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A convergence theory of multilevel additive Schwarz methods on unstructured meshes*

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We develop a convergence theory for two level and multilevel additive Schwarz domain decomposition methods for elliptic and parabolic problems on general unstructured meshes in two and three dimensions. The coarse and fine grids are assumed only to be shape regular, and the domains formed by the coarse and fine grids need not be identical. In this general setting, our convergence theory leads to completely local bounds for the condition numbers of two level additive Schwarz methods, which imply that these condition numbers are optimal, or independent of fine and coarse mesh sizes and subdomain sizes if the overlap amount of a subdomain with its neighbors varies proportionally to the subdomain size. In particular, we will show that additive Schwarz algorithms are still very efficient for non-selfadjoint parabolic problems with only symmetric, positive definite solvers both for local subproblems and for the global coarse problem. These conclusions for elliptic and parabolic problems improve our earlier results in [12, 15, 16]. Finally, the convergence theory is applied to multilevel additive Schwarz algorithms. Under some very weak assumptions on the fine mesh and coarser meshes, e.g., no requirements on the relation between neighboring coarse level meshes, we are able to derive a condition number bound of the order $O(\rho^2 L^2)$, where $\rho = \max_{1 \leqslant l \leqslant L}(h_l + h_{l-1})/\delta_l$, h_l is the element size of the lth level mesh, δ_l the overlap of subdomains on the lth level mesh, and L the number of mesh levels.

Keywords: convergence, multilevel additive methods, unstructured meshes.

AMS subject classification: 65N30, 65F10.

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1. Introduction

In recent years, unstructured meshes have become very popular in scientific computing. One of their advantages is the flexibility in adapting efficiently to complicated geometries and rapid changes in the solution field, cf. Barth [3] and Mavriplis [28].

Domain decomposition methods (DDMs) have proved to be very efficient for solving elliptic and parabolic problems on structured meshes. A natural question is: what is the efficiency of these methods for unstructured meshes? Our recent studies show that the existing well-developed convergence theory for overlapping DDMs on structured meshes can be fully extended to unstructured meshes. By unstructured meshes we mean here that the finite element meshes are highly non-quasi-uniform, and not generated by using the usual techniques of recursive refinement of coarser meshes. So in general, no coarse mesh nested to a unstructured one exists for the use in domain decomposition algorithms. Therefore, practical coarse meshes for DDMs on a fine unstructured mesh are generally nonnested to that fine mesh and the coarse domains formed by the coarse meshes are non-matching to the fine mesh domain. In this general setting, we have previously investigated the convergence of two level additive type DDMs for elliptic problems in Chan and Zou [15] and Chan et al. [12] (see also Cai [8], which assumed the quasi-uniform coarse mesh and the matching of the coarse domain boundary to the fine domain boundary), for parabolic problems in Chan and Zou [16]. All these results are valid both in two and three dimensions. Moreover, the multilevel additive Schwarz algorithm (see Bramble et al. [6] and Zhang [42]) on unstructured meshes was considered in our earlier technical report [14], but with some very restrictive assumptions on the meshes; for example, the fine mesh and all the coarse meshes were required to be quasi-uniform, each coarse element on every level had to contain sufficiently many fine elements, and coarser boundaries were required to be matching to the boundary of the original fine domain, etc. These restrictions will be removed in the present paper by using some new technical tools.

In this paper, by extending the convergence theory developed in Xu [36] for structured meshes (which assumes all involved local and coarser spaces are subspaces of a finest space) to unstructured meshes, we provide a unified convergence theory for additive type preconditioning iterative methods on unstructured meshes. Here we allow all involved local, coarser spaces and finest space to be independent of each other by introducing certain fine-to-coarse and coarse-to-fine operators. Applying this new analysis framework to two- and multi-level additive DDMs, we can unify our earlier convergence results in [12, 14–16] and greatly simplify those convergence proofs. Moreover, combining with some new technical lemmas, we can remove several previous restrictions on meshes required in [14], and are able to improve the conclusions in [12, 15, 16] by giving completely local condition number bounds, which imply immediately a practically important convergence phenomenon of the concerned domain decomposition algorithms, that is, the algorithms are still optimal in the sense that the corresponding condition numbers are independent of all mesh parameters as long as the overlap amount of a subdomain with its neighbors varies proportionally to the subdomain size.

For related multigrid methods, there are several existing theoretical works on nonnested spaces which are closely related to general unstructured meshes. For example, Bramble et al. [7] and Bramble [4] proposed a general multigrid convergence theory for symmetric multigrid methods with the same number of pre- and post-smoothings, and then applied the framework to obtain optimal convergence of multigrid algorithms with nonnested subspaces defined on quasi-unform meshes in two dimension. Zhang [40, 41] and Scott and Zhang [33] proved optimal convergence of multigrid algorithms with nonnested subspaces induced by non-quasi-uniform meshes in two dimension [40, 41] and quasi-uniform meshes in higher dimensions [33], where all grid domains on each level are required to have the same boundary. Douglas and Douglas [19] proposed a unified convergence theory for general multigrid or multilevel methods which use the recursive way to define coarse coefficient matrices by finer coefficient ones and are applicable to nonsymmetric, indefinite or singular linear systems of equations, and then Douglas et al. [20] extended and applied the theory in [19] to derive the convergence of nonnested multigrid methods for solving finite element equations and finite volume equations. Bank and Xu [1, 2] recently proposed an algorithm for coarsening unstructured meshes, which was then used for constructing a hierarchical basis multigrid method for unstructured meshes.

The content of the paper is arranged as follows: In section 2, we formulate the general framework for the development of the convergence theory for the additive type DDMs, and a perturbation extension of the result of section 2 will be presented in section 2.3 for the application of additive Schwarz algorithms to non-selfadjoint parabolic problems. In section 3 we introduce the fine and coarse finite element spaces, domain decompositions, the assumptions made for the fine and coarse finite element triangulations, and especially the L^2-optimal approximation and H^1-stability of the standard finite element interpolant and the Clément interpolant. Section 4 and section 5 will be devoted to the application of the abstract convergence theory developed in section 2 to selfadjoint elliptic problems and non-selfadjoint parabolic problems. Finally, in section 6 we apply the abstract theory of section 2 to the multilevel additive Schwarz methods.

We will focus only on the convergence theory of algorithms in the present paper, no attention will be paid to the numerical implementation of the algorithms. For details of the matrix representation of the algorithms, we refer to our earlier work [11, 12, 14, 15]; for numerical experiments, we refer to [9–12].

Throughout the paper, $\|\cdot\|_{m,r,\Omega}$ and $|\cdot|_{m,r,\Omega}$ denote the norm and semi-norm of the usual Sobolev spaces $W^{m,r}(\Omega)$ for any integer $m \geqslant 0$ and real number $r \geqslant 1$, but the subscript r will be omitted if $r = 2$, and so $W^{m,r}(\Omega) = H^m(\Omega)$. The sign $\|\cdot\|_A$ will be often used for the inner product $(A\cdot,\cdot)$-induced norm. The notation "$a \lesssim b$" ("$a \gtrsim b$" resp.) for real functions a and b which depend on a set of parameters (e.g., coarse and fine mesh sizes, subdomain sizes or time steps if any) means that there are two positive constants c_0 and c_1 independent of all the parameteres such that $c_0 a \lesssim c_1 b$ ($c_0 a \gtrsim c_1 b$ resp.).

2. Abstract framework of convergence analysis for additive preconditioners

This section is devoted to the abstract framework of convergence analysis for two and multilevel additive preconditioners.

2.1. *Additive preconditioners for symmetric positive definite operators*

Let V, and V^k, $0 \leqslant k \leqslant p$, be finite dimensional vector spaces with inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_k$, resp. All spaces V^k are not necessarily subspaces of V. The space V^0 is special, usually referring to the coarse grid space.

Given a symmetric positive definite (SPD in short) operator A on V and $f \in V$, we are interested in solving the equation

$$Au = f$$

on V, which arises from the discretization of elliptic or parabolic problems by using finite element methods. As A is ill-conditioned, our goal is to find a good preconditioner M for A such that MA is better conditioned than A, and the action of M is inexpensive to calculate. Then one can use iterative methods, like the Conjugate Gradient method, for $MAu = Mf$ instead of $Au = f$.

We will study in this paper preconditioners of the following additive type:

$$M = \sum_{k=0}^{p} I_k R_k Q_k, \tag{2.1}$$

where the "interpolation" operators $I_k : V^k \to V$ are linear, and the "projection" operators Q_k are the adjoints of I_k defined by

$$(Q_k u, v_k)_k = (u, I_k v_k), \quad \forall u \in V,\ v_k \in V^k, \tag{2.2}$$

and $R_k : V^k \to V^k$ are given SPD operators, approximating the inverses of the restrictions of A on V^k in some sense. It is easy to verify that M is an SPD operator on V.

We remark that the preconditioner form (2.1) is a natural extension of the one introduced by Xu [36] with nested subspaces. The awareness of this general form (2.1) was due to Griebel and Oswald [24], see also [12, 29, 34].

Following the theory of Xu [36] for structured meshes with all V^k being subspaces of V, one can similarly bound the condition number of MA for the present unstructured cases in terms of three parameters K_0, ω_0 and α_0 defined as follows (with a different definition for α_0 compared to [36]):

(P1) For any $u \in V$, there exist $u_k \in V^k$ $(0 \leqslant k \leqslant p)$ such that $u = \sum_{k=0}^{p} I_k u_k$ and

$$\sum_{k=0}^{p} \left(R_k^{-1} u_k, u_k\right)_k \leqslant K_0 (Au, u).$$

(P2) For any $u_k \in V^k$, $k = 0, 1, \ldots, p$,

$$(AI_k u_k, I_k u_k) \leqslant \omega_0 \left(R_k^{-1} u_k, u_k\right)_k.$$

(P3) For any $u \in V$ and $u_k \in V^k$ $(1 \leqslant k \leqslant p)$,

$$\sum_{k=1}^{p} (Au, I_k u_k) \leqslant \alpha_0^{1/2} (Au, u)^{1/2} \left(\sum_{k=1}^{p} (AI_k u_k, I_k u_k) \right)^{1/2}$$

Without loss of generality, we assume that $K_0 \geqslant 1$, $\omega_0 \geqslant 1$ and $\alpha_0 \geqslant 1$. From (2.1) we may write

$$MA = \sum_{k=0}^{p} I_k P_k, \quad P_k = R_k Q_k A. \tag{2.3}$$

We have the following theorem:

Theorem 1. Under the assumptions (P1)–(P3),

$$\kappa(MA) \leqslant \omega_0(\alpha_0 + 1) K_0.$$

Proof. Let us first estimate the minimum eigenvalue of MA, we obtain

$$\begin{aligned}
(Au, u) &= \left(Au, \sum_{k=0}^{p} I_k u_k \right) = \sum_{k=0}^{p} (Q_k Au, u_k)_k \quad \text{(by (P1) and (2.2))} \\
&= \sum_{k=0}^{p} \left(R_k^{-1} P_k u, u_k\right)_k \\
&\leqslant \left(\sum_{k=0}^{p} \left(R_k^{-1} u_k, u_k\right)_k \right)^{1/2} \left(\sum_{k=0}^{p} \left(R_k^{-1} P_k u, P_k u\right)_k \right)^{1/2} \\
&\quad \text{(Schwarz inequality)} \\
&\leqslant K_0^{1/2} (Au, u)^{1/2} \left(\sum_{k=0}^{p} (Q_k Au, P_k u)_k \right)^{1/2} = K_0^{1/2} (Au, u)^{1/2} (MAu, Au)^{1/2} \\
&\quad \text{((P1), (2.2) and (2.3))},
\end{aligned}$$

this implies $(MAu, Au) \geqslant K_0^{-1}(Au, u)$ which gives

$$\lambda_{\min}(MA) \geqslant K_0^{-1}.$$

Next we estimate $\lambda_{\max}(MA)$. We have for any $u_k \in V^k$ $(0 \leqslant k \leqslant p)$ that

$$\begin{aligned}\sum_{k=0}^{p}(Au, I_k u_k) &= (Au, I_0 u_0) + \sum_{k=1}^{p}(Au, I_k u_k)\\ &\leqslant \|u\|_A \|I_0 u_0\|_A + \alpha_0^{1/2}\|u\|_A\left(\sum_{k=1}^{p}\|I_k u_k\|_A^2\right)^{1/2}\\ &\qquad \text{(Schwarz inequality and (P3))}\\ &\leqslant (\alpha_0+1)^{1/2}\|u\|_A\left(\sum_{k=0}^{p}\|I_k u_k\|_A^2\right)^{1/2} \qquad (2.4)\\ &\qquad \text{(Schwarz inequality).}\end{aligned}$$

