt{\frac{|\delta|}{\varepsilon}}\right) \tag{3.18}$$

and, thus, prove (3.15).

We introduce the function $g(x, y_1, y_2) = f(x, y_2 + ny_1, my_1)$. Clearly, $g \in BV_x(\Omega \times T^2)$, and it is in $Lip(y_2)$ with a Lipschitz constant that equals the one of $f(x, \cdot, y_2)$. Moreover, one may easily verify that $\bar{g}(x) = \int_{T^2} g(x,y)dy = \bar{f}(x)$. With this, we have that

$$f(x, \frac{x}{\alpha\varepsilon+\delta}, \frac{x}{\varepsilon}) = g(x, \frac{x}{\eta_1}, \frac{x}{\eta_2}) \tag{3.19}$$

where

$$\eta_1 = m\varepsilon \quad \text{and} \quad \eta_2 = \frac{m\varepsilon(\alpha\varepsilon+\delta)}{(m-n\alpha)\varepsilon - n\delta} \ . \tag{3.20}$$

By our choice of m, n, ε and δ, we conclude, using (3.16)–(3.17), that

$$|(m-n\alpha)\varepsilon - n\delta| \le \frac{\varepsilon}{n} + n|\delta| = \frac{3\varepsilon}{n} \ ;$$

$$|(m-n\alpha)\varepsilon - n\delta| = n|\delta| - |m-n\alpha|\varepsilon \ge n|\delta| - \frac{\varepsilon}{n} = \frac{\varepsilon}{n} \ .$$

The above two estimates, together with (3.20) and (3.17), imply that

$$\left|\frac{\eta_1}{\eta_2}\right| = \left|\frac{(m-n\alpha)\varepsilon - n\delta}{\alpha\varepsilon+\delta}\right| \le \frac{Const}{n} = Const \cdot \sqrt{\frac{|\delta|}{\varepsilon}} \ ; \tag{3.21}$$

$$|\eta_2| = m\varepsilon \cdot \left|\frac{\alpha\varepsilon+\delta}{(m-n\alpha)\varepsilon - n\delta}\right| \le Const \cdot n^2\varepsilon = Const \cdot \frac{\varepsilon^2}{|\delta|} \ . \tag{3.22}$$

Hence, by Theorem 2.1,

$$\|f(x,\frac{x}{\alpha\varepsilon+\delta},\frac{x}{\varepsilon})-\bar{f}(x)\|_{W^{-1,\infty}(\Omega)}=\|g(x,\frac{x}{\eta_1},\frac{x}{\eta_2})-\bar{g}(x)\|_{W^{-1,\infty}(\Omega)}\le$$

$$Const\cdot\left(|\eta_2|+\left|\frac{\eta_1}{\eta_2}\right|\right)\le Const\cdot\left(\frac{\varepsilon^2}{|\delta|}+\sqrt{\frac{|\delta|}{\varepsilon}}\right)=Const\cdot\left(\frac{\varepsilon_2}{|r|}+\sqrt{|r|}\right)\ .$$

□

THEOREM 3.4. *Assume that* $r:=\frac{\varepsilon_1}{\varepsilon_2}-\alpha\to 0$, *where* α *is an irrational algebraic number, and that*

$$\frac{\varepsilon_2}{|r|}\le Const\cdot\varepsilon_2^p\qquad\text{for some } p>0\ . \tag{3.23}$$

Then if $f\in Lip(y_1)$, *the following error estimate holds for every positive number* $\nu>0$:

$$\|f_\varepsilon(x)-\bar{f}(x)\|_{W^{-1,\infty}(\Omega)}\le Const_\nu\cdot|r|^{-\nu}\cdot\left(\frac{\varepsilon_2}{|r|}+\sqrt{|r|}\right)\quad,\quad \bar{f}(x)=\int_{T^2}f(x,y)dy\ . \tag{3.24}$$

Remark. Note that assumption (3.23) guarantees that the error bound in (3.24) indeed tends to zero when $\varepsilon_1,\varepsilon_2\downarrow 0$, for sufficiently small values of ν. This error bound agrees with the one that we got for a subsequence in Theorem 3.3, modulo a spurious factor of $|r|^{-\nu}$.

Proof. Letting $g(x,y_1,y_2)$ and η_1,η_2 be the same as in the proof of Theorem 3.3, we get that

$$\|f_\varepsilon(x)-\bar{f}(x)\|_{W^{-1,\infty}(\Omega)}=\|g(x,\frac{x}{\eta_1},\frac{x}{\eta_2})-\bar{g}(x)\|_{W^{-1,\infty}(\Omega)}\le Const\cdot\left(|\eta_2|+\left|\frac{\eta_1}{\eta_2}\right|\right)\ . \tag{3.25}$$

We fix $0<q<1$. Then, as a straightforward consequence of Proposition 7.6, there exists a constant $c=c(\alpha,q)$ such that for all $\mu>0$

$$|n\alpha-m|\le\mu\quad\Longrightarrow\quad n\ge c\mu^{-q}\ . \tag{3.26}$$

Now, we set

$$\mu=\left(\frac{c}{2}\cdot\frac{|\delta|}{\varepsilon}\right)^{\frac{1}{1+q}}\ . \tag{3.27}$$

Hence, by [4, Theorem 36] and (3.26), there exists $\frac{m}{n}\in\mathbb{Q}$ such that

$$|n\alpha-m|\le\mu\quad\text{and}\quad c\mu^{-q}\le n\le\mu^{-1}\ . \tag{3.28}$$

Using (3.27) and (3.28), we get that

$$2\mu\varepsilon\le n|\delta|\le\frac{2}{c}\mu^q\varepsilon\quad\text{and}\quad |n\alpha-m|\varepsilon\le\mu\varepsilon\ . \tag{3.29}$$

Therefore, by (3.28)–(3.29), we conclude that η_1 and η_2, given in (3.20), satisfy:

$$(3.30)\qquad \left|\frac{\eta_1}{\eta_2}\right| = \left|\frac{(m-n\alpha)\varepsilon - n\delta}{\alpha\varepsilon+\delta}\right| \le$$

$$Const\cdot(\mu+\mu^q) \le Const\cdot\mu^q = Const\cdot\left(\frac{|\delta|}{\varepsilon}\right)^{\frac{1}{2}}\cdot\left(\frac{|\delta|}{\varepsilon}\right)^{\frac{q-1}{2(q+1)}} \quad ;$$

$$(3.31)\qquad |\eta_2| = m\varepsilon\cdot\left|\frac{\alpha\varepsilon+\delta}{(m-n\alpha)\varepsilon-n\delta}\right| \le Const\cdot\frac{\varepsilon}{\mu^2} = Const\cdot\frac{\varepsilon^2}{|\delta|}\cdot\left(\frac{|\delta|}{\varepsilon}\right)^{\frac{q-1}{q+1}} \quad .$$

Hence, by (3.25), (3.30) and (3.31), we arrive at the conclusion that for every $0<q<1$ there exists a constant, $Const_q$, for which the following error estimate holds:

$$(3.32)\qquad \|f_\varepsilon(x)-\bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le Const_q\cdot\left(\frac{\varepsilon^2}{|\delta|}+\sqrt{\frac{|\delta|}{\varepsilon}}\right)\cdot\left(\frac{|\delta|}{\varepsilon}\right)^{\frac{q-1}{q+1}} \quad .$$

Finally, let ν be a small positive number, $0<\nu\ll 1$. Then, by taking in (3.32) $q=\frac{1-\nu}{1+\nu}$ and recalling that $\frac{|\delta|}{\varepsilon}=|r|$ and $\frac{\varepsilon^2}{|\delta|}=\frac{\varepsilon_2}{|r|}$, we arrive at the desired error estimate (3.24). □

4. $W^{-1,\infty}$-Convergence Analysis with Multiple Scales. Here we study weak limits with respect to multiple number of scales and generalize the results of §2.

As a first step, we define an equivalence relation, $\sim$, on the set of scales, $\mathcal{S}=\{\varepsilon_i\}_{1\le i\le n}$. Later, we use this relation in order to reduce the problem of homogenization of f with respect to $\mathcal{S}$ to a problem of homogenization of another function with respect to the smaller set of scales, $\mathcal{S}/\sim$.

DEFINITION 4.1. *The two scales $\varepsilon_i(\varepsilon), \varepsilon_j(\varepsilon)\in\mathcal{S}$ are said to be equivalent, $\varepsilon_i\sim\varepsilon_j$, if there exist $\alpha\in\mathbb{Q}^*$ and $c\in\mathbb{R}$, such that*

$$(4.1)\qquad \frac{\frac{\varepsilon_i}{\varepsilon_j}-\alpha}{\varepsilon_j} \underset{\varepsilon\to 0^+}{\longrightarrow} c\ .$$

Remark. Theorem 2.4, which dealt with the case where $\lim\frac{\varepsilon_1}{\varepsilon_2}=\alpha\in\mathbb{Q}^*$, was separated into two subcases: when $\varepsilon_1\sim\varepsilon_2$, (2.25)–(2.26) hold, while otherwise (2.27)–(2.28) hold.

PROPOSITION 4.2. *The relation $\sim$ is an equivalence.*

Proof. The relation is clearly reflexive (with $\alpha=1$ and $c=0$). Next, we prove that it is symmetric. Let $\varepsilon_i,\varepsilon_j$ satisfy (4.1); then

$$(4.2)\qquad \frac{\varepsilon_i}{\varepsilon_j} = (c+r)\varepsilon_j+\alpha \quad \text{where } r \underset{\varepsilon\to 0^+}{\longrightarrow} 0\ .$$

Hence, using (4.2) and (4.1) we get that

$$\frac{\frac{\varepsilon_j}{\varepsilon_i}-\frac{1}{\alpha}}{\varepsilon_i}=\frac{\frac{\varepsilon_i}{\varepsilon_j}-\frac{1}{\alpha}\cdot\left(\frac{\varepsilon_i}{\varepsilon_j}\right)^2}{\varepsilon_j}\cdot\left(\frac{\varepsilon_j}{\varepsilon_i}\right)^3=\left[\frac{\frac{\varepsilon_i}{\varepsilon_j}-\alpha}{\varepsilon_j}-2(c+r)-\frac{(c+r)^2\varepsilon_j}{\alpha}\right]\cdot\left(\frac{\varepsilon_j}{\varepsilon_i}\right)^3\longrightarrow-\frac{c}{\alpha^3}\ .$$

This proves that $\varepsilon_j \sim \varepsilon_i$ and, thus, the symmetry of the relation. Finally, assume that

$$\frac{\frac{\varepsilon_i}{\varepsilon_j}-\alpha}{\varepsilon_j}\underset{\varepsilon\to0^+}{\longrightarrow}c \quad\text{and}\quad \frac{\frac{\varepsilon_j}{\varepsilon_k}-\beta}{\varepsilon_k}\underset{\varepsilon\to0^+}{\longrightarrow}d\ .$$

Then

$$\frac{\frac{\varepsilon_i}{\varepsilon_k}-\alpha\beta}{\varepsilon_k}=\frac{\frac{\varepsilon_i}{\varepsilon_j}-\alpha}{\varepsilon_j}\cdot\left(\frac{\varepsilon_j}{\varepsilon_k}\right)^2+\frac{\frac{\varepsilon_j}{\varepsilon_k}-\beta}{\varepsilon_k}\cdot\alpha\underset{\varepsilon\to0^+}{\longrightarrow}c\beta^2+d\alpha\ .$$

Hence, the relation is also transitive and, therefore, an equivalence. □

PROPOSITION 4.3. *Let $C=\{\varepsilon_1,\ldots,\varepsilon_k\}$, $k\geq 1$, be an equivalence class and*

$$\frac{\frac{\varepsilon_i}{\varepsilon_j}-\frac{n_j}{n_i}}{\varepsilon_j}\rightarrow c_{i,j}\qquad 1\leq i,j\leq k\ . \tag{4.3}$$

Let $f(x,y)\in BV_x(\Omega\times T^k)$ be $Lip(y_i)$ for all $1\leq i\leq k$ with the possible exception of $i=j$. Let $g(x,y_1)$ be defined as follows:

$$g(x,y_1):=f(x,z_1,\ldots,z_k)\quad\text{where}\quad z_i=n_iy_1-\frac{n_i^2c_{i,j}x}{n_j^2}\qquad 1\leq i\leq k\ . \tag{4.4}$$

