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Abstract

For weak solutions of the incompressible Euler equations, there is energy conservation if the velocity is in the Besov space B_s^3 with s greater than $1/3$. B_s^p consists of functions that are $Lip(s)$ (i.e., Hölder continuous with exponent s) measured in the L^p norm. Here this result is applied to velocities with a specified singularity spectrum which is spatially uniform in a sense made precise below. Such velocities are, roughly speaking, Lip(α) on sets of Hausdorff co-dimension $\kappa(\alpha)$ for a range of values of α. We show that the Frisch-Parisi multifractal formalism is valid for this function class, and that there is energy conservation if $\min_\alpha(3\alpha + \kappa(\alpha)) > 1$. Analogous conservation results are derived for the equations of incompressible ideal MHD (i.e., zero viscosity and resistivity) for both energy and helicity . In addition, a necessary condition is derived for singularity development in ideal MHD generalizing the Beale-Kato-Majda condition for ideal hydrodynamics.

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1 Introduction

In turbulent flow at high Reynolds number, the energy dissipation rate is observed to be approximately independent of the coefficient of viscosity. If the Euler equations for ideal hydrodynamics are to correctly describe the infinite Reynolds number limit for turbulent flow, which is a major open question of fluid mechanics, then energy dissipation and singularities must occur in their solutions.

The situation is similar for magneto-hydrodynamics (MHD) at high Reynolds and magnetic Reynolds number [3]. Although the available evidence is not as clear-cut, energy dissipation is apparently constant in the ideal limit. In contrast, according to the Taylor conjecture, magnetic helicity does not dissipate in the ideal limit. If the ideal MHD equations are to allow reasonable limits of incompressible MHD, these two observations must be reflected in properties of the solutions.

In 1949 Onsager [12] stated that energy is conserved for weak solutions $u \in Lip(\alpha)$ with $\alpha > 1/3$. This result was stated in a famous paper that initiated the statistical theory of point vortices, and was apparently overlooked until the work of Eyink [8], which gave it a rigorous mathematical proof in a certain function class. The proof was considerably simplified and extended to the Besov function space $B^p_s (= B^{p,\infty}_s)$ in subsequent work of Constantin, E and Titi [5].

In this note, we shall specialize the result of [5] to explicitly show the dependence on both the degree of singularity of the velocity and the dimension of the singular set. In particular, we consider a velocity with a specified ***uniform singularity spectrum***, in a sense that is defined precisely below. Roughly speaking, we assume that u is Lip(α) on sets of Hausdorff co-dimension $\kappa = \kappa(\alpha)$, in which α may have a discrete or continuous set of values.

Our main result for ideal hydrodynamics, which is stated formally in Corollary 2.2 below, is that there is energy conservation for weak solutions of the Euler equations if

$$\inf_{\alpha}(3\alpha + \kappa(\alpha)) > 1 \tag{1.1}$$

for all possible values of α. As shown below, this criterion is valid for negative, as well as positive, values of α.

In fact, we show that for this class of functions, the multifractal formalism of Frisch-Parisi [10] is valid, and that the functions are in the Besov space B^p_s for any $s > s_p = \inf_\alpha p\alpha + \kappa(\alpha)$ and for all $1 \leq p < \infty$. So the energy conservation criterion (1.1), using $p = 3$, then follows from [5]. In fact, the criterion (1.1) is correct whenever the Frisch-Parisi formalism is valid.

The energy conservation criterion (1.1) is implicit in the work of Eyink [9] on multifractals and Besov spaces. Nevertheless, we believe that an explicit statement of this criterion and its validation for a particular class of velocities is noteworthy. In particular, it should be helpful in predicting the type of singularities for Euler flows, and in assessing their fluid dynamic significance if they do occur.

We also believe that the definitions formulated here for uniform singularity spectrum may be useful in other contexts.

We then present two results on singularities and energy dissipation for ideal incompressible MHD. First, we derive criteria for energy conservation and helicity conservation for weak solutions of ideal MHD. Second, we show that if smooth initial data for the ideal MHD equations leads to a singularity at a finite time t_*, then

$$\int_0^{t_*} ||\omega||_\infty + ||J||_\infty dt = \infty \tag{1.2}$$

in which $\omega = \nabla \times u$ is the fluid vorticity, $J = \nabla \times B$ is the electrical current, and $|| \; ||_\infty$ is the L^∞ norm in space. This result is analogous to the theorem of Beale-Kato-Majda [2] for singularity formation in ideal hydrodynamics.

2 Uniform Singularity Spectrum

In this section we present precise definitions of the *uniform singularity spectrum* for a function and of two function classes for which the singularity spectrum is specified. Unlike the usual definition of singularity spectrum, this definition requires uniformity in the smoothness of the functions. This definition of uniform singularity spectrum will then be used to define a function class on which the Frisch-Parisi multifractal formalism is valid, so that the role of singularity type and dimension can be explicitly determined in the criterion for energy dissipation.

The Frisch-Parisi formalism was verified for a class of functions based on a wavelet construction by Eyink [9] and for functions obeying a refinement equation by Daubechies and Lagarias [7]. Bacry, Muzy and Arneodo [1] developed an alternative multifractal formalism based on wavelets. On the other hand, Jaffard [11] showed that neither of these two formalisms gives a generally correct characterization of the usual singularity spectrum.

Although the definition of uniform singularity spectrum given below is complicated, it has several attractive features: It is phrased directly in terms of function values rather than the wavelet transform, it is geometric and motivated by simple examples, and it does agree with the Frisch-Parisi formalism. Note also that we lack a procedure for computing the uniform singularity spectrum of a given function and that we do not know a norm corresponding to the function spaces defined below.

2.1 Examples and Motivations

As motivation, consider a function f_1, defined on a set $D \subset R^m$, and assume that f_1 is smooth except on a manifold S_0 of co-dimension κ (an integer) on which it is $Lip(\alpha)$; e.g., $f_1(x) = dist(x, S_0)^\alpha$. Define sets $S(r)$ consisting of points in D within distance r of S_0. Then

$$|S(r)| \equiv vol(S(r)) \leq ar^\kappa \tag{2.1}$$

for some constant a, which will be adjusted for use in subsequent bounds. Next consider the difference of $f_1(x)$ and $f_1(x+y)$ for two points x and $x+y$ that are at least distance r from S_0, i.e. with $x, x+y \in D - S(r)$. Since the derivative of f_1 blows up like $r^{-(1-\alpha)}$ then

$$|f_1(x) - f_1(x+y)| \leq ar^{-(1-\alpha)}|y|. \tag{2.2}$$

Alternatively, f_1 is everywhere $Lip(\alpha)$ if $\alpha \geq 0$, while f_1 is of size r^α if $\alpha < 0$; i.e.