Thus, we derive that

$$\begin{aligned}(MAu, Au) &= \sum_{k=0}^{p}(I_k P_k u, Au)\\ &\leqslant (\alpha_0+1)^{1/2}\|u\|_A\left(\sum_{k=0}^{p}\|I_k P_k u\|_A^2\right)^{1/2} \quad \text{(by (2.4))}\\ &\leqslant \omega_0^{1/2}(\alpha_0+1)^{1/2}(Au,u)^{1/2}\left(\sum_{k=0}^{p}\left(R_k^{-1}P_k u, P_k u\right)_k\right)^{1/2} \quad \text{(by (P2))}\\ &= \omega_0^{1/2}(\alpha_0+1)^{1/2}(Au,u)^{1/2}\left(\sum_{k=0}^{p}(Au, I_k P_k u)\right)^{1/2}\\ &\qquad \text{(by (2.3) and (2.2))}\\ &= \omega_0^{1/2}(\alpha_0+1)^{1/2}(Au,u)^{1/2}(MAu,Au)^{1/2} \quad \text{(by (2.3))},\end{aligned}$$

which indicates that

$$\lambda_{\max}(MA) \leqslant \omega_0(\alpha_0+1).$$

Combining the lower and upper bounds gives the desired bound on $\kappa(MA)$. □

Remark 1. For the related work to the abstract framework of additive Schwarz methods, we refer to Xu [36], Griebel and Oswald [24], Dryja and Widlund [22], Le Tallec [26] and Smith et al. [34].

(P1) and (P2) are natural extensions of assumptions from the theory for structured meshes by Xu [36], where all the spaces V^k, $0 \leqslant k \leqslant p$, are assumed to be subspaces of V. The kind of partition in (P1) was first introduced by Nepomnyaschikh [30] and Lions [27] with exact local solvers, i.e., R_k^{-1} equal to inverses of restrictions of A on V^k, and was then generalized by Xu [36] to allow for inexact local solvers. (P1) means that any function in V can be decomposed into a sum of functions in spaces V^k and this partition is stable with the "energy" norm in some sense, this ensures the lower bound of the smallest eigenvalue of MA. (P2) is equivalent to $\lambda_{\max}(R_k A_k) \leqslant \omega_0$ where $A_k = Q_k A I_k$ is the "restriction" of A on V^k, which means that the approximation of R_k to the inverse of A_k can not be "too bad". (P3) is a condition on the "local" properties of V^k $(1 \leqslant k \leqslant p)$, and requires that the overlapping of spaces V^k be bounded independent of p in terms of the energy norm.

Note that our (P3) is not the extension of the corresponding assumption used in [36]. It might be replaced by the extension of the so-called strengthened Cauchy–Schwarz inequality in [36] for nested subspaces with identity operators I_i $(1 \leqslant i \leqslant p)$:

(P3*) Let $\varepsilon_{ij} \in (0,1]$ be the smallest constants satisfying

$$(AI_i u_i, I_j u_j) \leqslant \varepsilon_{ij}(AI_i u_i, I_i u_i)^{1/2}(AI_j u_j, I_j u_j)^{1/2},$$
$$\forall u_i \in V^i, \quad u_j \in V^j, \qquad i,j = 1,\dots,p.$$

It is straightforward to prove that (P1), (P2) and (P3*) imply (P3). Thus (P3) is a weaker assumption than (P3*). We prefer (P3) to (P3*) as (P3) is more convenient to check than (P3*), especially for subspaces defined on unstructured meshes, and for multilevel additive type methods, see section 6 for more details.

2.2. *Multilevel additive preconditioners for SPD operators*

Though the formulation of preconditioners in this subsection may actually be included in (2.1) of the last section, we prefer to present a more detailed formulation of preconditioners and assumptions here for the convenience of the applications of these algorithms later on.

Let V and V^l $(0 \leqslant l \leqslant L)$ be given finite dimensional spaces with inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_l$ $(0 \leqslant l \leqslant L)$ respectively. All spaces V^l are not necessarily subspaces of V. We assume further that for each l: $1 \leqslant l \leqslant L$, the space V^l can be decomposed into a sum of subspaces V_k^l $(1 \leqslant k \leqslant N_l)$. Analogous to the last section, we are interested in the following type of preconditioners for a given SPD operator A defined on the space V:

$$M = \sum_{l=0}^{L}\sum_{k=1}^{N_l} I_k^l R_k^l Q_k^l, \qquad (2.5)$$

where the "interpolation" operators $I_k^l : V_k^l \to V$ are linear, and the "projection" operators Q_k^l are the adjoints of I_k^l defined by

$$\left(Q_k^l u, v_k^l\right)_l = \left(u, I_k^l v_k^l\right), \quad \forall u \in V,\ v_k^l \in V_k^l, \qquad (2.6)$$

and $R_k^l : V_k^l \to V_k^l$ are given SPD operators, approximating the inverses of the restrictions of A on V_k^l in some sense. It is easy to verify that M is an SPD operator on V. Note that for $l = 0$, we adopt the notation

$$N_0 = 1, \qquad I_k^0 = I^0, \qquad Q_k^0 = Q^0, \qquad R_k^0 = R^0.$$

As in the last section, the condition number of MA can be bounded in terms of three parameters K_0, ω and α_0 defined as follows:

(P1′) For any $u \in V$, there exist $u_k^l \in V_k^l$ $(0 \leqslant l \leqslant L,\ 1 \leqslant k \leqslant N_l)$ such that $u = \sum_{l=0}^{L}\sum_{k=1}^{N_l} I_k^l u_k^l$ and

$$\sum_{l=0}^{L}\sum_{k=1}^{N_l}\left((R_k^l)^{-1}u_k^l, u_k^l\right)_k \leqslant K_0(Au,u).$$

(P2′) For any $u_k^l \in V_k^l$, $0 \leqslant l \leqslant L$, $1 \leqslant k \leqslant N_l$,

$$(AI_k^l u_k^l, I_k^l u_k^l) \leqslant \omega_0\left((R_k^l)^{-1}u_k^l, u_k^l\right)_l.$$

(P3′) For any $u \in V$ and $u_k^l \in V_k^l$, $0 \leqslant l \leqslant L$, $1 \leqslant k \leqslant N_l$,

$$\sum_{l=0}^{L}\sum_{k=1}^{N_l}(Au, I_k^l u_k^l) \leqslant \alpha_0^{1/2}(Au,u)^{1/2}\left(\sum_{l=0}^{L}\sum_{k=1}^{N_l}(AI_k^l u_k^l, I_k^l u_k^l)\right)^{1/2}.$$

Analogous to theorem 1, we have

Theorem 2. Under the assumptions (P1′)–(P3′),

$$\kappa(MA) \leqslant \omega_0(\alpha_0+1)K_0.$$

2.3. *Additive preconditioners for small perturbations of SPD operators*

The results of this section are applicable to the systems arising from the Galerkin discretization of general non-symmetric parabolic problems. Let V be a finite dimensional space with the scalar product $(\cdot,\cdot)$, and E a non-symmetric operator on V which is a small perturbation of the SPD operator A, that is, $E = A + B$, and we solve the equation

$$Eu \equiv (A+B)u = f$$

on V. Our goal is to find a good preconditioner M for the non-symmetric operator E. Then we can use iterative methods, like GMRES or BiCGSTAB, to solve

$$MEu = Mf$$

instead of $Eu = f$. Let us consider the GMRES method. It is known (cf. [23]) that the convergence rate of GMRES depends on the following two parameters:

$$\beta_1 = \min_{u\neq 0}\frac{(u, MEu)_A}{(u,u)_A}, \qquad \beta_2 = \max_{u\neq 0}\frac{\|MEu\|_A}{\|u\|_A}. \tag{2.7}$$

If $\beta_1 > 0$, GMRES converges, and at the mth iteration the residual is bounded as (cf. [23])

$$\|Mf - MEu^m\|_A \leqslant \left(1 - \frac{\beta_1^2}{\beta_2^2}\right)^{m/2}\|Mf - MEu^0\|_A.$$

Suppose we are given finite dimensional spaces V^k $(1 \leqslant k \leqslant p)$ and the scalar products $(\cdot,\cdot)_k$, where V^k need not be subspaces of V. Let the linear operators $I_k : V^k \to V$, their adjoints Q_k and the SPD operators $R_k : V^k \to V^k$ be defined as in section 2.1. We are interested in the following additive type preconditioners

$$M = \sum_{k=0}^{p} I_k R_k Q_k, \tag{2.8}$$

for operator E. Note that we still use an SPD preconditioner M even though E is non-symmetric, this idea was earlier used by Xu and Cai [38] and Xu [37]. For later use, we write

$$ME = \sum_{k=0}^{p} I_k P_k, \quad P_k = R_k Q_k E. \tag{2.9}$$

We introduce two assumptions for the perturbation operator B:

(P4) For any $u \in V$ and $u_k \in V^k$, $1 \leqslant k \leqslant p$,

$$\sum_{k=1}^{p}(Bu, I_k u_k) \leqslant \alpha_1^{1/2}(Au,u)^{1/2}\left(\sum_{k=1}^{p}(AI_k u_k, I_k u_k)\right)^{1/2}.$$

(P5) There exists a constant $\mu_1 \in (0,1)$ such that for any $u, v \in V$,

$$|(Bu,v)| \leqslant \mu_1\|u\|_A\|v\|_A.$$

Remark 2. (P4) is the analogue of (P3) and means B is "bounded" by A in some sense, while (P5) means that the perturbation B is small relative to A and ensures the positive definiteness of E.

By the Cauchy–Schwarz inequality, (P4) and (P5) imply that for any $u \in V$ and $u_k \in V^k$ $(0 \leqslant k \leqslant p)$,

$$\sum_{k=0}^{p}(Bu, I_k u_k) \leqslant (\mu_1^2 + \alpha_1)^{1/2}(Au,u)^{1/2}\left(\sum_{k=0}^{p}(AI_k u_k, I_k u_k)\right)^{1/2} \tag{2.10}$$

Theorem 3. If in addition to (P1)–(P5), we assume further that

$$\mu_1^2 + \alpha_1 \leqslant \frac{(1-\mu_1)^2}{2\omega_0 K_0}, \tag{2.11}$$

then we have

$$\beta_1 = \min_{u\neq 0} \frac{(u, ME)_A}{(u,u)_A} \geqslant \frac{(1-\mu_1)^2}{4K_0},$$

$$\beta_2 = \max_{u\neq 0} \frac{\|MEu\|_A}{\|u\|_A} \leqslant 2\omega_0\alpha_0^{1/2}\left(1+\alpha_0+\alpha_1+\mu_1^2\right)^{1/2}.$$

Proof. We first estimate β_2. For any $u \in V$, let $w = \sum_{k=1}^p I_k P_k u$, then the following holds from (P3):

$$\|w\|_A^2 = \left(Aw, \sum_{k=1}^p I_k P_k u\right) \leqslant \alpha_0^{1/2}(Aw,w)^{1/2}\left(\sum_{k=1}^p (AI_kP_ku, I_kP_ku)\right)^{1/2},$$

which gives

$$\|w\|_A = \left\|\sum_{k=1}^p I_kP_ku\right\|_A \leqslant \alpha_0\left(\sum_{k=1}^p \|I_kP_ku\|_A^2\right)^{1/2}. \tag{2.12}$$

