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Abstract

Analysis and computations are presented for singularities in the solution of the steady Boussinesq equations for two-dimensional, stratified flow. The results show that for codimension 1 singularities, there are two generic singularity types for general solutions, and only one generic singularity type if there is a certain symmetry present. The analysis depends on a special choice of coordinates, which greatly simplifies the equations, showing that the type is exactly that of one dimensional Legendrian singularities, generalized so that the velocity

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can be infinite at the singularity. The solution is viewed as a surface in an appropriate compactified jet space. Smoothness of the solution surface is proved using the Cauchy-Kowalewski Theorem, which also shows that these singularity types are realizable. Numerical results from a special, highly accurate numerical method demonstrate the validity of this geometric analysis. An analysis of general Legendrian singularities with blowup, i.e. at which the derivative may be infinite, is also presented, using projective coordinates.

1 Introduction

The possibility of singularity formation from smooth initial data for the solution of the Euler equations for inviscid, incompressible fluid flow in three dimensions is one of the main unsolved problems of mathematical fluid mechanics. In spite of considerable effort on this problem from analytical, numerical and physical approaches, there is still little convincing evidence either for or against the possibility of singularity formation.

In this paper, singularities are analyzed for a related, but much simpler, system, the *steady Boussinesq equations:*

$$\begin{aligned} \boldsymbol{u} \cdot \nabla \rho &= 0 \\ \boldsymbol{u} \cdot \nabla \zeta &= -\rho_x \\ \nabla \times \boldsymbol{u} &= \zeta \\ \nabla \cdot \boldsymbol{u} &= 0. \end{aligned} \tag{1.1}$$

This system describes steady, two-dimensional, stratified (i.e. variable density), incompressible flow in which ρ is density, $\boldsymbol{u}$ is two dimensional velocity and ζ is vorticity. The buoyancy term ρ_x plays the role of the vortex stretching in the Euler equations. This system is derived in the "Boussinesq limit," in which the variation of density is important in the buoyancy terms but insignificant in the inertial terms.

For the steady Boussinesq equations (1.1) we will derive the generic (i.e. typical) form of singularities, under certain natural restrictions. Singularities for complex solutions and at complex spatial positions are considered in this study, but the results are also valid for real singularities of real solutions. In addition, we present numerical computations which confirm these generic singularity types.

The problem of singularity formation from smooth initial data for the (time-dependent) Euler or Boussinesq equations is the motivation for this work. Our reasons for studying *complex* singularities for *steady* flows are twofold:

First, this problem serves as a vehicle for development of methods that may be applicable to the singularity formation problem. For example, the present analysis includes solutions with infinite values of the the velocity and vorticity. Such infinite values in the dependent variables were excluded from earlier approaches [5].

Second, we believe that the form of steady complex singularities may be indicative of the form for dynamic real singularities or for nearly singular flow. One reason for this belief is that the genericity results for steady singularities can also be interpreted as genericity results for complex singularities that move in the complex plane without change in their structure. This is described in Appendix A. On the other hand, we have found that, for generic steady singularities, the density ρ is infinite. Since, infinite values of ρ cannot occur for singularity formation from smooth initial data, this would restrict the set of steady singularities that are relevant to singularity formation.

Further discussion of these points is presented in Section 8.

The main result of this paper can be informally stated as follows:

Consider singularities for the steady Boussinesq system (1.1) which have codimension 1 and for which the vorticity ζ is infinite. The generic form of such a singularity is of two types:

(1) $\psi \approx x^{3/2}$; $v \approx x^{1/2}$; $\zeta \approx x^{-1/2}$

(2) $\psi \approx x^{1/2}$; $v \approx x^{-1/2}$; $\zeta \approx x^{-3/2}$.

If in addition the stream function ψ is assumed to have an additional symmetry ($\psi(x,0) = \psi(-x,0)$), then the singularity type by

(3) $\psi \approx x^{2/3}$; $v \approx x^{-1/3}$; $\zeta \approx x^{-4/3}$.

In these formulas x and v denote some suitably chosen space and velocity coordinates. A precise statement of this result will be presented in Section 3.

The principal ingredient in this study will be a geometric approach to differential equations, which has been developed by two of the present authors and their co-workers for simpler systems in [5], as well as by Bryant, Griffiths and Hsu [17, 18]. In this approach the solution of a differential equation is viewed as a surface in an appropriate jet space (described in Section 3 5), and the PDE serves as a constraint on the possible form of this surface.

The main role of analysis in this approach is to show that such constrained surfaces are generically smooth, which is demonstrated using the Cauchy-Kowalewski Theorem for analytic solutions of a PDE. Geometric and algebraic methods, such as the singularity theory of Arnold [21], can then be used to analyze the generic types of singularities for this surface.

The background for this study consists of several analytical results and a numerical computation:

First, the mathematical significance of Euler or Boussinesq singularity formation (for the time-dependent equations), should it occur, is that it would limit the validity of mathematical existence theory. Beale, Kato and Majda [2] showed that if the initial velocity $\boldsymbol{u}_0$ is in Sobolev space H^s for some $s \geq 3$, but at time $t = t^* > 0$, $\boldsymbol{u}(t)$ is not in H^s, then

$$\int_0^{t^*} ||\omega(\cdot, t)||_\infty dt = \infty \tag{1.2}$$

in which $||\cdot||_\infty$ is the L^∞ norm in space and ω is the vorticity. Bardos and Benachour [1] proved a related, earlier result in an analytic function class; E and Shu [11] proved a similar result for the Boussinesq equations.

The physical significance of singularities is less clear and depends critically on the robustness and type of the singularities. Singularities might serve as an important means for transfer of energy from large to small scales. In this way they could be responsible for the onset of turbulent flow or even for the continuation of fully developed turbulence. A related analytic result, first stated by Onsager [15] and refined in [6, 10, 12], says that if a weak Euler solution does not conserve energy and it has Hölder exponent σ on a set of co-dimension κ, then $\sigma + \kappa/3 \leq 1/3$. This is physically meaningful, since there is clear evidence that the energy dissipation for the Navier Stokes solution remains nonzero in the zero viscosity limit.

The time-dependent Boussinesq equations are

$$\begin{aligned} \rho_t + \boldsymbol{u} \cdot \nabla\rho &= 0 \\ \zeta_t + \boldsymbol{u} \cdot \nabla\zeta &= -\rho_x \\ \nabla \times \boldsymbol{u} &= \zeta \\ \nabla \cdot \boldsymbol{u} &= 0. \end{aligned} \tag{1.3}$$

As pointed out by Childress et al. [9] and by Pumir and Siggia [16], these are very similar to the Euler equations for axi-symmetric flow with swirl. The latter equations also involve only two degrees of spatial variation (radius r and axial length z) and are the simplest form of the Euler equations with nontrivial vortex stretching. Pumir and Siggia [16] performed numerical computations for (1.3), which seemed to indicate singularity formation, but further computations by E and Shu [11] did not confirm their results.

A clearcut numerical demonstration of singularities for the Euler equation for axisymmetric flow with swirl, but for **complex** velocity, was performed by Caflisch [3] using a special form of the solution for which the numerical error is extremely small. The computed solutions had singularities of the simple form $\boldsymbol{u} \approx x^{-1/3}$, which suggested that this singularity might be generic.

These computations can be understood in two ways:
(1) The computations were performed for the exact Euler equations with no approximations but with complex initial data. In particular the initial data consists of only nonnegative wavenumbers in the axial variable z; i.e. it is upper analytic in z.
(2) These computations also correspond to Moore's approximation for the Euler equations with real initial data. In Moore's approximation, the upper and lower analytic components of the solution are decoupled. The upper analytic part satisfies the Euler equations, but with complex initial data, as in (1); the lower analytic part is its conjugate. The physical meaning of this approximation is to retain the forward cascade of energy while omitting the inverse cascade.

It has not yet been possible to relate the results of Moore's approximation back to the real Euler equations, but we conjecture that this approximation is valid, at least qualitatively, as long as the singularities are away from the real axis.

In this paper, these numerical singularity results are used instead to indicate the generic form of singularities for the steady flow problem. The singular solutions constructed in [3] were traveling waves (with imaginary speed), but by Galilean transformation they become steady solutions. In the present investigation, numerical solutions with complex singularities are constructed for the steady Boussinesq equations by a similar procedure as in [3].

Finally the singularity analysis of this study was motivated by a similar analysis of the generic singularity type for first order systems with at most two speeds by Caflisch, Ercolani, Hou and Landis [5]. That analysis used a generalization of the hodograph transformation to **unfold** the differential equations. While that *unfolding transformation* has been generalized [4] to a larger class of equations, including the Boussinesq equations or the Euler equations for axisymmetric flow with swirl, the analysis in the present study is effected through a simpler and more direct transformation of the equations.

The following is an outline of the remainder of the paper: In Section 2 the Boussinesq equations are transformed and scaled. The main analytic results on generic singularity type for steady Boussinesq are presented and proved in Section 3. The proof partly relies on an application of the Cauchy-Kowalewski theorem, presented in Section 4, to show that the solutions with the singularity types (1), (2) and (3) are actually realizable for the Boussinesq equations, and that the corresponding solution surfaces are smooth.