$$|f_1(x) - f_1(x+y)| \leq \begin{cases} a|y|^\alpha & \text{if } \alpha \geq 0 \\ ar^\alpha & \text{if } \alpha < 0 \end{cases} \tag{2.3}$$

for $x, x+y \in D - S(r)$. This can be generalized to a function that is $Lip(\alpha_0)$ in $D - S_0$, in which case the bounds can be combined as

$$|f_1(x) - f_1(x+y)| \leq \Delta_1(r, \alpha_0, \alpha_1) \tag{2.4}$$

if $x, x+y \in D - S(r)$, in which

$$\Delta_1(r, \alpha_0, \alpha_1) = \begin{cases} a|y|^{\alpha_0} r^{-(\alpha_0-\alpha_1)} & \text{if } |y| \leq r \\ a|y|^{\alpha_1} & \text{if } r < |y| \text{ and } \alpha_1 \geq 0 \\ ar^{\alpha_1} & \text{if } r < |y| \text{ and } \alpha_1 < 0 \end{cases} \tag{2.5}$$

In the next subsection bounds similar to (2.4) are used to characterize the uniform singularity spectrum.

Next consider an examples in which there are discrete values for α and κ. First some notation is needed. Fix a value of $\alpha_0 > 0$ and consider monotone sequences of n values of α, of r and of κ; i.e. assume that

$$\begin{aligned} 1 \geq \alpha_0 > \alpha_1 > \ldots > \alpha_n \\ 0 \leq r_1 \leq \ldots \leq r_n \\ 0 = \kappa_0 < \kappa_1 < \ldots < \kappa_n \leq m \end{aligned} \tag{2.6}$$

and define

$$\begin{aligned}
\boldsymbol{\alpha}_n &= (\alpha_0, \alpha_1, ..., \alpha_n) \\
\boldsymbol{r}_n &= (r_1, ..., r_n). \\
\boldsymbol{\kappa}_n &= (0, \kappa_1, \ldots, \kappa_n).
\end{aligned} \tag{2.7}$$

The α_i and κ_i are values of the smoothness degree for a function f and the codimension of the set S_i on which f has smoothness degree α_i. For an integer value of κ_i, the quantity r_i is the Euclidean distance from the set S_i, while for a noninteger value of κ_i, r_i is only a measure of this distance.

Next define a quantity Δ_n that will be a bound on the difference $f(x+y) - f(x)$. This will be defined sequentially, since away from the set S_n, the difference should be bounded by Δ_n times a coefficient $r_n^{-(\alpha_{n-1}-\alpha)}$ that blows up on S_n. Alternatively, the difference should be no worse that $|y|^{\alpha_n}$ if $\alpha_n \geq 0$, or $r_n^{\alpha_n}$ if $\alpha_n < 0$. These considerations lead to the following definition:

$$\Delta_n(\boldsymbol{r}_n, \boldsymbol{\alpha}_n) = \begin{cases} a r_n^{\alpha_n} & \text{if } r_n < |y| \text{ and } \alpha_{n-1} \geq 0 > \alpha_n \\ a|y|^{\alpha_n} & \text{if } r_n < |y| \text{ and } \alpha_n \geq 0 \\ \Delta_{n-1}(\boldsymbol{r}_{n-1}, \boldsymbol{\alpha}_{n-1}) r_n^{-(\alpha_{n-1}-\alpha_n)} & \text{otherwise} \end{cases} \tag{2.8}$$

in which Δ_1 is defined by (2.5).

Let κ_i be integers and let $S_i(0)$ be a nested set of manifolds of codimension κ_i. Denote $l_i(x) = distance(x, S_i)$ and

$$S_i(r) = \{x : l_i(x) \leq r\} \tag{2.9}$$

which satisfy

$$|S_i(r)| \leq a r^{\kappa_i}. \tag{2.10}$$

As a simple example, consider the function

$$f_n(x) = l_1^{\alpha_1} \prod_{i=2}^{n} l_i^{-(\alpha_{i-1}-\alpha_i)}. \tag{2.11}$$

Then

$$|f_n(x+y) - f_n(x)| \leq \Delta_n(\boldsymbol{r}_n, \boldsymbol{\alpha}_n) \tag{2.12}$$

if

$$x + y, x \in \cap_{i=1}^{n}(D - S_i(r_i)). \tag{2.13}$$

The bounds (2.2) and (2.12) on function differences and the bounds (2.1) and (2.9) on set size will be generalized in the subsequent definition of function classes and uniform singularity spectrum.

2.2 Definitions

Here we define a class of functions that are, roughly speaking, $Lip(\alpha_0)$ everywhere except for sets of Hausdorff co-dimension $\kappa(\alpha)$ on which they are $Lip(\alpha)$. At first the values of α will be assumed to be discrete, after which the definition will be extended to a continuous set of α values. The functions will be defined on a subset D of R^m.

Definition. Let $\boldsymbol{\alpha}_n$ and $\boldsymbol{\kappa}_n$ satisfy (2.6) and let $D \subset R^m$. Then $f \in Lip(\boldsymbol{\alpha}_n, \boldsymbol{\kappa}_n)$ if there is a constant a and a nested family of sets $S_i(r) \subset D$ for $0 < r \leq 1$ and $1 \leq i \leq n$, such that

(i)

$$S_i(r) \subset S_{i'}(r') \quad \text{if } r \leq r' \quad \text{and } i \geq i' \tag{2.14}$$

(ii)

$$S_i(1) = D \tag{2.15}$$

(iii) $|S_i(r)| = Vol(S_i(r))$ is a smooth function of r satisfying

$$|S_i(r)| < ar^{\kappa_i} \tag{2.16}$$

(iv) If $x \in D - S_1(1/2)$ then

$$|f(x)| < a. \tag{2.17}$$

(v) For all x and y with

$$x, x+y \in \bigcap_{i=1}^{n} (D - S_i(r_i)) \tag{2.18}$$

f satisfies

$$|f(x+y) - f(x)| < \Delta_n(\boldsymbol{r}_n, \boldsymbol{\alpha}_n). \tag{2.19}$$

Note. (1) Although this definition is complicated, it is motivated by the examples of the previous subsection; i.e. $f_1 \in Lip(\alpha_0, \alpha_1, 0, \kappa_1)$ and $f_n \in Lip(\boldsymbol{\alpha}_n, \boldsymbol{\kappa}_n)$.

(2) These function classes are naturally ordered so that

$$Lip(\boldsymbol{\alpha}_{n-1}, \boldsymbol{\kappa}_{n-1}) \subset Lip(\boldsymbol{\alpha}_n, \boldsymbol{\kappa}_n). \tag{2.20}$$

(3) Note that in the definition and the examples of the previous subsection, the values of α_i can be negative.