Using this, we can deduce that

$$\|MEu\|_A^2 = \left\|\sum_{k=0}^p I_kP_ku\right\|_A^2 \leqslant 2\|I_0P_0u\|_A^2 + 2\left\|\sum_{k=1}^p I_kP_ku\right\|_A^2$$

((2.9) and triangle inequality)

$$\leqslant 2\|I_0P_0u\|_A^2 + 2\alpha_0\sum_{k=1}^p \|I_kP_ku\|_A^2 \quad \text{(by (2.12))}$$

$$\leqslant 2\omega_0\alpha_0\sum_{k=0}^p \left(R_k^{-1}P_ku, P_ku\right)_k \quad \text{(by (P2))}, \tag{2.13}$$

but $R_k^{-1}P_k = Q_kE$ by definition of P_k, thus using $E = A+B$ we obtain that

$$\sum_{k=0}^p \left(R_k^{-1}P_ku, P_ku\right)_k = \sum_{k=0}^p (Q_kEu, P_ku)_k = \sum_{k=0}^p (Eu, I_kP_ku) \quad \text{(by (2.2))}$$

$$= \sum_{k=0}^p (Au, I_kP_ku) + \sum_{k=0}^p (Bu, I_kP_ku)$$

$$\leqslant \sqrt{2\left(1+\alpha_0+\alpha_1+\mu_1^2\right)}\|u\|_A\left(\sum_{k=0}^p \|I_kP_ku\|_A^2\right)^{1/2}$$

(by (2.5) and (2.10))

$$\leqslant \sqrt{2\left(1+\alpha_0+\alpha_1+\mu_1^2\right)}\omega_0^{1/2}\|u\|_A \left(\sum_{k=0}^p (R_k^{-1}P_ku, P_ku)_k\right)^{1/2} \quad \text{(by (P2))},$$

which indicates that

$$\sum_{k=0}^p \left(R_k^{-1}P_k, P_ku\right)_k \leqslant 2\omega_0\left(1+\alpha_0+\alpha_1+\mu_1^2\right)(Au,u).$$

This with (2.13) implies the bound for β_2.

Next we bound β_1. By (P5), we see that

$$(Au,u) \leqslant (Eu,u) - (Bu,u) \leqslant (Eu,u) + \mu_1(Au,u).$$

Then we deduce that

$$(1-\mu_1)(Au,u) \leqslant (Eu,u) = \sum_{k=0}^p (Eu, I_ku_k) \quad \text{(by (P1))}$$

$$= \sum_{k=0}^p (Q_kEu, u_k)_k = \sum_{k=0}^p \left(R_k^{-1}P_ku, u_k\right)_k \quad \text{(by (2.2) and (2.9))}$$

$$\leqslant K_0^{1/2}\|u\|_A\left(\sum_{k=0}^p (R_k^{-1}P_ku, P_ku)_k\right)^{1/2}$$

(Schwarz inequality and (P1)),

from which we obtain

$$(1-\mu_1)^2(Au,u) \leqslant K_0\sum_{k=0}^p \left(R_k^{-1}P_ku, P_ku\right)_k. \tag{2.14}$$

Finally we come to

$$(MEu,u)_A = \sum_{k=0}^p (I_kP_ku, Au)$$

$$= \sum_{k=0}^p (I_kP_ku, Eu) - \sum_{k=0}^p (I_kP_ku, Bu) \quad \text{(by } E = A+B\text{)}$$

$$= \sum_{k=0}^p \left(R_k^{-1}P_ku, P_ku\right)_k - \sum_{k=0}^p (Bu, I_kP_ku) \quad (P_k\text{'s definition})$$

$$\geqslant \sum_{k=0}^p \left(R_k^{-1}P_ku, P_ku\right)_k$$

$$-\,\omega_0^{1/2}\left(\mu_1^2+\alpha_1\right)^{1/2}\|u\|_A\left(\sum_{k=0}^p \left(R_k^{-1}P_ku, P_ku\right)_k\right)^{1/2}$$

$$\geqslant \frac{1}{2}\sum_{k=0}^{p}\left(R_k^{-1}P_k u, u\right)_k - \frac{1}{2}\omega_0\left(\mu_1^2+\alpha_1\right)(Au,u)$$

$$\geqslant \frac{(1-\mu_1)^2}{4K_0}(Au,u) \quad \text{(by (2.11) and (2.14))},$$

where we have used (P2) and (2.10) for the first inequality and the fact $xy \leqslant (x^2+y^2)/2$, $\forall x,y>0$, for the second inequality. □

3. Technical lemmas

In order to apply the general convergence framework of section 2 to elliptic and parabolic problems, one needs to verify three assumptions given in section 2. Here we develop some technical lemmas which will be used for such verifications in the subsequent sections. We first introduce the fine and coarse finite element spaces, domain decompositions and the assumptions required for the relations between the fine and coarse triangulations of the original domain Ω. Then we demonstrate the local L^2-optimal approximation and H^1-stability of the standard fine finite element interpolant on the coarse spaces, and the local L^2-optimal approximation and H^1-stability of the Clément interpolant.

3.1. *Finite elements and domain decompositions*

Given an open domain $\Omega \subset \mathbb{R}^d$ $(d=2,3)$ on which certain elliptic and parabolic problems are defined, we will solve these problems by finite element methods. Suppose we are given a family of triangulations $\{\mathcal{T}^h\}$ on Ω, consisting of simplices. We will not discuss the effects of approximating Ω but always assume in this paper that the triangulations $\{\mathcal{T}^h\}$ of Ω are exact, i.e., we assume Ω is polygonal or polyhedral, and

$$\Omega = \Omega^h \equiv \bigcup_{\tau^h \in \mathcal{T}^h} \tau^h.$$

Let $h_\tau = \operatorname{diam} \tau^h$, $\overline{h} = \max_{\tau^h\in\mathcal{T}^h} h_\tau$, $\underline{h} = \min_{\tau^h\in\mathcal{T}^h} h_\tau$, $\rho_\tau =$ the radius of the largest ball inscribed in τ^h. We say an element $\tau \in \mathcal{T}^h$ is σ_0-*shape regular* if

$$h_\tau/\rho_\tau \leqslant \sigma_0, \tag{3.1}$$

and $\mathcal{T}^h$ is σ_0-shape regular (or shape regular for short) if all its elements are σ_0-shape regular. Moreover, we say $\mathcal{T}^h$ is *quasi-uniform* if it is shape regular and satisfies

$$\overline{h} \leqslant \sigma_1 \underline{h},$$

with σ_0 and σ_1 fixed positive constants.

As quasi-uniformity is too restrictive for unstructured meshes, the present paper will assume only that all the fine and coarser triangulations used are shape regular, not necessarily quasi-uniform.

Let V^h be a piecewise linear finite element subspace of $H_0^1(\Omega)$ defined on $\mathcal{T}^h$ with the set of basis functions $\{\phi_i^h\}_{i=1}^n$, and $O_i^h = \operatorname{supp}\phi_i^h$. Later on we will use the following simple fact: if $\mathcal{T}^h$ is shape regular, there exist a constant $\nu_0>0$ and an integer $\nu_1>0$, both depending only on σ_0 in (3.1) and independent of h so that, for $1\leqslant i\leqslant n$,

$$\operatorname{diam} O_i^h \leqslant \nu_0 h_\tau \quad \forall \tau^h \subset O_i^h, \qquad \operatorname{card}\{\tau^h\in\mathcal{T}^h;\ \tau^h\subset O_i^h\} \leqslant \nu_1. \tag{3.2}$$

Decompose the domain Ω into p non-overlapping subdomains $\widetilde{\Omega}^k$ $(1\leqslant k\leqslant p)$, then extend each $\widetilde{\Omega}^k$ to a larger one Ω^k such that the distance between $\partial\Omega^k$ and $\partial\widetilde{\Omega}^k$ is bounded from below by $\delta_k>0$. We assume that $\partial\Omega^k$ does not cut through any element $\tau^h\in\mathcal{T}^h$. For the subdomains meeting the boundary $\partial\Omega$ we cut off the part of Ω^k which is outside $\bar{\Omega}$. We allow each Ω^k to be of quite different size and of quite different shape from other subdomains, but we make the following assumption:

(A1) Any point $x\in\Omega$ belongs to at most q_0 subdomains of $\{\Omega^k\}_{k=1}^p$ with $q_0>0$ an integer.

Define the subspaces $\{V^k\}_{k=1}^p$ of V^h corresponding to the subdomains $\{\Omega^k\}_{k=1}^p$ by

$$V^k = \{v\in V^h;\ v=0 \text{ on } \Omega\setminus\Omega^k\}. \tag{3.3}$$

To develop a two level method, we introduce a coarse grid $\mathcal{T}^H$ which forms a σ_0-shape regular triangulation of Ω, but otherwise has nothing to do with $\mathcal{T}^h$, i.e., none of the nodes of $\mathcal{T}^H$ need to be nodes of $\mathcal{T}^h$. Let Ω^0 be the coarse grid domain, i.e., $\Omega^0 = \bigcup_{\tau^H\in\mathcal{T}^H}\tau^H$, and $\{q_i^H\}$ the set of nodes of $\mathcal{T}^H$.

Denote by V^0 (resp. $\widehat{V}^0$) the subspace of $H_0^1(\Omega^0)$ (resp. $H^1(\Omega^0)$) consisting of piecewise linear functions defined on $\mathcal{T}^H$ with $\{\psi_k^H;\ q_k^H\in\Omega^0\}$ (resp. $\{\psi_k^H;\ q_k^H\in\bar{\Omega}^0\}$) the set of nodal basis functions. We remark that V^0 need not be piecewise linear as is V^h. Our linearity assumption on V^0 is just for the sake of simplicity. Note that Ω^0 usually does not match with Ω, and $V^0 \not\subset V^h$, cf. figure 1.

We need to impose a few reasonable assumptions on the coarse grid Ω^0:

(A2) For any $\tau^H\in\mathcal{T}^H$, the measure of all $\tau^h\in\mathcal{T}^h$: $\tau^h\cap\partial\tau^H\neq\varnothing$, is bounded by the measure of τ^H with a constant factor.

(A3) For any node $q_i^h\in\partial\Omega\cap(\Omega^0\setminus\Omega)$, one can construct two σ_0-shape regular simplices $\widehat{\tau}_i^h$ and $\widetilde{\tau}_i^h$ such that $q_i^h\in\widehat{\tau}_i^h$, $\widehat{\tau}_i^h\subset\widetilde{\tau}_i^h\cap\Omega^0$, $\operatorname{diam}(\widehat{\tau}_i^h)\approx\operatorname{diam}(\widetilde{\tau}_i^h)\approx\operatorname{diam}(O_i^h)$, and one face of $\widetilde{\tau}_i^h$ lies in $R^d\setminus\Omega^0$.

(A4) $\Omega\setminus\Omega^0 \subset \bigcup_{q_k^H\in\partial\Omega^0}\widehat{O}_k^H$, $\widehat{O}_k^H = B_{q_k^H}(\operatorname{diam} O_k^H)\cap\Omega$, where $O_k^H = \operatorname{supp}\psi_k^H$.

Note that (A3) means the coarse grid part outside the fine grid is of fine element sizes, while (A4) says the fine grid part outside the coarse grid is of coarse element sizes, cf. figure 2. Here $B_p(r)$ is a ball centered at point p with radius r.

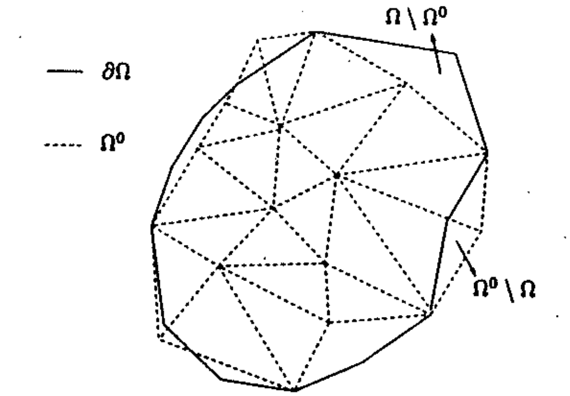

Figure 1. The fine domain Ω and non-matching coarse domain Ω^0.