Although the genericty results are derived directly from the Boussinesq equations, they are motivated by the theory of Legendrian singularities, and they could be derived from that theory. Section 5 presents a self-contained summary of the relevant aspects of the theory of Legendrian singularities (in section 5.1) as well as an extension of this theory to Legendrian blow-up (in section 5.2) which is new.

The analytic results of this study are motivated and validated by a numerical construction of singular solutions for the Boussinesq equations with *complex* velocities, which are described in Section 6 (the numerical method) and Section 7 (the numerical results).

Finally, some conclusions from this study, including a discussion of the significance of singularities for the time-dependent equations, are presented in Section 8.

While this presentation of the results is given in the simplest logical order to help the reader, the actual process by which these results were obtained was an interplay between analysis and numerical computation. Following the numerical solution of the Euler equations for axisymmetric flow with swirl, the Boussinesq equations were solved by the same method, and a solution of the form (3) was obtained. This motivated an analysis of the generic form of singularities which produced singularity types (1) and (2). Since the numerical results did not agree with (2) and since the analysis of the type (2) required some new singularity results for Legendrian surfaces with blowup of the derivative, we were at first skeptical of the validity of the form (2). Further consideration showed, however, that there was a symmetry ($\psi(x,0) = \psi(-x,0)$) in the original numerical solutions of Euler and Boussinesq that led to the form (3). Following this observation, we modified the computations to remove this symmetry and then found the correct asymmetric result (2). We also extended the analysis to show that the generic singularity form with symmetry is (3).

2 Unfolding the Steady Boussinesq Equation

The Boussinesq equations for steady, 2D, stratified, incompressible fluid flow are

$$
\begin{aligned}
\boldsymbol{u} \cdot \nabla \rho &= 0 && (2.1)\\
\boldsymbol{u} \cdot \nabla \zeta &= -\rho_x && (2.2)\\
\nabla \times \boldsymbol{u} &= \zeta && (2.3)\\
\nabla \cdot \boldsymbol{u} &= 0 && (2.4)
\end{aligned}
$$

in which $\boldsymbol{u} = (u, v)$ is velocity, ρ is density, ζ is vorticity. The incompressibility condition (2.4) implies the existence of a stream function ψ so that

$$\boldsymbol{u} = -\nabla \times \psi = (-\psi_y, \psi_x). \tag{2.5}$$

It is illuminating to recast the system (2.1) - (2.4) into an exterior differential system using differential forms. Since $d\psi = \psi_x dx + \psi_y dy = v dx - u dy$, then equation (2.5), which is equivalent to (2.4), can be written as

$$d\psi = v dx - u dy. \tag{2.6}$$

Similarly equations (2.1) and (2.2) are equivalent to

$$
\begin{aligned}
d\psi \wedge d\rho &= 0 && (2.7)\\
d\psi \wedge d\zeta &= dy \wedge d\rho. && (2.8)
\end{aligned}
$$

This simple reformulation of the equation is surprisingly potent: Equations (2.7) and (2.8) suggest that (y, ψ) would be convenient variables, in terms of which these equations become

$$
\begin{aligned}
\rho_y &= 0 && (2.9)\\
\zeta_y &= -\rho_\psi && (2.10)
\end{aligned}
$$

which has solution

$$
\begin{aligned}
\rho &= \rho(\psi) && (2.11)\\
\zeta &= \xi(\psi) - y\rho_\psi(\psi) && (2.12)
\end{aligned}
$$

in which ρ and ξ are arbitrary functions.

Next, consider the solution as a surface in (x, y, u, v, ψ) space Then equation (2.6) or the rescaled form

$$dx = \frac{1}{v}d\psi + \frac{u}{v}dy \tag{2.13}$$

provides a *contact constraint* on the solution; i.e. a differential form $\alpha = d\psi - vdx + udy$ that vanishes everywhere on the solution surface. This surface is a Legendrian surface due to the contact constraint, as discussed in the Section 5.

Finally, equations (2.3) and (2.4) can be written in terms of the new independent variables (y, ψ) as

$$\begin{aligned}\left(\frac{u}{v}\right)_\psi &= \left(\frac{1}{v}\right)_y &(2.14)\\ (u^2+v^2)_\psi &= 2u_y + 2\zeta. &(2.15)\end{aligned}$$

Equation (2.14) is equivalent to the contact constraint (2.13), while (2.15) provides a further constraint on the system.

Several further manipulations of the equations are worthwhile. If (u, v) are finite we can write (2.14), (2.15) as a first order system

$$\begin{pmatrix} u \\ v \end{pmatrix}_y + \begin{pmatrix} -u & -v \\ v & -u \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_\psi = \begin{pmatrix} -\zeta \\ 0 \end{pmatrix}. \tag{2.16}$$

In this case the previous classification of [5] describes the generic singularities for the system.

Second, suppose that (u, v) may become infinite. Following the Beale-Kato-Majda result [2], singularities with infinite vorticity are of most interest, so we will assume that ζ blows-up. According to the formula (2.12), blow-up of ζ occurs along fixed values of ψ. Therefore we unfold the singularity through unfolding only the ψ variable by a mapping $\psi = \psi(p)$. Then we look for a solution which is single valued in y and p. Furthermore we scale the velocity as

$$u = \frac{\mu}{\beta}, \quad v = \frac{\nu}{\beta} \tag{2.17}$$

in which $\beta = \beta(p)$ gives the rate of blow-up. As a consequence of (2.12) we may write ζ as

$$\zeta = (\eta - y\rho)_\psi \tag{2.18}$$

in which $\eta_\psi = \xi$.

Finally we assume that ζ is of the same order as $(u^2 + v^2)_\psi$ in (2.15) by scaling η and ρ as

$$\begin{aligned} \eta &= \beta^{-2}\kappa(p) \qquad (2.19) \\ \rho &= \beta^{-2}\gamma(p). \end{aligned}$$

The resulting pde's for μ and ν are

$$\left(\frac{\mu}{\nu}\right)_p = \beta\psi_p\left(\frac{1}{\nu}\right)_y \tag{2.20}$$

$$(\beta^{-2}(\mu^2 + \nu^2 - 2\kappa + 2y\gamma))_p = 2\beta^{-1}\psi_p\mu_y. \tag{2.21}$$

Integrate these with respect to p to obtain

$$\frac{\mu}{\nu}(p, y) = f(y) + \int_0^p \beta\psi_p(\nu^{-1})_y dp \tag{2.22}$$

$$(\mu^2 + \nu^2)(p, y) = 2(\kappa - y\gamma) + \beta^2 g(y) + \beta^2 \int^p \beta^{-1}\psi_p\mu_y dp \tag{2.23}$$

in which $f(y)$, $g(y)$, $\kappa(p)$, $\gamma(p)$, $\beta(p)$ and $\psi_p(p)$ are arbitrary functions. In Section 4, we shall show that these equations have a unique solution which is analytic in y for any choice of $f, g, \kappa, \gamma, \beta$ and ψ_p and that the solution depends continuously on these functions.

3 Generic Singularities for Boussinesq

In this section, the generic singularity types for the steady Boussinesq equations are analyzed under certain restrictions. The analysis here is performed directly on the integral form of the Boussinesq equations (2.22), (2.23), using the PDE results of Section 4. The genericity results are motivated by and could have been derived from Legendrian singularity theory, which is summarized in Section 5, but the direct derivation presented here is simpler.

The singularities under consideration are restricted as follows:

(i) The singularity set has codimension equal to 1. These are the singularities that are most likely to be observed.

(ii) As a function of (y, ψ) the solution depends smoothly on y. This is expected since ζ is a linear function of y for fixed ψ or p.

Under these restrictions, the generic singularity types for steady Boussinesq are described by the next two theorems. By *singularity type* we mean an equivalence class under analytic changes of variables. By *generic* we mean that two conditions are satisfied. First the singularity type should be stable with respect to perturbations. This is called the *stability* condition. Second, any solution with a singularity type which is not stable may be perturbed into one having only stable singularity types. This is called the *density* condition. More details concering these concepts are presented in Section 5.

Theorem 3.1 *(Generic singularities for Boussinesq).*

Under the restrictions (i), (ii) above, the two generic (i.e. stable and dense) singularity types for the steady Boussinesq equations are a curve of cusps and a curve of blow-up folds. Representatives of these two classes are

(i) Cusp. $x \approx \frac{1}{2}p^2, \quad v \approx p, \quad \psi \approx \frac{1}{3}p^3, \quad \zeta \approx p^{-1}$

(ii) Blow-up fold. $x \approx \frac{1}{2}p^2, \quad v \approx p^{-1}, \quad \psi \approx p, \quad \zeta \approx p^{-3}.$

Theorem 3.2 *(Generic singularities with symmetry for Boussinesq)*

In addition to restrictions (i), (ii), suppose that ψ is symmetric with repsect to reflection in x for some point on the singularity locus; i.e.

$$\psi(x_0 + x, y_0) = \psi(x_0 - x, y_0) \tag{3.1}$$

for some point (x_0, y_0) on the singularity locus. Then there is only one generic singularity type, a curve of blow-up cusps, *(iii) Blow-up Cusp.* $x \approx \frac{1}{3}p^3,\ \ v \approx p^{-1},\ \ \psi \approx \frac{1}{2}p^2,\ \ \zeta \approx p^{-4}$.

