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ASYMPTOTIC AND NUMERICAL APPROXIMATIONS OF THE ZEROS OF FOURIER INTEGRALS

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To the memory of Midge Bennahum

Abstract. The asymptotic behavior as $y \to +\infty$ of Fourier integrals of the form

$$f_n(y) = \int_{-\infty}^{\infty} e^{-t^{2n}+iyt}dt, \quad n \in \mathbb{N}, n \geqslant 2,$$

is derived via the method of steepest descents. A general formula is found for the coefficients of the expansion of $f_n(y)$ in the sector $|\arg y| < \frac{2n-1}{2n}\frac{\pi}{2}$ centered about the anti-Stokes line $y \in \mathbb{R}$. High order asymptotic approximations of the zeros of $f_n(y)$ which are all real is also obtained. A simple numerical method designed to compute the zeros of $f_n(y)$ is described. For $n = 2$ and $n = 3$ the asymptotic estimates of the zeros are compared to numerically computed values.

Key words. asymptotic expansions, steepest descents, Fourier integrals, Pearcey integral, zeros

AMS subject classifications. 30E15, 33B10, 41A60

1. Introduction.

In [20], Pólya showed that functions of the form

$$\int_{-\infty}^{\infty} e^{-at^{4n}+bt^{2n}+ct^2+iyt}dt, \qquad n \in \mathbb{N}, n \geqslant 1, a > 0, b \in \mathbb{R}, c \geqslant 0, \tag{1.1}$$

have only real zeros. Similar results are found in [19] concerning functions of the form

$$f_n(y) = \int_{-\infty}^{\infty} e^{-t^{2n}+iyt}dt, \qquad n \in \mathbb{N}, n \geqslant 2. \tag{1.2}$$

In [6], de Bruijn generalized Pólya's results to a larger class of functions whose zeros are real. Recently, Paris analyzed in [16] a generalized form of the Pearcey integral of which the "Pólya" functions given by (1.1) are particular cases. He studied the asymptotic behavior of

$$P_n'(X,Y) = \int_{-\infty}^{\infty} e^{i(u^{2n}+Xu^n+Yu)}du, \qquad n \in \mathbb{N}, n \geqslant 2, \tag{1.3}$$

as $|X| \to \infty$ or $|Y| \to \infty$. By rotation of the path of integration ($u = te^{\frac{\pi i}{4n}}$) and use of Jordan's lemma, it can be expressed as

$$P_n'(X,Y) = P_n(x,y) = e^{\frac{\pi i}{4n}} \int_{-\infty}^{\infty} e^{-t^{2n}-xt^n+iyt}dt, \tag{1.4}$$

with $x = Xe^{-\frac{\pi i}{4}}$ and $y = Ye^{\frac{\pi i}{4n}}$. For n even, $x > 0$, $P_n(x,y)$ falls into the category of functions (1.1) considered by Pólya. For $x = 0$, f_n and P_n are related by $f_n(y) =$

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$e^{-\frac{\pi i}{4n}}P_n(0,y)$. The Pearcey integral $(P_2'(X,Y))$ has been studied by Kaminski [14] and Paris [17], and references therein. The advantage of the method described by Paris is that it avoids the complicated, sometimes impossible task of finding a closed form expression for the saddle points.

Although a large portion of this work is devoted to the derivation of the asymptotic expansion of $f_n(y)$ as $y \to +\infty$, our main objective is the derivation of asymptotic approximations for the zeros of $f_n(y)$. The order of this real analytic even function which is defined as the positive number λ_n for which $\max_{|y|=r}|f_n(y)| \leqslant \exp(r^{\lambda_n+\epsilon})$, $\forall\epsilon > 0$ as soon as r is sufficiently large, is the rational number $1 < \lambda_n = \frac{2n}{2n-1} < 2$. The order being fractional, it is known that $f_n(y)$ has infinitely many zeros [2, 4], which, from the results of Pólya, must be real; thus we are mainly interested in the behavior of $P_n(0,y)$ for large real values of y. The expansion we find is an expression on the anti-Stokes line $\arg y = 0$ where two saddle points have equal contributions. Hence we first consider the real valued function $f_n(y) : \mathbb{R} \to \mathbb{R}$, which by the change of variable $t \to z(\frac{y}{2n})^{\frac{1}{2n-1}}$, for $|\arg y| < \frac{2n-1}{2n}\frac{\pi}{2}$, can be expressed in terms of another function $\mathcal{F}_n(\mu)$ defined as follows:

$$f_n(y) = \left(\frac{y}{2n}\right)^{\frac{1}{2n-1}} \mathcal{F}_n\left(\left(\frac{y}{2n}\right)^{\frac{2n}{2n-1}}\right), \tag{1.5}$$

where for $|\arg\mu| < \frac{\pi}{2}$

$$\mathcal{F}_n(\mu) = \int_{-\infty}^{\infty} e^{\mu(2niz-z^{2n})}dz = 2\int_0^{+\infty} \cos(2n\mu z)e^{-\mu z^{2n}}dz. \tag{1.6}$$

The advantage of introducing the function $\mathcal{F}_n(\mu)$ is that its saddle points are fixed to the unit disk, contrary to those of $f_n(y)$ which depend on the large variable y.

The expansion of functions of the type of $f_n(y)$ can be found as early as 1916 in the work of Brillouin in [5], and then in 1924 in the work of Burwell [8] who obtained first order asymptotics for the location of the zeros of such functions (see also [3]). Recently, Christ also characterized the zeros of similar functions in [9, lemma 2.1]. In [18, chap. 3], Paris and Wood investigate the asymptotic properties of high order differential equations whose solutions have integral representations closely related to (1.2). In their work, they derive recurrence relations to determine the coefficients of the asymptotic expansions (see for example [18, equ. 3.4.16]). We provide a different approach than Paris' and Wood's using the classical method of steepest descents, and we derive the full (generalized) asymptotic expansion of $\mathcal{F}_n(\mu)$ as $\mu \to +\infty$ valid in the sector $|\arg\mu| < \pi/2$, together with high order asymptotic approximations of its zeros. We provide a systematic way of calculating every coefficient of the expansion of $\mathcal{F}_n(\mu)$ via series reversion in terms of multinomial coefficients. This formulation can be compared to the results of Paris and Wood in [18] in which the coefficients are described by a $2n$-term recursion relation which is derived from ODE methods.

In the last section, we describe a simple numerical algorithm which computes the zeros of the function $\mathcal{F}_n(\mu)$. This algorithm is efficient for small values of the zeros, and for $n = 2$ and $n = 3$ it is implemented to gauge the accuracy of the asymptotic estimates. We use the following definitions and notations:

DEFINITION 1.1. *Compound asymptotic expansion (c.a.e.) of $f(z)$ with respect to the asymptotic sequences $\{\phi_n^1(z)\}$ and $\{\phi_n^2(z)\}$: we write*

$$f(z) \overset{z\to z_0}{\approx} g_1(z)\left[\sum_{n=0}^{\infty} f_n^1(z);\ \{\phi_n^1(z)\}\right] + g_2(z)\left[\sum_{n=0}^{\infty} f_n^2(z);\ \{\phi_n^2(z)\}\right],$$

where it is understood that

$$f(z) \sim g_1(z)\left[\sum_{n=0}^{\infty} f_n^1(z) + o(\phi_n^1(z))\right] + g_2(z)\left[\sum_{n=0}^{\infty} f_n^2(z) + o(\phi_n^2(z))\right] \quad as\ z \to z_0.$$

As an alternative to a c.a.e., it may be possible to express the asymptotic expansion of $f(z)$ as a generalized asymptotic expansion:

DEFINITION 1.2. *Generalized asymptotic expansion (g.a.e.) of $f(z)$ with respect to the asymptotic sequence $\{\phi_n(z)\}$: let $\{\phi_n(z)\}$ be an asymptotic sequence as $z \to z_0 \in R$, where R is a region in the complex plane, and $f(z)$, $f_n(z)$, $n = 0, 1, \cdots$, are functions such that for each positive integer m*

$$f(z) = \sum_{n=0}^{m-1} f_n(z) + \mathcal{O}(\phi_m(z)) \quad (z \to z_0 \in R).$$

Then we say that $\sum_n f_n(z)$ is a generalized asymptotic expansion with respect to the asymptotic sequence $\{\phi_n(z)\}$ and write

$$f(z) \sim \sum_{n=0}^{\infty} f_n(z); \quad \{\phi_n(z)\}\ as\ z \to z_0 \in R.$$

For convenience we also write it as

$$f(z) \underset{\{\phi_n(z)\}}{\overset{z\to z_0}{\sim}} \sum_{n=0}^{\infty} f_n(z).$$

We prove the following:

THEOREM 1.1. *Let $n \in \mathrm{N}, n \geqslant 2$, and for $|\arg \mu| < \pi/2$ let*

$$\mathcal{F}_n(\mu) = \int_{-\infty}^{\infty} e^{\mu(2niz - z^{2n})} dz.$$

The generalized asymptotic expansion of $\mathcal{F}_n(\mu)$ as $\mu \to +\infty$ with respect to the asymptotic sequence $\{\phi_j(\mu) = \mu^{-j}\}$, valid in the sector $|\arg \mu| < \frac{\pi}{2}$, is

$$\mathcal{F}_n(\mu) \underset{\{\mu^{-j}\}}{\overset{\mu\to+\infty}{\sim}} \sqrt{\frac{4\pi}{n(2n-1)\mu}} \exp\left\{-\mu(2n-1)\sin\left(\frac{\pi}{4n-2}\right)\right\} \mathcal{H}_n(\mu),$$

where

$$\mathcal{H}_n(\mu) = \sum_{j=0}^{\infty} \frac{\alpha_{n,j}}{\mu^j} \cos\left(\mu(2n-1)\cos\left(\frac{\pi}{4n-2}\right) + \frac{\pi}{4n-2}(1 - n(1+2j))\right),$$

and the coefficients $\alpha_{n,j}$ are normalized rational numbers ($\alpha_{n,0} = 1$) given by

$$\alpha_{n,j} = \frac{\Gamma(j+1/2)}{\sqrt{\pi}(n(2n-1))^j} \cdot \sum_{m=0}^{2j} \left\{ \frac{(1/2-j-m)_m}{(n(2n-1))^m} \cdot \sum_{\sigma}{}' \prod_{k=1}^{2n-2} \frac{1}{\sigma_k!} \binom{2n}{k+2}^{\sigma_k} \right\}.$$

The summation $\sum'_\sigma$ is to take place over all possible $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_{2n-2}) \in \mathrm{N}^{2n-2}$ such that $\sigma_1 + \sigma_2 + \cdots + \sigma_{2n-2} = m$, and $\sigma_1 + 2\sigma_2 + \cdots + (2n-2)\sigma_{2n-2} = 2j$. Moreover, *the first order approximation of the k-th ordered positive zero of $\mathcal{F}_n(\mu)$ is given by (for $k \geqslant 1$)*

$$\mu_{k,n}^{(0)} = \frac{\pi}{4n-2}\sec\left(\frac{\pi}{4n-2}\right)\left(\frac{n-1}{2n-1} - 1 + 2k\right) + \mathcal{O}\left(\frac{1}{k}\right) \quad \text{as } k \to +\infty.$$

Let

$$\mathcal{G}_n(\mu) = \mu + \frac{\sec(\frac{\pi}{4n-2})}{(2n-1)\,\mu}\left\{\alpha_{n,1}\sin\left(\frac{n\pi}{2n-1}\right) - \frac{\alpha_{n,1}^2 - 2\alpha_{n,2}}{2\mu}\sin\left(\frac{2n\pi}{2n-1}\right)\right.$$
$$\left. + \frac{\alpha_{n,1}^3 - 3\alpha_{n,1}\alpha_{n,2} + 3\alpha_{n,3}}{3\mu^2}\sin\left(\frac{3n\pi}{2n-1}\right) - \frac{\sec(\frac{\pi}{4n-2})}{(2n-1)}\frac{\alpha_{n,1}^2}{\mu^2}\sin^2\left(\frac{n\pi}{2n-1}\right)\right\},$$

then the fourth order approximation is given by

$$\mu_{k,n} = \mathcal{G}_n\left(\mu_{k,n}^{(0)}\right) + \mathcal{O}\left(\frac{1}{k^4}\right) \quad \text{as } k \to +\infty.$$

The corresponding $k-$th ordered zero $y_{k,n}$ of $f_n(y)$ is given by

$$y_{k,n} = \pm(2n)\,\mu_{k,n}^{\frac{2n-1}{2n}}. \tag{1.7}$$

Similarly, the corresponding expansion for the function $f_n(y)$ is obtained from the one of $\mathcal{F}_n(\mu)$ by the relation (1.5).

2. Asymptotic expansion of $F(\mu) = \int_{-\infty}^{\infty} e^{\mu(4iz-z^4)}dz$ as $\mu \to +\infty$. We first describe the procedure for $n = 2$ corresponding to a special case of the Pearcey integral, which we generalize in the following section to arbitrary $n \in \mathbb{N}$. The result in this section has been derived by Paris and Wood in [18, pp. 64-72] using differential equation methods, and will serve as comparison. The coefficients they derive corresponding to the coefficients $\alpha_{2,j}$ in Corollary 2.1 are given in terms of a recurrence relation, whereas we offer a different approach, and a different formulation in terms of elementary functions and combinatorial coefficients. This section serves as exposition for the general case $n \in \mathbb{N}$, and as such contains more details.