Then

$$\|f(x,\frac{x}{\varepsilon_1},\ldots,\frac{x}{\varepsilon_k})-g(x,\frac{x}{n_j\varepsilon_j})\|_{L^\infty(\Omega)}\leq Const\cdot\sum_{i\neq j}\left(\varepsilon_i+\left|\frac{\frac{\varepsilon_i}{\varepsilon_j}-\frac{n_j}{n_i}}{\varepsilon_j}-c_{i,j}\right|\right)\ . \tag{4.5}$$

Proof. Since f is Lipschitz-continuous with respect to y_i for all $1\leq i\leq k$, $i\neq j$, all we need to do is to estimate the difference in the corresponding arguments in the two functions on the left hand side of (4.5). Using the definition of g, (4.4), the difference in the ith argument is

$$E_i=\frac{n_ix}{n_i\varepsilon_i}-\left(\frac{n_ix}{n_j\varepsilon_j}-\frac{n_i^2c_{i,j}x}{n_j^2}\right)\ .$$

Note that since $c_{j,j}=0$, we have that $E_j=0$. Simple algebraic manipulations yield that

$$E_i=\frac{n_i^2x}{n_j}\cdot\left(\frac{c_{i,j}}{n_j}-\frac{\frac{\varepsilon_i}{\varepsilon_j}-\frac{n_j}{n_i}}{n_i\varepsilon_i}\right)\ . \tag{4.6}$$

Next, we observe that since $n_i\varepsilon_i = n_j\varepsilon_j + \mathcal{O}(\varepsilon_i^2)$,

$$(4.7)\qquad \frac{\frac{\varepsilon_i}{\varepsilon_j} - \frac{n_j}{n_i}}{n_i\varepsilon_i} = \frac{\frac{\varepsilon_i}{\varepsilon_j} - \frac{n_j}{n_i}}{n_j\varepsilon_j} + \mathcal{O}(\varepsilon_i) \ .$$

Therefore, by (4.6)–(4.7), we conclude that

$$(4.8)\qquad |E_i| \leq Const \cdot \left(\varepsilon_i + \left|\frac{\frac{\varepsilon_i}{\varepsilon_j} - \frac{n_j}{n_i}}{\varepsilon_j} - c_{i,j}\right|\right) \ ,$$

which proves our assertion. □

Proposition 4.3 enables us to unify equivalent scales into one scale in the following manner: instead of studying the weak limit of $f(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_k})$, where $\varepsilon_i \sim \varepsilon_j$ for all $1 \leq i, j \leq k$, we may study the weak limit of $g(x, \frac{x}{n_j\varepsilon_j})$ which depends only on one vanishing scale, $n_j\varepsilon_j$.

We would like to point out that if f is Lipschitz-continuous with respect to all of its y-variables, we may choose each of the quantities $n_j\varepsilon_j$, $1 \leq j \leq k$ as a representative for the equivalence class. Each choice will induce a different 2-variable function, $g_j(x, \frac{x}{n_j\varepsilon_j})$. However, it may be shown, along the lines of the remark after Theorem 2.4, that the weak limit of $g_j(x, \frac{x}{n_j\varepsilon_j})$, $\int_{T^1} g_j(x, y_1)dy_1$, is independent of j.

Assume that the equivalence relation $\sim$ defines ℓ equivalence classes in the set of scales, $\mathcal{S} = \{\varepsilon_i\}_{1\leq i\leq n}$, i.e.,

$$\mathcal{S}/\sim = \{\mathcal{C}_m\}_{1\leq m\leq \ell} \quad , \quad 1 \leq \ell \leq n \ ,$$

where

$$\mathcal{C}_m = \{\varepsilon_{m,1}, ..., \varepsilon_{m,k_m}\} \quad \text{and} \quad \frac{\varepsilon_{m,i}}{\varepsilon_{m,j}} \to \frac{n_{m,j}}{n_{m,i}} \quad 1 \leq i, j \leq k_m \ .$$

Defining a new set of scales,

$$\tilde{\varepsilon}_m = n_{m,i}\varepsilon_{m,i} \qquad 1 \leq m \leq \ell \ ,$$

where $\varepsilon_{m,i}$ is a representative of its class $\mathcal{C}_m$, $1 \leq i \leq k_m$, we may define a function $g(x, y) \in BV_x(\Omega \times T^\ell)$ such that

$$(4.9)\qquad \|f(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n}) - g(x, \frac{x}{\tilde{\varepsilon}_1}, ..., \frac{x}{\tilde{\varepsilon}_\ell})\|_{W^{-1,\infty}(\Omega)} \leq C(\varepsilon_1, ..., \varepsilon_n) \to 0 \ .$$

The exact expressions for the new function g and the bound $C(\varepsilon_1, ..., \varepsilon_n)$ may be obtained by applying Proposition 4.3 to each of the equivalence classes, separately. We omit further details to avoid tedious notations. In the reminder of this section we shall assume that $f(x, y)$ is Lipschitz-continuous with respect to all of its y-variables. It is easy to see that this implies that also g is Lipschitz-continuous in its y-variables.

In view of the above, the problem of finding the weak limit of $f(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n})$ reduces to finding the weak limit of $g(x, \frac{x}{\tilde{\varepsilon}_1}, ..., \frac{x}{\tilde{\varepsilon}_\ell})$. For conveniency, we elect to keep using the notations

f, ε_i and n, instead of g, $\tilde{\varepsilon}_i$ and ℓ. Hence, in the reminder of this section we assume that all scales are mutually non-equivalent. We shall also assume that all scales are proportional, i.e.,

$$\frac{\varepsilon_n}{\varepsilon_i} = \alpha_i > 0 \qquad 1 \le i \le n \ . \tag{4.10}$$

This assumption is made since our goal is to indicate a phenomenon of weak convergence to averages of the function on submanifolds of T^n; to this end, it suffices to concentrate on the simple case (4.10). After the complete study of the 2-scale case in §2, it would be bothersome to repeat the entire analysis in the multiscale setting as well. Indeed, assumption (4.10) may be avoided by separating scales of different order of magnitude along the lines of §2.1, and handling the case where the ratio between scales only *tends* to a positive number by applying similar methods to those used in §2.2 and §2.3.

In view of the above, we aim at finding the $W^{-1,\infty}$-weak limit of

$$f_\varepsilon(x) = f(x, \frac{\alpha_1 x}{\varepsilon}, \dots, \frac{\alpha_n x}{\varepsilon}) \ ,$$

where $\alpha_i > 0$, $1 \le i \le n$ and $\alpha_n = 1$. Note that since we assumed that no two scales are equivalent, α_i are irrational for $1 \le i \le n-1$. We now set $a = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ and define:

DEFINITION 4.4. *Let $a = (\alpha_1, \dots, \alpha_n)$ be a vector in $\mathbb{R}^n$. Then $\mathcal{M}(a)$ denotes the $\mathbb{Z}$-module of vectors in $\mathbb{Z}^n$ which are orthogonal to a, i.e.,*

$$\mathcal{M}(a) = \{(m_1, \dots, m_n) \in \mathbb{Z}^n \ : \ \sum_i m_i \alpha_i = 0\} \ . \tag{4.11}$$

If $\mathcal{M}(a) = 0$, $\{\alpha_i\}_{1 \le i \le n}$ are said to be linearly independent.

Let $\mathcal{M}_{\mathbb{R}}(a)$ denote the $\mathbb{R}$-subspace of $\mathbb{R}^n$ spanned by the vectors of $\mathcal{M}(a)$, and $\mathcal{M}_{\mathbb{R}}(a)^\perp$ be its orthogonal complement in $\mathbb{R}^n$. Since $\mathcal{M}_{\mathbb{R}}(a)$ has a basis of vectors in $\mathbb{Z}^n$, so does $\mathcal{M}_{\mathbb{R}}(a)^\perp$. Hence, denoting $k = \dim \mathcal{M}_{\mathbb{R}}(a)^\perp$, there exist $v_1, \dots, v_k \in \mathbb{Z}^n$ such that

$$\mathcal{M}_{\mathbb{R}}(a)^\perp = \{\sum_{j=1}^k y_j v_j \ : \ y_j \in \mathbb{R}\} \ . \tag{4.12}$$

Our statement is as follows:

THEOREM 4.5. *Under the above assumptions, if $f \in L^\infty(\Omega, H^s(T^n))$, $s > \frac{n}{2}$, then*

$$\|f_\varepsilon(x) - \bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \to 0 \ , \tag{4.13}$$

where

$$\bar{f}(x) = \int_{T^k} f(x, \sum_{j=1}^k y_j v_j) dy \quad , \quad y = (y_1, \dots, y_k) \ . \tag{4.14}$$

Before proving this theorem, we give three examples in order to clarify it:

Examples.

1. Let $n = 2$, α_1 be a positive irrational number and $\alpha_2 = 1$. Then this is the case covered by Lemma 3.1 according to which the weak limit of $f_\varepsilon(x)$ is $\int_{T^2} f(x,y)dy$. Indeed, here $a = (\alpha_1, 1)$ and $\mathcal{M}(a) = 0$; hence, $\mathcal{M}_{\mathbb{R}}(a)^\perp = \mathbb{R}^2$ and, therefore, (4.14) agrees with Lemma 3.1.

2. Let $n = 3$, $\alpha_1 = \pi^2$, $\alpha_2 = \pi$, $\alpha_3 = 1$ and $\varepsilon_i = \frac{\varepsilon}{\alpha_i}$, $i = 1,2,3$. Consider the function

$$(4.15)\quad f_\varepsilon(x) = f(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, \frac{x}{\varepsilon_3}) \text{ where } f(x, y_1, y_2, y_3) = \cos(4\pi(x + y_1))\cos(2\pi y_2)\cos(2\pi y_3).$$

Since π is transcendental, $\{\alpha_i\}_{1\le i\le 3}$ are linearly independent and therefore $\mathcal{M}(a) = 0$. Hence, $\mathcal{M}_{\mathbb{R}}(a)^\perp = \mathbb{R}^3$ and, by (4.14),

$$(4.16)\qquad \bar{f}(x) = \int_0^1\int_0^1\int_0^1 f(x, y_1, y_2, y_3)dy_1dy_2dy_3 = 0\ .$$

3. Consider the same problem with $\alpha_1 = \frac{1}{\sqrt{2}}$, $\alpha_2 = \frac{1}{1-\sqrt{2}}$ and $\alpha_3 = 1$. Here, $\mathcal{M}(a) = \{(2m, m, m)\ :\ m \in \mathbb{Z}\}$. Hence, $\mathcal{M}_{\mathbb{R}}(a) = \{(2t, t, t)\ :\ t \in \mathbb{R}\}$ and

$$\mathcal{M}_{\mathbb{R}}(a)^\perp = \{y_1v_1 + y_2v_2\ :\ y_1, y_2 \in \mathbb{R}\ ,\ v_1 = (-1, 2, 0)\ ,\ v_2 = (-1, 0, 2)\} =$$

$$\{(-y_1 - y_2, 2y_1, 2y_2)\ :\ y_1, y_2 \in \mathbb{R}\}\ .$$

Therefore,

$$(4.17)\qquad \bar{f}(x) = \int_0^1\int_0^1 f(x, -y_1 - y_2, 2y_1, 2y_2)dy_1dy_2 = \frac{\cos(4\pi x)}{4}\ .$$

♠

Remark. The integral in (4.14) does not depend on the choice of basis $v_1, ..., v_k \in \mathbb{Z}^n$; we omit the proof of that.

Proof of Theorem 4.5.