These singularities can be written in terms of x dependence (omitting constants), as follows:

Without symmetry

(i) $\psi \approx x^{3/2},\ \ v \approx x^{1/2},\ \ \zeta \approx x^{-1/2}$

(ii) $\psi \approx x^{1/2},\ \ v \approx x^{-1/2},\ \ \zeta \approx x^{-3/2}$

With symmetry

(iii) $\psi \approx x^{2/3},\ \ v \approx x^{-1/3},\ \ \zeta \approx x^{-4/3}$.

Proof of Theorem 3.1. We need to show that the singularity types (i) and (ii) are stable with respect to perturbation and that any other solution can be perturbed to one of these two types. The allowable perturbations are those of the solution as a complex surface in $\mathbb{C}^5$ with coordinates (x, y, μ, ν, ψ). This surface will in general not be closed. It may have the form of a surface with complex codimension 1 (real codimension 2) sets removed. These removed sets correspond to the loci where the Boussinesq solution has singularities and are described below. The natural setting for perturbation theory is not such surfaces per se but rather their simply connected universal covers. However, since our perturbation results, related to stability, are local it will suffice to consider surfaces in $\mathbb{C}^5$.

In Section 4 below, we show that the solution exists, as an analytic function for y in $\mathbb{C}$ and for p on a Riemann surface $\mathcal{R}$, for any choice of the "data" – i.e. the functions $\beta(p),\ \psi_p(p),\ \gamma(p),\ f(y),\ g(y)$ – except for a restriction on the location of the zeroes of β. In addition, the solution depends smoothly on this "data." So it suffices to consider perturbations of these functions. Since the stability results are local, only a neighborhood of $(y,p) = (0,0)$ need be considered.

The Riemann surface $\mathcal{R}$ has logarithmic branch points at each of the zeroes of β. For p near one of these branch points, say p_i of order n_i, and for any fixed y, the solution μ has the form

$$\mu(p) = \mu_0(p) + O((p-p_i)^{2n_i}\log(p-p_i)) \tag{3.2}$$

in which μ_0 is analytic for p near p_i. There is a similar form for ν. Since the logarithmic part of the solution is at such high order in p it does not affect the singularity type.

The zeroes of β (as well as those of ψ_p) are the places at which the Boussinesq solution has singularities. In a given neighborhood of the variable p, let the zeroes of β be $p_1, \ldots, p_m$. For the analytic results of Section 4 there there is a restriction that the spacing of the roots needs to be approximately uniform; i.e. for some constant c,

$$\max_{i \neq j} |p_i - p_j| \leq c \min_{i \neq j} |p_i - p_j|. \tag{3.3}$$

This class of functions β is dense with respect to the usual topology. It is also large enough to consider a neighborhood of a k-th order zero p_1 for β, and its generic perturbation to k simple zeroes $p_1, \ldots, p_k$. The Boussinesq solution is shown to be smooth with respect to such perturbations in Theorem 4.2.

Furthermore, any analytic functions $\psi_p(p)$, $\gamma(p)$, $f(y)$, $g(y)$ are allowed (it would not change anything if β, $\psi_p(p)$, $\gamma(p)$ were defined on $\mathcal{R}$ rather than $\mathbb{C}$). Actually for determination of the singularity type only the function $\beta(p)$ and $\psi_p(p)$ are significant. Generically, there are 3 cases:

(a) $\beta(0) \neq 0$, $\psi_p(0) \neq 0$. In this case there is no singularity.

(b) $\beta(0) \neq 0$; $\psi_p(0) = 0$, $\psi_{pp}(0) \neq 0$. Then

$$\psi = a_0 p^2 + a_1 p^3 + \ldots \tag{3.4}$$

for some constants a_0 and a_1. In this case, we may set $\beta = 1$ so that $\mu = u$ and $\nu = v$. Now generically $f(0) \neq 0$ and $g(0) \neq 0$. Thus $\frac{u}{v}$ is a unit and $u^2 + v^2$, for generic κ, is nonvanishing. It follows that

$$v = b_0 + b_1 p + \cdots \tag{3.5}$$

for some non-vanishing constants b_0 and b_1 depending on y. Since $v = \psi_x$, then

$$x_p = \psi_p / v = c_0 p + c_1 p^2 + \ldots \tag{3.6}$$

in which

$$c_0 = 2a_0/b_0, \ c_1 = 3a_1/b_0 - 2a_0 b_1/b_0^2 \tag{3.7}$$

so that

$$x = \frac{1}{2}c_0p^2 + \frac{1}{3}c_1p^3 + \cdots. \tag{3.8}$$

Under the transformation

$$\begin{aligned} \psi' &= \psi - 2(a_0/c_0)x \\ &= O(p^3) \end{aligned} \tag{3.9}$$

and

$$v' = v - b_0 = b_1p + \cdots \tag{3.10}$$

this is seen to be a singularity of type (i). In particular for generic η and ρ,

$$\begin{aligned} \zeta &= (\eta - y\rho)_\psi \\ &= (\eta - y\rho)_p/\psi_p \\ &= dp^{-1} + O(1) \end{aligned}$$

for some constant d.

(c) $\beta(0) = 0$, $\beta_p(0) \neq 0$, $\psi_p(0) \neq 0$. Then we may set

$$\beta = p + \ldots, \ \psi = p + \ldots. \tag{3.11}$$

Generically $f(0) \neq 0$ and $\eta(0) \neq 0$, so that $\binom{\mu}{\nu}$ is a unit and

$$\mu^2 + \nu^2 \approx \kappa. \tag{3.12}$$

It follows that generically μ and ν are $O(1)$ and that u, v are $O(\frac{1}{p})$. Then

$$\begin{aligned} x_p &= \frac{\psi_p}{v} = O(p) \\ x &= O(p^2) \\ \zeta &= (\beta^{-2}(\kappa - y\gamma))_p = O(p^{-3}). \end{aligned}$$

This is a singularity of type (ii), which finishes the proof of Theorem 3.1.

Proof of Theorem 3.2. Assume that $\psi(x) = \psi(-x)$ at $y = 0$. Then ψ is even in p at leading order, so that generically $\psi_p = O(p)$. Now, since μ and ν are generically nonvanishing as in the preceding proof,

$$\beta = O(\frac{1}{\psi_x}) = O(x_p/\psi_p). \tag{3.13}$$

If v blows up, i.e., if $\beta(0) = 0$, then generically $x = p^3 + \cdots$, which is singularity type (iii). If v does not blow up, then $\beta(0) = O(1)$ and $x = p + \cdots$, which is nonsingular. This completes the proof of Theorem 3.2.

4 Existence

In this section we construct solutions of the steady Boussinesq equations (2.20), (2.21) or the integral form (2.22), (2.23). The construction is within the class of analytic functions and follows the proof of the abstract Cauchy-Kowalewski Theorem [19]. The solution is unique for a given choice of the functions $\beta(p)$, $\psi(p)$, $f(y)$, $g(y)$, and depends continuously on the choice of this "data."

The significance of the Cauchy-Kowalewski Theorem is that, within the analytic function class, it effectively reduces the analysis of PDEs to algebraic conditions on the PDE and its data. Although for many problems the restriction to analytic solutions is too severe, it is natural here since singular solutions may be ill-posed in Sobolev space, complex singularities are of interest, and analytic solutions with singularities may serve as canonical examples of more general singularities.

Because equation (2.21) is singular at each value $p = p_i$ at which $\beta = 0$, its general solution is not analytic in p. Nevertheless, the abstract Cauchy-Kowalewski theorem is applicable, since it requires analyticity in the "space-like" variable y, but not in the "time-like" variable p.

Furthermore, the solution that we construct is analytic for p on a noncompact Riemann surface $\mathcal{R}$ = universal covering space of $\mathbb{C} - \{p_1, \ldots, p_m\}$. The fundamental group of $\mathbb{C} - \{p_1, \ldots, p_m\}$ is generated by m fundamental closed loops. The i^{th} such loop starts at a base point $p_0 \neq p_j$ for any $j = 1, \ldots, m$ and encircles p_i without encircling any other p_j and then returns to p_0. Denote this i^{th} fundamental loop by e_i. Every closed path on $\mathbb{C} - \{p_1, \ldots, p_m\}$ which is based at p_0 is homotopic to a path of the form

$$w = e_{i_1}^{k_1} \cdot e_{i_2}^{k_2} \cdot \ldots \cdot e_{i_n}^{k_n}, \tag{4.1}$$

for any finite n, where the product $a \cdot b$ denotes concatenation of loops; i.e., the loop constructed by tracing the path a followed by the path b. e_i^k corresponds to going k times around p_i . Thus the fundamental group of $\mathbb{C} - \{p_1, \ldots, p_m\}$ is isomorphic to the free group on the m generators $e_1, \ldots, e_m$. There is a 1-1 correspondence between homotopy classes of paths and words w (4.1) which represent elements of this free group. The *length* of such a word is the integer $k_1 + \ldots + k_n$.