We are interested in deriving an asymptotic expansion of $F(\mu)$ as $\mu \to +\infty$, where

$$F(\mu) = \int_{-\infty}^{\infty} e^{\mu w(z)}dz, \quad w(z) = 4iz - z^4.$$

The method we use to do so is a standard method for asymptotic expansions of integrals depending on a parameter (cf. [6, 11, 22]). This method, known as Debye's method of steepest descents, is based on deforming the original path of integration through the local extrema of the integrand. The new path is chosen in such a way that along it, the integrand does not oscillate; i.e. the imaginary part of $w(z)$ remains constant. If there are several extrema z_s, only those for which $\Re w(z_s)$ is greatest are taken into consideration. Those that qualify are called the contributing saddle points. In our analysis, we expect two such extrema which must satisfy the condition $\Re w(z_0) = \Re w(z_1)$. These two equally relevant saddle points allow for the cancellation which generates the zeros of $F(\mu)$.

2.1. Saddle points, steepest paths and contour deformation. We first locate the zeros of $w'(z)$ which we denote $z_s = \xi_s + i\,\eta_s$, and refer to as the saddle points of the integrand. For the quartic polynomial $w(z) = 4iz - z^4$, there are three saddle points:

$$0 = w'(z_s) = 4i - 4z_s^3 \Longrightarrow \{z_0, z_1, z_2\} = \left\{e^{\frac{\pi i}{6}}, e^{\frac{5\pi i}{6}}, e^{-\frac{\pi i}{2}}\right\}. \tag{2.1}$$

To determine which saddle points have a dominant contribution, we find $\Re w(z_s)$ for $s = 0, 1, 2$. Since $0 = \frac{z_s}{4}\,w'(z_s) = iz_s - z_s^4$, we find $w(z_s) = 4iz_s - z_s^4 = 3iz_s$, and therefore

$$\begin{aligned}\{w(z_0), w(z_1), w(z_2)\} &= \left\{3e^{\frac{2\pi i}{3}}, 3e^{\frac{4\pi i}{3}}, 3\right\}\\ &= \left\{-3/2 + i\,3\sqrt{3}/2, -3/2 - i\,3\sqrt{3}/2, 3\right\}.\end{aligned}$$

It would therefore seem that the dominant contribution comes from $z_2 = -i$. However we will see that it is not possible to deform the original integration path through z_2. It is also apparent that z_0 and z_1 are equally valid candidates for they have the same contribution:

$$\Re w(z_0) = \Re w(z_1) = -3/2.$$

It is in fact this symmetry which allows for the cancellation of the two asymptotic expansions generated by z_0 and z_1, which in turn will permit the determination of the asymptotic zeros of $F(\mu)$ with as much precision as necessary. Note that subsequently we often use the subscript s to state a property that is valid for both relevant saddle-points indexed by $s = 0, 1$. The deformed path of integration must satisfy the following conditions:

(i) The new path must go through a zero z_s of $w'(z)$
(ii) $\Im w(z) = \Im w(z_s)$ on the new path
(iii) $\Re w(z) \leqslant \Re w(z_s)$ on the new path

The next step consists of analyzing the hills, valleys, and paths of steepest descent and ascent of these saddle points. The level curves separating the hills and valleys of the saddle points z_s and the steepest paths emerging from them are given by

(i) Steepest Paths: $\Im\{\,w(z) - w(z_s)\,\} = 0$
(ii) Level curves: $\Re\{\,w(z) - w(z_s)\,\} = 0$

where

$$\begin{aligned}w(z) = w(\xi + i\eta) &= 4i(\xi + i\eta) - (\xi + i\eta)^4\\ &= -4\eta - \xi^4 + 6\eta^2\xi^2 - \eta^4 + 4i(\xi - \xi^3\eta + \eta^3\xi).\end{aligned}$$

The level curves that separate the hills and valleys above and below the saddle points are determined by the real branches of the following equations:

$$\begin{gathered}-4\eta - \xi^4 + 6\eta^2\xi^2 - \eta^4 = \Re w(z_s),\\ \{\Re w(z_0), \Re w(z_1), \Re w(z_2)\} = \{-3/2, -3/2, 1\}.\end{gathered}$$

Solving the bi-quadratic equation in ξ, for ξ as a function of η wherever it is permitted (both ξ and η are real variables), the asymptotic behavior of these curves as $\eta \to \pm\infty$

is given by $\xi(\eta) \sim \pm\sqrt{3 \pm 2\sqrt{2}}\,\eta$; that is, they all end at $\infty \exp(\frac{2k+1}{8}\pi i)$, for some $k \in \mathbb{N}$.

The steepest paths out of each of the saddle points are determined by the real branches of the following cubic equations:

$$\xi - \xi^3\eta + \xi\eta^3 = \Im w(z_s)/4,$$
$$\{\Im w(z_0), \Im w(z_1), \Im w(z_2)\} = \left\{3\sqrt{3}/2, -3\sqrt{3}/2, 0\right\}.$$

It can be shown that the (steepest) descent paths emerging from the saddle points go from $\infty e^{i\pi/2} \leftarrow z_0 \rightarrow +\infty$, and from $-\infty \leftarrow z_1 \rightarrow \infty e^{i\pi/2}$; the (steepest) ascent paths go from $\infty e^{-i\pi/4} \leftarrow z_0 \rightarrow \infty e^{i\pi/4}$, and from $\infty e^{i3\pi/4} \leftarrow z_1 \rightarrow \infty e^{i5\pi/4}$. The steepest descent path through $z_2 = -i$ is the imaginary axis ($\xi = 0$), and as such, we may not deform the original path through it. Therefore this saddle point does not contribute to the asymptotic expansion of $F(\mu)$. The convergence of the integral is preserved because the new paths always remain in the valleys of z_0 and z_1, and $w(e^{i\frac{\pi}{2}}z) = \mathcal{O}(-(iz)^4) = \mathcal{O}(-z^4)$ as $z \rightarrow +\infty$. The path deformation through z_0 and z_1 displayed in Fig. 2.1 is justified by a simple application of Cauchy's theorem. The solid lines represent the steepest paths, the dotted ones represent the level curves separating the hills and valleys above and below the saddle points z_0, z_1, z_2. The complete topography of the surface $u(\xi,\eta)$ is shown on Fig. 2.1.

Although we have just seen that it is possible to carry out the full (global) analysis of the steepest descent paths, it is not necessary do so. From a local analysis of the steepest directions at the saddle-points, one can choose a simple path that will have the desired properties. Let α_s be the steepest descent direction at the saddle-point z_s (also known as the axis of the saddle point z_s). It is determined by the inequality (cf. [6, chap. 5])

$$(z - z_s)^2 \frac{w''(z_s)}{2!} \leqslant 0 \Longrightarrow \arg\{z - z_s\} = \pm\pi/2 - \arg\{w''(z_s)\}.$$

Since $\alpha_s = \lim_{z \to z_s} \arg\{z - z_s\}$ along the steepest paths, where the correct choice of α_s is determined by the direction in which the saddle-point is crossed, we find that $\alpha_s = -(-1)^s\pi/6$. The path we consider is a combination of half-lines and line segments in the complex plane: $\gamma = \left(-\infty, -\sqrt{3}\right] \cup \mathcal{L}_1 \cup \mathcal{L}_0 \cup \left[\sqrt{3}, +\infty\right)$, where $\mathcal{L}_0$ and $\mathcal{L}_1$ are given by (see Fig. 2.2)

$$\begin{cases} \mathcal{L}_0: & z(t) = e^{i\pi/6} + e^{-i\pi/6}t \quad -1 \leqslant t \leqslant 1 \\ \mathcal{L}_1: & z(t) = e^{i5\pi/6} + e^{i\pi/6}t \quad -1 \leqslant t \leqslant 1 \end{cases}$$

Note that $\mathcal{L}_s$ is the parametrized line interval $z(t) = z_s + e^{i\alpha_s}t$ for $-1 \leqslant t \leqslant 1$, on which $z(0) = z_s$, $z(-1) = -\sqrt{3}$, $z(1) = i$ on $\mathcal{L}_1$, and $z(-1) = i$, $z(1) = \sqrt{3}$ on $\mathcal{L}_0$. Let $h_s(t) = \Re\{w(z(t)) - w(z_s)\} : [-1,1] \rightarrow \mathbb{R}$. Since $h_s(t) = -6t^2 - (-1)^s 2t^3 + t^4/2$, its only maximum for $t \in [-1,1]$ is located at $t = 0$ which corresponds to $z = z_s$. Hence the maximal contribution on the paths $\mathcal{L}_0, \mathcal{L}_1$ occurs at z_0, z_1. Moreover the contributions on the real intervals $\left(-\infty, -\sqrt{3}\right]$ and $\left[\sqrt{3}, +\infty\right)$ are negligible since $\Re w(z) = -z^4$. Thus the new path γ is admissible and we can still apply the series reversion process that follows. Indeed, we would only need to verify that the steepest descents path run from $\pm\infty$ to $i\infty$, and we would infer that the descent path γ is asymptotically equivalent to the steepest descent path Γ.

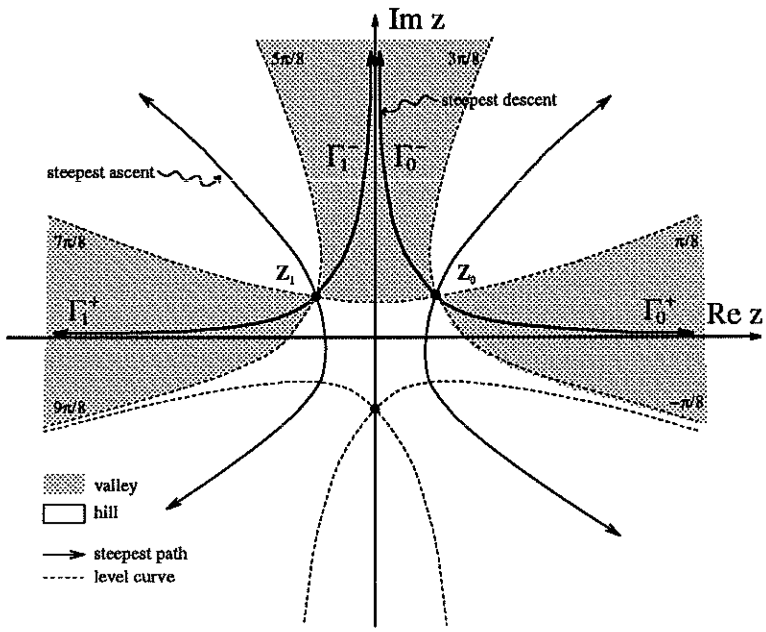

FIG. 2.1. *Hills, valleys, level curves and steepest paths of the saddle points $z_0 = e^{i\pi/6}$, $z_1 = e^{i5\pi/6}$ relevant to the expansion of $F(\mu) = \int_{-\infty}^{\infty} e^{\mu(4iz - z^4)} dz$ as $\mu \to +\infty$.*

Now that the path deformation is justified, we can proceed with the expansion regardless of whether we use the exact steepest descent path Γ, or just an approximate descent path γ. Since on the steepest descent paths $\Im\{w(z) - w(z_s)\} = 0$ for $s = 0, 1$, we have $w(z) - w(z_s) = -\tau$, $\tau \in \mathbb{R}^+$. Therefore,

(i) as $z \to e^{i\frac{k\pi}{2}}\infty$ for $k \in \mathrm{N}$, $\tau = w(z_s) - w(z) \to +\infty$,
(ii) as $z \to z_s$, $\tau = w(z_s) - w(z) \to 0$.