Step 1. Let $\hat{f}_m(x)$, $m = (m_1, ..., m_n)$ being a multi-index, denote the Fourier coefficients of $f(x,y)$, $y \in T^n$, i.e.,

$$(4.18)\qquad f(x,y) = \sum_{m\in\mathbb{Z}^n} \hat{f}_m(x)E_m(y) \qquad E_m(y) = e^{2\pi i m\cdot y}\ .$$

Then $\bar{f}(x)$, as given in (4.14), may be written as follows:

$$(4.19)\qquad \bar{f}(x) = \int_{T^k} f(x, \sum_{i=j}^{k} y_jv_j)dy = \sum_{m\in\mathcal{M}(a)} \hat{f}_m(x)\ .$$

Indeed, by (4.18),

$$(4.20)\quad f(x,\sum_{j=1}^{k}y_jv_j) = \sum_{m\in\mathbb{Z}^n}\hat{f}_m(x)E_m(\sum_{j=1}^{k}y_jv_j) = \sum_{m\in\mathcal{M}(a)}\hat{f}_m(x)\exp(2\pi i\sum_{j=1}^{k}c_j(m)y_j) +$$

$$\sum_{m\notin\mathcal{M}(a)}\hat{f}_m(x)\exp(2\pi i\sum_{j=1}^{k}c_j(m)y_j) = S_1(x,y)+S_2(x,y)\ ,$$

where $y=(y_1,...,y_k)\in T^k$ and $c_j(m)=m\cdot v_j$ are integers. In the first sum, $S_1(x,y)$, all $c_j(m)=0$ since v_j are orthogonal to every $m\in\mathcal{M}(a)$. Hence,

$$S_1(x,y)=\sum_{m\in\mathcal{M}(a)}\hat{f}_m(x)\ ,$$

and, consequently,

$$(4.21)\qquad \int_{T^k}S_1(x,y)dy=\sum_{m\in\mathcal{M}(a)}\hat{f}_m(x)\ .$$

On the other hand, for every $m\notin\mathcal{M}(a)$ there exists at least one j, $1\le j\le k$, for which $c_j(m)\neq 0$. Hence,

$$(4.22)\qquad \int_{T^k}S_2(x,y)dy=\sum_{m\notin\mathcal{M}(a)}\hat{f}_m(x)\prod_{j=1}^{k}\int_{T^1}\exp(2\pi i c_j(m)y_j)dy_j=0\ .$$

Equality (4.19) now follows from (4.20)–(4.22).

Step 2. Let f_N be the Nth order Fourier approximation of f,

$$(4.23)\qquad f_N(x,y)=\sum_{|m|\le N}\hat{f}_m(x)E_m(y)\qquad |m|=\max_i|m_i|\ .$$

The assumption $f\in L^\infty(\Omega,H^s(T^n))$, $s>\frac{n}{2}$, implies that

$$(4.24)\qquad \|f(x,y)-f_N(x,y)\|_{L^\infty(\Omega\times T^n)}\underset{N\to\infty}{\longrightarrow}0$$

(for more details, see in the proof of Theorem 2.7 and consult [9]). This implies that the sum on the right of (4.19) converges uniformly to $\bar{f}(x)$: denoting

$$(4.25)\qquad \bar{f}_N(x):=\int_{T^k}f_N(x,\sum_{j=1}^{k}y_jv_j)dy=\sum_{m\in\mathcal{M}(a),|m|\le N}\hat{f}_m(x)\ ,$$

(the proof of the last equality is similar to that of (4.19)), we get that

$$(4.26)\quad \|\bar{f}(x)-\bar{f}_N(x)\|_{L^\infty(\Omega)}\le\int_{T^k}\|f(x,\sum_{i=1}^{k}y_iv_i)-f_N(x,\sum_{i=1}^{k}y_iv_i)\|_{L^\infty(\Omega\times T^k)}dy\le$$

$$\|f(x,y)-f_N(x,y)\|_{L^\infty(\Omega\times T^n)} \underset{N\to\infty}{\longrightarrow} 0 \ .$$

Step 3. Let $\mu > 0$ be an arbitrary small positive number. Then, by (4.24) and (4.26), there exists $N > 0$, such that

$$\|f(x,y)-f_N(x,y)\|_{L^\infty(\Omega\times T^n)} + \|\bar{f}(x)-\bar{f}_N(x)\|_{L^\infty(\Omega)} \le \frac{\mu}{2|\Omega|} \ . \tag{4.27}$$

For this value of N,

$$\|f_\varepsilon(x)-\bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \le \|f(x,\frac{\alpha_1 x}{\varepsilon},...,\frac{\alpha_n x}{\varepsilon}) - f_N(x,\frac{\alpha_1 x}{\varepsilon},...,\frac{\alpha_n x}{\varepsilon})\|_{W^{-1,\infty}(\Omega)} + \tag{4.28}$$

$$\|f_N(x,\frac{\alpha_1 x}{\varepsilon},...,\frac{\alpha_n x}{\varepsilon}) - \bar{f}_N(x)\|_{W^{-1,\infty}(\Omega)} + \|\bar{f}_N(x)-\bar{f}(x)\|_{W^{-1,\infty}(\Omega)} \ .$$

The sum of the first and last terms on the right of (4.28) does not exceed $\frac{\mu}{2}$, in view of (4.27). Hence, it remains only to show that, by choosing ε sufficiently small, the second term on the right of (4.28) becomes also less than $\frac{\mu}{2}$. To this end, we observe that by (4.23) and (4.25),

$$f_N(x,\frac{\alpha_1 x}{\varepsilon},...,\frac{\alpha_n x}{\varepsilon}) - \bar{f}_N(x) =$$

$$\sum_{|m|\le N} \hat{f}_m(x)\exp\{2\pi i(m\cdot a)\frac{x}{\varepsilon}\} - \sum_{m\in\mathcal{M}(a),|m|\le N} \hat{f}_m(x) = \sum_{m\notin\mathcal{M}(a),|m|\le N} \hat{f}_m(x)\exp\{2\pi i(m\cdot a)\frac{x}{\varepsilon}\} \ .$$

Since $m\cdot a \neq 0$ for all $m \notin \mathcal{M}(a)$, (4.11), each of the terms in the last sum tends in $W^{-1,\infty}$ to zero when $\varepsilon \downarrow 0$, in view of Lemma 1.1. Hence, for sufficiently small ε,

$$\|f_N(x,\frac{\alpha_1 x}{\varepsilon},...,\frac{\alpha_n x}{\varepsilon}) - \bar{f}_N(x)\|_{W^{-1,\infty}(\Omega)} \le \frac{\mu}{2} \ . \tag{4.29}$$

This completes the proof. □

The case where $\{\alpha_i\}_{1\le i\le n}$ are linearly independent is of special interest. Here, $\mathcal{M}_{\mathbb{R}}(a) = 0$ and $\mathcal{M}_{\mathbb{R}}(a)^\perp = \mathbb{R}^n$. Hence, Theorem 4.5 implies:

COROLLARY 4.6. *If $\{\alpha_i\}_{1\le i\le n}$ are linearly independent, the weak limit of $f_\varepsilon(x)$ is*

$$\bar{f}(x) = \int_{T^n} f(x,y)dy \ . \tag{4.30}$$

5. Applications. In this section we provide two simple applications of our results. In §5.1 we study the motion of an n-dimensional harmonic oscillator (or, equivalently, of n independent penduli) and a 2-dimensional quasi-harmonic oscillator, while in §5.2 we describe briefly an application to convection-diffusion equations.

5.1. A dynamical system. For any fixed $a = (\alpha_1, ..., \alpha_n) \in (\mathbb{R}^*)^n$, $n \geq 2$, let $G(a) = \{g^{ta}\}_{t\in\mathbb{R}}$ denote the group of the following continuous one-to-one mappings of the torus T^n onto itself:

$$\Phi \in T^n \mapsto g^{ta}(\Phi) = P_n(\Phi + ta) , \tag{5.1}$$

P_n being the projection operator of $\mathbb{R}^n$ onto T^n. Let $\mathcal{L}_\Phi(a)$ be the $G(a)$-orbit in T^n which passes at $t = 0$ through the point Φ, i.e., $\mathcal{L}_\Phi(a) = G(a)\Phi = \{g^{ta}(\Phi) \ : \ t \in \mathbb{R}\}$. Then Theorem 4.5 implies the following:

THEOREM 5.1. *The orbit $\mathcal{L}_\Phi(a)$ is dense in the T^n-submanifold $\Sigma_\Phi(a) := P_n(\Phi + \mathcal{M}_\mathbb{R}(a)^\perp)$, where $\mathcal{M}_\mathbb{R}(a)^\perp$ is given in (4.12).*

Proof. Since $ta \in \mathcal{M}_\mathbb{R}(a)^\perp$ for all $t \in \mathbb{R}$, it follows that $\mathcal{L}_\Phi(a) \subset \Sigma_\Phi(a)$. Next, we prove that $\mathcal{L}_\Phi(a)$ is dense in $\Sigma_\Phi(a)$. Assume, by contradiction, that there is an open set of positive measure in $\Sigma_\Phi(a)$, S, such that $S \cap \mathcal{L}_\Phi(a) = \emptyset$. Let $f(y)$ be a smooth function on T^n such that $f(y) > 0$ for $y \in S$ and $f(y) = 0$ for $y \in \Sigma_\Phi(a) \setminus S$. By our assumption, the function $f_\varepsilon(x) = f(\Phi + \frac{x}{\varepsilon}a)$ is identically zero in $\mathbb{R}_x$ for all values of $\varepsilon > 0$. However, by Theorem 4.5, $f_\varepsilon(x)$ tends weakly to the average of f over $\Sigma_\Phi(a)$, which is positive. This establishes the contradiction and the proof is therefore complete. □

COROLLARY 5.2. *$\mathcal{L}_\Phi(a)$ is dense in T^n iff α_i are linearly independent over $\mathbb{Z}$ and it is a closed curve in T^n iff all α_i are rationally proportional, i.e., $\alpha_i = r_i\alpha_n$, where $r_i \in \mathbb{Q}^*$, $1 \leq i \leq n$.*

Remark. The case $n = 2$ is of special interest. Here, the orbits are dense in T^2 iff the ratio $\frac{\alpha_1}{\alpha_2}$ is irrational. This well-known result is a consequence of Poincare's Recurrence Theorem (consult [1, §16]).

Now, we use the above in order to study the motion of an n-dimensional harmonic oscillator,

$$\ddot{x}_i(t) = -\omega_i^2 x_i(t) \qquad 1 \leq i \leq n .$$

The general solution is given by

$$x_i(t) = A_i \cos(\omega_i t + \phi_i) \qquad 1 \leq i \leq n , \tag{5.2}$$

where $A_i \geq 0$ is the amplitude in the ith direction. Hence, the orbit of the oscillator,

$$X = \{x(t) := (x_1(t), ..., x_n(t)) \ : \ t \in \mathbb{R}\} , \tag{5.3}$$

is confined to the box $B^n = \prod_{i=1}^n [-A_i, A_i]$. Such orbits are called *Lissajous figures* (see [1, §5]).

Let F_i denote the one-to-one mapping of $[-A_i, A_i]$ onto $[0, 1]$,

$$F_i(x) = \frac{1}{\pi} \cos^{-1}\left(\frac{x}{A_i}\right) ,$$

and F_i^{-1} denote its inverse,

$$F_i^{-1}(y) = A_i \cos(\pi y) \ .$$

Furthermore, we denote by F the tensor product of F_i which maps B^n onto $[0,1]^n$ and, similarly, let F^{-1} denote the tensor product of F_i^{-1} which maps $[0,1]^n$ onto B^n.

We define the new variable $y(t) = F(x(t))$ and let Y denote the corresponding orbit in T^n, i.e., $Y = \{y(t) \ : \ t \in \mathbb{R}\}$. Then, using our previous notations,

$$y(t) = P_n(\Phi + ta) \quad \text{where} \quad \Phi = \frac{1}{\pi}(\phi_1, ..., \phi_n) \ , \ a = \frac{1}{\pi}(\omega_1, ..., \omega_n) \ , \tag{5.4}$$

and, hence, $Y = \mathcal{L}_\Phi(a)$. Since, by Theorem 5.1, Y is dense in the T^n-submanifold $\Sigma_\Phi(a)$, we conclude the following:

THEOREM 5.3. *The orbit of the n-dimensional harmonic oscillator (5.2)–(5.3) is dense in the B^n-submanifold $F^{-1}(\Sigma_\Phi(a))$, where Φ and a are given in (5.4).*

Examples.

1. The Lissajous figure of the 2-dimensional oscillator with

$$A_1 = 2 \ , \ A_2 = 1 \quad ; \quad \omega_1 = \pi \ , \ \omega_2 = 2\pi \quad ; \quad \phi_1 = \frac{3\pi}{5} \ , \ \phi_2 = 0 \ ,$$

is the closed curve in Figure 5.1 below.