A fundamental domain of $\mathcal{R}$ is homeomorphic to $\mathbb{C}$ cut along the rays $\overrightarrow{p_0 p_i}$ emanating from p_0 and terminating at p_i. (p_0 may be chosen so that these rays do not meet one another except at their common origin p_0.) $\mathcal{R}$ is the union of infinitely many copies of this fundamental domain (called sheets) labelled by the words w in the free group on m generators. Let $\mathcal{R}_w$ denote the sheet labelled by w. The Cayley diagram of this free group is a tree in which the vertices are the words and each edge has m branches emerging from it which correspond to the m fundamental loops. The different sheets correspond to these vertices and two sheets are identified along opposite sides of their respective i^{th} cuts if they correspond to two vertices connected by an edge corresponding to e_i. For further details on this type of construction we refer to [20].

A point p on sheet $\mathcal{R}_w$ of the Riemann surface $\mathcal{R}$ can be written as

$$p = (\tilde{p}, w) \tag{4.2}$$

in which $\tilde{p}$ is a point in $\mathbb{C} - \{p_1, \ldots, p_m\}$. We choose a metric on $\mathcal{R}$ which is in the conformality class of $\mathcal{R}$ [8]. For $p = (\tilde{p}, w)$, let $||p||$ denote the distance from $p^0 = (\tilde{p}_0, 1)$, where 1 denotes the null word (the identity in the free group), to p. On the other hand, let

$$|p| = |\tilde{p}|, \ |p - p_i| = |\tilde{p} - p_i| \tag{4.3}$$

denote the usual modulus of the projection into the complex plane. Define the neighborhood Ω_a of the origin in the Riemann surface $\mathcal{R}$ by

$$\Omega_a = \{p \ : \ ||p|| < a; \ |p - p_i| \cdot |\log(p - p_i)|^4 < a\}. \tag{4.4}$$

Note that since $|\log(p - p_i)|^2 = (\log|p - p_i|)^2 + (\arg(p - p_i))^2$ and $\arg(p - p_i)$ grows like the length of the word w, Ω_a has only a small intersection with a sheets for which the word w is long.

Define the function norms

$$
\begin{aligned}
|u|_{y_0} &= \sup_{|y|<y_0} |u(y)| \\
\|u\|_{q_0,\sigma} &= \sup_{p\in\Omega_{q_0}} |u(p,\cdot)|_{\sigma(q_0-\|p\|)}.
\end{aligned} \tag{4.5}
$$

Note that if $u = u(p)$ is independent of y, then

$$
\|u\|_{q_0,0} = \sup_{p\in\Omega_{q_0}} |u(p)|. \tag{4.6}
$$

Also define the function spaces

$$
\begin{aligned}
B_{y_0} &= \{u : u \text{ is analytic in } |y| < y_0, \text{ with } |u|_{y_0} < \infty\} \\
\mathcal{B}_{q_0,\sigma} &= \{u : u(p,\cdot) \in B_{\sigma(q_0-\|p\|)} \text{ for } p \in \Omega_{q_0} \text{ with } \|u\|_{q_0,\sigma} < \infty\}.
\end{aligned} \tag{4.7}
$$

Denote

$$
\begin{aligned}
F_1 &= F_1(y) = (f(y), g(y)) \\
F_2 &= F_2(p) = (\kappa(p), \gamma(p), \psi_p(p), \beta(p)).
\end{aligned} \tag{4.8}
$$

In the equations (2.22), (2.23) the functions F_1, F_2 can be prescribed arbitrarily. The analytic results depend on several assumptions:

(A1) Assume that

$$
\begin{aligned}
f_0 &= f(0) \neq 0 \\
\kappa_0 &= \kappa(0) \neq 0.
\end{aligned} \tag{4.9}
$$

(A2) Assume that

$$
\begin{aligned}
&F_1 \in B_{y_0} \\
&F_2 \in \mathcal{B}_{q_0,0} \\
&|F_1|_{y_0} + \|F_2\|_{q_0,0} < c_0
\end{aligned} \tag{4.10}
$$

for some y_0, q_0 and c_0.

(A3) Assume that for $|p| < q_0$, β has zeroes $p = p_1, \ldots, p_m$ of order $n_1, \ldots, n_m$; i.e.

$$
\beta = \tilde{\beta}(p) \sum_{i=1}^{m} (p - p_i)^{n_i} \tag{4.11}
$$

in which $\tilde{\beta}$ is analytic and non-vanishing in $|p| < q_0$. In addition assume that for some constant c_1

$$\max_{i\neq j} |p_i - p_j| \leq c_1 \min_{i\neq j} |p_i - p_j|. \tag{4.12}$$

The restriction (4.12) is further discussed after the proof. The main analytic results of this paper are the following:

Theorem 4.1 *(Existence and Uniqueness). Suppose that (F_1, F_2) satisfy assumptions (A.1), (A.2) and (A.3) for some constants y_0, q_0, c_0 and c_1. Then for some σ and c_2, there is a unique solution (μ, ν) of (2.22), (2.23) in $\mathcal{B}_{q_0,\sigma}$ with*

$$\|(\mu, \nu)\|_{q_0,\sigma} \leq c_2. \tag{4.13}$$

The constants σ and c_2 depend only on y_0, q_0, c_0 and c_1.

Theorem 4.2 *(Continuous Dependence on Data). Suppose that (F_1, F_2) and $(\tilde{F}_1, \tilde{F}_2)$ both satisfy assumptions (A.1), (A.2), (A.3) and let (μ, ν) and $(\tilde{\mu}, \tilde{\nu})$ be the corresponding solutions as in Theorem 4.1.*

Then

$$\|(\mu - \tilde{\mu}, \nu - \tilde{\nu})\|_{q_0,\sigma} \leq c_3(|F_1 - \tilde{F}_1|_{y_0} + |F_2 - \tilde{F}_2|_{q_0,0}). \tag{4.14}$$

The constant c_3 depends only on y_0, q_0, c_0 and c_1.

Proof of Theorem 4.1 Simplify somewhat by setting $p_1 = 0$, $\psi_p = 1$ and

$$\beta = \sum_{i=1}^{m} (p - p_i)^{n_i}. \tag{4.15}$$

Assume that $m \geq 2$; otherwise the proof can be simplified considerably. For other choices of p_0, ψ_p and β, the proof is only slightly different. Denote $N = \sum_{i=1}^{m} n_i$. Also define

$$\begin{aligned} a &= \frac{\mu}{\nu} \\ b &= \mu^2 + \nu^2 \end{aligned} \tag{4.16}$$

so that

$$\begin{aligned} \mu &= \left(\frac{b}{1+a^2}\right)^{\frac{1}{2}} a \\ \nu^{-1} &= \left(\frac{1+a^2}{b}\right)^{\frac{1}{2}}. \end{aligned} \tag{4.17}$$

Equations (2.22), (2.23) become the following equations for a, b:

$$\begin{aligned} a &= f + \int_0^p \beta(\nu^{-1})_y dp \\ b &= h + \beta^2 \int^p \beta^{-1}\mu_y dp \end{aligned} \tag{4.18}$$

in which

$$h = 2(\kappa - y\gamma) + \beta^2 g.$$

First, the dominant terms at $p = 0$ are extracted. Denote

$$\begin{aligned} a_0 &= f(y) \\ b_0 &= h + \beta^2 \int^p \beta^{-1}\mu_{0y} dp \end{aligned} \tag{4.19}$$

in which

$$\mu_0 = \mu(f, h) = \left(\frac{h}{1+f^2}\right)^{\frac{1}{2}} f. \tag{4.20}$$

The square root is non-singular in a neighborhood of $y = 0$, since $\kappa_0 \neq 0$. Note that a_0 is analytic for p in $\mathbb{C}$ and that the logarithmic part of b_0 is of size $O((p-p_i)^{2n_i}\log(p-p_i))$. The remaining terms are of even higher order.

Next denote

$$\begin{aligned} a_1 &= a - a_0 \\ b_1 &= b - b_0 \\ a_2 &= (\ell^2\beta)^{-1} a_{1p} \\ b_2 &= (\ell^2\beta)^{-1} b_{1p} \\ a_3 &= a_{1y} \\ b_3 &= b_{1y}. \\ \mu_1 &= \mu - \mu_0 \\ \ell &= \sup_{1 \le i \le m} |\log(p - p_i)|. \end{aligned} \tag{4.21}$$

Also set

$$\Gamma = \int_{-\infty}^{p} \beta^{-1}(p')dp' \tag{4.22}$$

in which the integration curve, which lies on the Riemann surface $\mathcal{R}$, must go around the zeroes p_i of β. Note that $\Gamma_p = \beta^{-1}$, and that Γ can be constructed explicitly using the partial fraction expansion for β.

The equations (4.18) for (a, b) become the following equations for a_1, a_2, a_3, b_1, b_2, b_3:

$$\begin{aligned}
a_1 &= \int_0^p \beta(\nu^{-1})_y dp \\
a_2 &= \ell^{-2}(\nu^{-1})_y \\
a_3 &= \int_0^p a_{1yp} dp = \int_0^p \ell^2 \beta a_{2y} dp \\
b_1 &= \beta^2 \int_0^p \beta^{-1}\mu_{1y} dp \\
b_2 &= 2\ell^{-2}\beta_p \int_0^p \beta^{-1}\mu_{1y} dp + \ell^{-2}\mu_{1y} \\
b_3 &= \int_0^p \ell^2 \beta b_{2y} dp.
\end{aligned} \tag{4.23}$$

The integration curve in each of these integrals lies on the Riemann surface $\mathcal{R}$, and goes around the zeroes p_i of β.