Next we define a function class $Lip([\alpha_-, \alpha_+], \kappa(\cdot))$ of functions with continuous uniform singularity spectrum $\kappa(\alpha)$. This function class consists, roughly speaking, of functions that are $Lip(\alpha)$ on sets of Hausdorf co-dimension $\kappa(\alpha)$

for every $\alpha \in [\alpha_-, \alpha_+]$. Such functions could be considered as multi-fractal, but without any self-similarity property.

Definition Let $1 \geq \alpha_+ > \max(0, \alpha_-)$ and $D \subset R^m$. Suppose that $\kappa(\alpha)$ is a continuous, monotone decreasing function defined for $\alpha \in [\alpha_-, \alpha_+]$, with $\kappa(\alpha_-) = 0$ and $\kappa(\alpha_+) \leq m$. Then $f \in Lip([\alpha_-, \alpha_+], \kappa(\cdot))$ if $f \in Lip(\boldsymbol{\alpha}_n, \boldsymbol{\kappa}_n)$ for any increasing set of $n+1$ values of $\alpha_i \in [\alpha_-, \alpha_+]$ with $\kappa_i = \kappa(\alpha_{i-1})$.

Finally define the uniform singularity spectrum for such functions.

Definition If $f \in Lip(\boldsymbol{\alpha}_n, \boldsymbol{\kappa}_n)$, the **discrete uniform singularity spectrum** of f is the discrete set $\boldsymbol{\alpha}_n, \boldsymbol{\kappa}_n$. If $f \in Lip([\alpha_-, \alpha_+], \kappa(\cdot))$, the **continuous uniform singularity spectrum** of f is the function $\kappa(\cdot)$.

2.3 L^p Estimates

Here we derive L^p estimates for any function in terms of its uniform singularity spectrum. These estimates show that such functions are in Besov space.

Lemma 2.1 *Let $f \in Lip(\boldsymbol{\alpha}_n, \boldsymbol{\kappa}_n)$ or $f \in Lip([\alpha_-, \alpha_+], \kappa(\cdot))$ and let $1 \leq p \leq \infty$. Define*

$$s_p = \min_{\alpha}(\alpha + \kappa(\alpha)/p). \tag{2.21}$$

and assume that $s_p > 0$. Then for any $s_p > s > 0$ there is a constant b (depending on $s_p - s$) such that

$$\|f(\cdot + y) - f(\cdot)\|_{L^p} < b|y|^s. \tag{2.22}$$

For $f \in Lip(\boldsymbol{\alpha}_n, \boldsymbol{\kappa}_n)$, the minimum in (2.21) is understood to be over the $n+1$ values α_i with co-dimension values of κ_i. In other words

$$s_p = \min_{0 \leq i \leq n}(\alpha_i + \kappa_i/p). \tag{2.23}$$

Proof of Lemma 2.1 for $Lip(\alpha_0, \alpha_1, 0, \kappa_1)$.

First assume that $\alpha_1 \geq 0$ and rewrite the defining inequality (2.19) with $n = 1$ in a smooth way as

$$|f(x+y) - f(x)| \leq \Delta_1(r) \equiv a(r + |y|)^{-\alpha_0 + \alpha_1}|y|^{\alpha_0} \tag{2.24}$$

for $x, x+y \in D - S(r)$. Also denote

$$\begin{aligned} V(r) &= \text{vol}(S(r)) \leq a(r + |y|)^{\kappa_1} \\ \tilde{V}(r) &= \text{vol}(S(r) \cup (S(r) - y)) \leq 2V(r). \end{aligned} \tag{2.25}$$

Write the integral of the Hölder difference as a Stieljes integral over r, then integrate by parts to estimate (omitting constant factors)

$$
\begin{aligned}
\int_D |f(x+y) &- f(x)|^p dx \le \int_D \Delta_1(r)^p dx \\
&= \int_0^1 \Delta_1(r)^p d\tilde{V}(r) \\
&= -\int_0^1 \frac{\partial}{\partial r}(\Delta_1(r)^p)\tilde{V}(r)dr + \Delta_1(1)^p\tilde{V}(1) \\
&\le |y|^{\alpha_0 p}\left\{\int_0^1 (r+|y|)^{-1-p(\alpha_0-\alpha_1)+\kappa_1} dr + 1\right\} \\
&\le |y|^{sp}\begin{cases} \log|y| & \text{if } \alpha_1+\kappa_1/p=\alpha_0 \\ 1 & \text{otherwise}\end{cases}
\end{aligned} \tag{2.26}
$$

in which $s = \min(\alpha_0, \alpha_1 + \kappa_1/p)$. This proves (2.22) for $\alpha_1 \ge 0$.

On the other hand, if $\alpha_1 < 0$ then

$$
\Delta_1(r) = \min(r^{-(\alpha_0-\alpha_1)}|y|^{\alpha_0}, r^{\alpha_1}) \tag{2.27}
$$

Then, repeating the first few steps of the previous estimation, the bound becomes

$$
\begin{aligned}
\int_D |f(x+y) &- f(x)|^p dx = -2\int_0^1 \frac{\partial}{\partial r}(\Delta_1(r)^p)V(r)dr + 2\Delta_1(1)^p V(1) \\
&\le \int_0^{|y|} r^{-1+p\alpha_1+\kappa_1} dr + |y|^{\alpha_0 p}\int_{|y|}^1 r^{-1-p(\alpha_0-\alpha_1)+\kappa_1} dr + a|y|^{\alpha_0} \\
&\le |y|^{sp}\begin{cases} \log|y| & \text{if } \alpha_1+\kappa_1/p=\alpha_0 \\ 1 & \text{otherwise}\end{cases}
\end{aligned} \tag{2.28}
$$

in which $s = \min(\alpha_0, \alpha_1 + \kappa_1/p) > 0$. This proves (2.22) for $f \in Lip(\alpha_0, \alpha_1, 0, \kappa_1)$.

The proof of Lemma 2.1 for $Lip(\alpha\ _n, \kappa\ _n)$ is a more complicated version of the proof for $f \in Lip(\alpha_0, \alpha_1, 0, \kappa_1)$ and is given in Appendix A. The proof for $Lip([\alpha_-, \alpha_+], \kappa(\cdot))$ follows directly from the result for $Lip(\alpha\ _n, \kappa\ _n)$ by discretization of the values of α.

The Besov spaces are characterized by the L^p bounds proved in Lemma 2.1, which leads to the following result:

Corollary 2.1 *Assume that function $f \in Lip(\alpha\ _n, \kappa\ _n)$ or $f \in Lip([\alpha_-, \alpha_+], \kappa(\cdot))$ and that $1 \le p < \infty$. Define*

$$
s_p = \min(\alpha + \kappa(\alpha)/p). \tag{2.29}
$$

If $s_p > 0$, then $f \in B^p_s$ for any $s_p \ge s > 0$.

This is exactly the formula for s_p in the Frisch-Parisi formalism, which shows validity of the Frisch-Parisi formalism for this function class.