Deforming the path of integration from the real axis to

$$\Gamma = \Gamma_0 \cup \Gamma_1 = \Gamma_1^- - \Gamma_1^+ + \Gamma_0^+ - \Gamma_0^-,$$

$F(\mu)$ can be written as

$$F(\mu) = \int_\Gamma e^{\mu w(z)}\, dz = \int_{\Gamma_1^- - \Gamma_1^+} e^{\mu w(z)}\, dz + \int_{\Gamma_0^+ - \Gamma_0^-} e^{\mu w(z)}\, dz.$$

Here $\Gamma_s^\pm$ are the respective steepest descent paths emerging from the saddle points z_s, where the $\pm$ signs refer to the corresponding branches $z_s^\pm(\tau)$ of the solution to the equation $\tau = w(z_s) - w(z)$ on Γ_s. The assignment of the correct branches $z_s^\pm(\tau)$ to the two steepest descent paths emerging from the saddle points z_s will follow once we have the series expansion for $z_s^\pm(\tau)$ about $\tau = 0$. Proceeding with the path deformation, we

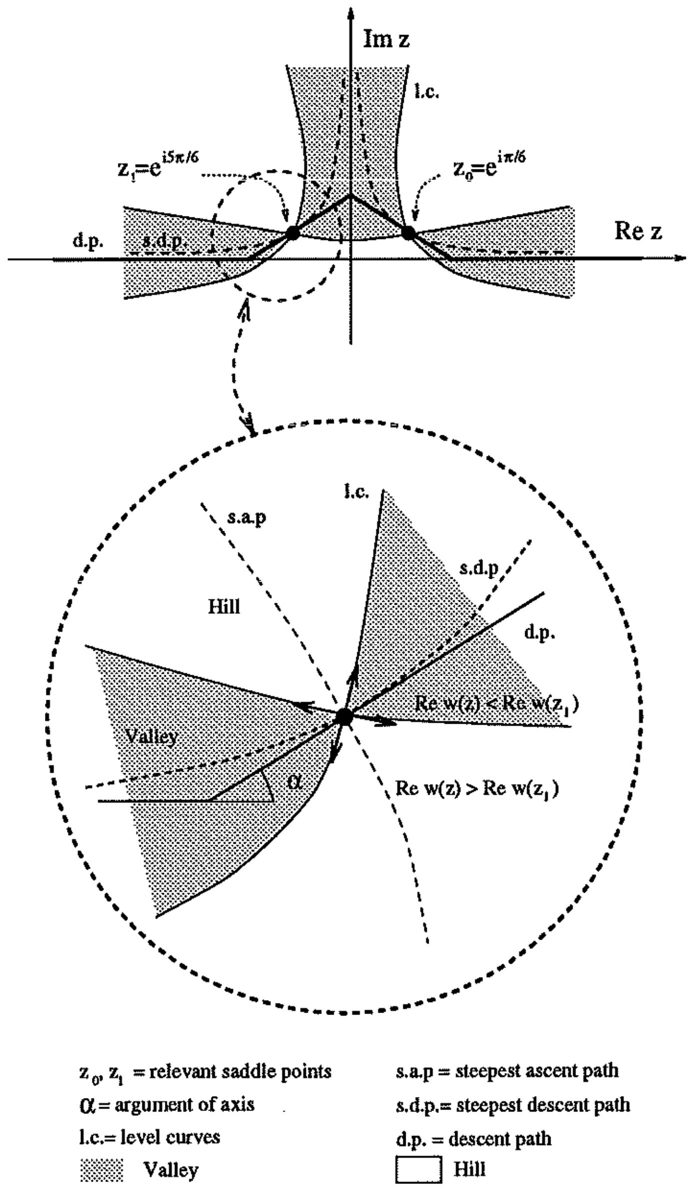

FIG. 2.2. *Comparison of the local/global analysis of the hills, valleys, level curves and steepest paths of the saddle points $z_0 = e^{i\pi/6}$, $z_1 = e^{i5\pi/6}$ relevant to the expansion of $F(\mu) = \int_{-\infty}^{\infty} e^{\mu(4iz-z^4)}dz$ as $\mu \to +\infty$.*

have

$$
\begin{aligned}
F(\mu) &= \int_{\Gamma_1^- - \Gamma_1^+} e^{\mu(w(z_1)-\tau)}\,dz + \int_{\Gamma_0^+ - \Gamma_0^-} e^{\mu(w(z_0)-\tau)}\,dz \\
&= \int_0^{+\infty} e^{\mu(w(z_1)-\tau)} \left(\frac{dz_1^-}{d\tau} - \frac{dz_1^+}{d\tau}\right)(\tau)\,d\tau \\
&\qquad + \int_0^{+\infty} e^{\mu(w(z_0)-\tau)} \left(\frac{dz_0^+}{d\tau} - \frac{dz_0^-}{d\tau}\right)(\tau)\,d\tau \\
&= \sum_{s=0,1} (-1)^s e^{\mu w(z_s)} \int_0^{+\infty} \Phi_s(\tau)\, e^{-\mu\tau}\,d\tau,
\end{aligned}
$$

where $\Phi_s(\tau) = \left(\frac{dz_s^+}{d\tau} - \frac{dz_s^-}{d\tau}\right)(\tau)$.

2.2. Series reversion. We have transformed the original integral into a sum of Laplace integrals. It is now necessary to find a series expansion in the sense of Watson (cf. [11, §22]) for $\Phi_s(\tau)$, and justify the use of Watson's Lemma in order to obtain the asymptotic expansion of $F(\mu)$ by term-by-term integration. From the Lagrange formula for the reversion of series (see [12]), we can invert the equation $\tau = w(z_s) - w(z)$ for z as a function of τ for τ near 0. We find that the two branches $z_s^{\pm}(\tau)$ corresponding to the steepest descent paths starting at $z = z_s^{\pm}(0) = z_s$ are given by convergent series in powers of $\sqrt{\tau}$ in a neighborhood of $\tau = 0$:

$$z_s^{\pm}(\tau) = z_s + \sum_{n=1}^{\infty} c_n(z_s)\left(\pm\sqrt{\tau}\right)^n, \tag{2.2}$$

where

$$c_n(z_s) = \frac{1}{n!}\lim_{z\to z_s}\frac{d^{n-1}}{dz^{n-1}}\left\{f(z)^{-n/2}\right\}, \tag{2.3}$$

and $f(z) = (w(z_s) - w(z))/(z - z_s)^2$ is defined by

$$\tau = w(z_s) - w(z) = (z - z_s)^2 f(z) = (z - z_s)^2\left(6z_s^2 + 4z_s(z - z_s) + (z - z_s)^2\right).$$

In the definition of the coefficients $c_n(z_s)$, we are taking the principal value of the square root of $f(z)$ for which $\sqrt{f(z_s)} = \sqrt{6}z_s$. We thus have

$$\frac{dz_s^{\pm}}{d\tau}(\tau) = \sum_{n=1}^{\infty}\frac{n}{2}c_n(z_s)(\pm 1)^n\tau^{n/2-1}, \tag{2.4}$$

so that

$$\Phi_s(\tau) = \frac{dz_s^+}{d\tau} - \frac{dz_s^-}{d\tau} = \sum_{n=1}^{\infty} nc_n(z_s)\left(\frac{1-(-1)^n}{2}\right)\tau^{n/2-1}.$$

We may now look into assigning different branches of $z_s^{\pm}(\tau)$ to the different paths $\Gamma_s^{\pm}$. The motion of $z_s^{\pm}(\tau)$ along the paths $\Gamma_s^{\pm}$ as τ increases from $\tau = 0$ to some $0 < \tau \ll 1$ is determined by

$$z_s^{\pm}(\tau) = z_s \pm c_1(z_s)\sqrt{\tau} + \mathcal{O}(\tau) \qquad \text{as } \tau \to 0^+. \tag{2.5}$$

Since $c_1(z_s) = \lim_{z\to z_s} f(z)^{-1/2} = (\sqrt{6}z_s)^{-1}$, we have $\arg(c_1(z_s)) = -\arg(z_s)$. Since $\arg(z_0) = \pi/6$, and $\arg(z_1) = 5\pi/6$, we have $\cos(\arg(z_1)) = \cos(5\pi/6) < 0$, and $\cos(\arg(z_0)) = \cos(\pi/6) > 0$. Hence $z_0^+(\tau)$ has increasing real part for τ increasing, and therefore $z_0^+(\tau)$ is the branch that goes from z_0 to $+\infty$ and conversely $z_0^-(\tau)$ is the branch that goes from z_0 to $e^{i\frac{\pi}{2}}\infty$. We name the branches respectively Γ_0^+, and Γ_0^-. Similarly, we find that $z_1^-(\tau)$ is the branch that goes from z_1 to $e^{i\frac{\pi}{2}}\infty$ and $z_1^+(\tau)$ is the branch that goes from z_1 to $-\infty$. We name the branches respectively Γ_1^- and Γ_1^+.

2.3. Watson's lemma and asymptotic development. In order to apply Watson's Lemma, we need to verify that

$$|\Phi_s(\tau)| = \left|\frac{dz_s^+}{d\tau} - \frac{dz_s^-}{d\tau}\right| < Ke^{b\tau}$$

for some positive K and b independent of τ when $\tau \geqslant \tau_0 > 0$ (see [11]):

$$\begin{aligned} &\tau = w(z_s) - w(z), \qquad \text{for } z \in \Gamma_s \\ \Longrightarrow\ &\frac{d\tau}{dz} = -w'(z) = 4(z^3 - i) = 4(z^3 - z_s^3) \\ \Longrightarrow\ &\Phi_s(\tau) = \frac{dz_s^+}{d\tau}(\tau) - \frac{dz_s^-}{d\tau}(\tau) = \frac{1}{4}\left(\frac{1}{z_s^+(\tau)^3 - z_s^3} - \frac{1}{z_s^-(\tau)^3 - z_s^3}\right). \end{aligned}$$

Since $|\Phi_s(\tau)| = \mathcal{O}(1)$ as $\tau \to +\infty$, using Watson's lemma we may substitute the series expansion of $\Phi_s(\tau)$ (in the sense of Watson) and integrate term by term to obtain a compound asymptotic expansion with respect to the asymptotic sequence $\{\phi_n(\mu) = \mu^{-n}\}$ (see Def. 1.1):

$$\begin{aligned} F(\mu) &\overset{\mu\to+\infty}{\sim} \sum_{s=0,1} (-1)^s e^{\mu w(z_s)} \sum_{n=1}^{\infty} n c_n(z_s) \left(\frac{1-(-1)^n}{2}\right) \Gamma\left(\frac{n}{2}\right) \mu^{-n/2} \\ &= \sum_{s=0,1} (-1)^s e^{\mu w(z_s)} \sum_{n=0}^{\infty} (2n+1) c_{2n+1}(z_s)\, \Gamma\left(n + \frac{1}{2}\right) \mu^{-(n+1/2)}. \end{aligned}$$

We let

$$a_n(z_s) = (2n+1)c_{2n+1}(z_s) = \frac{1}{(2n)!} \lim_{z\to z_s} \frac{d^{2n}}{dz^{2n}} \left\{f(z)^{-(n+1/2)}\right\}, \tag{2.6}$$

so that

$$F(\mu) \overset{\mu\to+\infty}{\sim} \frac{1}{\sqrt{\mu}} \sum_{s=0,1} (-1)^s e^{\mu w(z_s)} \sum_{n=0}^{\infty} a_n(z_s)\, \Gamma\left(n + \frac{1}{2}\right) \mu^{-n}. \tag{2.7}$$

We proceed with the explicit determination of the coefficients $a_n(z_s)$ in the expansion of F, where $a_n(z_s)$ is the $2n$-th coefficient in the Taylor expansion of $f(z)^{-(n+\frac{1}{2})}$ about $z = z_s$. Using the binomial theorem twice, we find

$$\begin{aligned} \left(\frac{f(z)}{6z_s^2}\right)^{-(n+1/2)} &= \left\{1 + \frac{2}{3z_s}(z - z_s)\left(1 + \frac{1}{4z_s}(z - z_s)\right)\right\}^{-(n+1/2)} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \binom{-n-\frac{1}{2}}{k} \binom{k}{j} \left(\frac{8}{3}\right)^k (4z_s)^{j-2k} (z - z_s)^{2k-j}. \end{aligned}$$

In order to take into account every term that contributes to $a_n(z_s)$, we find the range of k over which we sum by setting $2k - j = 2n$, that is $j = 2(k-n)$. Since j ranges from 0 to k, k ranges from n to $2n$. Thus summing over k from n to $2n$, we find

$$\begin{aligned} a_n(z_s) &= \frac{6^{-(n+1/2)}}{4^{2n}} z_s^{-(4n+1)} \sum_{k=n}^{2n} \binom{-n-\frac{1}{2}}{k} \binom{k}{2(k-n)} \left(\frac{8}{3}\right)^k \\ &= 6^{-(2n+1/2)} z_s^{-(4n+1)} \sum_{k=0}^{n} \binom{-n-\frac{1}{2}}{k+n} \binom{k+n}{2k} \left(\frac{8}{3}\right)^k. \end{aligned}$$

We introduce normalized coefficients ($\alpha_0 = 1$) which do not depend on z_s: let

$$\alpha_n = \Gamma(n+1/2)\sqrt{\frac{6}{\pi}}\, a_n(z_s)\, z_s^{4n+1},$$

then the compound asymptotic expansion of $F(\mu)$ with respect to the asymptotic sequence $\{\mu^{-n}\}$ as $\mu \to +\infty$ is

$$F(\mu) \overset{\mu\to+\infty}{\sim} \sqrt{\frac{\pi}{6\mu}} \sum_{s=0,1} (-1)^s e^{\mu w(z_s)} \sum_{n=0}^{\infty} \alpha_n z_s^{-(4n+1)} \mu^{-n},$$

where the rational coefficients α_n are given by

$$\alpha_n = \frac{\Gamma(\frac{1}{2}+n)}{6^{2n}\sqrt{\pi}} \sum_{k=0}^{n} \binom{-n-\frac{1}{2}}{k+n} \binom{k+n}{2k} \left(\frac{8}{3}\right)^k. \tag{2.8}$$

The first five values of α_n can easily be computed using a computer algebra system such as Mathematica [21], either directly from (2.8) or using the Mathematica code provided in Appendix C (see also Table C.1):

$$\alpha_0 = 1,\ \alpha_1 = \frac{7}{144},\ \alpha_2 = \frac{385}{41472},\ \alpha_3 = \frac{39655}{17915904},\ \alpha_4 = \frac{665665}{10319560704}. \tag{2.9}$$