The main difficulty in this analysis is the singular integral on the right side of the equations for b_1 and b_2. It is rewritten using integration by parts as

$$\int_0^p \beta^{-1}\mu_{1y} dp = \Gamma\mu_{1y}(p) - \int_0^p \Gamma\mu_{1yp} dp \tag{4.24}$$

using the fact that $\Gamma\mu_{1y}(p = 0) = 0$, which will be shown below.

Differentiate μ_1 and ν^{-1}, using (4.17), (4.19), (4.20), (4.21) and the fact that a_0 and b_0 have no zeroes in the domain of interest. The result can be summarized as

$$\begin{aligned}
(\nu^{-1})_y &= m_0 + m_1 a_3 + m_2 b_3 \\
\mu_{1y} &= n_0 + n_1 a_3 + n_2 b_3 \\
\mu_{1yp} &= k_0 + \ell^2\beta(k_1 a_{2y} + k_2 b_{2y})
\end{aligned} \tag{4.25}$$

in which the coefficients m_0, m_1, m_2, n_0, n_1, n_2, k_0, k_1, k_2 are analytic functions of p on $\mathcal{R}$, of y and of $a_1, a_2, a_3, b_1, b_2, b_3$, which are bounded in the solution domain.

The equations (4.23) then become the following:

$$
\begin{aligned}
a_1 &= \int_0^p \beta(m_0 + m_1 a_3 + m_2 b_3)dp \\
a_2 &= \ell^{-2}(m_0 + m_1 a_3 + m_2 b_3) \\
a_3 &= \int_0^p \ell^2 \beta a_{2y} dp \\
b_1 &= \beta^2 \Gamma(n_0 + n_1 a_3 + n_3 b_3) \\
&\qquad -2\beta^2 \int_0^p \Gamma(k_0 + \ell^2 \beta(k_1 a_{2y} + k_2 b_{2y}))dp \\
b_2 &= \ell^{-2}(2\beta_p \Gamma + 1)(n_0 + n_1 a_3 + n_2 b_3) \\
&\qquad -2\ell^{-2}\beta_p \int_0^p \Gamma(k_0 + \ell\beta(k_1 a_{2y} + k_2 b_{2y}))dp \\
b_3 &= \int_0^p \ell^2 \beta b_{2y} dp.
\end{aligned} \tag{4.26}
$$

At $p = p_i$, the factors β and ℓ^{-1} are zero and β_p is bounded or zero, but ℓ and Γ are infinite. In Lemma lemma7.1 below, we will show that

$$
\begin{gathered}
\ell^{-2}\beta_p \Gamma \\
\Gamma \ell^2 \beta \\
\ell^{-2}\beta_p \int_0^p |\Gamma| dp \beta
\end{gathered} \tag{4.27}
$$

are uniformly bounded in the domain Ω_{q_0} and go to zero as p goes to 0. Since $\beta^2 \leq c|\ell^{-2}\beta_p|$ for some constant c, the same is true for

$$
\begin{gathered}
\beta^2 \Gamma \\
\beta^2 \int_0^p |\Gamma| dp.
\end{gathered} \tag{4.28}
$$

In the bound on $\ell^{-2}\beta_p \Gamma$, the factor ℓ^{-2} is needed because of logarithmic terms in Γ. This is the reason for introducing this factor in the definition of a_2, b_2.

The system (4.26) can now be solved using the abstract Cauchy-Kowalewski Theorem, the best form of which is that of Safonov [19]. This theorem is usually stated as an existence theorem for a first order system of pde's, but the proof applies equally well to systems of the form (4.26) in which there is an integral over the "time" variable p. Denote $\phi = (a_1, a_2, a_3, b_1, b_2, b_3)$. Then (4.26) can be written as

$$
\phi(y,p) = G_0(y,p,\phi) + \int_0^p G_1(y,p,p',\phi,\phi_y) + G_2(y,p,p',\phi)dp' \tag{4.29}
$$

in which the integration is over the Riemann surface $\mathcal{R}$ and
(i) G_0, G_1 and G_2 are analytic in y, ϕ, ϕ_y near 0 and in p on $\mathcal{R}$.
(ii) $G_0, G_{0\phi}, G_1, G_{1\phi}$ and $G_{1\phi_y}$ are uniformly bounded, and $G_0, G_{0\phi}$ go to 0 as p goes to 0.
(iii) $\int_0^p |G_2| + |G_{2\phi}| + |G_{2\phi_y}| dp'$ is uniformly bounded and goes to 0 as p goes to 0.
In these three conditions, the bounds and the approach to 0 are uniform for y, ϕ, ϕ_y near 0, for p in Ω_{q_0}, and for data F_1, F_2 satisfying assumptions (A1)-(A3) with fixed y_0, q_0, c_0, c_1.

The Cauchy-Kowalewski proof for existence of a solution to (4.29) is based on iteration and estimation of ϕ_y in G_1 in terms of ϕ using the Cauchy estimates. The Cauchy estimates involve a factor that is infinite at the boundary of the existence domain. This large factor is bounded, however, when integrated over p. For this reason, the function G_1 and its derivatives need to be bounded, while the function G_2 and its derivatives need only be integrable.

Under these conditions, the proof of the abstract Cauchy-Kowalewski Theorem can be applied to produce a solution ϕ that is analytic in y and p (on $\mathcal{R}$). For some σ this solution is in $\mathcal{B}_{q_0,\sigma}$. The corresponding function μ, ν solve (2.22), (2.23) and satisfy the bound (4.13).

Thus the proof of Theorem 4.1 is finished once the following bounds are established:

Lemma 4.1 *Under assumption (A.3), there is a constant c_4 so that in Ω_{q_0}*

$$\begin{aligned} &|\Gamma \ell^3 \beta| < c_4 \\ &|\ell^{-1} \Gamma \beta_p| < c_4 \\ &|\beta_p| \int_0^p |\Gamma| dp' \le c_4. \end{aligned} \tag{4.30}$$

Furthermore

$$\begin{aligned} &\Gamma \ell^2 \beta \\ &\ell^{-2} \Gamma \beta_p \\ &\beta_p \ell^{-2} \int_0^p |\Gamma| dp' \end{aligned} \tag{4.31}$$

goes to 0 as p goes to 0.

Proof of Lemma 4.1. Denote

$$\Delta = \min_{1 \leq i \neq j \leq m} |p_i - p_j|. \tag{4.32}$$

Recall that $p_1 = 0$ and that $\max |p_i - p_j| \leq c\Delta$. The estimates are split into two cases: $p = O(\Delta)$ and $|p| >> \Delta$.

Using partial fractions for β^{-1}, one finds that

$$\begin{aligned} \beta^{-1} &= \sum_{i=1}^{m} \sum_{j=1}^{n_i} a_{ij}(p - p_i)^{-j} \\ \Gamma &= \sum_{i=1}^{m} a_{i1} \log(p - p_i) + \sum_{j=2}^{n_i} (1 - j)^{-1} a_{ij} (p - p_i)^{1-j} \end{aligned} \tag{4.33}$$

in which

$$|a_{ij}| \leq c\Delta^{-N+j}. \tag{4.34}$$

These formulas are used for $p = O(\Delta)$. For $p >> \Delta$, there are simpler bounds

$$\begin{aligned} \beta^{-1} &= O(p^{-N}) \\ \Gamma &= O(p^{1-N}). \end{aligned} \tag{4.35}$$

Estimate

$$\begin{aligned} |\beta| &= |\prod_{i=1}^{m} (p - p_i)^{n_i}| \\ &\leq \begin{cases} \min_i \Delta^{N-n_i} |p - p_i|^{n_i} & |p| = O(\Delta) \\ p^N & |p| >> \Delta \end{cases} \\ |\beta_p| &\leq \begin{cases} \min_i \Delta^{N-n_i} |p - p_i|^{n_i - 1} & |p| = O(\Delta) \\ p^{N-1} & |p| >> \Delta \end{cases} \\ |\Gamma| &\leq \begin{cases} \max_i (\Delta^{-N+1} |\log(p - p_i)| + \Delta^{-N+n_i} |p - p_i|^{1-n_i}) & |p| = O(\Delta) \\ p^{-N+1} & |p| >> \Delta \end{cases} \\ \int_0^p |\Gamma| dp' &\leq \begin{cases} \max_i (\Delta^{-N+1} |(p - p_i) \log(p - p_i)| & \\ \quad + \Delta^{-N+n_i} |p - p_i|^{2-n_i}) & |p| = O(\Delta) \\ p^{-N+2}. & |p| >> \Delta \end{cases} \end{aligned} \tag{4.36}$$

Then

$$
\begin{aligned}
\Gamma\beta_p &\leq \begin{cases} 1+\max_i|\log(p-p_i)| & |p| = \mathcal{O}(\Delta) \\ 1 & |p| >> \Delta \end{cases} \\
&\leq c\,\ell & (4.37) \\
\Gamma\ell^3\beta &\leq \begin{cases} \max_i|p-p_i|\;|\log(p-p_i)|^4 & |p| = \mathcal{O}(\Delta) \\ |p|\;|\log p|^3 & |p| >> \Delta \end{cases} \\
&\leq c & (4.38) \\
\beta_p\int_0^p|\Gamma|dp' &\leq \begin{cases} \max_i|(p-p_i)\log(p-p_i)| & |p| = \mathcal{O}(\Delta) \\ |p| & |p| >> \Delta \end{cases} \\
&\leq c. & (4.39)
\end{aligned}
$$

This proves (4.30), and since ℓ^{-1} goes to 0 as p goes to 0, (4.31) follows immediately.