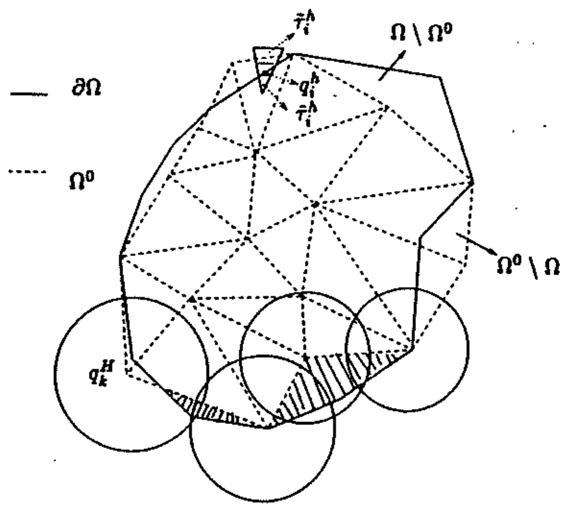

Figure 2. The shadow part belonging to $(\Omega \setminus \Omega^0)$ is covered by balls $B_{q_k^H}(r_k)$, $r_k = \text{diam}\, O_k^H$ as stated in (A4). q_i^h is a fine node in $\partial\Omega \cap (\Omega^0 \setminus \Omega)$, and $\hat{\tau}_i^h$ and $\bar{\tau}_i^h$ are two simplices stated in (A3).

3.2. H^1-stability and L^2-optimal approximation of two linear interpolants

Since $V^0 \not\subset V^h$ for our interest, the convergence proofs for overlapping domain decomposition methods require the existence of an operator $I_0 : V^0 \to V^h$ to satisfy the H^1-stability and L^2-optimal approximation properties.

We introduce notations: for $\tau^h \in \mathcal{T}^h$ and $\tau^H \in \mathcal{T}^H$,

$$N(\tau^h) = \bigcup_{\tau^{h'} \cap \tau^h \neq \emptyset} \tau^{h'}, \qquad h_k = \max_{\tau^h \subset \Omega^k} h_\tau, \qquad B_k = \bigcup_{\tau^H \cap \Omega^k \neq \emptyset} \tau^H,$$

$$N(\tau^H) = \bigcup_{\tau^{H'} \cap \tau^H \neq \emptyset} \tau^{H'}, \qquad H_k = \max_{\tau^H \subset B_k} H_\tau, \qquad S_k = \bigcup_{\tau^H \subset B_k} N(\tau^H).$$

Note that $N(\tau^h)$ (resp. $N(\tau^H)$) is the union of closest neighboring fine (resp. coarse) elements to τ^h (resp. τ^H). B_k is the union of all coarse elements having nonempty intersection with the subdomain Ω^k. We assume that

(A5) $h_k \lesssim H_k$ and $\text{card}\{\tau^H \in \mathcal{T}^H;\ \tau^H \subset B_k\} \leqslant q_0$ for $1 \leqslant k \leqslant p$ with $q_0 > 0$ an integer.

(A6) Any point $x \in \Omega^0$ belongs to at most q_0 subdomains of $\{S_k\}_{k=1}^p$.

Remark 3. (A5) means that the minimum number of coarse elements whose union covers the subdomain Ω^k is less than a fixed constant q_0.

Standard finite element interpolant

Define $\Pi_h^0 : V^0 \to V^h$ and $\Pi_h : V^0 \to H^1(\Omega)$ to be the standard finite element interpolants, i.e.,

$$\Pi_h^0 u = \sum_{q_i^h \in \Omega} u(q_i^h)\phi_i^h, \quad \Pi_h u = \sum_{q_i^h \in \bar{\Omega}} u(q_i^h)\phi_i^h \qquad \forall u \in V^0. \tag{3.4}$$

Note the only difference in the definitions of Π_h^0 and Π_h is that the former excludes the boundary nodes but the latter includes the boundary nodes. The following lemma gives the completely local properties of these two interpolants which are sharper than the ones in Chan et al. [12], where only global bounds are derived:

Lemma 4. With the assumptions (A2), (A3) and (A5), for any $u \in V^0$ and $s, t = 0, 1$, $s \leqslant t$,

$$|u - \Pi_h u|_{s,\Omega^k} \lesssim h_k^{t-s}\left(\frac{h_k}{H_k}\right)^{1-t} |u|_{t,\tau_k^H} \lesssim h_k^{t-s}\left(\frac{h_k}{H_k}\right)^{1-t} |u|_{t,B_k}, \quad 1 \leqslant k \leqslant p,$$

holds with $\tau_k^H \in \mathcal{T}^H$ such that $|u|_{1,\infty,\tau_k^H} = \max_{\tau^{H'} \subset B_k} |u|_{1,\infty,\tau^{H'}}$, and

$$|u - \Pi_h^0 u|_{s,\Omega^k} \lesssim h_k^{t-s}|u|_{t,B_k}, \quad 1 \leqslant k \leqslant p.$$

Proof. As $u \in V^0$, so $u \in W^{1,\infty}(R^d)$. For any $\tau^h \in \mathcal{T}^h$, one gets (cf. theorem 3.1.5 in [17])

$$|u - \Pi_h u|_{s,\tau^h}^2 \lesssim h_\tau^{2(1-s)} h_\tau^d |u|_{1,\infty,\tau^h}^2. \tag{3.5}$$

Using the analog of (3.2) for $\mathcal{T}^H$, we have for any $\tau^H \subset B_k$ that

$$\sum_{\substack{\tau^h\cap\tau^H\neq\emptyset\\ \tau^h\subset\Omega^k}} |u-\Pi_h u|^2_{s,\tau^h} \lesssim h_k^{2(1-s)} \sum_{\substack{\tau^h\cap\tau^H\neq\emptyset\\ \tau^h\subset\Omega^k}} h_\tau^d |u|^2_{1,\infty,\tau^h} \quad \text{(by (3.5))}$$

$$\lesssim h_k^{2(1-s)} H_k^d |u|^2_{1,\infty,\tau_k^H}$$

$$\lesssim h_k^{2(t-s)} \left(\frac{h_k}{H_k}\right)^{2(1-t)} |u|^2_{t,\tau_k^H}$$

(by (A2) and inverse inequality),

now the inequality for Π_h follows from (A5) and the relation:

$$|u-\Pi_h u|^2_{s,\Omega^k} \leqslant \sum_{\tau^H\subset B_k} \sum_{\substack{\tau^h\cap\tau^H\neq\emptyset\\ \tau^h\subset\Omega^k}} |u-\Pi_h u|^2_{s,\tau^h}.$$

For Π_h^0: let D^h be the set of boundary nodes of $\mathcal{T}^h$ which also belong to Ω^0, i.e., $D^h = \partial\Omega\cap(\Omega^0\setminus\Omega)$, we can write

$$u-\Pi_h^0 u = (u-\Pi_h u) + \widehat{\Pi}_h u, \quad \widehat{\Pi}_h u = \sum_{q_i^h\in D^h} u(q_i^h)\phi_i^h. \tag{3.6}$$

We have

$$|\widehat{\Pi}_h u|^2_{s,\Omega^k} \lesssim \sum_{q_i^h\in\bar\Omega^k\cap D^h} |u(q_i^h)\phi_i^h|^2_{s,O_i^h} \lesssim \sum_{q_i^h\in\bar\Omega^k\cap D^h} |\phi_i^h|^2_{s,O_i^h}\|u\|^2_{0,\infty,\widehat{\tau}_i^h} \quad \text{(by (A3))}$$

$$\lesssim \sum_{q_i^h\in\bar\Omega^k\cap D^h} h_k^{-2s}\|u\|^2_{0,\widehat{\tau}_i^h}$$

$$\leqslant \sum_{q_i^h\in\bar\Omega^k\cap D^h} h_k^{-2s}\|u\|^2_{0,\widehat{\tau}_i^h} \quad \text{(inverse inequality and (A3))}$$

$$\lesssim \sum_{q_i^h\in\bar\Omega^k\cap D^h} h_k^{2(t-s)}|u|^2_{t,\widehat{\tau}_i^h} \quad \text{((A3) and Poincaré inequality for } t=1\text{)}.$$

This with (3.6) and the triangle inequality implies the required result for Π_h^0. □

Clément's interpolant

We now introduce a locally defined interpolant R_H^0 proposed by Clément in [18]. Operators with similar properties to R_H^0 can be found in Scott and Zhang [32], see also Xu and Zou [39].

Definition 5. The mappings $R_H^0 : L^2(\Omega^0)\to V^0$ and $R_H : L^2(\Omega^0)\to\widehat{V}^0$ are defined by

$$R_H^0 u = \sum_{q_i^H\in\Omega^0} Q_i u(q_i^H)\psi_i^H, \quad R_H u = \sum_{q_i^H\in\bar\Omega^0} Q_i u(q_i^H)\psi_i^H, \qquad \forall u\in L^2(\Omega^0),$$

where $Q_i u\in\mathcal{P}_1(O_i^H)$, $O_i^H = \operatorname{supp}\psi_i^H$, satisfies

$$(Q_i u, p)_{0,O_i^H} = (u,p)_{0,O_i^H}, \quad \forall p\in\mathcal{P}_1(O_i^H).$$

Recall that V^0 and $\widehat{V}^0$ are defined in section 3.1. Clément [18] proved

Lemma 6. For any $u\in H^1(\Omega^0)$ and $s,t=0,1,\ s\leqslant t$,

$$|u-R_H u|_{s,\tau^H} \lesssim H_\tau^{t-s}|u|_{t,N(\tau^H)} \quad \forall\tau^H\in\mathcal{T}^H.$$

For the interpolant R_H^0, we have

Lemma 7. Under the assumption (A4), for any $u\in V^h$ and $s,t=0,1,\ s\leqslant t$,

$$\left|u-R_H^0 u\right|_{s,\Omega^k} \lesssim H_k^{t-s}|u|_{t,S_k} \quad (1\leqslant k\leqslant p), \qquad \left|R_H^0 u\right|_{s,\tau^H} \lesssim |u|_{s,N(\tau^H)},$$

if $\bar B_k\cap\partial\Omega^0=\emptyset$ and $\tau^H\in\mathcal{T}^H$: $\tau^H\cap\partial\Omega^0=\emptyset$; and

$$\left|u-R_H^0 u\right|^2_{s,\Omega^k} \lesssim H_k^{2(t-s)}|u|^2_{t,S_k} + \sum_{q_i^H} H_k^{2(t-s)}|u|^2_{t,\widehat{O}_i^H} \quad (1\leqslant k\leqslant p),$$

$$\left|R_H^0 u\right|^2_{s,\tau^H} \lesssim |u|^2_{s,N(\tau^H)} + \sum_{q_i^H\in\partial\Omega^0\cap\bar\tau^H} |u|^2_{s,\widehat{O}_i^H},$$

if $\bar B_k\cap\partial\Omega^0\neq\emptyset$ and $\tau^H\in\mathcal{T}^H$: $\tau^H\cap\partial\Omega^0\neq\emptyset$. Here $\widehat{O}_i^H$ is defined in (A4) and the above summand $\sum_{q_i^H}$ is made for all $q_i^H\in\partial\Omega^0$: $O_i^H\cap\Omega^k\neq\emptyset$.