One can compare the coefficients α_n to the coefficients c_n introduced by Paris and Wood in [17, p. 397] and [18, p. 72, equ. 3.4.16] which are found via a 4-term recurrence relation. It is easy to see that they are related by $\alpha_n = c_n/3^n$. Since

$$z_0 = e^{i\frac{\pi}{6}},\quad z_0^{4n} = e^{\frac{2n\pi}{3}i},\quad w(z_0) = w(e^{i\frac{\pi}{6}}) = -\frac{3}{2} + i\frac{3\sqrt{3}}{2},$$
$$z_1 = e^{i\frac{5\pi}{6}},\quad z_1^{4n} = e^{-\frac{2n\pi}{3}i},\quad w(z_1) = w(e^{i\frac{5\pi}{6}}) = -\frac{3}{2} - i\frac{3\sqrt{3}}{2},$$

we have

$$F(\mu) \overset{\mu\to+\infty}{\sim} \sqrt{\frac{\pi}{6\mu}} e^{-\frac{3}{2}\mu} \Bigg\{ e^{i(3\frac{\sqrt{3}}{2}\mu - \frac{\pi}{6})} \sum_{n=0}^{\infty} \alpha_n e^{-i\frac{2n\pi}{3}} \mu^{-n}$$
$$+\, e^{-i(3\frac{\sqrt{3}}{2}\mu - \frac{\pi}{6})} \sum_{n=0}^{\infty} \alpha_n e^{i\frac{2n\pi}{3}} \mu^{-n} \Bigg\}.$$

We can formulate this as a generalized asymptotic expansion with respect to the asymptotic sequence $\{\mu^{-n}\}$ (see Def. 1.2):

$$F(\mu) \underset{\{\mu^{-n}\}}{\overset{\mu\to+\infty}{\sim}} \sqrt{\frac{2\pi}{3\mu}} e^{-\frac{3}{2}\mu} \sum_{n=0}^{\infty} \alpha_n \cos\left(3\frac{\sqrt{3}}{2}\mu - \frac{\pi}{6} - \frac{2n\pi}{3}\right) \mu^{-n}. \tag{2.10}$$

2.4. Asymptotic zeros of $F(\mu)$. To determine the asymptotic zeros of $F(\mu)$, we make use of the compound asymptotic relation

$$h(\mu) = \sqrt{\frac{3\mu}{2\pi}} e^{\frac{3}{2}\mu} F(\mu) \overset{\mu\to+\infty}{\sim} \cos\left(3\frac{\sqrt{3}}{2}\mu - \frac{\pi}{6}\right) \sum_{n=0}^{\infty} \alpha_n \cos\left(\frac{2n\pi}{3}\right) \mu^{-n} \tag{2.11}$$
$$+ \sin\left(3\frac{\sqrt{3}}{2}\mu - \frac{\pi}{6}\right) \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{2n\pi}{3}\right) \mu^{-n}.$$

Let $h_m(\mu)$ be the partial sum

$$h_m(\mu) = \cos\left(3\frac{\sqrt{3}}{2}\mu - \frac{\pi}{6}\right)\sum_{n=0}^{m} \alpha_n \cos\left(\frac{2n\pi}{3}\right)\mu^{-n} + \sin\left(3\frac{\sqrt{3}}{2}\mu - \frac{\pi}{6}\right)\sum_{n=1}^{m} \alpha_n \sin\left(\frac{2n\pi}{3}\right)\mu^{-n}, \tag{2.12}$$

then solving the equation $h_m(\mu) = 0$ is equivalent to solving

$$\tan\left(3\frac{\sqrt{3}}{2}\mu - \frac{\pi}{6}\right) = -\frac{\sum_{n=0}^{m} \alpha_n \cos\left(\frac{2n\pi}{3}\right)\mu^{-n}}{\sum_{n=1}^{m} \alpha_n \sin\left(\frac{2n\pi}{3}\right)\mu^{-n}}; \tag{2.13}$$

that is

$$3\frac{\sqrt{3}}{2}\mu - \frac{\pi}{6} = k\,\pi - \tan^{-1}\left(\frac{\sum_{n=0}^{m} \alpha_n \cos\left(\frac{2n\pi}{3}\right)\mu^{-n}}{\sum_{n=1}^{m} \alpha_n \sin\left(\frac{2n\pi}{3}\right)\mu^{-n}}\right), \tag{2.14}$$

where $|\tan^{-1} x| < \pi/2$. For m sufficiently large, we have as $\mu \to +\infty$,

$$\tan^{-1}\left(\frac{\sum_{n=0}^{m} \alpha_n \cos\left(\frac{2n\pi}{3}\right)\mu^{-n}}{\sum_{n=1}^{m} \alpha_n \sin\left(\frac{2n\pi}{3}\right)\mu^{-n}}\right) = \frac{\pi}{2} - \frac{7}{96\sqrt{3}\,\mu} + \frac{7}{576\sqrt{3}\,\mu^2} + \frac{49}{497664\sqrt{3}\,\mu^4} - \frac{379351}{268738560\sqrt{3}\,\mu^5} + \mathcal{O}\left(\frac{1}{\mu^6}\right). \tag{2.15}$$

Thus we have as $\mu \to +\infty$

$$3\frac{\sqrt{3}}{2}\mu - \frac{\pi}{6} = k\,\pi - \left\{\frac{\pi}{2} - \frac{7}{96\sqrt{3}\,\mu} + \frac{7}{576\sqrt{3}\,\mu^2} + \frac{49}{497664\sqrt{3}\,\mu^4} - \frac{379351}{268738560\sqrt{3}\,\mu^5}\right\} + \mathcal{O}\left(\frac{1}{\mu^6}\right). \tag{2.16}$$

Letting

$$\mu_k^{(0)} = \frac{2\pi}{3\sqrt{3}}(k - 1/3), \qquad k \geqslant 1 \tag{2.17}$$

we have

$$\mu = \mu_k^{(0)} + \frac{7}{432\,\mu}\left\{1 - \frac{1}{6\,\mu} - \frac{7}{5184\,\mu^3} + \frac{54193}{2799360\,\mu^4}\right\} + \mathcal{O}\left(\frac{1}{\mu^6}\right). \tag{2.18}$$

We now state a lemma which enables us to improve the first order asymptotic estimate to a higher order one. Its proof is based on the reversion of (asymptotic) series by either Lagrange's formula (see [15, p. 21,§8.4]), or using the method of successive re-substitution (see Appendix A):

LEMMA 2.1. *If $\mu^{(0)} = \mathcal{O}(k)$ as $k \to +\infty$, and*

$$\mu = \mu^{(0)} + \frac{a_1}{\mu}\left(a_2 + \frac{a_3}{\mu} + \frac{a_4}{\mu^2} + \frac{a_5}{\mu^3} + \frac{a_6}{\mu^4}\right) + \mathcal{O}\left(\frac{1}{\mu^6}\right)$$

is an asymptotic relation which holds as $\mu \to +\infty$, *then*

$$\mu = \mu^{(0)} + \frac{a_1}{\mu^{(0)}}\left(a_2 + \frac{a_3}{\mu^{(0)}} + \frac{a_4 - a_1 a_2^2}{\mu^{(0)^2}} + \frac{a_5 - 3a_1a_2a_3}{\mu^{(0)^3}} + \frac{a_6 - 2a_1a_3^2 + 2a_1^2a_2^3 - 4a_1a_2a_4}{\mu^{(0)^4}}\right) + \mathcal{O}\left(\frac{1}{k^6}\right) \qquad as\ k \to +\infty.$$

Combining (2.8), (2.17), (2.18) and Lemma 2.1 we have proven

COROLLARY 2.1. *For* $n = 2$, *the rational coefficients* $\alpha_{2,j}$ *in the expansion of* $\mathcal{F}_2(\mu) = \int_{-\infty}^{\infty} e^{\mu(4ix - x^4)}dx$ *are*

$$\alpha_{2,j} = \frac{\Gamma(\frac{1}{2}+j)}{6^{2j}\sqrt{\pi}} \sum_{k=0}^{j} \binom{-j-\frac{1}{2}}{k+j}\binom{k+j}{2k}\left(\frac{8}{3}\right)^k .$$

For $k \geqslant 1$, *the approximation of the k-th ordered positive zero* $\mu_{k,2}$ *of* $\mathcal{F}_2(\mu)$ *is given by*

$$\mu_{k,2}^{(0)} = \frac{2\pi}{3\sqrt{3}}\left(k - \frac{1}{3}\right) + \mathcal{O}\left(\frac{1}{k}\right) \qquad as\ k \to +\infty.$$

The sixth order approximation is

$$\mu_{k,2} = \mathcal{G}_2\left(\mu_{k,2}^{(0)}\right) + \mathcal{O}\left(\frac{1}{k^6}\right) \qquad as\ k \to +\infty,$$

$$\mathcal{G}_2(\mu) = \mu + \frac{7}{432\mu}\left(1 - \frac{1}{6\mu}\left(1 + \frac{7}{72\mu}\left(1 - \frac{5}{12\mu}\left(1 + \frac{53143}{18900\mu}\right)\right)\right)\right).$$

The fact that this is actually the expansion of the k-th ordered zero of $F(\mu)$ can be proved by the argument principle (see [12, 15]).

3. Asymptotic expansion of $\mathcal{F}_n(\mu)$ as $\mu \to +\infty$. For $n \geqslant 2$ and $|\arg\mu| < \pi/2$, we consider the function introduced in (1.6):

$$\mathcal{F}_n(\mu) = \int_{-\infty}^{\infty} e^{\mu(2niz - z^{2n})}dz.$$

The saddle points z_s of the integrand and their contributions are given by

$$\left\{\begin{array}{l} w_n(z) = 2niz - z^{2n} \\ w_n'(z_s) = 2niz_s - 2nz_s^{2n-1} = 0 \\ 0 = \frac{z_s}{2n}w_n'(z_s) = iz_s - z_s^{2n} \end{array}\right. \implies \left\{\begin{array}{l} z_s = \exp\left(\frac{i\pi}{4n-2}(1+4k)\right) \\ w_n(z_s) = (2n-1)iz_s \\ \Re w_n(z_s) = -(2n-1)\Im z_s \end{array}\right.$$

Thus

$$\Re w_n(z_s) = -(2n-1)\sin\left(\frac{\pi}{4n-2}(1+4k)\right), \qquad k = 0, 1, \cdots, 2n-2.$$

On the two steepest descent paths emerging from relevant saddle points we have

$$w_n(z) - w_n(z_s) = -(z - z_s)^2 f_n(z) = -\tau \leqslant 0,$$

$$f_n(z) = -\sum_{k=0}^{2n-2} \frac{w_n^{(k+2)}(z_s)}{(k+2)!}(z - z_s)^k.$$

As before we expect to have contributions from two equally relevant saddle points which come in symmetric pairs satisfying the relation $z_0 = -\overline{z_1}$ (see section 3.2). The assignment of the branches $z_s^{\pm}(\tau)$ to the steepest descent paths must be dealt with carefully. From Lagrange's formula, we express $z_s^{\pm}(\tau)$ as convergent series in powers of $\sqrt{\tau}$ in a neighborhood of $\tau = 0$:

$$z_s^{\pm}(\tau) = z_s + \sum_{j=1}^{\infty} \mathfrak{c}_{n,j}(z_s)(\pm\sqrt{\tau})^j,$$

with

$$\mathfrak{c}_{n,j}(z_s) = \frac{1}{j!} \lim_{z\to z_s} \frac{d^{j-1}}{dz^{j-1}} \left\{ f_n(z)^{-j/2} \right\}.$$

Since $\mathfrak{c}_{n,1}(z_s) = f_n(z_s)^{-1/2} = \binom{2n}{2}^{-1/2} z_s^{1-n}$, the behavior of $z_s^{\pm}(\tau)$ in a neighborhood of $\tau = 0$ is determined by

$$z_s^{\pm}(\tau) = z_s \pm \frac{z_s^{1-n}}{\sqrt{\binom{2n}{2}}}\sqrt{\tau} + \mathcal{O}(\tau) \qquad \text{as } \tau \to 0^+.$$

In the definition of $\mathfrak{c}_{n,j}(z_s)$, we have taken the principal value of $\sqrt{f_n(z)}$ for which $\sqrt{f_n(z_s)} = \sqrt{\binom{2n}{2}} z_s^{n-1}$. In what follows, we assume that the pair of relevant saddle points $\{z_0, z_1 = -\overline{z_0}\}$ is the first pair with smallest positive imaginary part, that is, $z_0 = \exp\left(\frac{i\pi}{4n-2}\right)$ and $z_1 = -\exp\left(-\frac{i\pi}{4n-2}\right)$. This is so because it is the only pair whose steepest descent paths are admissible, in the sense that the original path of integration cannot be deformed through any of the steepest descent paths emerging from the other saddle points with negative imaginary part. Indeed, such saddle points would yield an incorrect increasing exponential behavior since we would then have $\Re w_n(z_s) = -(2n-1)\Im z_s > 0$. None of the other saddle points with positive imaginary part (all saddle points come in symmetric pair $\{z_s, -\overline{z_s}\}$ except those for which $\Re z_s = 0$) have admissible steepest descent paths. Even if it was possible to deform the path of integration through another pair, their contribution would be exponentially smaller than that of the pair $\{z_0, -\overline{z_0}\}$. On the steepest descent paths $\Gamma_1^{\pm}$ corresponding to the equation $\tau = w(z_1) - w(z)$, we have

$$z_1^{\pm}(\tau) = z_1 \mp (-1)^n \frac{e^{i\frac{n-1}{4n-2}\pi}}{\sqrt{\binom{2n}{2}}}\sqrt{\tau} + \mathcal{O}(\tau) \qquad \text{as } \tau \to 0^+.$$

Following the motion of $z_1^{\pm}(\tau)$ along $\Gamma_1^{\pm}$ for increasing $\tau > 0$ as in (2.5), one can correctly choose the branches $z_1^{\pm}(\tau)$. Hence we see that the assignment of the branches $z_1^{\pm}(\tau)$ changes from the upper branch to the lower one (as shown in Fig. 3.1) depending on the parity of the index n. Note that this feature is not present in the case of $z_0^{\pm}(\tau)$. In the case $n = 3$ the paths of steepest descent labeled $\Gamma_0^{\pm}$ go from $e^{\frac{\pi i}{3}}\infty \leftarrow z_0 = e^{\frac{\pi i}{10}} \to +\infty$; the ones labeled $\Gamma_1^{\pm}$ go from $-\infty \leftarrow z_1 = -\overline{z_0} = e^{\frac{9\pi i}{10}} \to e^{\frac{2\pi i}{3}}\infty$. There is a third path labeled Γ_2 which connects Γ_1^{+} and Γ_0^{-}. This third path remains in the common valley of the saddle points z_0 and $-\overline{z_0}$ and is subdominant with respect to the other paths. In other words, its contribution is exponentially small compared to the contributions of Γ_0 and Γ_1. In the general case the topography remains similar.