The proofs of Lemma 4.1 and of Theorem 4.1 are now finished.

Proof of Theorem 4.2 The proof of Theorem 4.2 exactly follows that of Theorem 4.1. Let (F_1, F_2) and $(\tilde{F}_1, \tilde{F}_2)$ both satisfy assumptions (A.1), (A.2) and (A.3), and let (μ, ν) and $(\tilde{\mu}, \tilde{\nu})$ denote the corresponding solutions from Theorem 4.1. Define

$$
\begin{aligned}
\hat{\mu} &= D^{-1}(\mu - \tilde{\mu}) \\
\hat{\nu} &= D^{-1}(\nu - \tilde{\nu}) \qquad (4.40)
\end{aligned}
$$

in which

$$
D = |F_1 - \tilde{F}_1|_{y_0} + |F_2 - \tilde{F}_2|_{q_0,0}. \qquad (4.41)
$$

In the same way, define

$$
\begin{aligned}
\hat{a}_i &= D^{-1}(a_i - \tilde{a}_i) \\
\hat{b}_i &= D^{-1}(b_i - \tilde{b}_i) \qquad (4.42) \\
\hat{G}_i &= D^{-1}(G_i - \tilde{G}_i).
\end{aligned}
$$

Then the functions $\hat{G}_i$ satisfy the bounds (i), (ii) and (iii) above. It follows that (μ, ν) and $(\tilde{\mu}, \tilde{\nu})$ satisfy the bound (4.14).

Note: (1) Validity of the abstract Cauchy-Kowalewski Theorem without analyticity in the "time" variable (p here) was first observed by Nirenberg

[14] although it has not been used in an essential way before, to the best of our knowledge. Here p is analytic only on a nontrivial Reimann surface $\mathcal{R}$, so that the classical proofs using power series expansions would fail.

(2) Although it is not presented here, further analysis in a neighborhood of $p = p_i$ which is a zero of order n for β, shows that the solution is analytic in the variables y, p and $(p-p_i)^{2n}\log(p-p_i)$. On the other hand, we have not succeeded in extending this representation to an neighborhood of p containing several zeroes $(p_1,\ldots,p_m)$ of order $(n_1,\ldots,n_m)$ respectively for β.

(3) The restriction(4.12) on the relative locations of the zeroes p_i for β was used to simplify the cancellation between β and Γ in Lemma 4.1. This suffices for the genericity results, but it should be possible to remove the restriction.

5 Legendrian Singularities and Blowup

This section contains a detailed exposition of the key idea introduced in the Section 2; namely, the notion of a contact structure for the Boussinesq PDE system and the interpretation of the solution of this system as a constrained Legendrian surface. In previous sections these ideas were introduced as motivation for the transformation of the Boussinesq system into a particularly convenient and elegant form, which reduces the construction and analysis of singular solutions to the study of an integral equation. However, this geometric approach is more than just a convenient device; it clarifies the influence of incompressibility on the generic singularity type. This understanding enabled us to predict the generic singularity types, which could then be analytically verified.

The first step in a geometric analysis of a PDE system is to consider the graph of a solution as the fundamental object, rather than just the solution function. For the Boussinesq system, (2.1)-(2.4), the graph of the solution is naturally situated in the large space with coordinates $(x, y, u, v, \zeta, \rho)$. Here $\boldsymbol{x} = (x, y)$ are the independent variables and the other variables are dependent, so that the geometric solution is a (2-dimensional) surface, which is a graph over $\boldsymbol{x}$ in this six-dimensional space, if the solution is "classical", i.e. smooth.

Some of the equations in the Boussinesq system have a fundamental geometric character which enables one to reduce the size of this ambient space. In particular equation (2.4), which is the incompressibility condition, implies the existence of a stream function ψ whose partial derivatives are u and v as described in (2.5). Furthermore, equation (2.1) implies that ρ is a (arbitrary) function of ψ as noted in (2.11). Equation (2.3) defines ζ as a second derivative of ψ. This reduction shows that the graph of ψ contains all the information needed for the solution of the Boussinesq system.

The notion of a Legendrian surface, which is fundamental to our singularity classification, requires the introduction of a contact structure encoding the differential relation between ψ and (u, v). Thus, the correct ambient space for our analysis is a space with local coordinates (x, y, u, v, ψ) and the solution surface is the lifting of the graph of ψ to this space. This space is classically known as the space of 1-jets associated to functions ψ and will be described further below. The lifting of the solution surface to this space is called the 1-jet prolongation of the solution surface.

For solution surfaces that are graphs, there are no singularities. The geometric setting just described, however, easily admits generalization of a solution graph to a solution surface which is a smooth surface (not necessarily a graph) in the 1-jet space satisfying the contact constraint. Such a solution may be interpreted as a *multivalued* classical solution–classical in the sense that the surface will be assumed to be smooth or even analytic. The Legendrian singularities are the branch curves of these surfaces, under projection to $\boldsymbol{x}$.

In the first subsection below we will define all of this precisely and review the standard definition of a Legendrian manifold and a Legendrian singularity. In the second subsection we introduce a directional compactification of the jet space, which extends the theory to singularities at which the velocities, (u, v), blow up. Although this uses standard methods of projectivization, their application in the context of solving PDE's is novel.

5.1 Singularities of Legendrian Mappings

In order to systematically define and analyze *multivalued* solutions of the Boussinesq system, it is convenient to rewrite the incompressibility condition in the differential form

$$d\psi = \boldsymbol{u}^{\perp} \cdot d\boldsymbol{x} \tag{5.1}$$

in which $\boldsymbol{u}^{\perp} = (-v, u)$.

This representation of incompressibility has the following interpretation. If $\Gamma := (\boldsymbol{x}, \psi(\boldsymbol{x}))$ denotes the graph of ψ in $\mathbb{C}^3$, then the surface Γ must satisfy a differential constraint on its tangent planes; namely, that the one-form

$$\alpha = d\psi - \boldsymbol{u}^{\perp} \cdot d\boldsymbol{x} \tag{5.2}$$

must annihilate the tangent planes to the graph Γ; i.e.,

$$\alpha|_{T\Gamma} \equiv 0\,, \tag{5.3}$$

where $T\Gamma$ denotes the tangent bundle to Γ.

The usefulness of this representation is that a multivalued analytic stream function may be defined as an analytic surface

$$S := \{(\boldsymbol{x}, \psi) | f(\boldsymbol{x}, \psi) = 0\} \tag{5.4}$$

where f is an analytic function and, most importantly,

$$\alpha|_{TS} \equiv 0\,. \tag{5.5}$$

The sheets of this surface, under projection onto $\boldsymbol{x}$,

$$p : (\boldsymbol{x}, \psi) \to \boldsymbol{x}\ , \tag{5.6}$$

then give the *branches* of the *multi-valued* stream function ψ.

More intrinsically an odd dimensional complex manifold equipped with an analytic 1-form α satisfying the conditions

1. $d\alpha$ is closed, and

2. $d\alpha$ is nondegenerate as a two-form

is called a *contact manifold.* The 1-form is usually called a *contact form.* Any two-form, such as $d\alpha$, which satisfies the above two conditions is called a *symplectic form.* In our application the manifold is $\mathbb{C}^5$ and the form α is defined in (5.2).

In geometric optics there is a concrete realization of a contact manifold which provides some intuition about these structures and is now briefly reviewed. It consists of an n-dimensional *observation manifold*, Ω, parameterized by $\boldsymbol{x} = (x_1, \ldots x_n)$, on which the rays propagate. $\mathcal{B} = \partial\Omega$ is called the *source manifold*, parameterized by $\boldsymbol{s} = (s_1, \ldots s_{n-1})$. The rays emerge perpendicular to $\mathcal{B}$ pointing into Ω. The *wavefronts* are the level sets of the distance function where distance is measured from the boundary $\mathcal{B}$. More precisely, let $F(\boldsymbol{s}, \boldsymbol{x})$ denote the distance between a point $\boldsymbol{s}$ on $\mathcal{B}$ and $\boldsymbol{x}$ on Ω. Then the distance function is

$$\psi(\boldsymbol{x}) = min_{\boldsymbol{s} \in \mathcal{B}} F(\boldsymbol{s}, \boldsymbol{x}). \tag{5.7}$$

For small values of the distance, ψ will always be single-valued. On the other hand, for sufficiently large distances ψ will be multi-valued generically. Moreover, the wavefronts, which are the level curves of ψ, acquire "wavefront singularities" along the *caustics* of the rays. We will now reinterpret these familiar constructions from geometric optics in terms of a contact structure, which provides a mathematical definition of wavefront singularity.