2.4 Energy Conservation for Ideal Hydrodynamics

For simplicity assume that $D = [0,1]^3$ with periodic boundary conditions. A weak solution of the incompressible Euler equation is a function $\boldsymbol{u} = (u_1, u_2, u_3)$ satisfying

$$\begin{aligned} \int_0^T \int_0 u_j \partial_t \psi_j + (\partial_i \psi_j) u_i u_j - (\partial_i \psi_i) p dx\, dt &= \int_0 u_j \psi_j (t=0) dx \\ \int_D u_j (\partial_j \varphi) dx &= 0 \end{aligned} \tag{2.30}$$

for all test functions $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3) \in C^\infty(D \times \boldsymbol{R}^+)$ and $\varphi \in C^\infty(D)$ with compact support. Energy is conserved for an Euler solution if

$$\int_D |\boldsymbol{u}(x,t)|^2 dx = \int_D |\boldsymbol{u}(x,0)|^2 dx \tag{2.31}$$

for $t \in [0,T]$.

The following energy conservation theorem for ideal hydrodynamics is a consequence of Corollary 2.1 and the theorem of [5].

Corollary 2.2 *(Energy Conservation for Euler). Let $\boldsymbol{u}$ be a weak solution of the Euler equations on $D = [0,1]^3$. Suppose that $\boldsymbol{u} \in C([0,T], B(D))$ in which $B(D) = Lip(\boldsymbol{\alpha}_n, \boldsymbol{\kappa}_n)$ or $L = Lip([\alpha_-, \alpha_+], \kappa(\cdot))$. Then energy is conserved if*

$$\min_\alpha (3\alpha + \kappa(\alpha)) > 1. \tag{2.32}$$

Note that here and in the next section, the function space $C([0,T], B(D))$ could be replaced by $L^3([0,T], B(D)) \cap C([0,T], L^2(D))$ or something similar, as in [5].

3 Energy Conservation for Ideal MHD

The energy conservation results of [5] can be extended to ideal MHD in a straightforward manner. The equations for ideal MHD are

$$(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p - \frac{1}{2}\nabla \boldsymbol{b}^2 + \boldsymbol{b} \cdot \nabla \boldsymbol{b}$$

$$\begin{aligned}(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{b} &= \boldsymbol{b} \cdot \nabla \boldsymbol{u} \\ \nabla \cdot \boldsymbol{u} &= \nabla \cdot \boldsymbol{b} = 0.\end{aligned} \tag{3.1}$$

Actually, incompressibility of $\boldsymbol{b}$ ($\nabla \cdot \boldsymbol{b} = 0$) need only be required at $t = 0$, and it then holds for all t. Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ be functions satisfying the weak form of the ideal MHD equations, namely,

$$\begin{aligned}\int_0^T \int_D \left[u_j \partial_t \psi_j^{(1)} + (b_i b_j - u_i u_j)\partial_i \psi_j^{(1)} - (p + b^2/2)\partial_i \psi_i^{(1)}\right] dx\ dt &= \int_{D,t=0} u_j \psi_j^{(1)} dx \\ \int_0^T \int_D \left[b_j \partial_t \psi_j^{(2)} + (\epsilon_{jkl} u_k b_l)(\epsilon_{jmn}\partial_m \psi_n^{(2)})\right] dx\ dt &= \int_{D,t=0} b_j \psi_j^{(2)} dx \\ \int_D u_j \partial_j \xi^{(1)} dx = 0 \quad \int_D b_j \partial_j \xi^{(2)} dx &= 0\end{aligned}$$

for all test function $\boldsymbol{\psi}^{(\beta)} = (\psi_1^{(\beta)}, \psi_1^{(\beta)}, \psi_1^{(\beta)}) \in C^\infty(D \times R^+)$ and $\xi^{(\beta)} \in C^\infty(D)$, with $\beta = 1, 2$. Again, the incompressibility condition on $\boldsymbol{b}$ need only be imposed at $t = 0$ and it then follows for all t. In analogy to the conservation of energy for the Euler equations, energy conservation for ideal MHD holds if

$$\int_D \left(|\boldsymbol{u}(\boldsymbol{x}, t)|^2 + |\boldsymbol{b}(\boldsymbol{x}, t)|^2\right) dx = \int_D \left(|\boldsymbol{u}(\boldsymbol{x}, 0)|^2 + |\boldsymbol{b}(\boldsymbol{x}, 0)|^2\right) dx \tag{3.2}$$

for $t \in [0, T]$. For simplicity we assume that $D = [0, 1]^n$. Whereas singularity formation and energy dissipation is only possible for three-dimensional hydrodynamics, for MHD it is a possibility for dimension $n = 2$ or $n = 3$.

Theorem 3.1 *(Energy Conservation for Ideal MHD). Let $\boldsymbol{u}$ and $\boldsymbol{b}$ be a weak solution of the ideal MHD equations in $D = [0, 1]^n$. Suppose that $\boldsymbol{u} \in C([0, T], B_3^{\alpha_1})$ and $\boldsymbol{b} \in C([0, T], B_3^{\alpha_2})$. If*

$$\begin{aligned}\alpha_1 &> 1/3 \\ \alpha_1 + 2\alpha_2 &> 1\end{aligned} \tag{3.3}$$

then (3.2) holds.

Proof. The proof follows that of [5] but will be briefly repeated here. Define $\varphi_\epsilon(\boldsymbol{x}) = (1/\epsilon^n)\varphi(\boldsymbol{x}/\epsilon)$ to be a positive, smooth mollifier with support in $B(0, 1)$ and total mass 1. We make use of the definitions

$$\begin{aligned}r_\epsilon(f, g)(\boldsymbol{x}) &= \int \varphi^\epsilon(y)(\delta_y f(\boldsymbol{x}) \otimes \delta_y g(\boldsymbol{x}))dy \\ q_\epsilon(f, g)(\boldsymbol{x}) &= \int \varphi^\epsilon(y)(\delta_y f(\boldsymbol{x}) \times \delta_y g(\boldsymbol{x}))dy\end{aligned}$$

where $\delta_y h(x) = h(x - y) - h(x)$. The proof relies critically on the following identities (first observed in [5]):

$$(f \otimes g)^\epsilon = f^\epsilon \otimes g^\epsilon + r_\epsilon(f,g) - (f - f^\epsilon) \otimes (g - g^\epsilon) \tag{3.4}$$
$$(f \times g)^\epsilon = f^\epsilon \times g^\epsilon + q_\epsilon(f,g) - (f - f^\epsilon) \times (g - g^\epsilon). \tag{3.5}$$