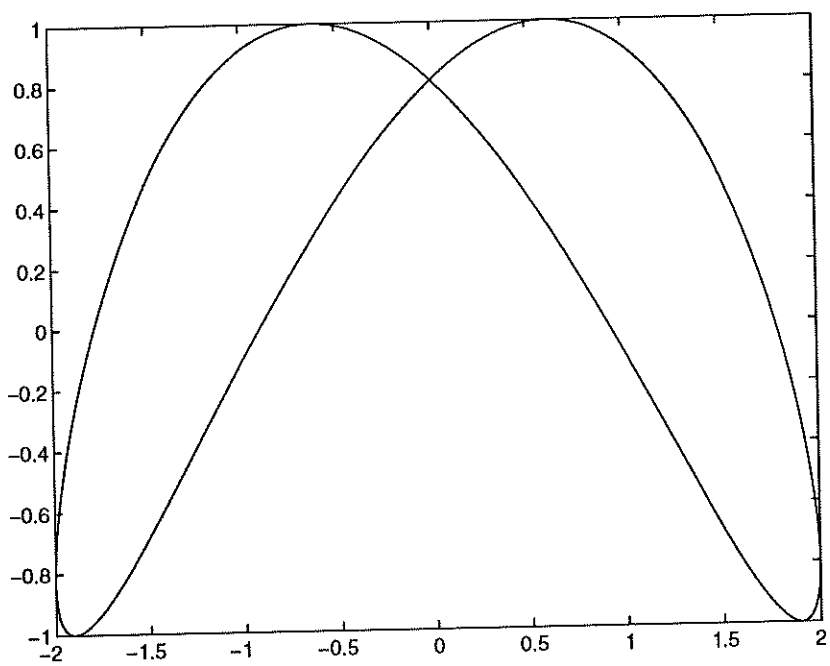

Figure 5.1

2. Figure 5.2 describes the orbit of the 2-dimensional oscillator with

$$A_1 = A_2 = 1 \quad ; \quad \omega_1 = 1 \ , \ \omega_2 = 2\pi \quad ; \quad \phi_1 = 0 \ , \ \phi_2 = \frac{\pi}{2} \ ,$$

for $0 \leq t \leq 20$. When $t \to \infty$, this curve becomes dense in the box $B^2 = [-1,1]^2$.

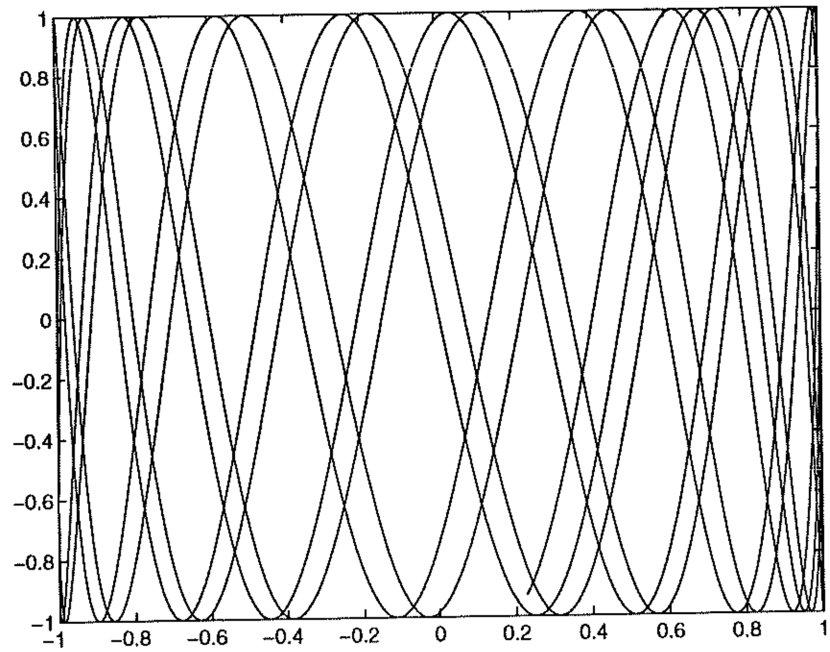

Figure 5.2

3. Consider the 3-dimensional oscillator

$$x(t) = (\cos(\pi t), \cos(\pi t - \frac{\pi}{2}), \cos(t)) \ .$$

Here, $a = (1, 1, \frac{1}{\pi})$ and, therefore, $\mathcal{M}_{\mathbb{R}}(a)^{\perp} = \{(t_1, t_1, t_2) \ : \ t_1, t_2 \in \mathbb{R}\}$. Since $\Phi = (0, -\frac{1}{2}, 0)$ in this case, the orbit is dense in the following submanifold of $B^3 = [-1,1]^3$,

$$F^{-1}(\Sigma_{\Phi}(a)) = \{(\cos(\pi t_1), \cos(\pi t_1 - \frac{\pi}{2}), \cos(\pi t_2)) \ : \ t_1, t_2 \in T^1\} \ ,$$

which is the cylinderical manifold

$$\{x \in B^3 \ : \ x_1^2 + x_2^2 = 1\} \ .$$

Next, we would like to study the motion of a quasi-harmonic oscillator,

$$\text{(5.5)} \qquad x_i(t) = A_i \cos(f_i(t)) \ , \quad f_i(t) \underset{t\to\infty}{\longrightarrow} \infty \qquad 1 \leq i \leq n \ .$$

The harmonic oscillator, (5.2), is a special case of (5.5) where $f_i(t)$ are all linear. We concentrate on the 2-dimensional case and examine the orbits of such oscillators,

$$\text{(5.6)} \qquad X = \{x(t) := (x_1(t), x_2(t)) \ : \ t \in \mathbb{R}\} \ .$$

Let us assume that $\lim_{t\to\infty} \frac{f_1(t)}{f_2(t)} = \alpha$. Then, by §2, we conclude (along the lines of the proof of Theorem 5.1) that when α is zero or irrational, X is dense in $B^2 = [-A_1, A_1] \times [-A_2, A_2]$. The case of a nonzero rational limit, $\alpha = \frac{m}{n}$ (m and n are mutually prime), is more interesting. In this case,

$$f_1(t) = \frac{m}{n} \cdot f_2(t) + g(t) \quad \text{where} \quad \frac{g(t)}{f_2(t)} \underset{t\to\infty}{\longrightarrow} 0 \ . \tag{5.7}$$

We consider three subcases:

1. $\lim_{t\to\infty} |g(t)| = \infty$. Here, part (2) of Theorem 2.4 may be applied in order to conclude that X is dense in B^2. See Figure 5.3 where the orbit of the quasi-harmonic oscillator with $f_1(t) = \pi \cdot (t + 0.6)$ and $f_2(t) = 2\pi t + t^{0.2}$ is depicted for $0 \leq t \leq 20$.

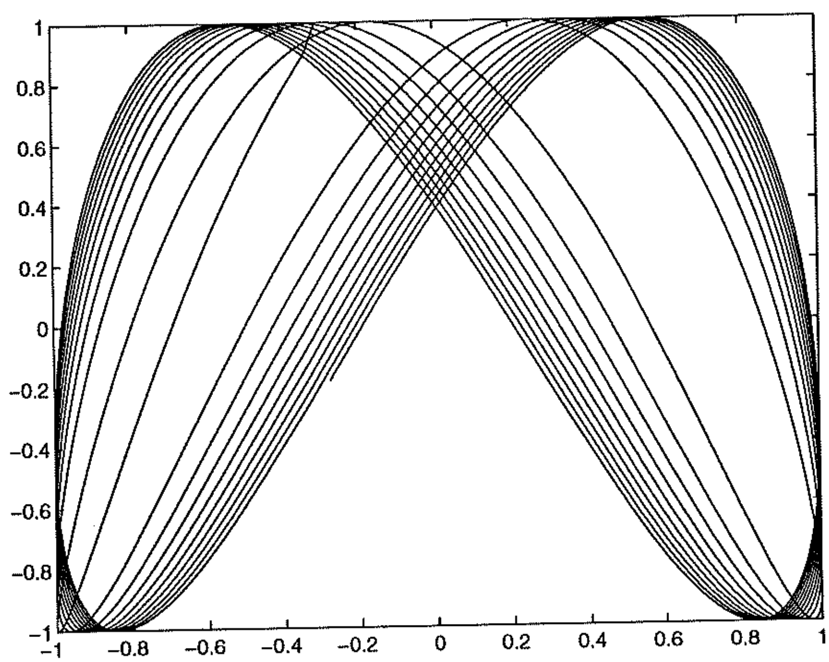

Figure 5.3

2. $\lim_{t\to\infty} g(t) = g_\infty$. In this case, the orbit $\Xi(m, n; g_\infty) = \{(\xi_1(t), \xi_2(t)) : \ t \in \mathbb{R}\}$ of the harmonic oscillator

$$\xi_1(t) = \cos(mt + g_\infty) \quad , \quad \xi_2(t) = \cos(nt) \ ,$$

serves as an attractor for X when $t \to \infty$. Figure 5.4 depicts the orbit which corresponds to $f_1(t) = \pi \cdot (t + 0.6)$ and $f_2(t) = 2\pi t + (1 + t)^{-1}$ whose attractor is the one in Figure 5.1.

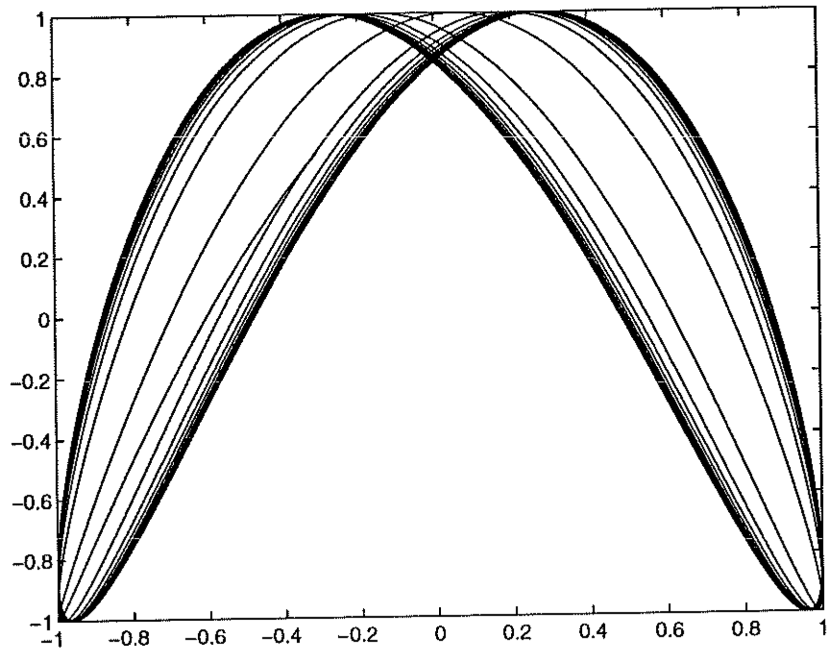

Figure 5.4

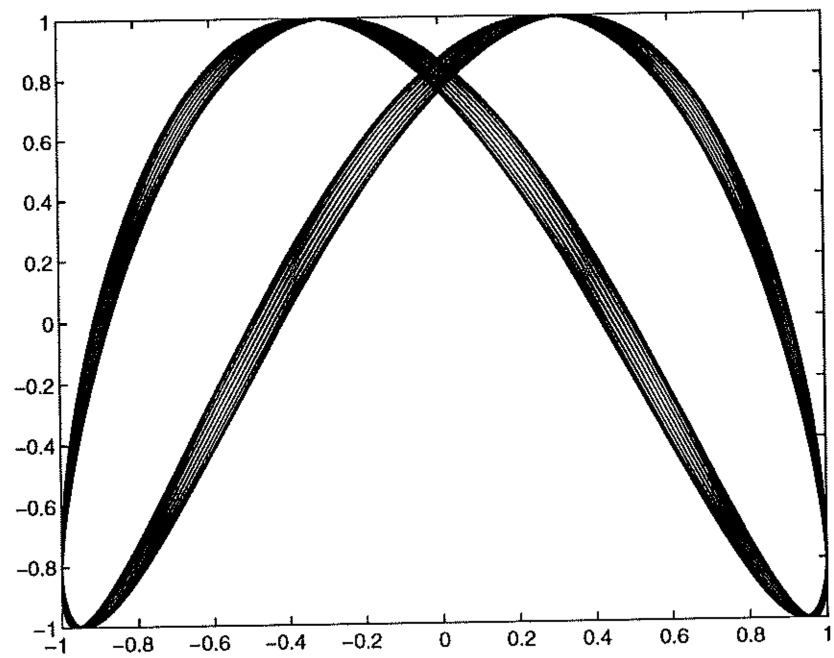

Figure 5.5

3. $g_1 \leq g(t) \leq g_2$. Here,

$$X \subset \cup_{\gamma \in [g_1, g_2]} \Xi(m, n; \gamma) , \tag{5.8}$$

where $\Xi(m, n; \gamma)$ is, as before, the orbit of the harmonic oscillator

$$\xi_1(t) = \cos(mt + \gamma) \quad , \quad \xi_2(t) = \cos(nt) .$$

We note in passing that if $|g_2 - g_1| \geq \frac{1}{n}$, the union in (5.8) covers the entire box B^2. This union of orbits is very apparent in Figure 5.5 which corresponds to $f_1(t) = \pi \cdot (t + 0.6)$ and $f_2(t) = 2\pi t + 0.1 \cdot \sin(t)$.