To this end we introduce a *generating function*, $F(\boldsymbol{s}, \boldsymbol{x})$ on $\mathcal{B} \times \Omega$. In the geometric optics model, as mentioned above, F is the distance function. The contact manifold is $M = \Omega \times \mathbb{C}^n \times \mathbb{C}$ with coordinates $(\boldsymbol{x}, \boldsymbol{w}, \psi)$ and the contact form is $\alpha = d\psi - \boldsymbol{w} \cdot d\boldsymbol{x}$. Since $d\alpha = d\boldsymbol{x} \wedge d\boldsymbol{w}$, this two-form is certainly symplectic. The manifold M in this case is often referred to as the space of 1-jets, denoted $J^1(\Omega)$. Two scalar-valued functions F_1 and F_2 on Ω are said to have k^{th} order contact at $\boldsymbol{x} \in \Omega$ if the Taylor expansions of F_1 and F_2 at $\boldsymbol{x}$ agree up to order k. $J^k(\Omega)_{(\boldsymbol{x}, f)}$ denotes the equivalence class of functions F with $F(\boldsymbol{x}) = f$ under the equivalence relation of k^{th} order contact. In this way $J^k(\Omega)$ is naturally a bundle over $J^0(\Omega)$ ($= \Omega \times \mathbb{C}$). The bundle projection $p : J^1(\Omega) \to J^0(\Omega)$ amounts to forgetting the first derivative in the the first order Taylor polynomial. Thus p is just the map $p(\boldsymbol{x}, \boldsymbol{w}, \psi) = (\boldsymbol{x}, \psi)$.

An n-dimensional submanifold $\mathcal{L}$ of M is called *Legendrian* if $\alpha|_{T\mathcal{L}} \equiv 0$. The generating function $F(\boldsymbol{s}, \boldsymbol{x})$ *generates* a Legendrian submanifold of the form

$$\mathcal{L} = \{(\boldsymbol{x}, \boldsymbol{w}, \psi) \in M \mid \partial F/\partial \boldsymbol{s} = 0; \boldsymbol{w} = \partial F/\partial \boldsymbol{x}; \psi = F(\boldsymbol{s}, \boldsymbol{x})\}. \tag{5.8}$$

In the geometric optics context, $\mathcal{L}$ is the "manifold of phases" for the wavefronts; the first condition says that the rays are perpendicular to the boundary, $\mathcal{B}$, while the *phase vector* $\boldsymbol{w}$ is normal to the wavefronts. In this optical model $\alpha|_{T\mathcal{L}} = d\psi - \boldsymbol{w} \cdot d\boldsymbol{x} = dF - \partial F/\partial \boldsymbol{x} \cdot d\boldsymbol{x} \equiv 0$ by the chain rule. Hence $\mathcal{L}$ is manifestly Legendrian.

A *Legendrian projection* $P : M \to N$, where N is an (n+1)-dimensional manifold, is a submersion [1] whose fibers, $P^{-1}(\eta)$ for $\eta \in N$, are Legendrian submanifolds of M. In the geometric optics model, the projection map $p : J^1(\Omega) \to J^0(\Omega)$ described above and defined by $p(\boldsymbol{x}, \boldsymbol{w}, \psi) = (\boldsymbol{x}, \psi)$ is Legendrian because the fibers are defined by $\boldsymbol{x}$ = constant and ψ = constant so that α vanishes identically.

A *Legendrian mapping* is the restriction of a Legendrian projection to a Legendrian manifold $\mathcal{L}$. In geometric optics, Legendrian mappings arise as the projection of the manifold of phases onto the (possibly multi-valued) graph, Γ, of the stream function ψ in $\Omega \times \mathbb{C}$.

A *Legendrian singularity* of a Legendrian mapping occurs at places where the mapping fails to be an immersion[2]; i.e., at the critical points of the mapping. In geometric optics these are places where the graph of the stream function acquires a "cuspidal" singularity. Since the level sets of this graph are just the wavefronts, they inherit the singularities of the graph. Thus the mathematical definition of Legendrian singularity precisely encodes and generalizes the intuitive notion of a wavefront singularity.

We can now say what is meant by a Legendrian singularity "type" and furthermore what it means for a type to be "generic". First, define the equivalence of two Legendrian mappings. This is effectively summarized in the diagram of Figure 1 where M_1 and M_2 are two contact manifolds of the same dimension.

We say that two Legendrian mappings, $p_1|_{\mathcal{L}_1}$ and $p_2|_{\mathcal{L}_2}$, are equivalent if there exist diffeomorphisms H and h making the diagram of Figure Fig3.1 commute, with $\mathcal{L}_2 = H(\mathcal{L}_1)$, and preserving the respective contact structures (i.e. the pullback $H^*(\alpha_2) = \alpha_1$). This definition can be localized at a point in the usual way by passing to germs of Legendrian mappings and germs of diffeomorphisms [21].

A *Legendrian singularity type* is an equivalence class of germs of Legendrian mappings. The equivalence class of nonsingular germs is also a singularity type by this definition; in what follows, however, we will only be inter-

[1] A *submersion* is a mapping whose derivative at each point is surjective.

[2] An *immersion* is a mapping whose derivative at each point is injective.

$$\begin{tikzcd}
\mathcal{L}_1 \subset (M_1, \alpha_1) \arrow[r, "H"] \arrow[d, "p_1"] & (M_2, \alpha_2) \supset \mathcal{L}_2 \arrow[d, "p_2"'] \\
\Omega_1 \times \mathbb{C} \arrow[r, "h"] & \Omega_2 \times \mathbb{C}
\end{tikzcd}$$

Figure 1: Diagram for Equivalence of Legendrian Mappings

ested in singularity types whose representative germs are actually singular. Since these equivalence classes are preserved under coordinate changes that preserve the contact structure, a singularity type has an intrinsic geometric significance.

There is a natural topology on the space of Legendrian mappings which is induced from the C^∞ topology on the space of smooth mappings [21]. With respect to this topology we say that a germ of a Legendrian mapping $p|_{\mathcal{L}}$ is *stable* if any Legendrian map germ in a neighborhood of this map is equivalent to the germ of $p|_{\mathcal{L}}$. In other words the equivalence class of this germ is an open set in the space of Legendrian map germs. If this open set is dense as well, we say that the singularity type is *generic.*

Although it would appear that the classification problem for Legendrian singularities is quite complicated because of all the structures involved, there are ways to reduce it to a tractable singularity calculation. This is particularly true for the following example:

Consider the classification for $n = 1$, which serves as an illustration and is be used later. In this case, M is the jet space $J^1(\equiv \mathbb{C}^3)$ with coordinates (x, v, ψ), the contact form is $\alpha = d\psi - v dx$, and the Legendrian submanifolds are curves.

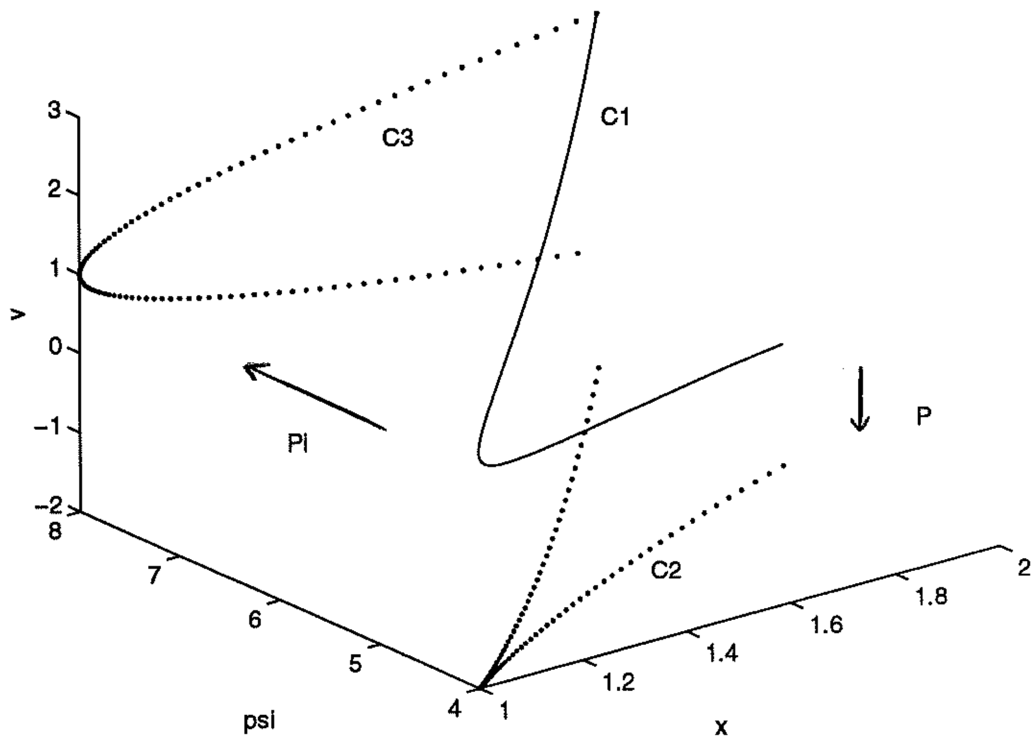

Figure 2: Generic Legendrian curve and its projection. The curves are labeled $C1$ for $\mathcal{L}$, $C2$ for $P(\mathcal{L})$ and $C3$ for $\pi(\mathcal{L})$.