In addition the following estimates hold for functions in B_3^α:

$$||f(\cdot + y) - f(\cdot)||_{L^3} \leq c|y|^\alpha \tag{3.6}$$
$$||\nabla f_\epsilon||_{L^3} \leq C\epsilon^{\alpha-1}||f||_{L^3} \tag{3.7}$$
$$||f - f_\epsilon||_{L^3} \leq C\epsilon^{\alpha}||f||_{L^3}. \tag{3.8}$$

Using $\psi^{(1)\epsilon}(x) = \int \varphi^\epsilon(y - x) u^\epsilon(y,t)dy$ and $\psi^{(2)\epsilon}(x) = \int \varphi^\epsilon(y - x) b^\epsilon(y,t)dy$ as test functions results in the equations

$$\int_D |u^\epsilon(x,t)|^2 dx - \int_D |u^\epsilon(x,0)|^2 dx = \int_0^t \int_D Tr\left[(u \otimes u)^\epsilon \nabla u^\epsilon - (b \otimes b)^\epsilon \nabla u^\epsilon\right](x,t) dx\, dt$$
$$\int_D |b^\epsilon(x,t)|^2 dx - \int_D |b^\epsilon(x,0)|^2 dx = \int_0^t \int_D \left[(u \times b)^\epsilon \cdot \nabla \times b^\epsilon\right](x,t) dx\, dt.$$

The identities (3.6), (3.7) and (3.8) then yield the estimates

$$\begin{aligned}
\left|\int_D |u^\epsilon(x,t)|^2 + |b^\epsilon(x,t)|^2 dx - \int_D |u^\epsilon(x,0)|^2 + |b^\epsilon(x,0)|^2 dx\right| \\
\leq \int_0^t \int_D |Tr\left[(r_\epsilon(u,u) - r_\epsilon(b,b) - (u - u^\epsilon) \otimes (u - u^\epsilon)\right. \\
\left. +(b - b^\epsilon) \otimes (b - b^\epsilon)) \nabla u^\epsilon\right]| dx\, d\tau \\
+ \int_0^t \int_D |(q_\epsilon(u,b) - (u - u^\epsilon) \times (b - b^\epsilon)) \cdot \nabla \times b^\epsilon| dx\, d\tau \\
\leq \int_0^t \left[\left(||r_\epsilon(u,u)||_{3/2}^{2/3} + ||r_\epsilon(b,b)||_{3/2}^{2/3} + ||u - u^\epsilon||_{3/2}^{2/3} + ||b - b^\epsilon||_{3/2}^{2/3}\right) ||\nabla u^\epsilon||_3^{1/3}\right. \\
\left. + \left(||q_\epsilon(u,b)||_{3/2}^{2/3} + ||u - u^\epsilon||_{3/2}^{1/3}||b - b^\epsilon||_{3/2}^{1/3}\right) ||\nabla u^\epsilon||_3^{1/3}\right] d\tau \\
\leq C_1\epsilon^{3\alpha_1 - 1} + C_2\epsilon^{\alpha_1 + 2\alpha_2 - 1}.
\end{aligned}$$

The result (3.2) follows in the limit $\epsilon \to 0$, which finishes the proof of Theorem 3.1.

A similar theorem for magnetic helicity can be proven. The time evolution of the magnetic helicity for smooth ideal MHD is given by

$$\begin{aligned}
\frac{d}{dt}\int_D [a_t \cdot b + a \cdot b_t] dx &= \int_D [b \cdot (u \times b) + b \cdot \nabla\alpha + a \cdot \nabla \times (u \times b)] dx \\
&= \int_D [b \cdot \nabla\alpha + a \cdot \nabla \times (u \times b)] dx \\
&= 0
\end{aligned}$$

where a is some smooth function and $b = \nabla \times a$. Then for $\psi \in C^\infty(D \times R^+)$

$$\int_0^T \int_D (\nabla \times \psi(x,t)) \cdot (u(x,t) \times b(x,t)) dx\, dt = 0 \qquad (3.9)$$

implies weak conservation of helicity. Using arguments identical to those of the previous proof we obtain

Theorem 3.2 *(Magnetic Helicity Conservation for Ideal MHD). Let u and b be a weak solution of the ideal MHD equations in $D = [0,1]^n$. Suppose that $u \in C([0,T], B_3^{\alpha_1})$ and $b \in C([0,T], B_3^{\alpha_2})$. If $\alpha_1 + 2\alpha_2 > 0$, then (3.9) holds.*

In 2 dimensions the magnetic helicity vanishes identically. In its place the quantity $\int_D a^2 dx$ serves as a non-trivial invariant. In 2 dimensions, a satisfies (up to a gradient)

$$a_t + u \cdot \nabla a = 0$$

and we have

Theorem 3.3 . *Let u and b be a weak solution of the ideal MHD equations in $D = [0,1]^2$. Suppose that $u \in C([0,T], B_3^{\alpha_1})$and $a \in C([0,T], B_3^{\alpha_2+1})$. If $\alpha_1 + 2\alpha_2 > -1$, then $\int_D a^2 dx$ is conserved.*

We remark that Theorems 3.1,3.2, and 3.3 specialize easily to functions u and b with specified uniform singularity spectrum, as in Corollary 2.2. In these cases the bounds of Theorem 3.1 become $s_1 > 1/3$, $s_1 + 2s_2 > 1$, the bound for Theorem 3.2 becomes $s_1 + 2s_2 > 0$, and the bound for Theorem 3.3 becomes $s_1 + 2s_2 > -1$. Here

$$\begin{aligned} s_1 &= \min_{\alpha_1}(\alpha_1 + \kappa_1(\alpha_1)/3) \\ s_2 &= \min_{\alpha_2}(\alpha_2 + \kappa_2(\alpha_2)/3) \end{aligned}$$

where κ_1, κ_2 are defined as in the introduction. For the commonly observed phenomenon of codimension 1 current sheets, $\kappa_2 = 1$ so that $s_1 + 2\alpha_2 > 1/3$ implies energy conservation and $s_1 + 2\alpha_2 > -2/3$ implies helicity conservation ($-5/3$ in 2D).

Analogous results can be obtained in terms of the Elsasser (characteristic) variables $z^\pm = u \pm b$ for the MHD equations. The system (3.1) can be rewritten as

$$\begin{aligned} (\partial_t + z^+ \cdot \nabla) z^- &= -\nabla \Pi \\ (\partial_t + z^- \cdot \nabla) z^+ &= -\nabla \Pi \qquad (3.10) \\ \nabla \cdot z^\pm &= 0 \end{aligned}$$

in which $\Pi = p + \frac{1}{2}b^{\ 2}$.

The following theorem gives two variants of the previous energy conservation result for MHD.