5.2. Homogenization of nonlinear convection-diffusion equations. Here, we combine our analysis with the homogenization theory of [10] (see there for more details).

Assume that $u_0(x, y)$ and $h(x, y, t)$ (t is a parameter) are functions in $BV_x(\Omega \times T^n)$ which are constant for $x \notin \Omega$, and let

$$u_0^\varepsilon(x) = u_0(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_n}) \quad , \quad h^\varepsilon(x, t) = u_0(x, \frac{x}{\tilde{\varepsilon}_1}, \dots, \frac{x}{\tilde{\varepsilon}_n}, t) \ , \tag{5.9}$$

where $\varepsilon_i = \varepsilon_i(\varepsilon) > 0$ and $\tilde{\varepsilon}_i = \tilde{\varepsilon}_i(\varepsilon) > 0$ vanish when $\varepsilon \to 0+$. Let $\bar{u}_0(x)$ and $\bar{h}(x,t)$ denote, respectively, the $W^{-1,\infty}$-weak limits of $u_0^\varepsilon(x)$ and $h^\varepsilon(x,t)$. Consider now the convection-diffusion problem

$$u_t^\varepsilon = K(u^\varepsilon, u_x^\varepsilon)_x + h^\varepsilon(x, t) \ , \qquad u^\varepsilon(x, 0) = u_0^\varepsilon(x), \tag{5.10}$$

with modulated initial and forcing data, $u_0^\varepsilon(x)$ and $h^\varepsilon(x,t)$, as given in (5.9). Here, $K = K(u, p)$ is a non-decreasing function in p and $u^\varepsilon(x,t)$ is the unique entropy solution of the problem, namely, that which corresponds to $K^\delta(u,p) = K(u,p) + \delta p$, $\delta \downarrow 0$. Then, according to [10, Theorem 2.3], $u^\varepsilon(\cdot, t)$, $t \geq 0$, tends weakly in $W^{-1,\infty}$ to $u(\cdot, t)$, the entropy solution of the homogenized problem,

$$u_t = K(u, u_x)_x + \bar{h}(x, t) \ , \qquad u(x, 0) = \bar{u}_0(x) \ . \tag{5.11}$$

Moreover, if the equation is $W^{s,r}$-regular (in the sense that its solution operator maps bounded sets in L^∞ into bounded sets in the regularity spaces, $W^{s,r}_{loc}$, $s > 0$, $1 \leq r \leq \infty$), this type of weak convergence may be translated in positive times into a strong one; namely,

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^p(\Omega)} \underset{\varepsilon \to 0}{\longrightarrow} 0 \quad , \quad t > 0 \ , \tag{5.12}$$

for some values of $p \in [1, \infty]$, consult [10, Theorem 3.1].

As examples, we mention convex hyperbolic conservation laws, $K(u,p) = -f(u)$, $f'' \geq Const > 0$, which posses $W^{1,1}$-regularity and the subquadratic porous media equation, $K(u,p) = mu^{m-1}p$, $1 \leq m \leq 2$, $u \geq 0$, which possesses $W^{2,1}$-regularity [10, Propositions 4.1 & 5.1].

6. Graphical demonstrations. Here we provide visual illustrations of the results of our analysis. These convincing graphs not only confirm the analysis but even reveal some other interesting phenomena in the behavior of the oscillatory function when the scales tend to zero.

We start with demonstrating the results of §2. To this end, let

$$f(y_1, y_2) = \cos(2\pi y_1)\cos(2\pi y_2) \quad \text{and} \quad f_\varepsilon(x) = f(\frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}) \ . \tag{6.1}$$

Denoting $\varepsilon = \varepsilon_2$, we consider five cases:

1. $\varepsilon_1 = \varepsilon^2$;
2. $\varepsilon_1 = \varepsilon$;
3. $\varepsilon_1 = \varepsilon + \varepsilon^2$;
4. $\varepsilon_1 = \varepsilon + \varepsilon^{1.5}$;
5. $\varepsilon_1 = \pi\varepsilon$.

In case 1 the weak limit is, according to Theorem 2.1,

$$\int_0^1 \int_0^1 f(y_1, y_2) dy_1 dy_2 = 0 \ . \tag{6.2}$$

The first part of Theorem 2.4 implies that the weak limit in case 2 is

$$\int_0^1 f(y_1, y_1) dy_1 = \frac{1}{2} \ , \tag{6.3}$$

while in case 3 (where $r = \frac{\varepsilon_1}{\varepsilon_2} - 1 = \varepsilon_2$) it is

$$\int_0^1 f(y_1 - x, y_1) dy_1 = \frac{1}{2} \cos(2\pi x) \ . \tag{6.4}$$

Finally, the weak limit in cases 4 and 5 is as in (6.2), as implied by the second part of Theorem 2.4 and by Theorem 2.7.

Figures 6.1–6.5 depict $f_\varepsilon(x)$, and the corresponding weak limits, in cases 1–5 for the following two values of ε:

a. $\varepsilon = 0.0408$;
b. $\varepsilon = 0.00273$.

Next, we demonstrate our results in the multiscale setting for the $BV_x([0,1] \times T^3)$-function given in (4.15). Denoting $\varepsilon = \varepsilon_3$, we consider here three cases:

6. $\varepsilon_1 = \varepsilon + \varepsilon^2$, $\varepsilon_2 = \varepsilon + 2\varepsilon^2$;
7. $\varepsilon_1 = \frac{\varepsilon}{\pi^2}$, $\varepsilon_2 = \frac{\varepsilon}{\pi}$;
8. $\varepsilon_1 = \sqrt{2} \cdot \varepsilon$, $\varepsilon_2 = (1 - \sqrt{2}) \cdot \varepsilon$.

In case 6, all the scales are equivalent in the sense of Definition 4.1. Using the notations of Proposition 4.3, $n_1 = n_2 = n_3 = 1$ and $c_{1,3} = 1, c_{2,3} = 2, c_{3,3} = 0$. Hence, this Proposition asserts that the weak limit of $f_\varepsilon(x)$ coincides with that of

$$g_\varepsilon(x) = g(x, \frac{x}{\varepsilon}) \quad \text{where} \quad g(x, y) = f(x, y - x, y - 2x, y),$$

which equals

$$\int_0^1 g(x, y) dy = \frac{1}{4} \cos(4\pi x) \ .$$

The weak limits in cases 7 and 8 were already studied and are given in (4.16) and (4.17), respectively. The graphs of $f_\varepsilon(x)$ in these cases, for the same two values of ε as before, are given in Figures 6.6–6.8.

Finally, we consider the following initial value problem for the hyperbolic Burgers' equation:

$$u_t^\varepsilon + \frac{1}{2}((u^\varepsilon)^2)_x = f(\frac{x}{\varepsilon}, \frac{x}{\varepsilon}) \quad , \quad t > 0 \; ; \tag{6.5}$$

$$u^\varepsilon(x, 0) = f(\frac{x}{\varepsilon + \varepsilon^2}, \frac{x}{\varepsilon}) \; , \tag{6.6}$$

where $f(\cdot, \cdot)$ is given in (6.1). The weak limits, when $\varepsilon \downarrow 0$, of the right hand side of (6.5) and the initial value (6.6), are given, respectively, in (6.3) and (6.4). Hence, according to §5.2, the entropy solution of (6.5)–(6.6), $u^\varepsilon(\cdot, t)$, tends weakly in $W^{-1,\infty}$ to $u(\cdot, t)$, the entropy solution of the homogenized problem,

$$u_t + \frac{1}{2}(u^2)_x = \frac{1}{2} \quad , \quad t > 0 \; ; \tag{6.7}$$

$$u(x, 0) = \frac{1}{2}\cos(2\pi x) \; . \tag{6.8}$$

Moreover, apart from an initial layer of width $\mathcal{O}(\varepsilon)$, $u^\varepsilon(\cdot, t)$ converges strongly to $u(\cdot, t)$. In Figures 6.9a-d we plot $u^\varepsilon(\cdot, t)$, with $\varepsilon = 0.0408$, and $u(\cdot, t)$ for the following time values: $t_a = 3.75 * 10^{-4}$, $t_b = 7.5 * 10^{-4}$, $t_c = 1.5 * 10^{-3}$ and $t_d = 0.04$ (u^ε is described by the solid line and u by the broken one). We see that the compact solution operator of the nonlinear equation cancels out the oscillations and that the convergence of $u^\varepsilon(\cdot, t)$ to $u(\cdot, t)$, for $t > 0$, is in the strong sense. Note that at $t_a, t_b, t_c \ll \varepsilon$ the oscillations in u^ε are still apparent, while at $t_d \approx \varepsilon$ they no longer exist.