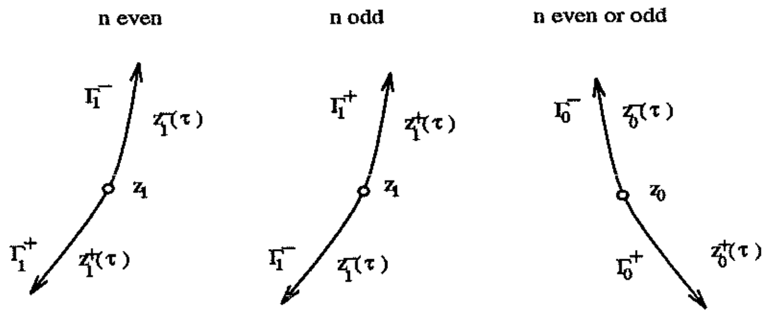

FIG. 3.1. *Interchange of the branch assignment $z_1^{\pm}(\tau)$ according to the parity of the index n.*

We expect the paths of steepest descent emerging from the saddle points at $z_0 = e^{\frac{\pi i}{4n-2}}$ and $z_1 = -\overline{z_0}$ to end in respective valleys. Let

$$\Gamma_n = \Gamma_0^+ - \Gamma_0^- + \Gamma_1^{(-)^{n+1}} - \Gamma_1^{(-)^n} + \Gamma_2$$

denote the new path of integration, where

$$\Gamma_1^{(-)^n} = \begin{cases} \Gamma_1^+ & \text{if } n \text{ even} \\ \Gamma_1^- & \text{if } n \text{ odd} \end{cases}$$

and reciprocally for $\Gamma_1^{(-)^{n+1}}$. The asymptotic behavior of these paths is as follows:

$$\begin{aligned} &\Gamma_0^+ : z_0 \to +\infty, \qquad \Gamma_0^- : z_0 \to \infty\, e^{i\pi/n} \\ &\Gamma_1^{(-)^{n+1}} : z_1 \to \infty\, e^{i(\pi-\pi/n)}, \qquad \Gamma_1^{(-)^n} : z_1 \to -\infty, \\ &\Gamma_2 : \infty\, e^{i(\pi-\pi/n)} \to \infty\, e^{i\pi/n}. \end{aligned}$$

One can also choose a simple descent path which is the straight line $\gamma_n : \Im z = \Im z_0 = \sin(\frac{\pi}{4n-2})$ in the complex plane going through both saddle points z_0 and $-\overline{z_0}$ parallel to the real axis depicted in Fig. 3.2 as a dashed path (see the argument in [9, Lemma 2.1]). We now deform the original contour of integration along the path Γ_n or γ_n as in Fig. 3.2, and we take into account the interchange of the branches $z_1^{\pm}(\tau)$ based on the parity of the index n by including a factor $(-1)^{s\cdot n}$, $s = 0, 1$ in (2.7). We notice that $|\Phi_s(\tau)| = |dz_s^+/d\tau - dz_s^-/d\tau| = \mathcal{O}(1)$ as $\tau \to +\infty$, so we can appeal to Watson's lemma to find a compound asymptotic expansion for $\mathcal{F}_n(\mu)$ with respect to the asymptotic sequence $\{\phi_j(\mu) = \mu^{-j}\}$:

$$\mathcal{F}_n(\mu) \overset{\mu\to+\infty}{\sim} \frac{1}{\sqrt{\mu}} \sum_{s=0,1} (-1)^{s(1+n)} e^{\mu w_n(z_s)} \sum_{j=0}^{\infty} \mathfrak{a}_{n,j}(z_s)\, \Gamma\Big(j+\frac{1}{2}\Big)\, \mu^{-j}, \tag{3.1}$$

$$\mathfrak{a}_{n,j}(z_s) = (2j+1)\, \mathfrak{c}_{n,2j+1}(z_s) = \frac{1}{(2j)!} \lim_{z\to z_s} \frac{d^{2j}}{dz^{2j}} \left\{ f_n(z)^{-(j+1/2)} \right\}. \tag{3.2}$$

3.1. Coefficients of the expansion.

Let

$$\mathfrak{a}_{n,j}(z_s) = \frac{1}{(2j)!} \lim_{z\to z_s} \frac{d^{2j}}{dz^{2j}} \{ g_j(f_n(z)) \},$$

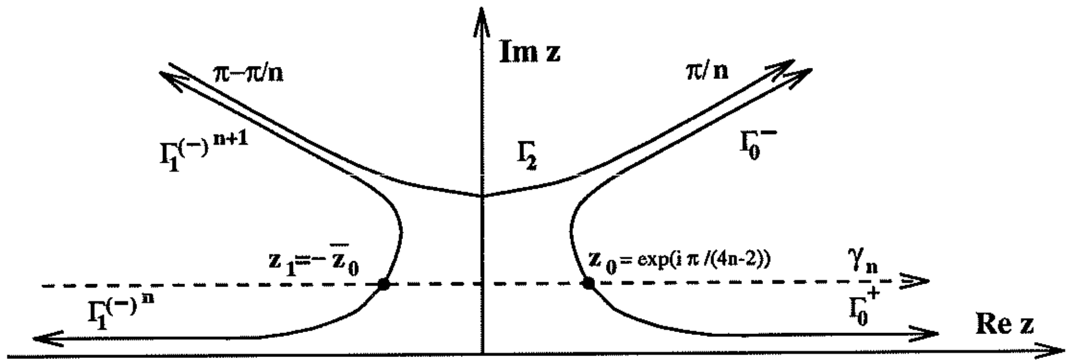

FIG. 3.2. *Steepest descent path (solid)* $\Gamma_n = \Gamma_0^+ - \Gamma_0^- + \Gamma_1^{(-)^{n+1}} - \Gamma_1^{(-)^n} + \Gamma_2$, *and alternate path (dashed)* $\gamma_n : \Im z = \Im z_0 = sin(\frac{\pi}{4n-2})$ *in the expansion of* $\mathcal{F}_n(\mu) = \int_{-\infty}^{\infty} e^{\mu(2niz - z^{2n})}\, dz$ *as* $\mu \to +\infty$ *for* $n \geqslant 3$.

where

$$g_j(z) = z^{-j-1/2}, \quad f_n(z) = \sum_{k=0}^{2n-2} \binom{2n}{k+2} z_s^{2n-2-k} (z - z_s)^k.$$

According to the definition of Pochhammer's symbol $(z)_n$, we let $(1/2 - j - m)_m = \Gamma(1/2 - j)/\Gamma(1/2 - j - m) = (-j - 1/2)\cdot(-j - 3/2)\cdots(1/2 - j - m)$ for $m \geqslant 1$, and $(1/2 - j)_0 = 1$, so that we may write

$$g_j^{(m)}(f_n(z_s)) = \left(n(2n-1)\, z_s^{2n-2}\right)^{-j-1/2-m} (1/2 - j - m)_m \tag{3.3}$$

$$\frac{f_n^{(k)}(z_s)}{k!} = \left\{ \begin{array}{ll} \binom{2n}{k+2} z_s^{2n-2-k} & 0 \leqslant k \leqslant 2n-2 \\ 0 & k \geqslant 2n-1 \end{array} \right. \tag{3.4}$$

Using (3.3) and (3.4) in Faà di Bruno's formula (see (B.1)), we find

$$\begin{aligned} \mathfrak{a}_{n,j}(z_s) = \sum_{m=0}^{2j} \Bigg\{ & \left(n(2n-1)\, z_s^{2n-2}\right)^{-j-\frac{1}{2}-m} (1/2 - j - m)_m \\ & \times \sum_{\sigma}{}' \prod_{k=1}^{2j} \frac{1}{\sigma_k!} \left(\binom{2n}{k+2} z_s^{2n-2-k} \right)^{\sigma_k} \Bigg\}. \end{aligned} \tag{3.5}$$

The summation $\sum'_\sigma$ is taken over all $2j$-vectors $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_{2j}) \in \mathbb{N}^{2j}$ such that the following conditions are simultaneously satisfied:

$$\left\{ \begin{array}{c} \sigma_1 + \sigma_2 + \cdots + \sigma_{2j} = m \\ \sigma_1 + 2\sigma_2 + \cdots + 2j\sigma_{2j} = 2j \\ \sigma_k = 0, \quad \forall\, k \geqslant 2n-1 \end{array} \right\}. \tag{3.6}$$

The last condition $\sigma_k = 0, \forall\, k \geqslant 2n-1$ arises from the fact that $f_n(z)$ is a polynomial of order $2n-2$ and therefore any derivative of order $k \geqslant 2n-1$ of $f_n(z)$ is zero. In order for the product to be non-trivial, we must set the corresponding powers σ_k to zero when $k \geqslant 2n-1$ (see (B.1)). This amounts to using the truncated $(2n-2)$-vector $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_{2n-2}) \in \mathbb{N}^{2n-2}$ in the last product, whereby we can reduce conditions (3.6) to a new set of conditions

$$\left\{ \begin{array}{c} \sigma_1 + \sigma_2 + \cdots + \sigma_{2n-2} = m \\ \sigma_1 + 2\sigma_2 + \cdots + (2n-2)\sigma_{2n-2} = 2j \\ \sigma_k = 0, \quad \forall\, k \geqslant 2n-1 \end{array} \right\}. \tag{3.7}$$

We express (3.5) as

$$\mathfrak{a}_{n,j}(z_s) = \sum_{m=0}^{2j} \Bigg\{ \left(n(2n-1)\, z_s^{2n-2}\right)^{-j-\frac{1}{2}-m} (1/2-j-m)_m \times \sum_{\sigma}{}' \prod_{k=1}^{2n-2} \frac{1}{\sigma_k!} \left(\binom{2n}{k+2} z_s^{2n-2-k} \right)^{\sigma_k} \Bigg\}. \tag{3.8}$$

Using the first and second condition in (3.7), we notice that

$$\prod_{k=1}^{2n-2} z_s^{(2n-2-k)\cdot\sigma_k} = z_s^{(2n-2)\sum_{k=1}^{2n-2}\sigma_k - \sum_{k=1}^{2n-2} k\sigma_k} = z_s^{(2n-2)m-2j}.$$

We can therefore extract the z_s dependency from the summation signs in (3.8):

$$\mathfrak{a}_{n,j}(z_s) = \frac{z_s^{1-n(1+2j)}}{\left(n(2n-1)\right)^{j+1/2}} \sum_{m=0}^{2j} \Bigg\{ \frac{(1/2-j-m)_m}{\left(n(2n-1)\right)^m} \times \sum_{\sigma}{}' \prod_{k=1}^{2n-2} \frac{1}{\sigma_k!} \binom{2n}{k+2}^{\sigma_k} \Bigg\}. \tag{3.9}$$

We introduce normalized coefficients ($\alpha_{n,0} = 1$) which do not depend on the saddle points z_s:

$$\alpha_{n,j} = \Gamma(j+1/2) \sqrt{\frac{n(2n-1)}{\pi}}\, z_s^{n(1+2j)-1}\, \mathfrak{a}_{n,j}(z_s). \tag{3.10}$$

The j-th coefficient for $j \geqslant 0$ is then a rational number given by

$$\alpha_{n,j} = \frac{\Gamma(j+1/2)}{\sqrt{\pi}\left(n(2n-1)\right)^j} \sum_{m=0}^{2j} \Bigg\{ \frac{(1/2-j-m)_m}{\left(n(2n-1)\right)^m} \times \sum_{\sigma}{}' \prod_{k=1}^{2n-2} \frac{1}{\sigma_k!} \binom{2n}{k+2}^{\sigma_k} \Bigg\}, \tag{3.11}$$

where the summation $\sum_\sigma'$ is taken over all possible $\sigma \in \mathrm{N}^{2n-2}$ such that

$$\left\{ \begin{array}{c} \sigma_1 + \sigma_2 + \cdots + \sigma_{2n-2} = m \\ \sigma_1 + 2\sigma_2 + \cdots + (2n-2)\sigma_{2n-2} = 2j \end{array} \right\}. \tag{3.12}$$

A Mathematica code is provided for the reader's convenience in Appendix C to compute the coefficients $\alpha_{n,j}$ from (3.11) and (3.12) (see also Table C.1).