Theorem 3.4 *(Energy Conservation for Ideal MHD in Characteristic Variables). For a weak solution of the MHD equations in $[0,1]^n$, there is energy conservation if either of the following conditions are satisfied:*

(i) For some p, q with values in $(1, \infty)$ and with $1/p + 2/q = 1$

$$\begin{aligned} u &\in C([0,T], B_3^{\alpha_0} \cap B_p^{\alpha_1}) \\ b &\in C([0,T], B_q^{\alpha_2}) \end{aligned} \tag{3.11}$$

in which

$$\begin{aligned} &3\alpha_0 > 1 \\ &\alpha_1 + 2\alpha_2 > 1. \end{aligned} \tag{3.12}$$

(ii) For some p_i, q_i $(i = 1, 2)$ with values in $(1, \infty)$ and with $1/p_1 + 2/q_1 = 2/p_2 + 1/q_2 = 1$,

$$\begin{aligned} &z^{\ +} \in C([0,T], B_{p_1}^{\alpha_1} \cap B_{p_2}^{\alpha_2}) \\ &z^{\ -} \in C([0,T], B_{q_1}^{\beta_1} \cap B_{q_2}^{\beta_2}) \end{aligned} \tag{3.13}$$

in which

$$\begin{aligned} &\alpha_1 + 2\beta_1 > 1. \\ &2\alpha_2 + \beta_2 > 1. \end{aligned} \tag{3.14}$$

Similar statements can be made with regards to magnetic helicity.

4 Singularity Formation for Ideal MHD

We will show the analogue of the Beale-Kato-Majda theorem for ideal MHD.

Theorem 4.1 *For the system (3.1) with initial data $u_{\ 0}, b_{\ 0} \in H^s$, with $s \geq 3$, the solution $u\ (t), b\ (t)$ is in the class*

$$C([0,T], H^s) \cap C^1([0,T], H^{s-1})$$

as long as

$$\int_0^T |\omega\ (t)|_\infty + |j\ (t)|_\infty dt < \infty.$$

and

$$\int_0^T |\nabla \times z^+|_\infty + |\nabla \times z^-|_\infty dt < \infty.$$

(The 2 inequalities are in fact equivalent.)

Here $j\ = \nabla \times b$ and H^s is the L_2 Sobolev space. The approach closely follows that of [2]. Assume that

$$\int_0^T |\nabla \times z\ ^+|_\infty + |\nabla \times z\ ^-|_\infty dt = M < \infty. \tag{4.1}$$

The proof consists of three parts: First, we derive energy estimates on $|z\ ^{\pm}|_s$ in terms of $|\nabla z\ ^{\pm}|_\infty$. Second, we estimate $|\nabla \times z\ ^{\pm}|_{L_2}$. Finally, we utilize an inequality derived in [2] and Gronwall's lemma to bound $|z\ ^{\pm}|_s$.

4.1 Energy Estimates

We begin by deriving energy estimates for the system (3.10) with $t \in [0, T]$. Let α be a multi-index with $|\alpha| \leq s$. Let $\eta\ = D_x^\alpha z\ ^+$. Apply D_x^α to the second equation in (3.10) to obtain

$$(\partial_t + z\ ^- \cdot \nabla)\eta\ = -\nabla \Pi' - F$$

in which $\Pi' = D_x^\alpha \Pi$ and

$$F = D^\alpha[(z\ ^- \cdot \nabla z\ ^+)] - z\ ^- \cdot D^\alpha \nabla z\ ^+.$$

A bound on F in the L_2 norm can be based on the general inequality

$$|D^\alpha(fg) - fD^\alpha g|_{L^2} \leq c(|f|_s |g|_\infty + |\nabla f|_\infty |g|_{s-1}) \ .$$

which was derived in [2] based on the Gagliardo-Nirenberg inequalities. Application of this to F yields

$$|F|_{L^2} \leq c(|z\ ^-|_s |\nabla z\ ^+|_\infty + |\nabla z\ ^-|_\infty |\nabla z\ ^+|_{s-1}). \tag{4.2}$$

This leads to the following bound on η

$$\frac{d}{dt}|\eta\ |_{L^2}^2 \ \leq \ c(|z\ ^-|_s |\nabla z\ ^+|_\infty + |\nabla z\ ^-|_\infty |\nabla z\ ^+|_{s-1})|\eta\ |_{L^2}.$$

Summing over α leads to

$$\frac{d}{dt}|z^+|_s^2 \leq c(|z^-|_s|\nabla z^+|_\infty + |\nabla z^-|_\infty|z^+|_s)|z^+|_s. \tag{4.3}$$

There is a similar result for z^-; i.e.,

$$\frac{d}{dt}|z^-|_s^2 \leq c(|z^+|_s|\nabla z^-|_\infty + |\nabla z^+|_\infty|z^-|_s)|z^-|_s. \tag{4.4}$$

Add these two inequalities to obtain

$$\frac{d}{dt}(|z^-|_s^2 + |z^+|_s^2) \leq c(|\nabla z^+|_\infty + |\nabla z^-|_\infty)(|z^+|_s^2 + |z^-|_s^2), \tag{4.5}$$

and thus

$$|z^+|_s^2 + |z^-|_s^2 \leq (|z_0^+|_s^2 + |z_0^-|_s^2)\exp\left(C\int_0^t(|\nabla z^+|_\infty + |\nabla z^-|_\infty)d\tau\right). \tag{4.6}$$

4.2 L^2 Bounds on $\nabla \times z_\pm$

Next, bound

$$|\nabla z^+(t,\cdot)|_\infty + |\nabla z^-(t,\cdot)|_\infty.$$

Take the curl of (3.10) to obtain

$$\begin{aligned}(\partial_t + z^+ \cdot \nabla)\zeta^- &= \nabla z^+ A \nabla z^- \\ (\partial_t + z^- \cdot \nabla)\zeta^+ &= \nabla z^- A \nabla z^+\end{aligned} \tag{4.7}$$

where $\zeta^\pm = \nabla \times z^\pm$ and A is a constant matrix. Multiplying the first equation in (4.7) by ζ^- and integrating gives

$$\begin{aligned}\frac{d}{dt}|\zeta^-|_{L^2}^2 &\leq C\int |\nabla z^+|\,|\nabla z^-|\,|\zeta^-|\,dx \\ &\leq C|\zeta^-|_\infty(|\nabla z^+|_{L^2}|\nabla z^-|_{L^2}) \\ &\leq C|\zeta^-|_\infty(|\nabla z^+|_{L^2}^2 + |\nabla z^-|_{L^2}^2).\end{aligned} \tag{4.8}$$

Since $\nabla \cdot z^\pm = 0$, $z^\pm$ and $\zeta^\pm$ are related by

$$z^\pm = -\nabla \times (\Delta^{-1}\zeta^\pm)$$

and their Fourier transforms are related by $(\nabla z^\pm)(k) = S(k)\zeta^\pm(k)$ where $S(k)$ is bounded independent of k. Thus $|\nabla z^\pm|_{L^2} \leq C|\zeta^\pm|_{L^2}$, so that (4.8) leads to

$$\frac{d}{dt}|\zeta^-|_{L^2}^2 \leq C|\zeta^-|_\infty(|\zeta^+|_{L^2}^2 + |\zeta^-|_{L^2}^2).$$