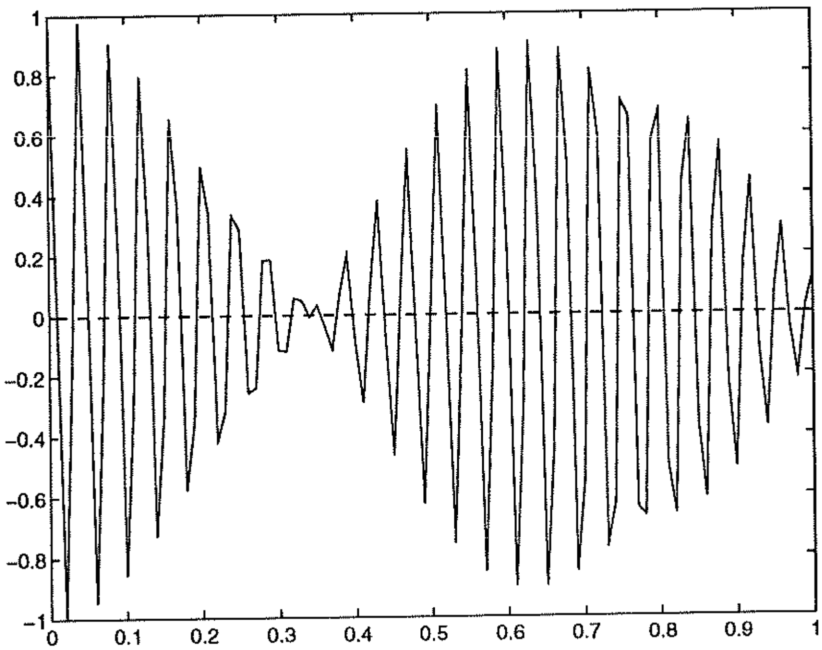

Figure 6.1a

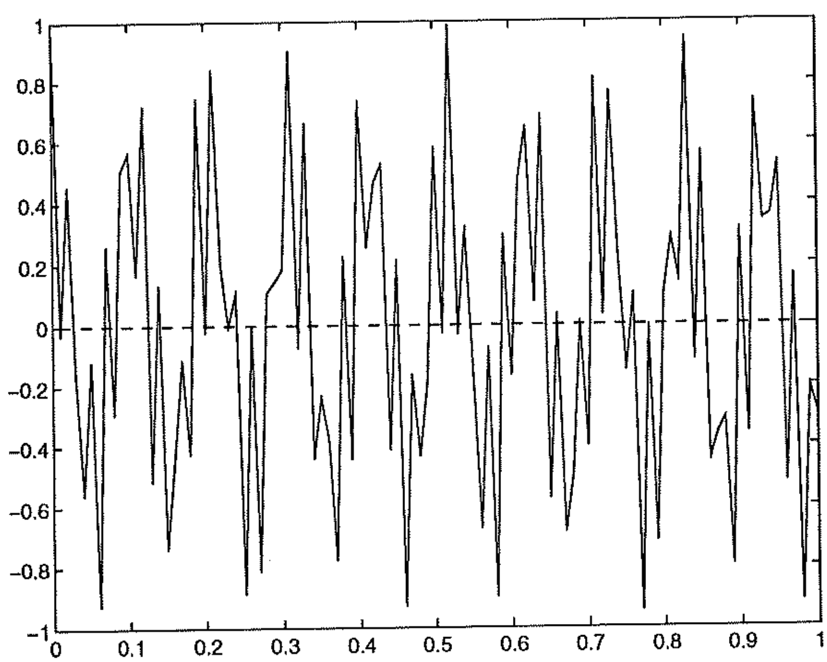

Figure 6.1b

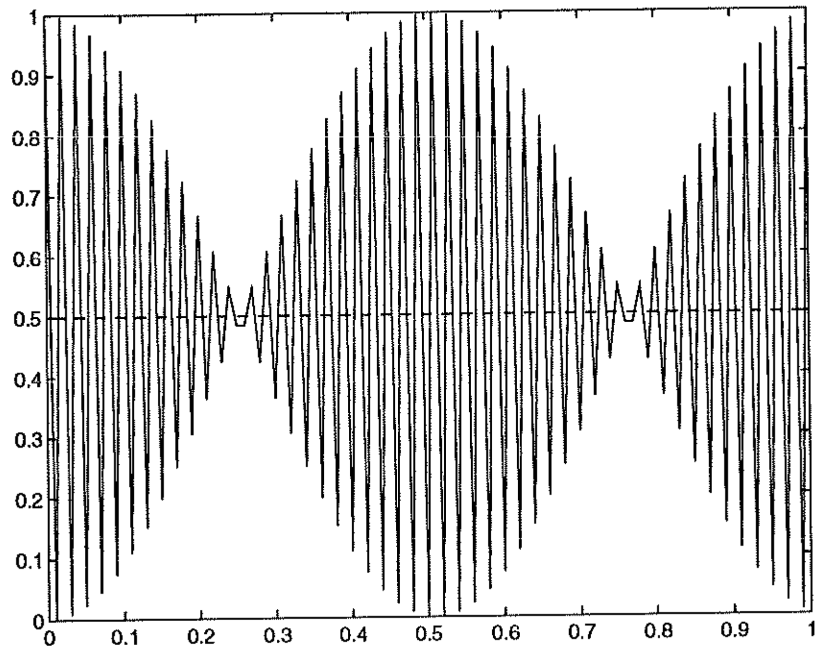

Figure 6.2a

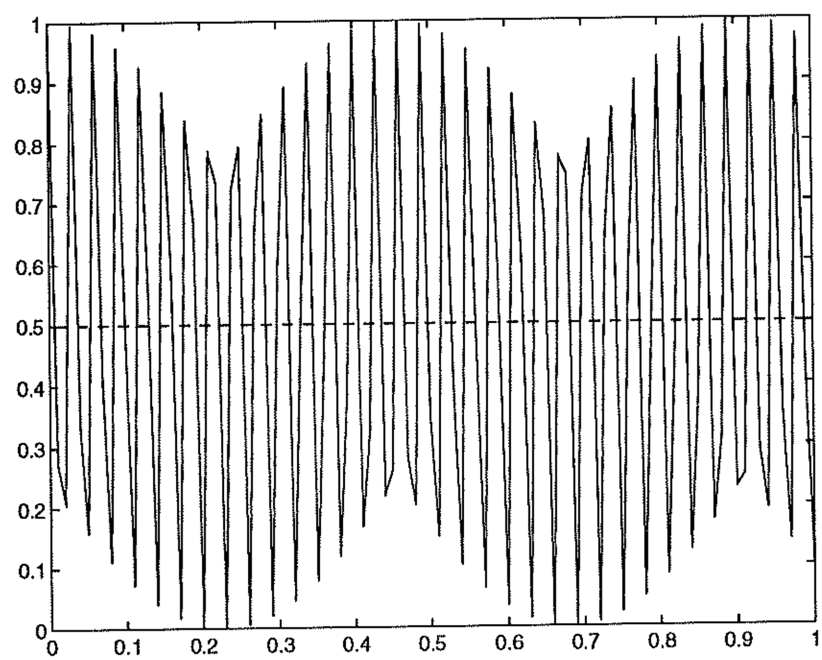

Figure 6.2b

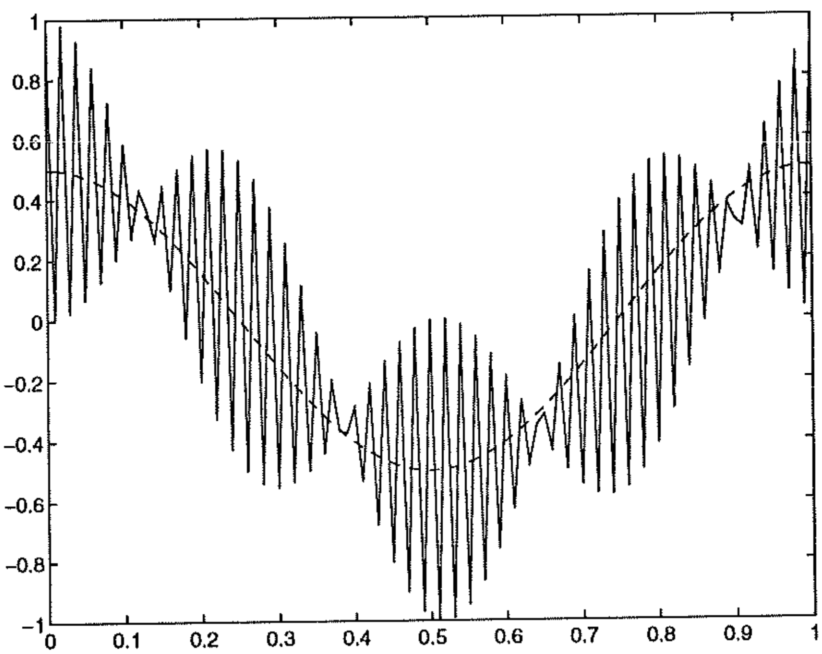

Figure 6.3a

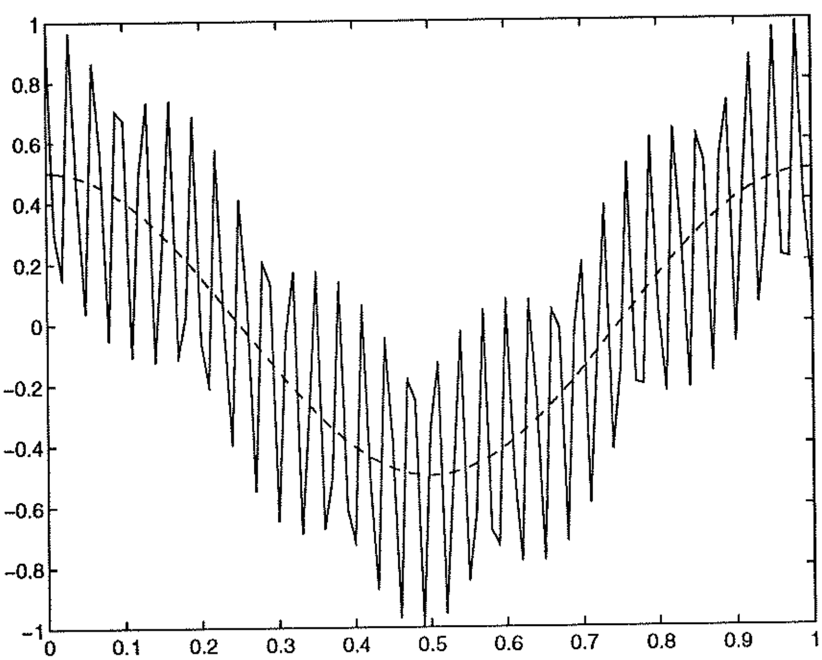

Figure 6.3b

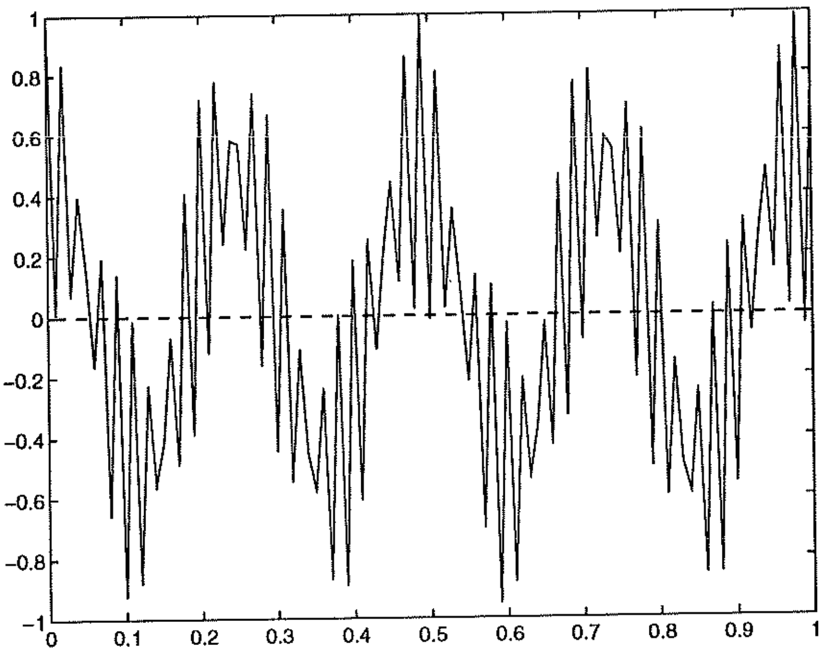

Figure 6.4a

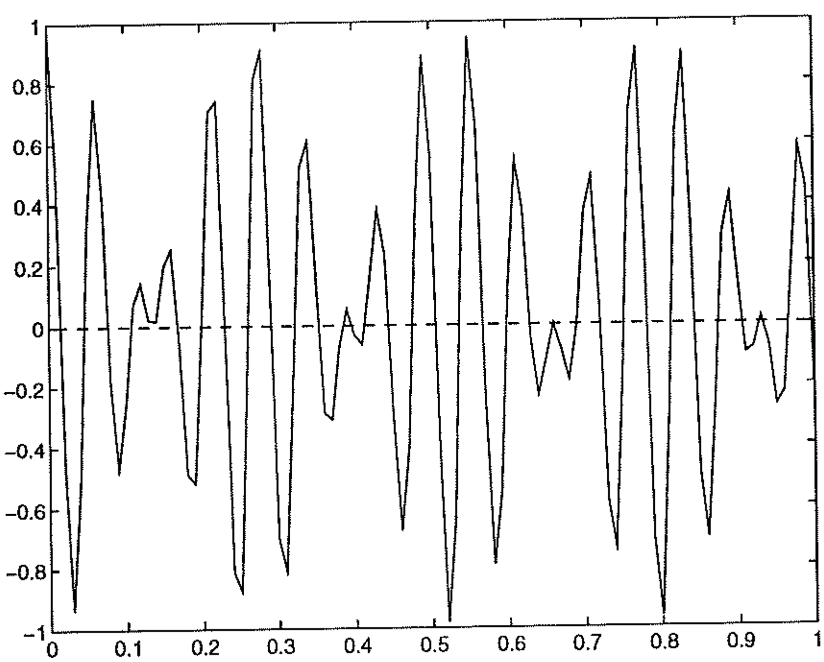

Figure 6.4b

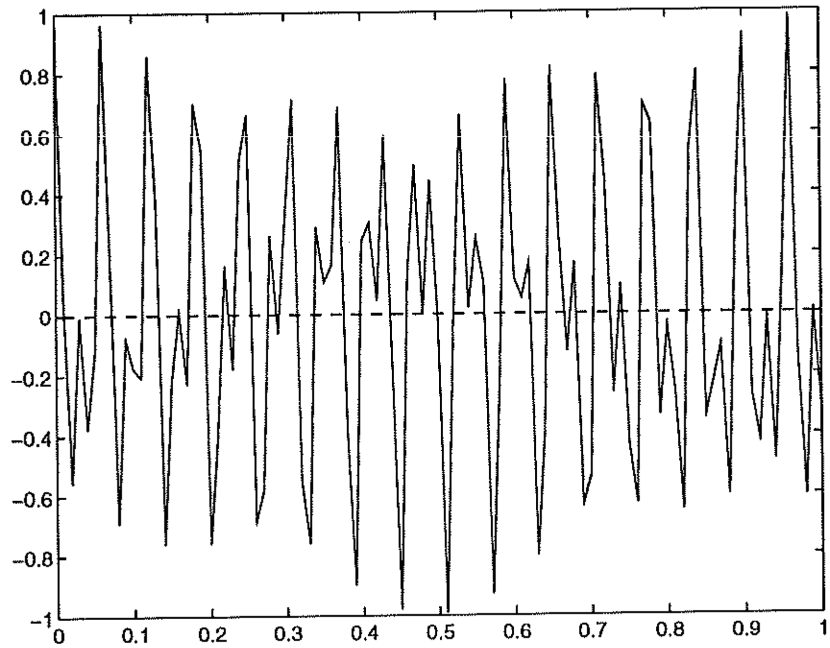

Figure 6.5a

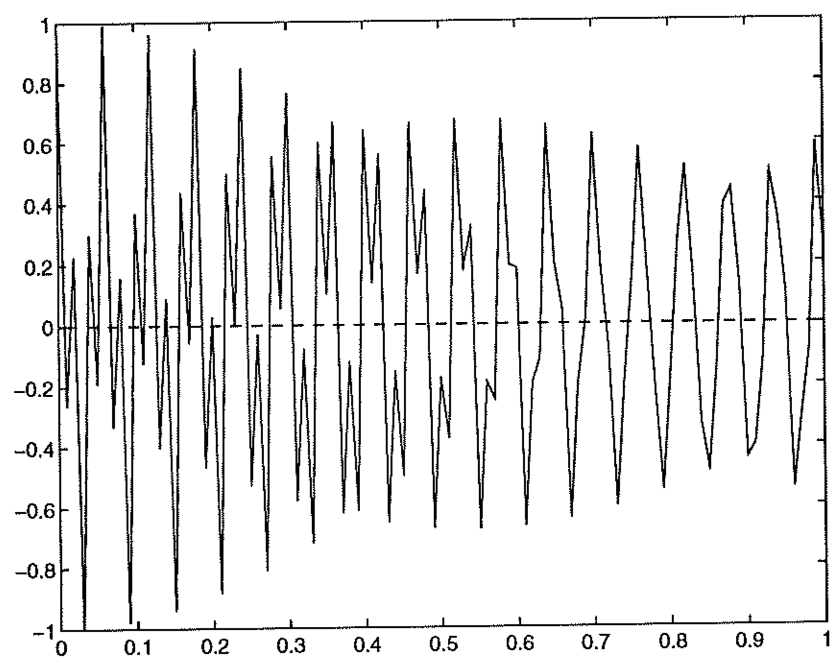

Figure 6.5b

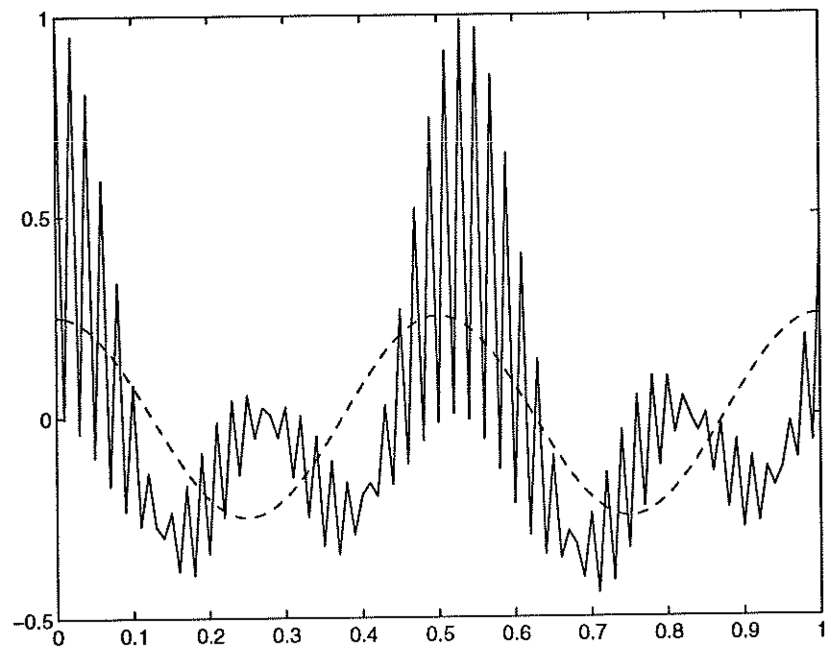

Figure 6.6a

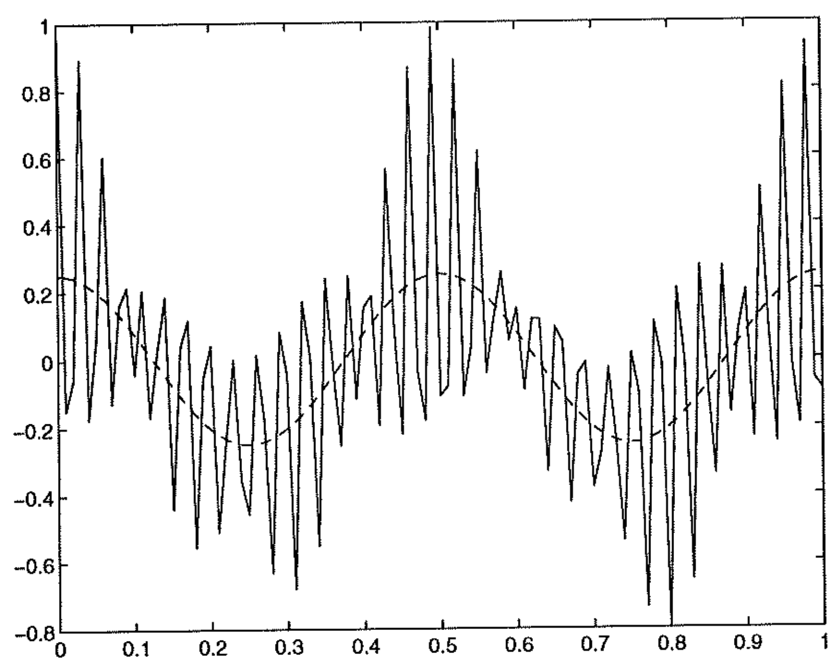

Figure 6.6b

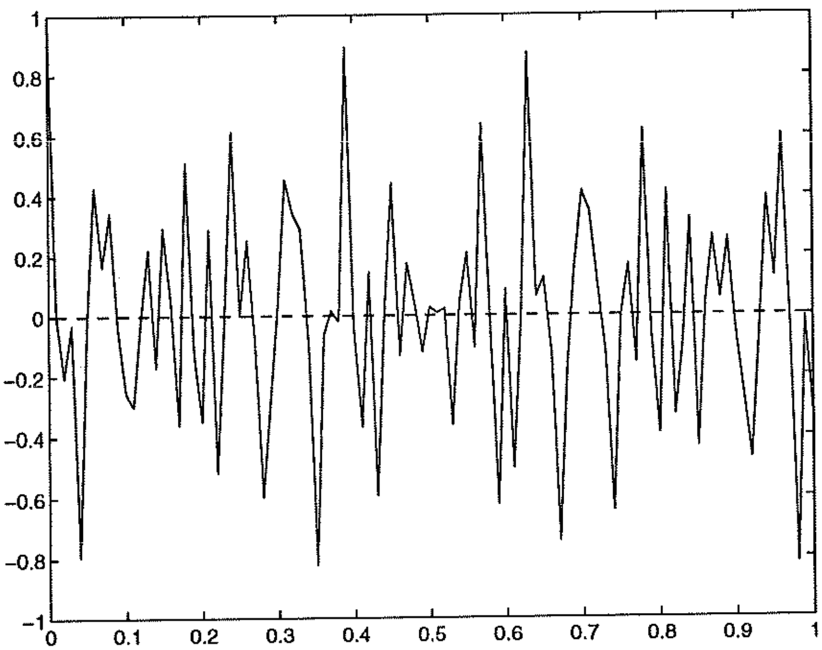

Figure 6.7a

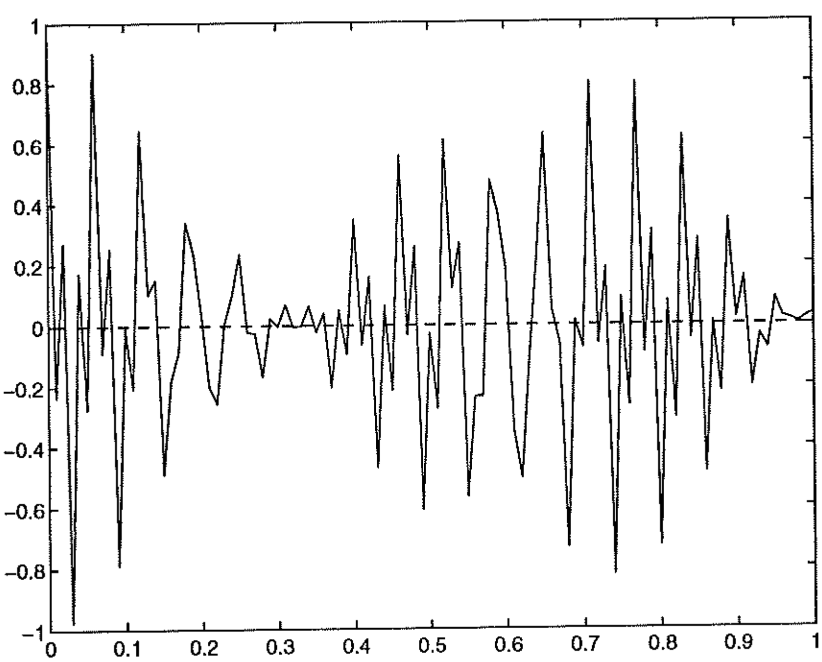

Figure 6.7b

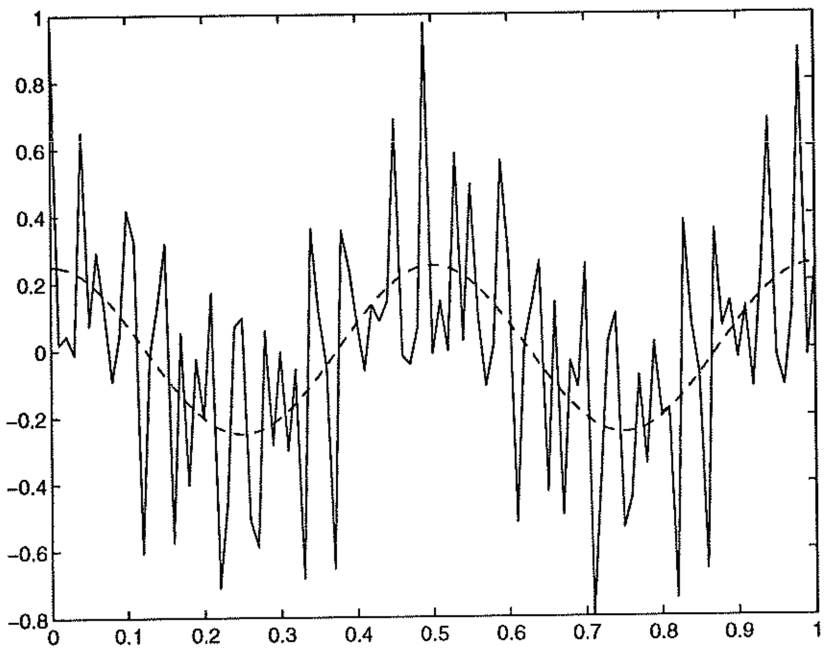

Figure 6.8a

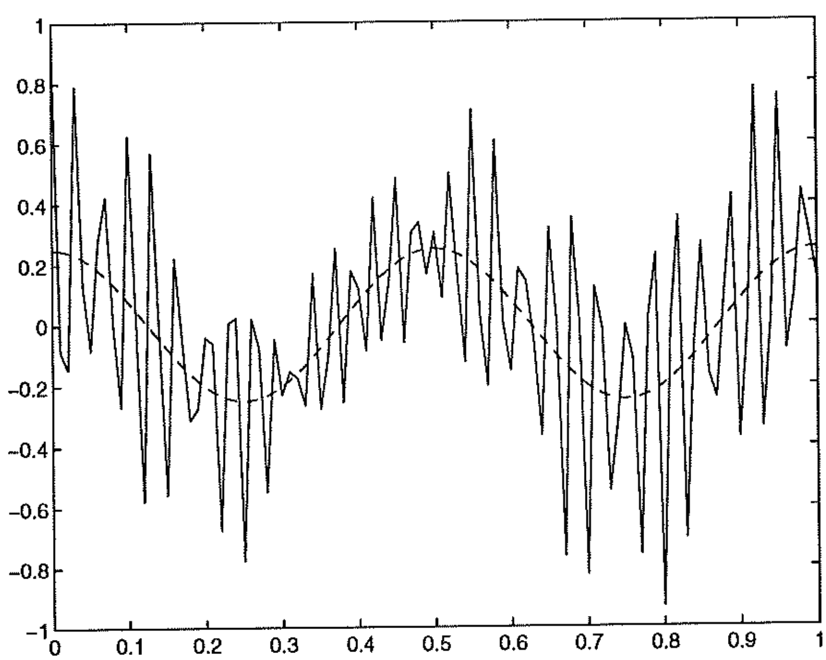

Figure 6.8b

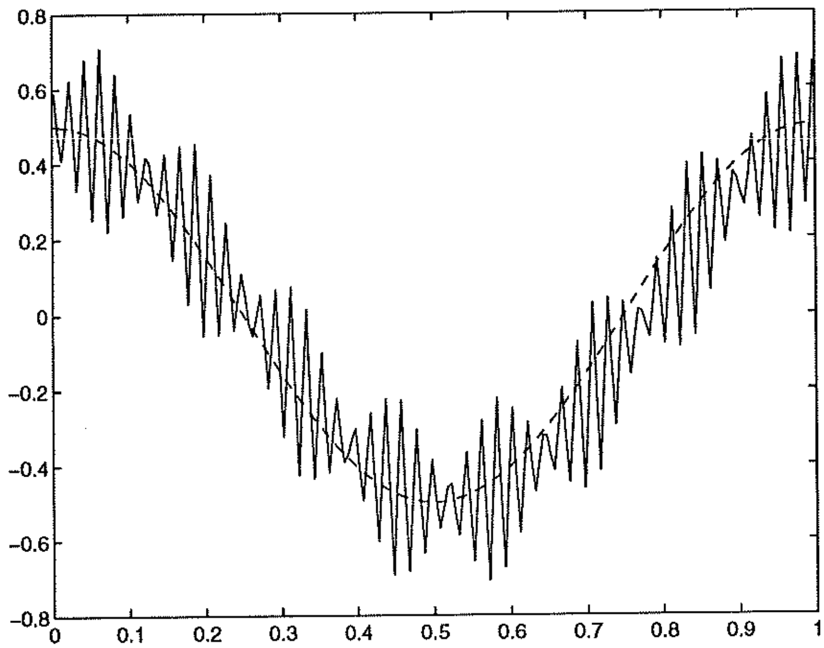

Figure 6.9a

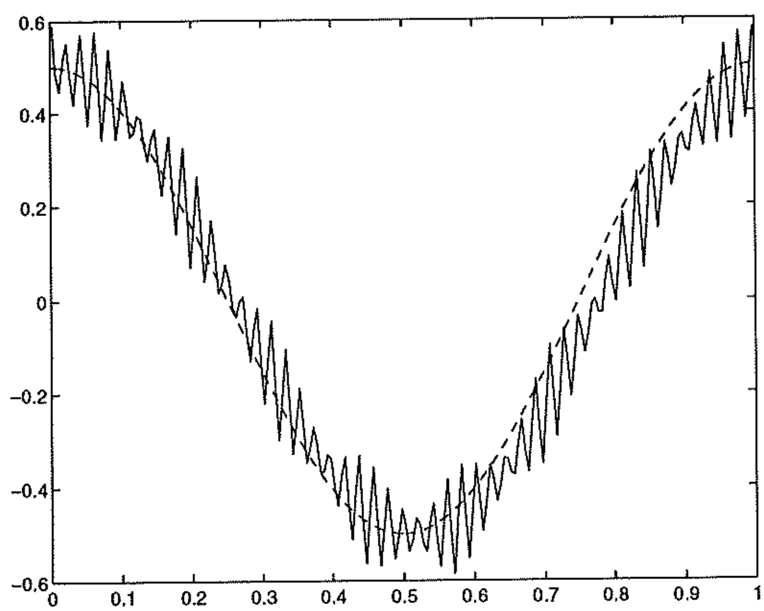

Figure 6.9b

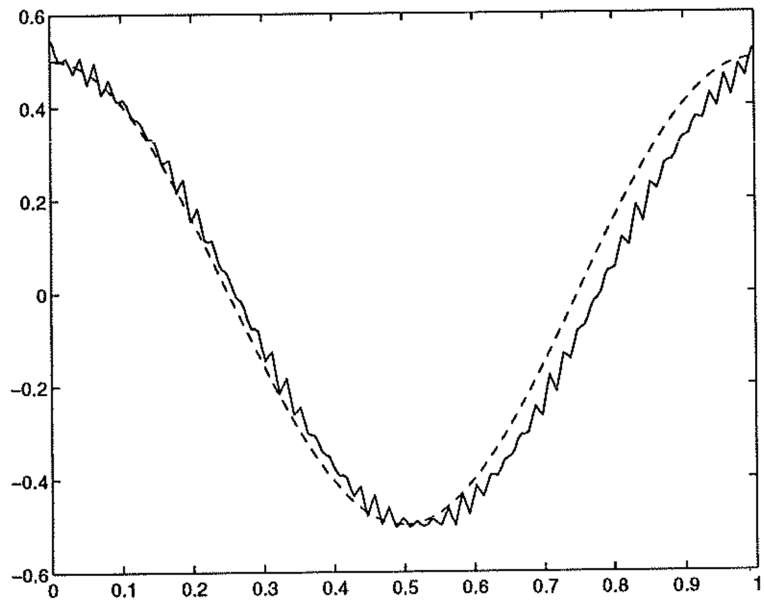

Figure 6.9c

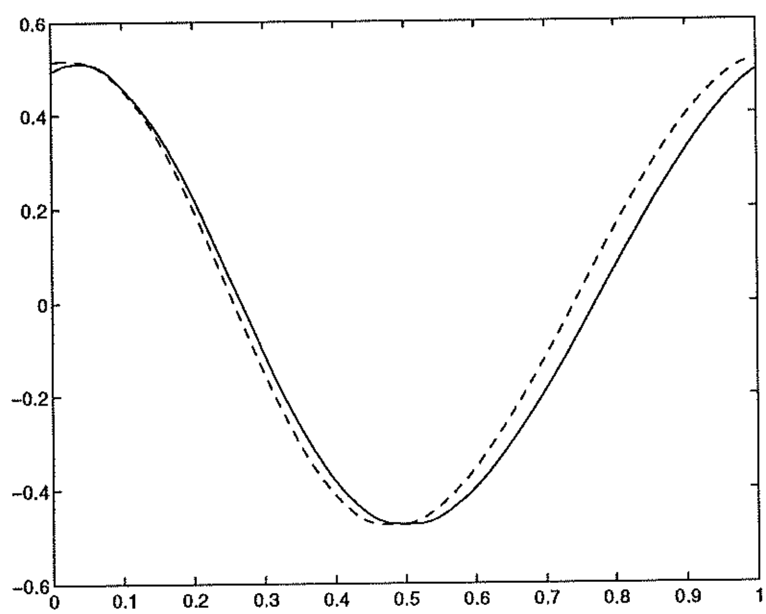

Figure 6.9d

7. Appendix. Here we provide a brief review of some results from the theory of quasi-Monte Carlo numerical integration methods and related topics in number theory, [6, 7].

Let $f(y)$ be a function in $BV[0,1]$ and $\{x_n\}$ be a sequence of points in $[0,1]$. Then, a quasi-Monte Carlo approximation for the integral of f is given by

$$\int_0^1 f(y)dy \approx \frac{1}{N}\sum_{n=1}^{N} f(x_n) . \tag{7.1}$$

In order for the approximation to converge, the sequence of points must be "well-distributed" in the interval of integration. The *discrepancy* of the sequence, defined as

$$D_N(x_1, x_2, ..., x_N) = \sup_{0\le r\le 1} \left| \frac{\#\{x_i :\ 1 \le i \le N,\ x_i \in [0,r]\}}{N} - r \right| , \tag{7.2}$$

is a mean to quantify how well the sequence is distributed. With this definition, the following error estimate holds [6, Theorem 2.9]:

$$\left| \int_0^1 f(y)dy - \frac{1}{N}\sum_{n=1}^{N} f(x_n) \right| \le D_N(x_1, x_2, ..., x_N) \cdot \|f\|_{BV[0,1]} . \tag{7.3}$$