3.2. Asymptotic expansion of $\mathcal{F}_n(\mu)$ as $\mu \to +\infty$. From equation (3.1) we find

$$\mathcal{F}_n(\mu) \overset{\mu\to+\infty}{\sim} \sqrt{\frac{\pi}{n(2n-1)\mu}} \sum_{s=0,1} (-1)^{s(1+n)} e^{\mu w_n(z_s)} \sum_{j=0}^{\infty} \alpha_{n,j}\, z_s^{1-n(1+2j)} \mu^{-j}.$$

Since $z_0 = -\overline{z_1} \in \Delta = \{z \in \mathbb{C} : |z| = 1\}$, and $w_n(z_0) = \overline{w_n(z_1)}$, we find

$$\mathcal{F}_n(\mu) \underset{\{\mu^{-j}\}}{\overset{\mu\to+\infty}{\sim}} \sqrt{\frac{4\pi}{n(2n-1)\mu}}\, e^{\mu \Re w_n(z_0)}\, \mathcal{H}_n(\mu), \tag{3.13}$$

where $\mathcal{H}_n(\mu)$ should be interpreted as a generalized asymptotic expansion with respect to the asymptotic sequence $\{\mu^{-j}\}$:

$$\mathcal{H}_n(\mu) = \sum_{j=0}^{\infty} \frac{\alpha_{n,j}}{\mu^j} \cos\left(\mu\, \Im w_n(z_0) + (1 - n(1+2j)) \arg(z_0)\right). \tag{3.14}$$

Since $w_n(z_0) = (2n-1)iz_0$ and $z_0 = e^{\frac{\pi i}{4n-2}}$,

$$\Re w_n(z_0) = -(2n-1)\Im z_0 = -(2n-1)\sin\left(\frac{\pi}{4n-2}\right),$$
$$\Im w_n(z_0) = (2n-1)\Re z_0 = (2n-1)\cos\left(\frac{\pi}{4n-2}\right).$$

Thus (3.13) is

$$\mathcal{F}_n(\mu) \underset{\{\mu^{-j}\}}{\overset{\mu\to+\infty}{\sim}} \sqrt{\frac{4\pi}{n(2n-1)\mu}} \exp\left\{-\mu\,(2n-1)\sin\left(\frac{\pi}{4n-2}\right)\right\} \mathcal{H}_n(\mu), \tag{3.15}$$

and (3.14) becomes

$$\mathcal{H}_n(\mu) = \sum_{j=0}^{\infty} \frac{\alpha_{n,j}}{\mu^j} \cos\left(\mu\,(2n-1)\cos\left(\frac{\pi}{4n-2}\right) + \pi\,\frac{1-n(1+2j)}{4n-2}\right). \tag{3.16}$$

3.3. Asymptotic zeros of $\mathcal{F}_n(\mu)$. $\mathcal{H}_n(\mu)$ is the component of the expansion of $\mathcal{F}_n(\mu)$ that determines its zeros, and its m-th partial sum $\mathcal{H}_{n,m}(\mu)$ can also be expressed as a compound asymptotic expansion (see Def. 1.1):

$$\begin{aligned}\mathcal{H}_{n,m}(\mu) &= \cos\left(\mu\,(2n-1)\cos\left(\frac{\pi}{4n-2}\right) - \pi\,\frac{n-1}{4n-2}\right)\sum_{j=0}^{m}\frac{\alpha_{n,j}}{\mu^j}\cos\left(\frac{nj\pi}{2n-1}\right)\\ &+ \sin\left(\mu\,(2n-1)\cos\left(\frac{\pi}{4n-2}\right) - \pi\,\frac{n-1}{4n-2}\right)\sum_{j=1}^{m}\frac{\alpha_{n,j}}{\mu^j}\sin\left(\frac{nj\pi}{2n-1}\right).\end{aligned} \tag{3.17}$$

The first order approximation for the zeros of $\mathcal{H}_{n,m}(\mu)$ is found immediately by setting $\cos\left(\mu\,(2n-1)\cos\left(\frac{\pi}{4n-2}\right) - \pi\,\frac{n-1}{4n-2}\right) = 0$. Thus we find that the k-th ordered positive zero $\mu_{k,n}^{(0)}$ of $\mathcal{F}_n(\mu)$ is given by (for $k \geqslant 1$ so that $\mu_{k,n}^{(0)} > 0$)

$$\mu_{k,n}^{(0)} = \frac{\pi}{4n-2}\sec\left(\frac{\pi}{4n-2}\right)\left(\frac{n-1}{2n-1} - 1 + 2k\right) + \mathcal{O}\left(\frac{1}{k}\right), \tag{3.18}$$

as $k \to +\infty$. Solving the equation $\mathcal{H}_{n,m}(\mu) = 0$ yields

$$\begin{aligned}\mu(2n-1)\cos\left(\frac{\pi}{4n-2}\right) - \pi\frac{n-1}{4n-2}\\ = k\pi - \tan^{-1}\left(\frac{\sum_{j=0}^{m}\frac{\alpha_{n,j}}{\mu^j}\cos\left(\frac{nj\pi}{2n-1}\right)}{\sum_{j=1}^{m}\frac{\alpha_{n,j}}{\mu^j}\sin\left(\frac{nj\pi}{2n-1}\right)}\right).\end{aligned} \tag{3.19}$$

We expand the $\tan^{-1}$ for large μ and sufficiently large m, and combine it with (3.18) and (3.19) to find

$$\begin{aligned}\mu = \mu_{k,n}^{(0)} + \frac{\sec(\frac{\pi}{4n-2})}{(2n-1)\,\mu}\Big\{\alpha_{n,1}\sin\left(\frac{n\pi}{2n-1}\right) - (\alpha_{n,1}^2 - 2\alpha_{n,2})\sin\left(\frac{2n\pi}{2n-1}\right)\frac{1}{2\mu} \\ + (\alpha_{n,1}^3 - 3\alpha_{n,1}\alpha_{n,2} + 3\alpha_{n,3})\sin\left(\frac{3n\pi}{2n-1}\right)\frac{1}{3\mu^2} \\ - (\alpha_{n,1}^4 - 4\alpha_{n,1}^2\alpha_{n,2} + 2\alpha_{n,2}^2 + 4\alpha_{n,1}\alpha_{n,3} - 4\alpha_{n,4})\sin\left(\frac{4n\pi}{2n-1}\right)\frac{1}{4\mu^3} \\ + (\alpha_{n,1}^5 - 5\alpha_{n,1}^3\alpha_{n,2} + 5\alpha_{n,1}\alpha_{n,2}^2 + 5\alpha_{n,1}^2\alpha_{n,3} - 5\alpha_{n,2}\alpha_{n,3} \\ - 5\alpha_{n,1}\alpha_{n,4} + 5\alpha_{n,5})\sin\left(\frac{5n\pi}{2n-1}\right)\frac{1}{5\mu^4}\Big\} + \mathcal{O}\left(\frac{1}{\mu^6}\right).\end{aligned}$$

Appealing to Lemma 2.1, we define

$$\begin{aligned}(3.20)\quad \mathcal{G}_n(\mu) = \mu + \frac{\sec(\frac{\pi}{4n-2})}{(2n-1)\,\mu}\Big\{\alpha_{n,1}\sin\left(\frac{n\pi}{2n-1}\right) - \frac{\alpha_{n,1}^2 - 2\alpha_{n,2}}{2\mu}\sin\left(\frac{2n\pi}{2n-1}\right) \\ + \frac{\alpha_{n,1}^3 - 3\alpha_{n,1}\alpha_{n,2} + 3\alpha_{n,3}}{3\mu^2}\sin\left(\frac{3n\pi}{2n-1}\right) - \frac{\sec(\frac{\pi}{4n-2})}{(2n-1)}\frac{\alpha_{n,1}^2}{\mu^2}\sin^2\left(\frac{n\pi}{2n-1}\right)\Big\}.\end{aligned}$$

Let $\mu_{k,n}$ denote the k-th ordered positive zero of $\mathcal{F}_n(\mu)$, so that the fourth order approximation of $\mu_{k,n}$ is given by

$$\mu_{k,n} = \mathcal{G}_n\left(\mu_{k,n}^{(0)}\right) + +\mathcal{O}\left(\frac{1}{k^4}\right) \quad \text{as } k \to +\infty.$$

Combining (3.11), (3.12), (3.15), (3.16), (3.18), and (3.20), Theorem 1.1 is proved. □

3.4. Coefficients $\alpha_{2,j}$. For $n = 2$ we verify the validity of formula (3.11) by finding the corresponding coefficients $\alpha_{2,j}$ which should match the coefficients α_j of section 2. The conditions (3.7) on $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_{2j})$ come out to be

$$\begin{cases} \sum_{k=1}^{2j} \sigma_k = m \\ \sum_{k=1}^{2j} k\sigma_k = 2j \\ \sigma_k = 0 \quad \forall\, k \geqslant 3 \end{cases}$$

From the third condition, we have that the only non-zero coefficients are σ_1 and σ_2. From the first and second condition, they satisfy the 2×2 system

$$\begin{cases} \sigma_1 + \sigma_2 = m \\ \sigma_1 + 2\sigma_2 = 2j \end{cases}$$

whose unique solution is $\boldsymbol{\sigma} = (\sigma_1 = 2(m-j), \sigma_2 = 2j - m)$. Using $n = 2$ we have

$$\sum_{\boldsymbol{\sigma}}{}' \prod_{k=1}^{2n-2} \frac{1}{\sigma_k!}\binom{2n}{k+2}^{\sigma_k} = \frac{4^{2(m-j)}}{(2(m-j))!}\cdot\frac{1}{(2j-m)!}.$$

Equation (3.11) becomes

$$\alpha_{2,j} = \frac{\Gamma(j+1/2)}{\sqrt{\pi}\,6^j}\sum_{m=0}^{2j}\frac{(1/2-j-m)_m}{6^m}\cdot\frac{4^{2(m-j)}}{(2(m-j))!}\cdot\frac{1}{(2j-m)!}.$$

We discard all terms $m < j$, thus we let $m = k + j$, so that k ranges from 0 to j:

$$\begin{aligned}\alpha_{2,j} &= \frac{\Gamma(j+1/2)}{\sqrt{\pi}\,6^{2j}} \sum_{k=0}^{j} \frac{(k+j)!\binom{-j-1/2}{k+j}}{6^k} \cdot \frac{16^k}{(2k)!} \cdot \frac{1}{(j-k)!} \\ &= \frac{\Gamma(j+1/2)}{\sqrt{\pi}\,6^{2j}} \sum_{k=0}^{j} \binom{-j-1/2}{k+j} \binom{k+j}{2k} \left(\frac{8}{3}\right)^k \\ &= \alpha_j \quad \text{see (2.8)}\end{aligned}$$

3.5. Asymptotic zeros of $\mathcal{F}_3(\mu)$. For $n = 3$, the first four coefficients $\alpha_{3,j}, j = 1, \cdots, 4$ are given by (see Appendix C and Table C.1):

$$\alpha_{3,0} = 1,\ \alpha_{3,1} = \frac{11}{180},\ \alpha_{3,2} = \frac{517}{64800},\ \alpha_{3,3} = \frac{-22253}{174960000}. \tag{3.21}$$

In order to describe the asymptotic approximations of the zeros of $\mathcal{F}_3(\mu)$, we need the following trigonometric expressions:

$$\begin{aligned}&\sin(\pi/10) = -\cos(3\pi/5) = \frac{\sqrt{5}-1}{4}, \quad \cos(6\pi/5) = -\frac{1+\sqrt{5}}{4}, \\ &\sin(6\pi/5) = -\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}, \quad \cos(\pi/10) = \sin(3\pi/5) = \frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}, \\ &\sin(9\pi/5) = -\sin(\pi/5) = -2\sin(\pi/10)\cos(\pi/10).\end{aligned}$$

Using (3.11), the first order approximation is

$$\mu_{k,3}^{(0)} = \frac{2\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}\left(k - \frac{3}{10}\right), \qquad k \geqslant 1 \qquad (\mu > 0) \tag{3.22}$$

and combining (3.20) and (3.21), the fourth order approximation is given in the following Corrolary:

COROLLARY 3.1. *For $n = 3$ and $k \geqslant 1$, the approximation of the k-th ordered positive zero $\mu_{k,3}$ of $\mathcal{F}_3(\mu) = \int_{-\infty}^{\infty} e^{\mu(6iz - z^6)}dz$ is given by*

$$\mu_{k,3}^{(0)} = \frac{2\pi}{5}\sqrt{\frac{2}{5+\sqrt{5}}}\left(k - \frac{3}{10}\right) + \mathcal{O}\left(\frac{1}{k}\right) \quad \text{as } k \to +\infty.$$

The fourth order approximation is

$$\mu_{k,3} = \mathcal{G}_3\left(\mu_{k,3}^{(0)}\right) + \mathcal{O}\left(\frac{1}{k^4}\right) \quad \text{as } k \to +\infty,$$

$$\mathcal{G}_3(\mu) = \mu + \frac{11}{900\,\mu}\left(1 - \frac{\sqrt{5}-1}{20\mu} - \frac{11}{900}\left(1 - \frac{119}{165}\frac{\sqrt{5}-1}{2}\right)\frac{1}{\mu^2}\right).$$

4. Numerical evaluation of the zeros of $\mathcal{F}_n(\mu)$. In this section, a numerical method is designed to compute the zeros of the function $\mathcal{F}_n(\mu)$. The purpose of including such an analysis is to judge the accuracy of the asymptotic approximations. This numerical algorithm shows that the high accuracy of the asymptotic predictions is attained for moderatly large zeros, thereby confirming the strength of the asymptotics.