Proof. The first case is an immediate consequence of lemma 6 and the fact $R_H^0 u = R_H u$ on $\tau^H\subset B_k$. Next we show only the case that $B_k\cap\partial\Omega^0\neq\emptyset$, the result for $\tau^H\in\mathcal{T}^H$: $\tau^H\cap\partial\Omega^0\neq\emptyset$ can be proved analagously. Obviously,

$$\left|u-R_H^0 u\right|^2_{s,\Omega^k} = \left|u-R_H^0 u\right|^2_{s,\Omega^k\cap\Omega^0} + \left|u-R_H^0 u\right|^2_{s,\Omega^k\cap(\Omega\setminus\Omega^0)}. \tag{3.7}$$

One obtains by (A4) and the Poincaré inequality that

$$\left|u-R_H^0 u\right|^2_{s,\Omega^k\cap(\Omega\setminus\Omega^0)} = |u|^2_{s,\Omega^k\cap(\bar\Omega\setminus\Omega^0)} \leqslant \sum_{q_j^H\in\partial\Omega^0} |u|^2_{s,\widehat{O}_j^H\cap\Omega^k}$$

$$\lesssim \sum_{q_j^H\in\partial\Omega^0} \left(\operatorname{diam} O_j^H\right)^{2(t-s)}|u|^2_{t,\widehat{O}_j^H\cap\Omega^k} \lesssim H_k^{2(t-s)}|u|^2_{t,\Omega^k}.$$

Using

$$u - R_H^0 u = u - R_H u + \sum_{q_i^H \in \partial\Omega^0} Q_i u(q_i^H)\psi_i^H,$$

we have

$$|u - R_H^0 u|^2_{s,\Omega^k\cap\Omega^0} \lesssim \sum_{\tau^H \subset B_k} |u - R_H u|^2_{s,\tau^H} + \sum_{q_i^H} |Q_i u(q_i^H)\psi_i^H|^2_{s,\Omega^k\cap\Omega^0},$$

where the second summand is made for all $q_i^H \in \partial\Omega^0$: $O_i^H \cap \Omega^k \neq \varnothing$. To estimate this term, for any such coarse node q_i^H, one derives

$$\begin{aligned}
|Q_i u(q_i^H)\psi_i^H|^2_{s,\Omega^k\cap\Omega^0} &\leqslant \sum_{\tau^H \subset O_i^H} |\psi_i^H|^2_{s,O_i^H} \|Q_i u\|^2_{0,\infty,\tau^H} \\
&\lesssim \sum_{\tau^H \subset O_i^H} H_\tau^{-2s} \|Q_i u\|^2_{0,\tau^H} \\
&\leqslant H_\tau^{-2s} \|Q_i u\|^2_{0,O_i^H} \quad \text{(inverse inequality)} \\
&\lesssim H_\tau^{-2s} \|u\|^2_{0,O_i^H} \leqslant H_\tau^{-2s} \|u\|^2_{0,\widehat{O}_i^H} \quad \text{((A4) and } Q_i\text{'s definition)} \\
&\lesssim H_k^{2(t-s)} |u|^2_{t,\widehat{O}_i^H} \quad \text{(Poincaré inequality).}
\end{aligned}$$

This with lemma 6 gives

$$|u - R_H^0 u|^2_{s,\Omega^k\cap\Omega^0} \lesssim H_k^{2(t-s)} |u|^2_{t,S_k} + \sum_{q_i^H} H_k^{2(t-s)} |u|^2_{t,\widehat{O}_i^H}.$$

Lemma 7 follows from (3.7) and the above. □

4. Two level additive Schwarz method for elliptic problems

In this and the next two sections, we apply the general theory of section 2 to second order elliptic and parabolic problems. For simplicity, we restrict ourselves only to pure Dirichlet or pure Neumann boundary conditions. First, in this section, we consider the following selfadjoint elliptic problem:

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + bu = f \quad \text{in } \Omega \tag{4.1}$$

with Dirichlet boundary condition: $u = 0$ on $\partial\Omega$. Here $\Omega \subset \mathbf{R}^d$ $(d = 2, 3)$ as described in section 3.1, $(a_{ij}(x))$ is symmetric, uniformly positive definite, and $b(x) \geqslant 0$ in Ω.

The purpose of this section is to present more simplified proofs than our old ones in [12, 15] by means of the present framework and also give an improved local bound on the condition number. The previous bounds of the condition numbers are global.

The weak formulation of the above problem is: Find $u \in H_0^1(\Omega)$ such that

$$A_\Omega(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega),$$

with

$$A_\Omega(u, v) = \int_\Omega \left(\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} + buv \right) dx.$$

All notations of this section are inherited from section 3.

The finite element problem is: Find $u \in V^h$ such that

$$A_\Omega(u, v) = (f, v), \quad \forall v \in V^h. \tag{4.2}$$

Based on the local finite element spaces V^k $(1 \leqslant k \leqslant p)$ and the coarse space $V^0 = V^H$ defined in section 3.1, we shall use the abstract theory of section 2 to construct the two level overlapping Schwarz methods on unstructured meshes.

We define scalar products $(\cdot,\cdot)_k = (\cdot,\cdot)_{0,\Omega^k}$ on V^k for $1 \leqslant k \leqslant p$ and $(\cdot,\cdot) = (\cdot,\cdot)_{0,\Omega}$ on V^h, and then define an SPD operator A on V^h and a coarse operator A_0 by

$$(Au, v) = A_\Omega(u, v), \quad \forall u, v \in V^h, \qquad (A_0 u, v)_0 = A_{\Omega^0}(u, v), \quad \forall u, v \in V^0,$$

and local operators A_k, $1 \leqslant k \leqslant p$ by

$$(A_k u, v)_k = A_{\Omega^k}(u, v), \quad \forall u, v \in V^k.$$

Since $V^k \subset V^h$, $1 \leqslant k \leqslant p$, we define $I_k : V^k \to V^h$ to be the natural injection operator. Note $V^0 \not\subset V^h$, define $I_0 : V^0 \to V^h$ to be the natural interpolant Π_h^0 in (3.4). One may also use other choices of I_0, e.g., the Clément interpolant R_h^0. Choose the local solvers R_k, $0 \leqslant k \leqslant p$ to be exact solvers, i.e., $R_k^{-1} = A_k$. Then the preconditioner M in (2.1) for A is

$$M = \sum_{k=0}^{p} I_k A_k^{-1} Q_k.$$

In order to apply theorem 1 for the estimate of $\kappa(MA)$, we need a partition lemma for the finite element space V^h:

$$V^h = I_0 V^0 + V^1 + \cdots + V^p,$$

whose proof will be given at the end of the section.

Lemma 8. With the assumptions (A1)–(A6), for any $u \in V^h$, there exist $u_k \in V^k$ $(1 \leqslant k \leqslant p)$ and $u_0 = R_H^0 u \in V^0$ such that $u = \sum_{k=0}^p I_k u_k = I_0 R_H^0 u + \sum_{k=1}^p u_k$ and

$$\sum_{k=1}^p |u_k|_{1,\Omega}^2 \lesssim \left(\max_{1\leqslant k\leqslant p} \frac{H_k^2}{\delta_k^2}\right)|u|_{1,\Omega}^2, \quad \sum_{k=1}^p \|u_k\|_{0,\Omega}^2 \lesssim \left(\max_{1\leqslant k\leqslant p} \frac{h_k^2}{\delta_k^2}\right)\|u\|_{0,\Omega}^2, \tag{4.3}$$

$$|u_0|_{s,\Omega^0} \lesssim |u|_{s,\Omega}, \quad s = 0, 1. \tag{4.4}$$

The following theorem gives the bound of the condition number $\kappa(MA)$.

Theorem 9. Under the assumptions (A1)–(A6), we have

$$\kappa(MA) \lesssim \max_{1\leqslant k\leqslant p} \frac{H_k^2}{\delta_k^2}.$$

Note that from theorem 9 one can expect an optimal condition number if the local overlap δ_k is proportional to the size H_k for each k $(1 \leqslant k \leqslant p)$.

Proof. It suffices by theorem 1 to verify (P1)–(P3). For (P1): by lemma 8, $\forall u \in V^h$, there exist $u_k \in V^k$ and $u_0 = R_H^0 u$ such that $u = I_0 u_0 + \sum_{k=1}^p u_k$. We derive

$$\sum_{k=0}^p \left(R_k^{-1} u_k, u_k\right)_k = A_{\Omega^0}(u_0, u_0) + \sum_{k=1}^p A_{\Omega^k}(u_k, u_k) \lesssim \|u_0\|_{1,\Omega^0}^2 + \sum_{k=1}^p \|u_k\|_{1,\Omega}^2$$

$$\lesssim \left(\max_{1\leqslant k\leqslant p} \frac{H_k^2}{\delta_k^2}\right)\|u\|_{1,\Omega}^2 \lesssim \left(\max_{1\leqslant k\leqslant p} \frac{H_k^2}{\delta_k^2}\right)(Au, u),$$

which indicates $K_0 \lesssim \max_{1\leqslant k\leqslant p} H_k^2/\delta_k^2$.

For (P2): we get for any $u_k \in V^k$ $(0 \leqslant k \leqslant p)$,

$$(AI_k u_k, I_k u_k) \lesssim \|I_k u_k\|_{1,\Omega}^2 = \|u_k\|_{1,\Omega}^2 \lesssim (Au_k, u_k) = (A_k u_k, u_k)_k \quad (k \neq 0),$$

$$(AI_0 u_0, I_0 u_0) \lesssim \|I_0 u_0\|_{1,\Omega}^2 \lesssim \sum_{k=1}^p \|I_0 u_0\|_{1,\Omega^k}^2 \quad (\text{by } \Omega = \textstyle\bigcup_{k=1}^p \Omega^k)$$

$$\lesssim \sum_{k=1}^p \|u_0\|_{1,B_k}^2 \lesssim \|u_0\|_{1,\Omega^0}^2 \lesssim (A_0 u_0, u_0)_0 \quad \text{(lemma 4 and (A6))},$$

which implies $\omega_0 \lesssim 1$.

For (P3): $\forall u \in V^h$ and $u_k \in V^k$,

$$\sum_{k=1}^p (Au, I_k u_k) = \sum_{k=1}^p A_{\Omega^k}(u, u_k) \lesssim \left(\sum_{k=1}^p A_{\Omega^k}(u, u)\right)^{1/2} \left(\sum_{k=1}^p A_{\Omega^k}(u_k, u_k)\right)^{1/2}$$

$$\lesssim (Au, u)^{1/2} \left(\sum_{k=1}^p (Au_k, u_k)\right)^{1/2} \quad \text{(by (A1))},$$

thus $\alpha_0 \lesssim 1$. This completes the proof of theorem 9 by using theorem 1. □

Proof of lemma 8. (4.4) follows directly from lemma 7. We now prove (4.3). For any $u \in V^h$, choose $u_0 = R_H^0 u$ and let $v_0 = I_0 u_0$. Then lemmas 4 and 7 imply for $s = 0, 1$ that

$$|u - v_0|_{s,\Omega^k}^2 \leqslant |u - u_0|_{s,\Omega^k}^2 + |u_0 - I_0 u_0|_{s,\Omega^k}^2$$

$$\lesssim H_k^{2(1-s)}\left(|u|_{1,S_k}^2 + \sum_{q_i^H} |u|_{1,\widehat{O}_i^H}^2\right) + h_k^{2(1-s)}|u_0|_{1,B_k}^2.$$

The rest of the proof is quite routine, we refer to Chan and Zou [15]. □

5. Two level additive Schwarz method for parabolic problems

Consider the following non-selfadjoint parabolic problems:

$$\frac{\partial u}{\partial t} + Lu = f \quad \text{in } \Omega \times (0, T),$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ as described in section 3.1, and

$$Lu = -\sum_{i=1}^d \frac{\partial}{\partial x_i}\left(\sum_{j=1}^d a_{ij}\frac{\partial u}{\partial x_j} + d_i u\right) + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i} + cu$$

with $(a_{ij}(x,t))$ symmetric, uniformly positive definite and continuous on $\bar\Omega \times [0, T]$, the functions $b_i(x,t), d_i(x,t)$ and $c(x,t)$ continuous on $\bar\Omega \times [0, T]$. The initial and boundary conditions are

$$u(x, 0) = u_0(x) \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \times (0, T).$$

After discretizing the variational problem corresponding to the above parabolic problem by using some implicit finite difference schemes in time with a time step size τ and the finite element space V^h in space, the resulting discrete system may be formulated: Find $u \in V^h$ such that

$$E_\Omega(u, v) \equiv A_\Omega(u, v) + B_\Omega(u, v) = (f, v), \quad \forall v \in V^h, \tag{5.1}$$

where

$$A_\Omega(u, v) = \int_\Omega uv\,\mathrm{d}x + \tau \int_\Omega \sum_{i,j=1}^d a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,\mathrm{d}x + \tau\int_\Omega cuv\,\mathrm{d}x$$

and

$$B_\Omega(u, v) = \tau \int_\Omega \sum_{i=1}^d \left(b_i \frac{\partial u}{\partial x_i} v + d_i u \frac{\partial v}{\partial x_i}\right)\mathrm{d}x.$$

It is known that (5.1) has a unique solution for sufficient small (relative to the coefficients) time step τ, which will be assumed implicitly throughout this section.