We obtain a similar result for $\zeta^{\,+}$; that is

$$\frac{d}{dt}|\zeta^{\,+}|_{L^2}^2 \leq C|\zeta^{\,+}|_\infty(|\zeta^{\,+}|_{L^2}^2+|\zeta^{\,-}|_{L^2}^2).$$

Add these two equations to obtain

$$\frac{d}{dt}(|\zeta^{\,+}|_{L^2}^2+|\zeta^{\,-}|_{L^2}^2) \quad \leq \quad c(|\zeta^{\,+}|_\infty+|\zeta^{\,-}|_\infty)(|\zeta^{\,+}|_{L^2}^2+|\zeta^{\,-}|_{L^2}^2)$$

so that

$$|\zeta^{\,+}|_{L^2}^2+|\zeta^{\,-}|_{L^2}^2 \leq (|\zeta_{\,0}^{\,+}|_{L^2}^2+|\zeta_{\,0}^{\,-}|_{L^2}^2 \exp\left(C\int_0^t(|\zeta^{\,+}(\tau)|_\infty+|\zeta^{\,-}(\tau)|_\infty)d\tau\right).$$

By Assumption (4.1) we have

$$|\zeta^{\,+}|_{L^2}^2+|\zeta^{\,-}|_{L^2}^2 \leq \overline{M}(|\zeta_{\,0}^{\,+}|_{L^2}^2+|\zeta_{\,0}^{\,-}|_{L^2}^2) \tag{4.9}$$

where $\overline{M}=\exp(CM)$.

4.3 Final Estimates

In [2] it was proved, via the Biot-Savart law, that

$$|\nabla f|_\infty \leq C\{1+(1+\log^+|f|_3)|\nabla\times f|_\infty+|\nabla\times f|_{L^2}\} \tag{4.10}$$

where

$$\log^+ a = \begin{cases} \log a & \text{if } a\geq 1 \\ 0 & \text{otherwise.}\end{cases} \tag{4.11}$$

Thus

$$\begin{aligned}|\nabla z^{\,+}|_\infty+|\nabla z^{\,-}|_\infty \leq C\{1 \quad &+ \quad (1+\log^+|z^{\,+}|_3)|\zeta^{\,+}|_\infty+|\zeta^{\,+}|_{L^2} \\ &+ \quad (1+\log^+|z^{\,-}|_3)|\zeta^{\,-}|_\infty+|\zeta^{\,-}|_{L^2}\}.\end{aligned}$$

Using the result (4.9) from Section 4.2, we have

$$|\nabla z^{\,+}|_\infty+|\nabla z^{\,-}|_\infty \quad \leq \quad C\{1+(|\zeta^{\,+}|_\infty+|\zeta^{\,-}|_\infty)(\log^+|z^{\,+}|_3+\log^+|z^{\,-}|_3+2)$$

Combining this with the result (4.6) from Section 4.1 gives

$$\begin{aligned}|z^{\,+}|_s+|z^{\,-}|_s \leq c(|z_0^+|+|z_0^-|)\exp\Big\{C\int_0^t &\left[1+(|\zeta^{\,+}|_\infty+|\zeta^{\,-}|_\infty)\right.\\ &\left.(\log(|z^{\,+}|_3+e)+\log(|z^{\,-}|_3+e))\right]d\tau\Big\}\end{aligned}$$

Let $y^\pm(t)=\log(|z^{\,\pm}|_s+e)$ then

$$\begin{aligned}y^+(t)+y^-(t) \quad \leq \quad &\log c(|z_{\,0}^{\,+}|_s+|z_{\,0}^{\,-}|)_s \\ &+C\int_0^t(1+(|\zeta^{\,+}|_\infty+|\zeta^{\,-}|_\infty)(y^+(\tau)+y^-(\tau))d\tau.\end{aligned}$$

Application of Gronwall's lemma then shows that $y^+(t)+y^-(t)$ is bounded by a constant which depends only on M, T and $|z^{\pm}(0,\cdot)|_s$. This concludes the proof of Theorem 4.1.

5 Conclusions

At present, there are only a few analytical results on singularities in ideal hydrodynamics: The Beale-Kato-Majda theorem is a necessary condition for the formation of singularities from smooth initial data. Constantin [4] and Constantin & Fefferman [6] have obtained additional necessary conditions in terms of the geometry of the vorticity field. Finally, Onsager's energy conservation criterion provides a necessary condition for energy dissipation due to singularities in an ideal fluid.

The first part of this paper has refined Onsager's criterion by explicitly showing the effect of singularity type and dimension on the necessary condition for energy dissipation. The result is an example of the Frisch-Parisi multifractal formalism, which has been proved to be valid for functions with specified uniform singularity spectrum, as defined here.

These two analytical results–the Beale-Kato-Majda theorem and Onsager's energy conservation theorem–have also been extended to ideal MHD. Since energy dissipation but helicity conservation are expected, this suggests a limited range of values for the uniform singularity spectrum in MHD. The appearance of the Elsasser variables z^+ and z^- in the extension of the Beale-Kato-Majda inequality should also be noted.

We expect these results to be useful in two ways: First, as a sufficient condition for regularity of ideal hydrodynamic and MHD solutions. They should also serve as a guide in investigation of possible singularities and their physical significance. For example in 3D hydrodynamics with singularities of type α on a smooth set S, nonzero energy dissipation requires $\alpha \leq 0$ for a 2D singularity surface ($\kappa = 1$), $\alpha \leq -1/3$ for a curve of singularities ($\kappa = 2$), and $\alpha \leq -2/3$ for a point singularity ($\kappa = 3$). In particular, in the point and curve cases, infinite velocities are required.

These results also help to indicate the relation between the smoothness of $\boldsymbol{b}$ and that of $\boldsymbol{u}$. Theorem 4.1 suggests that $\boldsymbol{b}$ and $\boldsymbol{u}$ should have the same degree of smoothness, while Theorems 3.1,3.2, and 3.3 suggest a tradeoff between smoothness of $\boldsymbol{u}$ and that of $\boldsymbol{b}$.

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6 Appendix

Proof of Lemma 2.1 for $Lip(\boldsymbol{\alpha}_n, \boldsymbol{\kappa}_n)$.