The function $\mathcal{F}_n(\mu)$ is approximated using Simpson's rule and extrapolation, to which we apply the secant method to locate the zeros. The asymptotic approximations of the zeros $\mu_{k,2}$ and $\mu_{k,3}$ of $\mathcal{F}_2(\mu)$ and $\mathcal{F}_3(\mu)$ derived in the previous sections are compared to their numerically calculated values. We also compare these estimates to the zeros of $h_{10}(\mu) = \mathcal{H}_{2,10}(\mu)$ and $\mathcal{H}_{3,10}(\mu)$ (see (2.12) and (3.17)) which are computed with the secant method.

4.1. Numerical approximation $\mathcal{F}_n^{m,l}(\mu)$ of $\mathcal{F}_n(\mu)$ by Simpson's rule. The numerical evaluation of Pearcey-type integrals has been studied by Connor and Curtis in [10]. However, since we consider a special case of Pearcey integrals, we devise a simple algorithm to numerically evaluate $\mathcal{F}_n(\mu)$. Using the alternate expression for $\mathcal{F}_n(\mu)$ (see (1.6)) given by

$$\mathcal{F}_n(\mu) = 2\int_0^{+\infty} \cos(2n\mu y)\, e^{-\mu y^{2n}}\, dy,$$

we construct the approximation by dividing the range of integration into subintervals over which the integrand does not oscillate. Let

$$x_{-1} = 0, \quad x_k = x_k(\mu) = \frac{(k+1/2)\pi}{2n\mu} \quad \text{for } k \in \mathbb{N},$$

$$g_n(\mu, y) = 2\cos(2n\mu y)\exp(-\mu y^{2n}), \qquad \mathcal{I}_n^k(\mu) = (-1)^k \int_{x_{k-1}}^{x_k} |g_n(\mu, y)|\, dy,$$

so that

$$\mathcal{F}_n(\mu) = \sum_{k=0}^{\infty} \mathcal{I}_n^k(\mu) = \overbrace{\sum_{k=0}^{m} \mathcal{I}_n^k(\mu)}^{\mathcal{Q}_n^m(\mu)} + \overbrace{\sum_{k=m+1}^{\infty} \mathcal{I}_n^k(\mu)}^{\mathcal{R}_n^m(\mu)},$$

where

$$\mathcal{Q}_n^m(\mu) = \int_0^{x_m} g_n(\mu, y)\, dy, \qquad \mathcal{R}_n^m(\mu) = \int_{x_m}^{+\infty} g_n(\mu, y)\, dy.$$

We first estimate the remainder $\mathcal{R}_n^m(\mu)$:

$$|\mathcal{R}_n^m(\mu)| \leqslant 2\int_{x_m}^{+\infty} e^{-\mu y^{2n}}\, dy = \frac{\Gamma\left(\frac{1}{2n}, \mu x_m^{2n}\right)}{n\mu^{\frac{1}{2n}}},$$

where $\Gamma(a, x)$ is the incomplete Gamma function defined for $\Re a > 0$. Since $\Gamma(a, x) \sim e^{-x} x^{a-1}$ as $x \to +\infty$ (see [1]), we find that

$$|\mathcal{R}_n^m(\mu)| \leqslant \frac{e^{-\mu x_m^{2n}}}{n\mu x_m^{2n-1}} \qquad \text{for sufficiently large } \mu x_m^{2n}.$$

The endpoint $x_m(\mu) = (m+1/2)\pi/(2n\mu)$ is chosen in such a way that the contribution from the remainder $\mathcal{R}_n^m(\mu)$ is negligible for a fixed (bounded) μ. If we require that

$$\exp\left(-\mu x_m^{2n}\right) < \varepsilon = 10^{-\kappa}, \quad \kappa \in \mathbb{N}, \tag{4.1}$$

then it yields a good initial choice for m given by

$$m = m[\kappa; n, \mu_{max}] = \mathrm{Int}\left[\frac{2n}{\pi}\mu_{max}^{\frac{2n-1}{2n}}(\kappa \log 10)^{\frac{1}{2n}} - \frac{1}{2}\right] + 1, \tag{4.2}$$

where Int $[x]$ denotes the integer part of x, and μ_{max} is a bound on the largest zero we wish compute. It is clear from this analysis that the larger μ_{max}, the larger m will need to be, which is why this algorithm is practical only for small roots $\mu_{k,n}$.

We now approximate $\mathcal{Q}_n^m(\mu)$ by $\mathcal{Q}_n^{m,l}(\mu)$ for large l and moderate m (due to the rapid decay of the integrand),

$$\mathcal{Q}_n^m(\mu) = \sum_{k=0}^{m} \mathcal{I}_n^k(\mu) \approx \mathcal{Q}_n^{m,l}(\mu) = \sum_{k=0}^{m} \mathcal{I}_n^{k,l}(\mu),$$

where each integral $\mathcal{I}_n^k(\mu)$ is approximated by $\mathcal{I}_n^{k,l}(\mu)$ using Simpson's rule: l is the number of gridpoints and the spacing h is defined by

$$h = \frac{\Delta x_k}{l} = \frac{x_{k+1} - x_k}{l} = \frac{\pi}{2n\mu l}.$$

Since

$$\mathcal{I}_n^k(\mu) = \mathcal{I}_n^{k,l}(\mu) + \mathcal{O}\left(\frac{1}{l^4}\right), \quad \mathcal{Q}_n^m(\mu) = \mathcal{Q}_n^{m,l}(\mu) + \mathcal{O}\left(\frac{m}{l^4}\right),$$

and using extrapolation we define

$$\mathcal{F}_n^{m,l}(\mu) = \mathcal{Q}_n^{m,2l}(\mu) + \frac{\mathcal{Q}_n^{m,2l}(\mu) - \mathcal{Q}_n^{m,l}(\mu)}{2^4 - 1} \Longrightarrow \mathcal{Q}_n^m(\mu) = \mathcal{F}_n^{m,l}(\mu) + \mathcal{O}\left(\frac{m}{l^5}\right).$$

Thus the final approximation is

$$\mathcal{F}_n(\mu) = \mathcal{F}_n^{m,l}(\mu) + \mathcal{O}\left(\frac{m}{l^5}\right) + \mathcal{O}\left(\frac{e^{-\mu x_m^{2n}}}{\mu x_m^{2n-1}}\right)$$

as $\mu x_m^{2n} \to +\infty$ and $l \to +\infty$. Clearly the constraint on this algorithm arises from the choice of l since a moderately large value of $m \ll l$ is sufficient to make the remainder $\mathcal{R}_n^m(\mu)$ as small as desired. Moreover it is difficult to estimate the asymptotic constant in the term $\mathcal{O}(m/l^5)$ which may be large since it involves $\frac{\partial^4}{\partial y^4} g_n(\mu, y)$. Hence the choice for l is made by doubling its value until two successive values of all the zeros $\mu_{k,n} < \mu_{max}$ agree to 10 significant digits.

4.2. Numerical approximation of the zeros of $\mathcal{F}_n^{m,l}(\mu)$ and $\mathcal{H}_{n,m}(\mu)$ by the secant method. We use the secant method to approximate the zeros of $\mathcal{F}_n^{m,l}(\mu)$ and $\mathcal{H}_{n,m}(\mu)$ which appear in (respectively) the "Numerical values" column and the column "$\mathcal{H}_{n,m}^{-1}(0)$" of Tables 4.1 and 4.2. From (3.17), we express $\mathcal{H}_{n,m}(\mu)$ as

$$\mathcal{H}_{n,m}(\mu) = \sum_{j=0}^{m} \alpha_{n,j} \cos\left(\mu\,(2n-1)\cos\left(\frac{\pi}{4n-2}\right) + \pi\,\frac{1 - n(1+2j)}{4n-2}\right)\mu^{-j},$$

and let $\mathcal{K}(\mu)$ stand for either $\mathcal{F}_n^{m,l}(\mu)$ or $\mathcal{H}_{n,m}(\mu)$. Then the procedure consists in successively evaluating for any $k \geqslant 1$ $(\mu > 0)$

$$\mu_{k,n}^0 = \frac{\pi}{4n-2}\sec\left(\frac{\pi}{4n-2}\right)\left(\frac{n-1}{2n-1} - 1 + 2k\right)$$

$$\mu_{k,n}^1 = \mu_{k,n}^0 - \frac{2\,\delta\mu}{\mathcal{K}(\mu_{k,n}^0 + \delta\mu) - \mathcal{K}(\mu_{k,n}^0 - \delta\mu)} \cdot \mathcal{K}(\mu_{k,n}^0) \qquad (\delta\mu = 10^{-2})$$

$$\mu_{k,n}^{j+1} = \mu_{k,n}^j - \frac{\mu_{k,n}^j - \mu_{k,n}^{j-1}}{\mathcal{K}(\mu_{k,n}^j) - \mathcal{K}(\mu_{k,n}^{j-1})} \cdot \mathcal{K}(\mu_{k,n}^j) \qquad j \geqslant 1$$

$\mu_{k,2}$	Numerical zeros	$\mu_{k,2}^{(0)}$ (2.17)	$\mu_{k,2}^{(5)}$ (Cor. 2.1)	$\mathcal{H}_{2,10}^{-1}(0)$ (3.17)
$\mu_{1,2}$	0.8221037147	0.8061330508	0.8227392717	0.8240052094
$\mu_{2,2}$	2.0226889660	2.0153326269	2.0226917275	2.0226893916
$\mu_{3,2}$	3.2292915284	3.2245322031	3.2292916648	3.2292915324
$\mu_{4,2}$	4.4372464748	4.4337317792	4.4372464915	4.4372464749
$\mu_{5,2}$	5.6457167459	5.6429313554	5.6457167492	5.6457167459
$\mu_{6,2}$	6.8544374340	6.8521309316	6.8544374349	6.8544374340
$\mu_{7,2}$	8.0632985369	8.0613305077	8.0632985372	8.0632985369
$\mu_{8,2}$	9.2722462225	9.2705300839	9.2722462225	9.2722462225
$\mu_{9,2}$	10.4812510476	10.4797296601	10.4812510479	10.4812510479

TABLE 4.1
Numerical approximation of the zeros $\mu_{k,2}$ of $\mathcal{F}_2(\mu) = \int_{-\infty}^{\infty} e^{\mu(4iz-z^4)}dz$.

until the convergence of $\mu_{k,n}^j \to \mu_{k,n}$ which is based upon a relative error test of the form

$$\left|\frac{\mu_{k,n}^{j+1} - \mu_{k,n}^j}{\mu_{k,n}^{j+1}}\right| < \text{tol } = 10^{-10}$$

4.3. n=2. If m is chosen so as to satisfy (4.1), then a crude initial choice for l is $l = 10^{\kappa/5}$ (typically $\kappa = 12 \Rightarrow l \approx 250$). We take $\mu_{max} = 11$ to be a bound for the largest zero we wish to compute, and $\kappa = 12$ so that from (4.2), we find that $m = m[12;2,11] = 18$. Starting from $l = 10^{\kappa/5} \approx 250$, we double the value of l until all ten significant figures in the column "Numerical zeros" of Table 4.1 do not change. The first such value is $l = 1000$. Note that for $k \leqslant 5$ ($\mu_{max} \leqslant 6$), $m = 12$ is sufficient. One can see in Table 4.1 that the values computed from the asymptotic approximations are very good. Notice that the first zero $\mu_{1,2}$ is not well approximated by any of the asymptotic predictions since it is less than 1. Beyond the first zero, the asymptotic approximations improve with increasing index k. For $5 \leqslant k \leqslant 8$, $\mathcal{H}_{2,10}^{-1}(0)$ agrees with the numerical values up to 10 digits. For $k \geqslant 5$, $\mu_{k,2}^{(5)}$ and the numerical values agree up to 8 digits. For $k \geqslant 8$, the numerical and asymptotic values grow apart due to the lack of accuracy of the numerical procedure (see the comment following equation (4.2)). Note also that for $k \geqslant 8$, $\mu_{k,2}^{(5)}$ and $\mathcal{H}_{2,10}^{-1}(0)$ agree up to 10 digits ($\mathcal{H}_{2,10}^{-1}(0)$ is computed for the sake of comparison of the asymptotic and numerical estimates). In computing $\mathcal{H}_{2,10}(\mu)$, the 10 coefficients $\alpha_{2,j}, j = 1, \cdots, 10$ are determined using Appendix C. The same is done for $\mathcal{H}_{3,10}(\mu)$ below.

4.4. n=3. Once again we take $\mu_{max} = 11$ to be a bound for the largest zero we wish to compute, and $\kappa = 12$ so that from (4.2), we find that $m = m[12;3,11] = 25$. As in the case $n = 2$, starting from $l = 250$, we double l until all ten significant figures in the column "Numerical zeros" of Table 4.2 do not change. The first such value is $l = 1000$. For $k \geqslant 10$, there is 6 digit accuracy when we compare $\mu_{k,3}^{(3)}$ (see (3.1)) and the numerical values; for $10 \leqslant k \leqslant 14$, there is also 10 digit accuracy when comparing the numerical values with $\mathcal{H}_{3,10}^{-1}(0)$, and 7 digit accuracy between $\mu_{k,3}^{(3)}$ and $\mathcal{H}_{3,10}^{-1}(0)$ for $k \geqslant 16$. These results are reported in Table 4.2.