Based on the local finite element spaces V^k $(1 \leqslant k \leqslant p)$ and the coarse space V^0 defined in section 3.1, we now construct the two level additive type preconditioners for the non-symmetric operator E by using the perturbation theory of section 2.3.

Define two non-symmetric operators E and B and a symmetric operator A on V^h by

$$(Eu, v) = E_\Omega(u, v); \quad (Bu, v) = B_\Omega(u, v); \quad (Au, v) = A_\Omega(u, v), \qquad \forall u, v \in V^h,$$

where and in the sequel, $(\cdot,\cdot) = (\cdot,\cdot)_{0,\Omega}$ and $(\cdot,\cdot)_k = (\cdot,\cdot)_{0,\Omega^k}$ for $1 \leqslant k \leqslant p$, and define local operators A_k for $1 \leqslant k \leqslant p$ and the coarse operator A_0 by

$$(A_k u, v)_k = A_{\Omega^k}(u, v), \quad \forall u, v \in V^k.$$

Let the operators $I_k : V^k \to V^h$ be defined as in section 4, and $R_k = A_k^{-1} : V^k \to V^k$, then (2.8) gives preconditioner M for the non-symmetric operator E:

$$M = \sum_{k=0}^{p} I_k A_k^{-1} Q_k.$$

Remark 4. Note that in the definition of the preconditioner M above we use SPD solvers both for the coarse space V^0 and for local subspaces V^k $(1 \leqslant k \leqslant p)$, although the original operator E to be preconditioned is non-symmetric.

As stated in section 2.3, instead of solving $Eu \equiv Au + Bu = f$ from (5.1), one may solve the following preconditioned system

$$MEu = Mf$$

by GMRES method, the convergence rate is $(1 - \beta_1^2/\beta_2^2)^{1/2}$, where β_1 and β_2 can be bounded as follows:

Theorem 10. In addition to (A1)–(A6), we assume further that

$$\tau \lesssim \min_k \frac{\delta_k^2}{H_k^2}.$$

Then we have

$$\beta_1 = \min_{u \neq 0} \frac{(u, MEu)_A}{(u, u)_A} \gtrsim \min_{1 \leqslant k \leqslant p} \frac{\delta_k^2}{H_k^2}, \qquad \beta_2 = \max_{u \neq 0} \frac{\|MEu\|_A}{\|u\|_A} \lesssim 1. \tag{5.2}$$

Proof. Denote $\|\cdot\|_A^2 = A_\Omega(\cdot,\cdot) = (A\cdot,\cdot)$, $\|\cdot\|_{A,\Omega^k}^2 = A_{\Omega^k}(\cdot,\cdot)$ $(1 \leqslant k \leqslant p)$. By theorem 3, it suffices to verify (P1)–(P5) and (2.11). (P1)–(P3) can be shown in the same way as in theorem 9 with minor natural modification. Thus we know $1 \leqslant K_0 \lesssim \max_{1 \leqslant k \leqslant p} H_k^2/\delta_k^2$, $\omega_0 \approx 1$ and $\alpha_0 \approx 1$.

To prove (P5), we first see by the definition of A, B and the Cauchy–Schwarz inequality that

$$(Au, u) \lesssim \|u\|_{0,\Omega}^2 + \tau |u|_{1,\Omega}^2 \lesssim (Au, u), \quad \forall u \in H^1(\Omega),$$

hence for any $u, v \in H^1(\Omega)$, using the Cauchy–Schwarz inequality again indicates

$$|(Bu, v)| = \sqrt{\tau} \left| \int_\Omega \sum_{i=1}^{d} \left[\left(\sqrt{\tau} \frac{\partial u}{\partial x_i} \right)(b_i v) + (d_i u) \left(\sqrt{\tau} \frac{\partial v}{\partial x_i} \right) \right] \mathrm{d}x \right| \lesssim \sqrt{\tau} \|u\|_A \|v\|_A,$$

which shows (P5) with $\mu_1 \lesssim \sqrt{\tau}$. The same reasoning gives

$$|(Bu, u_k)| \lesssim \sqrt{\tau} \|u\|_{A,\Omega^k} \|u_k\|_{A,\Omega^k}, \quad \forall u \in V^h, \ u_k \in V^k, \ 1 \leqslant k \leqslant p, \tag{5.3}$$

so we can derive

$$\begin{aligned} \sum_{k=1}^{p} (Bu, I_k u_k) &= \sum_{k=1}^{p} B_\Omega(u, u_k) \lesssim \sqrt{\tau} \sum_{k=1}^{p} \|u\|_{A,\Omega^k} \|u_k\|_{A,\Omega^k} \quad \text{(by (5.3))} \\ &\lesssim \sqrt{\tau} \left(\sum_{k=1}^{p} A_{\Omega^k}(u, u) \right)^{1/2} \left(\sum_{k=1}^{p} A_{\Omega^k}(u_k, u_k) \right)^{1/2} \\ &\quad \text{(Schwarz inequality)} \\ &\lesssim \sqrt{\tau} \left(\sum_{k=1}^{p} A_{\Omega^k}(u_k, u_k) \right)^{1/2} \quad \text{(by (A1))}, \end{aligned}$$

therefore (P4) holds with $\alpha_1 \lesssim \tau$.

From the above, it is easily seen that (2.11) holds if $\tau \lesssim \min_{1 \leqslant k \leqslant p} \delta_k^2/H_k^2$. Now by theorem 3,

$$\beta_1 \geqslant \frac{(1 - \mu_1)^2}{4K_0} \gtrsim \min_{1 \leqslant k \leqslant p} \frac{\delta_k^2}{H_k^2}, \qquad \beta_2 \leqslant 2\omega_0 \alpha_0^{1/2} (2 + \alpha_0 + \alpha_1)^{1/2} \lesssim 1,$$

which ends the proof of theorem 10. □

6. Multilevel additive Schwarz method for elliptic problems

For simplicity of exposition, we consider only the two dimensional polygonal domain Ω and the elliptic problem

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + bu = f \quad \text{in } \Omega$$

with Neumann boundary condition

$$\sum_{i,j=1}^{2} a_{ij}\frac{\partial u}{\partial x_j}n_i = 0 \quad \text{on } \partial\Omega,$$

where $b > 0$ and continuous on $\bar{\Omega}$ while (a_{ij}) is symmetric, uniformly positive definite and continuous on $\bar{\Omega}$. It is straightforward to generalize the results of this section to three dimensions.

We remark that in sections 4 and 5, where the Dirichlet boundary conditins are considered, we allow that the coarse domain may partly cover the fine domain or partly be contained in the fine domain. But for the Neumann boundary conditions in this section, we require the coarser grid domains completely cover the fine grid domain. We refer to Chan et al. [9] for some numerical experiments in the general case.

The finite element problem is formulated as: Find $u \in V^h$ such that

$$A_\Omega(u,v) = (f,v), \quad \forall v \in V^h, \tag{6.1}$$

where $V^h \subset H^1(\Omega)$ consists of piecewise linear functions defined on $\mathcal{T}^h$. In this section, we will use the theory of section 2.2 to construct multilevel additive type preconditioners for the operator A corresponding to $A_\Omega(\cdot,\cdot)$.

Let $\{\mathcal{T}^l\}_{l=0}^L$ be a not necessarily nested sequence of shape regular triangulations on Ω with h_l the maximum diameter of all elements in $\mathcal{T}^l$. $\mathcal{T}^L = \mathcal{T}^h$ is the finest triangulation on which the finite element space V^h and the finite element problem (6.1) are defined. Denote the coarser domains corresponding to the coarser triangulations $\mathcal{T}^l$, $0 \leqslant l \leqslant L-1$, by Ω^l. As we indicated above, the coarse grid domain is required to cover the original fine grid domain Ω (cf. figure 3), i.e., we assume

$$\bar{\Omega}^L = \bar{\Omega} \subset \bar{\Omega}^l = \bigcup_{\tau^l \in \mathcal{T}^l} \bar{\tau}^l, \quad 0 \leqslant l < L.$$

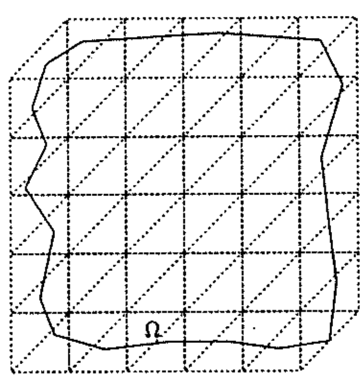

Figure 3. The fine domain Ω and lth coarse domain Ω^l.

For $0 \leqslant l < L$, let $V^l \subset H^1(\Omega^l)$ be the piecewise linear finite element space defined on $\mathcal{T}^l$, and $\{q_i^l\}_{i=1}^{m_l}$ and $\{\phi_i^l\}_{i=1}^{m_l}$ the set of nodal points of the triangulation $\mathcal{T}^l$ and the set of nodal basis functions of V^l, resp. We will define some linear coarse-to-fine grid transfer mappings such that spaces V^l are subspaces of V^h under these mappings. For that purpose, we always assume that the dimensions of all coarser spaces V^l $(l < L)$ are less than the dimension of the fine space V^L.

To define the multilevel additive preconditioner, we use the idea of the two-level algorithm on each level, i.e., decompose each domain Ω^l for $1 \leqslant l \leqslant L$ into overlapping subdomains of Ω^l. Thus we assume that for each level $l = 1,2,\ldots,L$, without the coarsest level, $\{\Omega_k^l\}_{k=1}^{N_l}$ is an overlapping domain decomposition of Ω^l, obtained by extending a given nonoverlapping subdomain covering $\{\widetilde{\Omega}_k^l\}_{k=1}^{N_l}$ of Ω^l such that $\operatorname{dist}(\partial\widetilde{\Omega}_k^l, \partial\Omega_k^l \cap \Omega^l) \geqslant \delta_l > 0$, $1 \leqslant k \leqslant N_l$. δ_l is called the lth level overlapping ratio. Here the boundaries of the subdomains Ω_k^l are required to align with the boundaries of the lth level elements in $\mathcal{T}^l$. We make the following very reasonable hypotheses about the triangulations:

(H1) For any coarse element $\tau^l \in \mathcal{T}^l$, $0 \leqslant l < L$, let $m(\tau^l)$ be the measure of all $\tau^h \in \mathcal{T}^h$: $\tau^h \cap \partial\tau^l \neq \varnothing$, then $m(\tau^l) \lesssim |\tau^l|$.

(H2) Any point in Ω^l is covered by at most q_0 subdomains of $\{\Omega_k^l\}_{k=1}^{N_l}$.

(H3) Any coarse boundary element τ^l in $\mathcal{T}^l$ $(l < L)$, i.e., $\tau^l \cap \partial\Omega^l \neq \varnothing$, has a significant part inside the original domain Ω. More accurately, one can construct a σ_0-shape regular simplex τ_0^l such that $\tau_0^l \subset \tau^l \cap \Omega$ and $\operatorname{diam} \tau_0^l \approx \operatorname{diam} \tau^l$.

For each coarse space V^l $(1 \leqslant l \leqslant p)$, we define a mapping $I_l : V^l \to V^h$ to be the standard finite element interpolant Π_h, i.e.,

$$I_l u \equiv \Pi_h u = \sum_{q_i^h \in \bar{\Omega}} u(q_i^h)\phi_i^h, \quad \forall u \in V^l, \tag{6.2}$$

and for each subdomain Ω_k^l on the lth level, we define a local subspace by

$$V_k^l = \{v \in V^l;\ v = 0 \text{ on } \partial\Omega_k^l \cap \Omega^l\} \subset V^l$$

and a prolongation operator $I_k^l : V_k^l \to V^h$ to be I_l, but I_k^l's adjoint $Q_k^l : V^h \to V_k^l$ by

$$(Q_k^l u, v_k^l)_l = (u, I_k^l v_k^l), \quad \forall u \in V^h,\ v_k^h \in V_k^l,$$

where $(\cdot,\cdot)_l = (\cdot,\cdot)_{0,\Omega^l}$ is the scalar product in $L^2(\Omega^l)$.