Hence, in order to obtain an error estimate for the quasi-Monte Carlo method, one must upper-bound the discrepancy of the corresponding sequence of points. Since it was proved that for any sequence

$$D_N \ge 0.06 \cdot N^{-1} \log N \qquad \text{for infinitely many } N ,$$

we cannot hope for an error estimate better than $\mathcal{O}(N^{-1}\log N)$, unless we assume more on the smoothness of f.

We are interested here in sequences of the form $x_n = P_1(n\alpha)$, where α is an irrational number and P_1 denotes, as before, the projection of $\mathbb{R}$ onto T^1 (namely, $P_1(x)$ is the fractional part of x). For such sequences we may apply the ergodic theorem of equi-partition modulo 1 (Bohl-Serpinskii-Weyl) [2], which implies that

$$\left| \int_0^1 f(y)dy - \frac{1}{N}\sum_{n=1}^{N} f(x_n) \right| \underset{N\to\infty}{\longrightarrow} 0 . \tag{7.4}$$

Convergence rate estimates are available in some special cases. We cite below two of the more important results in this direction.

PROPOSITION 7.1. *If α is a proper irrational number (defined below) then $D_N = \mathcal{O}(N^{-1}\log N)$, where D_N is the discrepancy of the sequence $x_n = P_1(n\alpha)$.*

An irrational number, α, is called *proper* if the partial quotients a_i in its (unique) continued fraction expansion,

$$\alpha = a_0 + 1/(a_1 + 1/(a_2 + ...)) \ , \quad a_i \in \mathbb{Z} \ , \ a_i \ge 1 \ \ \forall i \ge 1 \ ,$$

are such that $\sum_{i=1}^{m} a_i = \mathcal{O}(m)$.

In view of (7.3) and Proposition 7.1 we conclude:

PROPOSITION 7.2. *If $f \in BV[0,1]$ and $x_n = P_1(n\alpha)$, α being a proper irrational number, then*

$$\left| \int_0^1 f(y)dy - \frac{1}{N} \sum_{n=1}^{N} f(x_n) \right| \leq Const \cdot N^{-1} \log N \ . \tag{7.5}$$

By further assumptions on the smoothness of f, we may obtain an $\mathcal{O}(N^{-1})$-error estimate. To this end we define the following:

DEFINITION 7.3. *Let α be an irrational number and let $S = S_\alpha$ be defined as*