The proof is a more complicated version of the proof for $f \in Lip(\alpha_0, \alpha_1, 0, \kappa_1)$. As above, denote

$$\begin{aligned} V_n(\boldsymbol{\alpha}, \boldsymbol{r}) &= \operatorname{vol}(\cap_{i=1}^n S(\alpha_i, r_i)) \\ \tilde{V}_n(\boldsymbol{\alpha}, \boldsymbol{r}) &= \operatorname{vol}((\cap_{i=1}^n S(\alpha_i, r_i)) \cup (y + \cap_{i=1}^n S(\alpha_i, r_i))) \\ &\leq 2V_n(\boldsymbol{\alpha}, \boldsymbol{r}). \end{aligned} \tag{6.1}$$

Note that

$$V_n(\boldsymbol{\alpha}, \boldsymbol{r}) \leq \inf_i a r_i^{\kappa_i}. \tag{6.2}$$

The function $\Delta_n(\boldsymbol{\alpha}_n, \boldsymbol{r}_n)$ can be rewritten as follows: Denote l_1 and l_2 to satisfy

$$\begin{aligned} &\alpha_{l_1 - 1} \geq 0 > \alpha_{l_1} \\ &r_{l_2} < |y| \leq r_{l_2+1}. \end{aligned} \tag{6.3}$$

Then

$$\Delta_n(\boldsymbol{\alpha}_n, \boldsymbol{r}_n) = \begin{cases} r_{l_1}^{\alpha_{l_1}} \prod_{i=l_1+1}^n r_i^{-(\alpha_{i-1} - \alpha_i)} & \text{if } l_2 < l_1 \\ |y|^{\alpha_{l_2}} \prod_{i=l_2+1}^n r_i^{-(\alpha_{i-1} - \alpha_i)} & \text{if } l_1 \leq l_2. \end{cases} \tag{6.4}$$

Moreover the function Δ_n can be extended to non-monotone sequences $\boldsymbol{r}_n$ as follows:

$$\Delta_n(\boldsymbol{\alpha}_n, \boldsymbol{r}_n) = \Delta_n(\boldsymbol{\alpha}_n, \tilde{\boldsymbol{r}}_n) \tag{6.5}$$

in which

$$(\tilde{\boldsymbol{r}}_n)_k = \max(r_k, \ldots, r_n). \tag{6.6}$$

Note that by the extended definition of the function $\Delta_n(\boldsymbol{\alpha}_n, \boldsymbol{r}_n)$, if $r_{j+1} = 1$ for some j then

$$\Delta_n(\boldsymbol{\alpha}_n, \boldsymbol{r}_n) = \Delta_j(\boldsymbol{\alpha}_j, \boldsymbol{r}_j). \tag{6.7}$$

In particular Δ_n is independent of r_i for $i > j$.

As above denote $I = [0, 1]$. Also the integral of the Hölder difference can again be represented as a Stieljes integral, and through repeated integration by parts, we find that

$$\begin{aligned} \int |f(x+y) &- f(x)|^p dx \leq \int_D \Delta_n(\boldsymbol{r}, \boldsymbol{\alpha})^p dx \\ &= \int_{I^n} \Delta_n(r)^p d\tilde{V}_n(r) \\ &= \sum_{N'} \int_{I^{n'}} (-1)^{n'} \left(\frac{\partial^{n'}}{\partial r^{(N')}} \Delta_n(r)^p \right) \tilde{V}_n(r) dr^{(N')} \big|_{r^{(N'')} = 1^{(N'')}} \end{aligned} \tag{6.8}$$

In (6.8), $N' = (i_1, \ldots, i_{n'})$ is any subsequence of the integers $(1, \ldots, n)$ and the sum is over all such subsequence. Also denote $N'' = \{1, \ldots, n\} - N'$, $n' = |N'|$, $n'' = |N''| = n - n'$,

$$\begin{aligned} r^{(N')} &= (r_{i_1}, \ldots, r_{i_{n'}}) \\ \partial r^{(N')} &= \partial r_{i_1} \ldots \partial r_{i_{n'}} \\ r^{(N'')} &= (r_i)_{i \in N''} \end{aligned} \tag{6.9}$$

and $1^{(N'')}$ is the vector of length n'' consisting of all 1's.

The definition of Δ_n implies that this integrand is nonzero if and only if $N' = N_m = (1, \ldots, m)$ for some m and $r_{\ m}$ is a monotone increasing sequence. Thus the only surviving terms are the m terms

$$\int_{I^m} \left(\frac{\partial^m}{\partial r_1 \ldots \partial r_m} \Delta_m^p \right) V_m dr^m \tag{6.10}$$

for some $1 \leq m \leq n$. As before, positive and negative values of α_i must be treated differently. Suppose that

$$\alpha_1 > \ldots > \alpha_k \geq 0 > \alpha_{k+1} > \ldots > \alpha_{m+1} \tag{6.11}$$

Then the integral in (6.10) is bounded by

$$|y|^{\alpha_1} \prod_{i=1}^{k-1} J_i \prod_{i=k}^{m-1} K_i \ L_m \tag{6.12}$$

in which J_i (for $\alpha_{i+1} \geq 0$) and K_i (for $\alpha_{i+1} < 0$) are given by

$$\begin{aligned} J_i &= \int_0^1 \frac{\partial}{\partial r} \left(\min(|y|^{-\alpha_{i-1}} r^{\alpha_i}, r^{-(\alpha_{i-1} - \alpha_i)})^p \right) dr \\ &= \int_0^{|y|} |y|^{-\alpha_{i-1}p} r^{\alpha_i p - 1} dr + \int_{|y|}^1 r^{-(\alpha_{i-1} - \alpha_i)p - 1} dr \\ &\leq |y|^{-(\alpha_{i-1} - \alpha_i)p} \\ K_i &= \int_0^1 \frac{\partial}{\partial r} \left(\min(|y|^{-(\alpha_{i-1} - \alpha_i)}, r^{-(\alpha_{i-1} - \alpha_i)})^p \right) dr \\ &= \int_{|y|}^1 r^{-(\alpha_{i-1} - \alpha_i)p - 1} dr. \\ &\leq |y|^{-(\alpha_{i-1} - \alpha_i)p} \end{aligned} \tag{6.13}$$

with constant factors omitted. If $\alpha_n \geq 0$, then L_m is given by

$$L_m = \int_0^{|y|} |y|^{-\alpha_{m-1}p} r^{\alpha_m p - 1 + \kappa_m} dr + \int_{|y|}^1 r^{-(\alpha_{m-1} - \alpha_m)p - 1 + \kappa_m} dr \tag{6.14}$$

while if $\alpha_m < 0$, then L_m is given by

$$L_m = \int_{|y|}^{1} r^{-(\alpha_{m-1}-\alpha_m)p-1+\kappa_m}\,dr \tag{6.15}$$

both of which satisfy

$$L_m \leq |y|^{-(\alpha_{m-1}-\alpha_m)p+\kappa_m} \tag{6.16}$$

Taking the product then gives

$$I \leq |y|^{\alpha_m p+\kappa_m}. \tag{6.17}$$

This finishes the proof of (2.22) for a function in $Lip(\alpha_n, \kappa_n)$.