Furthermore, we define local operators $A_k^l : V_k^l \to V_k^l$ by

$$(A_k^l u, v)_l = A_{\Omega_k^l}(u,v), \quad \forall u, v \in V_k^l,$$

and let $R_k^l = (A_k^l)^{-1}$ for simplicity of exposition. Then we may construct the additive Schwarz preconditioner as in (2.5) by

$$M = \sum_{l=0}^{L}\sum_{k=1}^{N_l} I_k^l R_k^l Q_k^l \equiv \sum_{l=0}^{L}\sum_{k=1}^{N_l} I_k^l (A_k^l)^{-1} Q_k^l.$$

To estimate the condition number $\kappa(MA)$, we need only to verify the conditions (P1′)–(P3′) by theorem 2. For this purpose, we will use the following three lemmas. The first one indicates that the standard nodal value interpolation Π_h is H^1 stable and has the L^2 optimal approximation when it is restricted to the coarser finite element subspaces, whose proof is the same as the one for lemma 4.

Lemma 11. Under the assumption (H1), for any coarser triangulation $\mathcal{T}^l$, $0 \leqslant l < L$, and any $u \in V^l$, we have

$$|\Pi_h u|_{s,\Omega} \lesssim |u|_{s,\Omega^l}, \quad s = 0, 1; \qquad \|u - \Pi_h u\|_{0,\Omega} \lesssim h|u|_{1,\Omega^l}.$$

The second lemma gives the properties of Clément's interpolant, cf. definition 5, lemma 6 and Clément [18].

Lemma 12. Let $R_l : L^2(\Omega) \to V^l$ be Clément's interpolant operator corresponding to $\mathcal{T}^l$, $0 \leqslant l < L$, then for $s = 0, 1$, $t = 1, 2$, $s \leqslant t$ and any $u \in H^t(\Omega^l)$, we have

$$|u - R_l u|_{s,\Omega^l} \lesssim h_l^{t-s}|u|_{t,\Omega^l}. \tag{6.3}$$

Moreover, let $\widehat{V}^l \subset H^2(\Omega^l)$ be a higher order finite element space defined on $\mathcal{T}^l$ (e.g., Argyris and Clough and Tocher finite elements) and $\widehat{R}_l : L^2(\Omega^l) \to \widehat{V}^l$ the corresponding Clément interpolant, then $\widehat{R}_l$ satisfies (6.3) and

$$|\widehat{R}_l u|_{2,\Omega^l} \lesssim |u|_{2,\Omega^l}.$$

Define an orthogonal projection $P^l : H^1(\Omega) \to I_l V^l$ with $I_l = \Pi_h$ by

$$A_\Omega(P^l u, v) = A_\Omega(u, v), \quad \forall u \in H^1(\Omega),\ v \in I_l V^l.$$

Note that $I_l = \Pi_h$ is defined in (6.2).

Lemma 13. Suppose that the domain Ω is convex, then for $0 \leqslant l < L$,

$$\|v - P^l v\|_{0,\Omega} \lesssim h_l \|v\|_{1,\Omega}, \quad \forall v \in V^h.$$

Proof. We apply the Aubin–Nitsche trick to prove lemma 13, and first show that for any $u \in H^2(\Omega)$, there exists $u^l \in V^l$ such that

$$\|u - I_l u^l\|_{1,\Omega} \lesssim h_l \|u\|_{2,\Omega}. \tag{6.4}$$

Let $\widehat{\Omega}$ be an open domain in R^d large enough such that $\Omega \subset \Omega^0 \subset\subset \widehat{\Omega}$. It is well-known (cf. Stein [35]) that there exists a linear continuous extension operator $E : H^2(\Omega) \to H^2(\widehat{\Omega})$ such that $Eu = u$ in Ω and

$$\|Eu\|_{2,\widehat{\Omega}} \lesssim \|u\|_{2,\Omega}. \tag{6.5}$$

We show the function $u^l = R_l Eu \in V^l$ satisfies (6.4). The triangle inequality gives

$$\|u - I_l u^l\|_{1,\Omega} \leqslant \|Eu - R_l Eu\|_{1,\Omega} + \|R_l Eu - \widehat{R}_l Eu\|_{1,\Omega} + \|\widehat{R}_l Eu - I_l R_l Eu\|_{1,\Omega}.$$

By lemmas 11 and 12, we obtain $\|Eu - R_l Eu\|_{1,\Omega} \lesssim h_l |Eu|_{2,\Omega^l}$, and

$$\begin{aligned}
\|R_l Eu - \widehat{R}_l Eu\|_{1,\Omega} &\leqslant \|Eu - R_l Eu\|_{1,\Omega} + \|Eu - \widehat{R}_l Eu\|_{1,\Omega} \quad \text{(by lemma 12)}\\
&\lesssim h_l |Eu|_{2,\Omega^l} \lesssim h_l |u|_{2,\Omega} \quad \text{(by (6.5))},\\
\|\widehat{R}_l Eu - I_l R_l Eu\|_{1,\Omega} &\lesssim \|\widehat{R}_l Eu - I_l \widehat{R}_l Eu\|_{1,\Omega} + \|I_l(\widehat{R}_l Eu - R_l Eu)\|_{1,\Omega}\\
&\lesssim h_l |\widehat{R}_l Eu|_{2,\Omega^l} + \|\widehat{R}_l Eu - R_l Eu\|_{1,\Omega^l}\\
&\qquad \text{(by the interpolation result and lemma 11)}\\
&\lesssim h_l \|Eu\|_{2,\Omega^l} \lesssim h_l \|u\|_{2,\Omega} \quad \text{(by (6.5))},
\end{aligned}$$

which proves (6.4).

Now the Aubin–Nitsche trick will give the final result. For any $v \in V^h$, let $w \in H^1(\Omega)$ be the solution of the problem:

$$A_\Omega(w, u) = (v - P^l v, u), \quad \forall u \in H^1(\Omega),$$

and w_h the finite element solution of w in $I_l V^l$:

$$A_\Omega(w_h, u) = (v - P^l v, u), \quad \forall u \in I_l V^l.$$

Since Ω is convex, we know $w \in H^2(\Omega)$, and $\|w\|_{2,\Omega} \lesssim \|v - P^l v\|_{0,\Omega}$ (cf. Grisvard [25]). Hence the previous result (6.4) says there exists $w^l \in V^l$ such that

$$\|w - I_l w^l\|_{a,\Omega} \lesssim h_l \|w\|_{2,\Omega},$$

from this and the definitions of w and P^l, we obtain

$$\begin{aligned}
\|v - P^l v\|_{0,\Omega}^2 &= A_\Omega(w, v - P^l v) = A_\Omega(w - w_h, v - P^l v)\\
&\lesssim \|w - w_h\|_{A,\Omega}\|v - P^l v\|_{A,\Omega} \leqslant \|w - I_l w^l\|_{A,\Omega}\|v\|_{A,\Omega}\\
&\lesssim h_l \|w\|_{2,\Omega}\|v\|_{A,\Omega} \lesssim h_l \|v - P^l v\|_{0,\Omega}\|v\|_{A,\Omega},
\end{aligned}$$

which implies lemma 13. Here $\|\cdot\|_{A,\Omega}$ is the $A_\Omega(\cdot,\cdot)$-induced norm. □

Lemma 4 says the interpolant Π_h is L^2- and H^1-stable in the coarse finite element spaces. In the following lemma, we show that the inverse of Π_h is also L^2- and H^1-stable. Due to technicalities in our proof of the lemma, we classify the triangulations of all coarser levels into two groups: triangulations "far away" from and "very close" to the fine triangulation. That is, for some positive interger $l_0 < L$, the triangulations $\mathcal{T}^l$ with $0 \leqslant l \leqslant l_0$ (resp. $l_0 < l < L$) are regarded as far away from (resp. very close to) the fine triangulation. For these two groups of coarser meshes, we assume that

(H4) The set of nodes of each coarse triangulation which is very close to the fine triangulation is a subset of the nodes of the fine triangulation. And for any coarse element from these triangulations, say τ^l in $\mathcal{T}^l$ ($l_0 < l < L$), all fine elements which have non-empty intersections with τ^l are roughly of the same sizes as τ^l. More accurately, for any fine element τ^h of $\mathcal{T}^h$ satisfying $\tau^h \cap \tau^l \neq \emptyset$, there are two positive constants κ_0 and κ_1 such that $\kappa_0 \operatorname{diam}(\tau^h) \leqslant \operatorname{diam}(\tau^l) \leqslant \kappa_1 \operatorname{diam}(\tau^h)$.

(H5) For each coarse triangulation $\mathcal{T}^l$ which is far away from the fine one, i.e., $0 \leqslant l \leqslant l_0$, and any fine element τ^h in $\mathcal{T}^h$, if τ^h has a non-empty intersection with some coarse element in $\mathcal{T}^l$, say $\tau^l \in \mathcal{T}^l$, then there exists another fine element τ_0^h (possibly τ^h itself) which is completely located inside τ^l and only an $O(h)$ (or element-size) distance away from τ^h, i.e., $\operatorname{dist}(\tau^h, \tau_0^h) \leqslant \kappa_2 \operatorname{diam}(\tau^h)$ with κ_2 a positive constant.

Remark 5. Asssumptions (H4) and (H5) are both practically reasonable. (H4) means that element sizes of the fine triangulation and those coarser triangulations which are close enough to the fine one are of the same magnitude locally (*not* globally). (H5) means that each coarse element of those triangulations which are far away from the fine one contains one or more fine elements.

We only require that the sets of nodes of the coarse triangulations which are close enough to the fine triangulation are subsets of fine nodes; the corresponding coarse domains are allowed to be non-nested to the fine domain. Many existing coarse grid generating algorithms possess this property, see [1, 2] and [9, 11, 13]. In practice, $l_0 = L-3$ or $L-4$ would be enough, i.e., the third or fourth coarsening of the fine triangulation.

The two assumptions (H4) and (H5) are posed for technical reasons. Other less restrictive assumptions are possible.

Now we can prove

Lemma 14. With the asumptions (H1), (H4) and (H5), we have for $0 \leqslant l < L$ and $s = 0, 1$ that

$$|u^l|_{s,\Omega^l} \lesssim |\Pi_h u^l|_{s,\Omega}, \quad \forall u^l \in V^l. \tag{6.6}$$

Proof. We prove the lemma in two steps: $l \leqslant l_0$ and $l_0 < l$.