$$S = \{\sigma \ : \ \exists c = c(\alpha, \sigma) \ \text{such that} \ \ dist(\alpha n, \mathbb{Z}) \geq \frac{c}{n^\sigma} \quad \forall n \in \mathbb{N}\} \ .$$

Then if $S \neq \emptyset$, α is said to be of type η, where $\eta = \inf S$.

DEFINITION 7.4. *Let f be a 1-periodic function and assume that $|\hat{f}_n| \leq \mathcal{O}(|n|^{-k})$ for all $n \neq 0$, where $\hat{f}_n$ are the Fourier coefficients of f, $f(y) = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{2\pi i n y}$. Then f is said to be of class $\mathcal{E}^k$.*

PROPOSITION 7.5. *Let f be a 1-periodic function of class $\mathcal{E}^k$ and α be an irrational number of type $\eta < k$. Then the following error estimate holds:*

$$\left| \int_0^1 f(y)dy - \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) \right| \leq Const \cdot N^{-1} \ .$$

We conclude this brief review with some remarks on the type of irrational numbers. Whenever the type $\eta = \eta(\alpha)$ is defined, it is greater than or equal to 1, as implied by Proposition 7.7 below. Moreover, if α is an algebraic number, $\eta(\alpha) = 1$; this may be stated in the following straightforward manner:

PROPOSITION 7.6. *Let α be an irrational algebraic number. Then for any $\sigma > 1$ there exists $c = c(\alpha, \sigma)$ such that $|\alpha n - m| \geq c n^{-\sigma}$ for all $\frac{m}{n} \in \mathbb{Q}$.*

This property of irrational algebraic numbers, which we use in our convergence analysis, is an immediate consequence of a theorem by Roth [8] which asserts that for irrational algebraic numbers α and any $c > 0$ and $\sigma > 1$, there exist only finitely many $\frac{m}{n} \in \mathbb{Q}$ for which $|\alpha n - m| \leq c n^{-\sigma}$. This statement is no longer true when $\sigma = 1$; for this value of the power σ we have the following proposition, [4, Theorem 185], which holds for all real numbers α:

PROPOSITION 7.7. *For any $\alpha \in \mathbb{R}$, there exist infinitely many $\frac{m}{n} \in \mathbb{Q}$, such that $|\alpha n - m| \leq n^{-1}$.*

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