$\mu_{k,3}$	Numerical zeros	$\mu_{k,3}^{(0)}$ (3.22)	$\mu_{k,3}^{(3)}$ (Cor. 3.1)	$\mathcal{H}_{3,10}^{-1}(0)$ (3.17)
$\mu_{1,3}$	0.5006640277	0.4624572398	0.4845169688	0.46750721075
$\mu_{2,3}$	1.1311965433	1.1231104397	1.1333562062	1.1332534896
$\mu_{3,3}$	1.7905548747	1.7837636396	1.7903635764	1.7903439964
$\mu_{4,3}$	2.4492569634	2.4444168394	2.4492848081	2.4492788273
$\mu_{5,3}$	3.1089250327	3.1050700392	3.1089251416	3.1089227904
$\mu_{6,3}$	3.7689127436	3.7657232391	3.7689140713	3.7689129739
$\mu_{7,3}$	4.4290976016	4.4263764389	4.4290981557	4.4290975781
$\mu_{8,3}$	5.0894021100	5.0870296388	5.0894024443	5.0894021124
$\mu_{9,3}$	5.7497857943	5.7476828386	5.7497859980	5.7497857940
$\mu_{10,3}$	6.4102244359	6.4083360384	6.4102245680	6.4102244359
$\mu_{11,3}$	7.0707027897	7.0689892382	7.0707028789	7.0707027897
$\mu_{12,3}$	7.7312107680	7.7296424381	7.7312108304	7.7312107680
$\mu_{13,3}$	8.3917414319	8.3902956379	8.3917414769	8.3917414319
$\mu_{14,3}$	9.0522898522	9.0509488377	9.0522898854	9.0522898522
$\mu_{15,3}$	9.7128524305	9.7116020376	9.7128524558	9.7128524307
$\mu_{16,3}$	10.373426479	10.372255237	10.373426499	10.373426480

TABLE 4.2
Numerical approximation of the zeros $\mu_{k,3}$ of $\mathcal{F}_3(\mu) = \int_{-\infty}^{\infty} e^{\mu(6iz-z^6)}dz$.

A. Proof of Lemma 2.1. To prove this lemma, we succesively substitute higher estimates in the equation: Let $\zeta = \mu^{(0)}$, then the asymptotic relation reads

$$\mu = \zeta + \frac{a_1}{\mu}\left(a_2 + \frac{a_3}{\mu} + \frac{a_4}{\mu^2} + \frac{a_5}{\mu^3} + \frac{a_6}{\mu^4}\right) + \mathcal{O}\left(\frac{1}{\mu^6}\right).$$

We have

$$\mu = \zeta + \frac{a_1 a_2}{\zeta} + \mathcal{O}\left(\frac{1}{\zeta^2}\right),$$

followed by

$$\mu = \zeta + \frac{a_1 a_2}{\zeta} + \frac{a_1 a_3}{\zeta^2} + \mathcal{O}\left(\frac{1}{\zeta^3}\right).$$

We now have

$$\frac{a_1 a_2}{\mu} = \frac{a_1 a_2}{\zeta}\left(1 - \frac{a_1 a_2}{\zeta^2} - \frac{a_1 a_3}{\zeta^3}\right) + \mathcal{O}\left(\frac{1}{\zeta^5}\right),$$
$$\frac{a_1 a_3}{\mu^2} = \frac{a_1 a_3}{\zeta^2}\left(1 - 2\frac{a_1 a_2}{\zeta^2}\right) + \mathcal{O}\left(\frac{1}{\zeta^5}\right),$$

so that there is a $-(a_1 a_2)^2/\zeta^3$ and a $-3a_1^2 a_2 a_3/\zeta^4$ correction term:

$$\mu = \zeta + \frac{a_1 a_2}{\zeta} + \frac{a_1 a_3}{\zeta^2} + \frac{a_1 a_4 - (a_1 a_2)^2}{\zeta^3} + \frac{a_1 a_5 - 3a_1^2 a_2 a_3}{\zeta^4} + \mathcal{O}\left(\frac{1}{\zeta^5}\right).$$

Finally, we use

$$\begin{aligned}
\frac{a_1a_2}{\mu} &= \frac{a_1a_2}{\zeta}\left(1-\frac{a_1a_2}{\zeta^2}-\frac{a_1a_3}{\zeta^3}-\frac{a_1a_4-(a_1a_2)^2}{\zeta^4}+\frac{(a_1a_2)^2}{\zeta^4}\right)+\mathcal{O}\left(\frac{1}{\zeta^6}\right),\\
\frac{a_1a_3}{\mu^2} &= \frac{a_1a_3}{\zeta^2}\left(1-2\left(\frac{a_1a_2}{\zeta^2}+\frac{a_1a_3}{\zeta^3}\right)\right)+\mathcal{O}\left(\frac{1}{\zeta^6}\right),\\
\frac{a_1a_4}{\mu^3} &= \frac{a_1a_4}{\zeta^3}\left(1-3\frac{a_1a_2}{\zeta^2}\right)+\mathcal{O}\left(\frac{1}{\zeta^6}\right),
\end{aligned}$$

so that we must add a $\left(-4a_1^2a_2a_4+2a_1^3a_2^3-2a_1^2a_3^2\right)/\zeta^5$ correction term. Thus we find

$$\begin{aligned}
\mu = \zeta + \frac{a_1}{\zeta}\Bigg(a_2+\frac{a_3}{\zeta}+\frac{a_4-a_1a_2^2}{\zeta^2}+\frac{a_5-3a_1a_2a_3}{\zeta^3}\\
+\frac{a_6-2a_1a_3^2+2a_1^2a_2^3-4a_1a_2a_4}{\zeta^4}\Bigg)+\mathcal{O}\left(\frac{1}{\zeta^6}\right) \quad \text{as } \zeta=\mu^{(0)}\to+\infty.
\end{aligned}$$

B. Faa di Bruno's formula. For $\boldsymbol{a}=(a_1,a_2,\cdots,a_n)\in \mathrm{N}^n$, following the notation in [1], we define the multinomial coefficients

$$\begin{aligned}
(n;a_1,a_2,\cdots,a_n) &= \frac{n!}{a_1!a_2!\cdots a_n!},\\
(n;a_1,a_2,\cdots,a_n)' &= \frac{n!}{(1!)^{a_1}a_1!(2!)^{a_2}a_2!\cdots(n!)^{a_n}a_n!}.
\end{aligned}$$

The n-th derivative of the composition of two functions is given by Faà di Bruno's formula in [1, §24.1.2] and [13]:

$$\begin{aligned}
\frac{d^n}{dx^n}g(f(x)) &= \sum_{m=0}^{n} g^{(m)}(f(x))\cdot{\sum_{\boldsymbol{a}\in \mathrm{N}^n}}'(n;a_1,a_2,\cdots,a_n)'\cdot\prod_{k=1}^{n}\left\{f^{(k)}(x)\right\}^{a_k}\\
&= \sum_{m=0}^{n} g^{(m)}(f(x))\cdot{\sum_{\boldsymbol{a}\in \mathrm{N}^n}}'\frac{n!}{a_1!a_2!\cdots a_n!}\cdot\prod_{k=1}^{n}\left\{\frac{f^{(k)}(x)}{k!}\right\}^{a_k}, \qquad \text{(B.1)}
\end{aligned}$$

where the second summation sign ${\sum'}_{\boldsymbol{a}\in \mathrm{N}^n}$ is taken over all integer n-vectors $\boldsymbol{a}=(a_1,a_2,\cdots,a_n)\in \mathrm{N}^n$ such that $\sum_k ka_k = a_1+2a_2+\cdots+na_n=n$ and $|\boldsymbol{a}|=\sum_k a_k = a_1+a_2+\cdots+a_n=m$.

C. Mathematica code for the computation of the coefficients $\alpha_{n,j}$.

```
<< DiscreteMath'Combinatorica';

vector[n_, j_, m_] :=
 Module[ {dim,k2},
  dim=2n-2;  k2=2j;
  If[ j==0, 1,
    Apply[ Plus, Map[ (Apply [Times, Flatten[
      MapIndexed[((Binomial[2n, #2+2]^#1)/#1!)&, #],1]])&,
        Select[ Flatten[ Map[ Permutations, Select[
          Map[ (Join[ Table[0, {dim-Length[#]}],#])&, Partitions[m]],
          (Length[#] == dim)&]],1], (Range[dim] . # == k2)& ] ] ] ] ];
```

$\alpha_{n,j}$	1	2	3	4	5
2	$\frac{7}{144}$	$\frac{385}{41472}$	$\frac{39655}{17915904}$	$\frac{665665}{10319560704}$	$\frac{-1375739365}{1486016741376}$
3	$\frac{11}{180}$	$\frac{517}{64800}$	$\frac{-22253}{174960000}$	$\frac{-158440051}{125971200000}$	$\frac{-3797666873}{4534963200000}$
4	$\frac{15}{224}$	$\frac{705}{100352}$	$\frac{-23595}{22478848}$	$\frac{-26196885}{20141047808}$	$\frac{-9089431065}{31581162962944}$
5	$\frac{19}{270}$	$\frac{931}{145800}$	$\frac{-111587}{73811250}$	$\frac{-761484451}{637729200000}$	$\frac{6741607873}{172186884000000}$
6	$\frac{115}{1584}$	$\frac{29785}{5018112}$	$\frac{-42479045}{23846068224}$	$\frac{-163420180175}{151088688267264}$	$\frac{56283394450535}{239324482215346176}$
7	$\frac{27}{364}$	$\frac{1485}{264992}$	$\frac{-188595}{96457088}$	$\frac{-138332205}{140441520128}$	$\frac{128908298475}{357844993286144}$
8	$\frac{217}{2880}$	$\frac{88753}{16588800}$	$\frac{-1487341219}{716636160000}$	$\frac{-7471144611931}{8255648563200000}$	$\cdots$
9	$\frac{35}{459}$	$\frac{1085}{210681}$	$\frac{-295295}{136521288}$	$\frac{-1787240455}{2130551220528}$	$\cdots$
10	$\frac{117}{1520}$	$\frac{23049}{4620800}$	$\frac{-78289029}{35118080000}$	$\cdots$	$\cdots$

TABLE C.1
Coefficients $\alpha_{n,j}$ for $n = 2, \cdots, 10$, and $j = 1, \cdots, 5$.

```
Alpha[n_,j_] :=  Gamma[j+1/2] / (Sqrt[Pi] (n(2n-1))^j) *
  Sum[ Pochhammer[1/2-j-m,m] / (n(2n-1))^m * vector[n,j,m], {m,0,2j}];
```

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REFERENCES

[1] M. ABRAMOWITZ AND I.A. STEGUN, *Handbook of Mathematical Functions*, Dover, 1965.
[2] L.V. AHLFORS, *Complex Analysis*, Mc.Graw Hill, 3rd. edition, 1979.
[3] N.G. BAKHOOM, *Asymptotic expansions of the function $F_k(x) = \int_0^\infty \exp(xu - u^k)du$*, Proc. London Math. Soc. (2) **35** (1933), pp. 83-100.
[4] R.P. BOAS, *Entire Functions*, Academic Press Inc., 1954.
[5] L. BRILLOUIN, *Sur une méthode de calcul approchée de certaines intégrales, dite méthode de col*, Ann. Ec. Norm. **3**, Vol. 33 (1916), pp. 17-69.
[6] N.G. DE BRUIJN, *Asymptotic Methods in Analysis*, Dover, 1981.
[7] N.G. DE BRUIJN, *The roots of trigonometric integrals*, Duke Mathematical Journal **17** (1950), pp. 197-226.
[8] W.R. BURWELL, *Asymptotic expansions of generalized hypergeometric functions*, Proc. Lond. Math. Soc. **2**, Vol. 22 (1924), pp. 57-72.
[9] M. CHRIST, *Examples of analytic non-hypoellipticity of $\overline{\partial_b}$*, Comm. P.D.E. **19** (5 & 6) (1994), pp. 911-941.

[10] J.N.L. Connor and P.R. Curtis, *A method for the numerical evaluation of the oscillatory integrals associated with the cuspoid catastrophes: application to Pearcey's integral and its derivatives*, J. Phys. A **15** (1982), pp. 1179-1190.
[11] E.T. Copson, *Asymptotic Expansions*, Cambridge University Press, 1965.
[12] E.T. Copson, *An Introduction to the Theory of Functions of a Complex Variable*, Oxford University Press, 1957.
[13] C.F. Faà di Bruno, *Note sur une nouvelle formule du calcul differentiel*, Quart. J. Math 1 (1855), pp. 359-360.
[14] D. Kaminski, *Asymptotic expansion of the Pearcey integral near the caustic*, SIAM J. Math. Anal., **20** (1989), pp. 987-1005.
[15] F.W.J. Olver, *Asymptotics and Special Functions*, Academic Press, 1974.
[16] R.B. Paris, *A generalization of Pearcey's integral*, SIAM J. Math. Anal., **25**, No. 2 (1994), pp. 630-645.
[17] R.B. Paris, *The asymptotic behavior of Pearcey's integral for complex variables*, Proc. Roy. Soc. London, Ser. A **432** (1991), pp. 391-426.
[18] R.B. Paris and A.D. Wood, *Asymptotics of higher order differential equations*, Pitman Research Notes in Mathematics Series, vol. 129, London: Longman, 1986.
[19] G. Pólya, *Über trigonometrische integrale mit nur reellen nullstellen*, Journal für die reine und angewandte Mathematik **158** (1927), pp. 6-18.
[20] G. Pólya, *On the zeros of an integral function represented by Fourier's integral*, Messenger of Math. **52** (1923), pp. 185-188.
[21] Wolfram Research, Inc., *Mathematica*, Version 2.2, Champaign, Illinois, 1993.
[22] R. Wong, *Asymptotic Approximations of Integrals*, Academic Press, 1989.