(a) For the case $l \leqslant l_0$: let τ^h be any fine element in $\mathcal{T}^h$, we have

$$\|u^l\|_{0,\tau^h}^2 \lesssim h_\tau^2 \|u^l\|_{0,\infty,\tau^h}^2 \equiv h_\tau^2 \, |u^l(x_0)|^2, \tag{6.7}$$

where $x_0 \in \tau^h$. As $x_0 \in \Omega \subset \Omega^l$, x_0 must be in some element τ^l of $\mathcal{T}^l$. By (H5), there exists another fine element τ_0^h inside τ^l such that $\operatorname{dist}(\tau^h, \tau_0^h) \leqslant \kappa_2 \operatorname{diam}(\tau^h)$. Let z_i, $i = 1, 2, 3$, be the vertices of τ_0^h, by Taylor expansion we have

$$u^l(x_0) = u^l(z_1) + \nabla u^l(x)(x_0 - z_1), \quad x \in \tau_0^h.$$

Squaring both sides and integrating over τ^h gives

$$h_\tau^2 |u^l(x_0)|^2 \leqslant h_\tau^2 |u^l(z_1)|^2 + h_\tau^2 \|\nabla u^l\|_{0,\tau_0^h}^2.$$

Using the shape regularity we know $\|\Pi_h u^l\|_{0,\tau_0^h}^2 \approx h_\tau^2 \sum_{i=1}^3 (u^l(z_i))^2$, this with the inverse inequality and $\Pi_h u^l = u^l$ on τ_0^h implies

$$h_\tau^2 |u^l(x_0)|^2 \lesssim \|\Pi_h u^l\|_{0,\tau_0^h}^2 + \|u^l\|_{0,\tau_0^h}^2 \approx \|\Pi_h u^l\|_{0,\tau_0^h}^2.$$

Thus we obtain by using this last relation and summing over all τ^h in (6.7) that

$$\|u^l\|_{0,\Omega}^2 \lesssim \|\Pi_h u^l\|_{0,\Omega}^2. \tag{6.8}$$

Now consider any boundary element τ^l in $\mathcal{T}^l$, i.e., $\tau^l \cap \partial\Omega^l \neq \emptyset$. By noting the linearity of u^l in τ^l and (H3), we obtain

$$\|u^l\|_{s,\tau^l}^2 \lesssim \|u^l\|_{s,\tau_0^l}^2 \quad \text{for } s = 0, 1. \tag{6.9}$$

This with (6.8) indicates (6.6) holds with $s = 0$. To show (6.6) holds with $s = 1$, taking any fine element τ^h of $\mathcal{T}^h$, we know from the standard interpolation result (cf. [17]) that

$$|u^l - \Pi_h u^l|_{1,\tau^h}^2 \lesssim h_\tau^2 |u^l|_{1,\infty,\tau^h}^2 \equiv h_\tau^2 |\nabla u^l(x_0)|^2, \quad x_0 \in \tau^h. \tag{6.10}$$

As above, we can find an element τ^l of $\mathcal{T}^l$ and another fine element τ_0^h inside τ^l such that $\operatorname{dist}(\tau^h, \tau_0^h) \leqslant \kappa_2 \operatorname{diam}(\tau^h)$. Since u^l is linear in τ^l and $\Pi_h u^l = u^l$ on τ^l, we obtain by (6.10) and inverse inequality that

$$|u^l - \Pi_h u^l|_{1,\tau^h}^2 \lesssim h_\tau^2 |u^l|_{1,\infty,\tau_0^h}^2 \lesssim h_\tau^{-2} \|u^l\|_{0,\tau_0^h}^2 = h_\tau^{-2} \|\Pi_h u^l\|_{0,\tau_0^h}^2. \tag{6.11}$$

Noting $(u^l - \Pi_h u^l)$ does not change by replacing u^l by $u^l + c$ for any constant c, we derive

$$|u^l - \Pi_h u^l|_{1,\tau^h}^2 \lesssim |\Pi_h u^l|_{1,\tau_0^h}^2,$$

this combined with the triangle inequality and (6.9) implies (6.6) for $s = 1$.

(b) For the case $l_0 < l \leqslant L$. Taking any coarse element $\tau^l \subset \Omega$, we have by (H4) and the shape regularities of $\mathcal{T}^h$ and $\mathcal{T}^l$ that

$$\sum_{\tau^h \cap \tau^l \neq \emptyset} \|\Pi_h u^l\|_{0,\tau^h}^2 \gtrsim \sum_{\tau^h \cap \tau^l \neq \emptyset} (\operatorname{diam}(\tau^h))^2 \sum_{x_i \in \mathcal{N}(\tau^h)} (u^l(x_i))^2$$
$$\gtrsim (\operatorname{diam}(\tau^l))^2 \sum_{x_i \in \mathcal{N}(\tau^l)} (u^l(x_i))^2 \gtrsim \|u^l\|_{0,\tau^l}^2.$$

Here $\mathcal{N}(\tau^h)$ is the set of nodes of τ^h, the same for $\mathcal{N}(\tau^l)$. This with (6.9) shows (6.6) for $s = 0$.

For $s = 1$: the shape regularities of $\mathcal{T}^l$ and $\mathcal{T}^h$, and the assumption (H4) imply

$$\begin{aligned} |u^l|^2_{1,\tau^l} &\lesssim \sum_{x_i, x_j \in \mathcal{N}(\tau^l)} \left(u^l(x_i) - u^l(x_j)\right)^2 \\ &\lesssim \sum_{\tau^h \cap \tau^l \neq \emptyset} \sum_{x_i, x_j \in \mathcal{N}(\tau^h)} \left(u^l(x_i) - u^l(x_j)\right)^2 \\ &\lesssim \sum_{\tau^h \cap \tau^l \neq \emptyset} |\Pi_h u^l|^2_{1,\tau^h}, \end{aligned}$$

which combined with (6.9) implies (6.6) for $s = 1$. This completes the proof of lemma 14. □

We are now in a position to estimate the condition number $\kappa(MA)$. For convex domain Ω, we have

Theorem 15. With the assumptions (H1)–(H5),

$$\kappa(MA) \lesssim \rho^2 L^2,$$

where $\rho = \max_{1 \leqslant l \leqslant L}(h_l + h_{l-1})/\delta_l$.

Proof. It suffices by theorem 2.2 to verify (P1′)–(P3′). (P2′) is obvious with $\omega_0 \lesssim 1$ from the definition of I^l_k, R^l_k and lemma 11.

To prove (P3′), for any $u^l_k \in V^l_k$ defined on Ω^l_k, let $\widehat{\Omega}^l_k = \mathrm{supp} I^l_k u^l_k \subset \bar{\Omega}^l_k \cap \bar{\Omega}$. Recall $I^l_k = \Pi_h$, then (H2) and the Cauchy–Schwarz inequality lead to

$$\begin{aligned} \sum_{l=0}^{L}\sum_{k=1}^{N_l}(Au, I^l_k u^l_k) &= \sum_{l=0}^{L}\sum_{k=1}^{N_l} A_{\widehat{\Omega}^l_k}(u, I^l_k u^l_k) \\ &\leqslant \left(\sum_{l=0}^{L}\sum_{k=1}^{N_l} A_{\widehat{\Omega}^l_k}(u,u)\right)^{1/2} \left(\sum_{l=0}^{L}\sum_{k=1}^{N_l} A_{\widehat{\Omega}^l_k}(I^l_k u^l_k, I^l_k u^l_k)\right)^{1/2} \\ &\lesssim L^{1/2}\left(A_\Omega(u,u)\right)^{1/2}\left(\sum_{l=0}^{L}\sum_{k=1}^{N_l} A_\Omega(I^l_k u^l_k, I^l_k u^l_k)\right)^{1/2}, \end{aligned}$$

which says (P3′) holds with $\alpha_0 = L$.

Finally we verify (P1′). To do so, we need a proper decomposition for any finite element function u in V^h. As in our earlier report [14], we use the following

decomposition:

$$\begin{aligned} u^0 &= P^0 u, \\ u^1 &= P^1(u - u^0), \\ u^2 &= P^2(u - u^0 - u^1), \\ &\vdots \\ u^l &= P^l(u - u^0 - u^1 - \cdots - u^{l-1}), \\ &\vdots \\ u^L &= P^h(u - u^0 - u^1 - \cdots - u^{L-1}) = u - u^0 - u^1 - \cdots - u^{L-1}. \end{aligned}$$

It is readily seen that $u^l \in I_l V^l$ and $u = \sum_{l=0}^L u^l$.

Let $w^l = u - \sum_{i=0}^{l-1} u^i$ for $1 \leqslant l \leqslant L$ but $w^0 = u$. Then $u^l = P^l w^l$ and

$$w^l = w^{l-1} - u^{l-1} = w^{l-1} - P^{l-1}w^{l-1}. \tag{6.12}$$

By the definition of P^l, we derive for $0 \leqslant l \leqslant L$ that

$$\|u^l\|_{A,\Omega} \leqslant \|w^l\|_{A,\Omega} \leqslant \|w^{l-1}\|_{A,\Omega} \leqslant \cdots \leqslant \|w^1\|_{A,\Omega} \leqslant \|u\|_{A,\Omega}, \tag{6.13}$$

and we get

$$\begin{aligned} \|u^l\|_{0,\Omega} = \|P^l w^l\|_{0,\Omega} &\leqslant \|w^l\|_{0,\Omega} + \|w^l - P^l w^l\|_{0,\Omega} \quad \text{(triangle inequality)} \\ &= \|w^{l-1} - P^{l-1}w^{l-1}\|_{0,\Omega} + \|w^l - P^l w^l\|_{0,\Omega} \quad \text{(by (6.12))} \\ &\lesssim (h_{l-1} + h_l)\|u\|_{A,\Omega}, \quad 1 \leqslant l \leqslant L \quad \text{(lemma 13)}, \end{aligned} \tag{6.14}$$

$$\|u^0\|_{0,\Omega} \lesssim \|u^0\|_{A,\Omega} \leqslant \|u\|_{A,\Omega}. \tag{6.15}$$

As $u^l \in I_l V^l$, we can write $u^l = I_l v^l$, $v^l \in V^l$. We further decompose v^l. It is known (cf. Dryja and Widlund [21], Bramble et al. [5]) that there exists a partition $\{\theta^l_k\}_{k=1}^{N_l}$ of unity for Ω^l related to the subdomains $\{\Omega^l_k\}$ such that $\sum_{k=1}^{N_l}\theta^l_k(x) = 1$ on Ω^l and for $1 \leqslant k \leqslant N_l$,

$$\mathrm{supp}\,\theta^l_k \subset \Omega^l_k \cup \partial\Omega, \qquad 0 \leqslant \theta^l_k \leqslant 1, \qquad \|\nabla\theta^l_k\|_{L^\infty(\Omega_i)} \lesssim \delta_l^{-1}. \tag{6.16}$$

Using this partition of unity, we can decompose v^l as

$$v^l = \sum_{k=1}^{N_l}\Pi_l(\theta^l_k v^l) \equiv \sum_{k=1}^{N_l} v^l_k, \quad v^l_k \in V^l_k,$$

where Π_l is the standard nodal value interpolant corresponding to V^l. This gives

$$u = u^0 + \sum_{l=1}^{L} u^l = u^0 + \sum_{l=1}^{L}\sum_{k=1}^{N_l} I_l v_k^l.$$

By the standard proof (cf. Xu [36], or Chan and Zou [15]), we have

$$\sum_{k=1}^{N_l} \|v_k^l\|_{1,\Omega^l}^2 \lesssim \left(|v^l|_{1,\Omega^l}^2 + \delta_l^{-2}\|v^l\|_{0,\Omega^l}^2\right). \tag{6.17}$$

Thus, we deduce from (6.16)–(6.17) that

$$\begin{aligned}\sum_{l=0}^{L}\sum_{k=1}^{N_l}\left(A_k^l v_k^l, v_k^l\right) &= A_{\Omega^0}(v^0, v^0) + \sum_{l=1}^{L}\sum_{k=1}^{N_l} A_{\Omega_k^l}\left(v_k^l, v_k^l\right) \\ &\lesssim \sum_{l=0}^{L}\left(|v^l|_{1,\Omega^l}^2 + \delta_l^{-2}\|v^l\|_{0,\Omega^l}^2\right) \quad (\text{set } \delta_0 = 1 \text{ and by (6.17)}) \\ &\lesssim \sum_{l=0}^{L}\left(|u^l|_{1,\Omega}^2 + \delta_l^{-2}\|u^l\|_{0,\Omega}^2\right) \quad (\text{by lemma 14}) \\ &\lesssim \rho^2 L(Au, u) \quad (\text{by (6.13)–(6.15)}),\end{aligned}$$

which shows that (P1′) holds with $K_0 = \rho^2 L$. Now theorem 15 follows from theorem 1. □

Remark 6. The condition number bound given in theorem 15 grows like L^2. It is known that in the structured case, one can remove this dependence on L and obtain an optimal condition number (cf. Zhang [42] and Oswald [31]). At this point, we do not know how to obtain a similar optimal bound for our unstructured case.

Remark 7. The convexity assumption on the domain Ω is only needed in the proof of lemma 13 for the technical requirement by use of the Aubin–Nitsche trick. We do not know if lemma 13 holds for non-convex domains. But our numerical experiments demonstrated very satisfactory results also for non-convex domains, e.g., airfoil-shaped domains in [9–12].

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