This classification is tractable because a Legendrian curve, $\mathcal{L}$, is completely determined by its projection onto the (x, v) plane.

As in Figure 2, let $\pi : (x, v, \psi) \rightarrow (x, v)$ denote the projection onto the (x, v)-plane and let $P : (x, v, \psi) \rightarrow (x, \psi)$ be the Legendrian projection from J^1 to J^0 introduced earlier. Along the projected curve, $\pi(\mathcal{L})$, one may implicitly solve for $v = v(x)$. Since $\alpha = 0$ along the curve $\mathcal{L}$, then $d\psi = v(x)dx$, i.e. $v = \psi_x$, and

$$\psi(x) = \int^x v(\xi)d\xi. \tag{5.9}$$

Thus the curve $\mathcal{L}$ is parameterized by x as $(x, v(x), \psi(x))$.

Notice that if the mapping from $\pi(\mathcal{L})$ to the x-axis given by $(x, v(x)) \rightarrow x$ is nonsingular at a point, then the map $P : \mathcal{L} \rightarrow J^0$ is nonsingular at the corresponding point. This is a consequence of (5.9). Moreover, the singularities of the Legendrian mapping P from $\mathcal{L}$ to the (x, ψ) plane are in one-to-one correspondence with the singularities of the projection from $\pi(\mathcal{L})$ to the x-axis. But singularities of this latter projection are in 1:1 correspondence with critical points of the "inverse" function $x(v)$. We know from Morse theory [13] that the generic critical points of scalar functions are just the simple critical points; i.e., $dx/dv = 0$ but $d^2x/dv^2 \neq 0$. Any higher order critical point breaks up into several simple points under an arbitrarily small perturbation of $x(v)$. Moreover, it is straightforward to show that there is a local change of variables such that this function has the normal form $x(v) = v^2$.

Therefore generic singularities have the normal form $v(x) = x^{1/2}$ and, by (5.9), $\psi(x) = \frac{2}{3}x^{3/2}$. Thus the image of a Legendrian curve in the (x, ψ) plane in the vicinity of a generic singularity has the form of a simple *isolated cusp.*

5.2 Legendrian Blowup

The standard theory of Legendrian singularities is extended here to allow infinite velocities. One says that a Legendrian manifold $\mathcal{L}$ *blows up* if there is a sequence of points $(\boldsymbol{x}_m, \boldsymbol{w}_m, \psi_m) \in \mathcal{L}$ such that $\boldsymbol{w}_m \to \infty$ as $m \to \infty$, but such that the closure of $P(\mathcal{L})$ in $\Omega \times \mathbb{C}$ is a hypersurface, possibly singular, with a vertical (i.e. parallel to the ψ-axis) tangent plane over $\bar{\boldsymbol{x}} = \lim \boldsymbol{x}_m$. So the limit point exists in $\Omega \times \mathbb{C}$ although not in M. This is remedied by a directional compactification of M in the direction of the fibers (coordinatized by $\boldsymbol{w}$) so that the contact structure extends to this compactification. This compactification is again illustrated in the case when Ω is one-dimensional $(n = 1)$. As described in Section 3.1, a Legendrian manifold for $n = 1$ is a curve $\mathcal{L}$ with parameterization $x \to (x, v(x), \psi(x))$. A blowup is a point on this curve whose v-coordinate becomes infinite. The projection of this curve into J^0 (Figure 3a) has coordinates (x, ψ), and may be interpreted as the (possibly multi-valued) graph of ψ over the x-axis. The blowup corresponds to a point on this image curve which has a vertical tangent (i.e. parallel to the ψ-axis).

The directional compactification is achieved by projectivizing the fibers of the map $P : J^1 \to J^0$, replacing the coordinates on J^1 by $(x, [v_0 : v_1], \psi)$, in

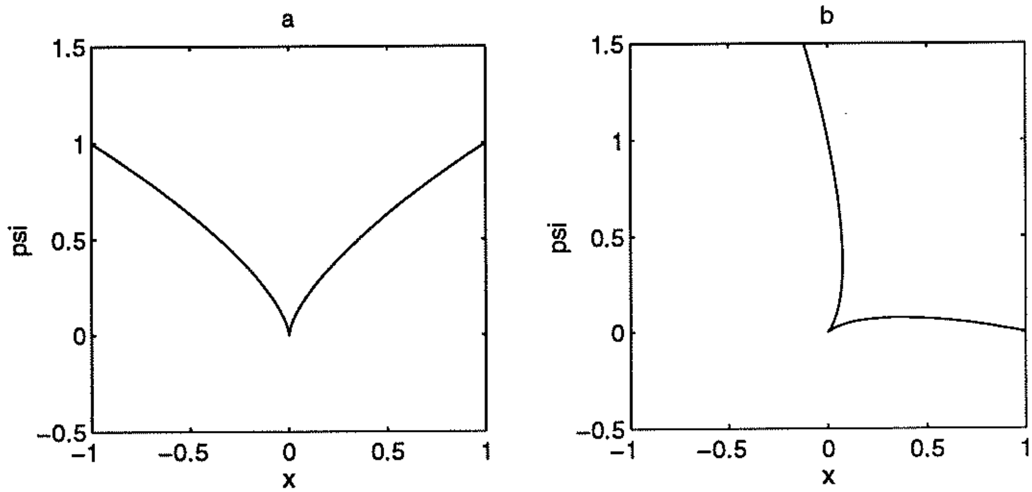

Figure 3: Generic blowup for Legendrian curve: symmetric (a) and asymmetric (b).

which $[v_0 : v_1]$ are projective coordinates. In this representation $[v_0 : v_1]$ and $[w_0 : w_1]$ denote the same point iff there is a nonzero constant λ such that $v_0 = \lambda w_0$ and $v_1 = \lambda w_1$, and the contact form is written as $\alpha = v_0 d\psi - v_1 dx$. The original J^1 consists of the points with $v_0 \neq 0$ so that $[v_0 : v_1] = [1 : v_1/v_0]$. The projectivized J^1, which we denote by $\tilde{J}^1$ contains the additional points $(\bar{x}, [0:1], \psi(\bar{x}))$ for each fiber, which correspond to places where the original velocity is infinite. This gives a one point compactification of each fiber and thus a directional compactification of J^1.

The contact structure remains nondegenerate under this extension. Set $w = v_0/v_1$ and $\tilde{\alpha} = dx - w d\psi$. Then $d\tilde{\alpha} = -dw \wedge d\psi$ which is still nondegenerate in a neighborhood of $(\bar{x}, [0 : 1], \psi(\bar{x}))$. Thus the curve remains Legendrian in $\tilde{J}^1$. Notice, however, that the roles of x and ψ have been interchanged and it is natural to regard x as the "stream function" in these coordinates.

The presence of a blowup point is generic since a small Legendrian perturbation of $\pi(\mathcal{L})$ cannot remove a vertical tangent. One might first expect that the generic blowup for $n = 1$ would correspond to the inverse function of the cusp, i.e. to $x = \psi^{3/2}$, which is graphed in Figure 3a. This blowup point coincides with a Legendrian singularity, however, which is not generic as shown next.

It is straightforward to check that the curve given by the parameterization $p \to (\psi = \frac{3}{2}p^2, w = p, x = p^3)$ is Legendrian with both a cuspidal singularity and a blowup at $p = 0$. Its normal form is $v(x) = x^{-1/3}$ $(v = 1/w)$ and $\psi(x) = \frac{3}{2}x^{2/3}$. Figure 3a depicts the graph of this ψ which is the projection of the Legendrian curve under the extended $P : \tilde{J}^1 \to J^0$. Notice that in this special example the location of the vertical tangent (Legendrian blowup) coincides with the cuspidal point (Legendrian singularity) at $p = x = 0$.

However, the perturbation $p \to (\psi = \frac{3}{2}p^2, w = p + \frac{2}{3}\epsilon, x = p^3 + \epsilon p^2)$ is also Legendrian with a Legendrian singularity at $p = x = 0$. The blow-up occurs where $w = 0$, i.e. at the point of vertical tangent, $p = -\frac{2}{3}\epsilon$ or $x = \frac{4}{27}\epsilon^3$. The projection $p \to x$ is given by $x = p^3 + \epsilon p^2$ whose derivative is $p(3p + 2\epsilon)$. Thus $w = p + \frac{2}{3}\epsilon = 0$ corresponds to a *simple* critical point of this projection and thus the normal form for the blowup is $\psi = x^{1/2}$ which is not a Legendrian singularity. The corresponding P-projection of the Legendrian curve is depicted in Figure 3b.

So generically a blowup does not coincide with a singularity. However, if there is an additional symmetry imposed on the system, a blowup may be forced to coincide with a Legendrian singularity.

We summarize the results of this section in the following:

Proposition 5.1 *The generic type of a codimension one Legendrian singularity is*

$$\psi(\boldsymbol{x}\ ) = x_1^{3/2}.$$

The generic type of a codimension one Legendrian blowup is

$$\psi(\boldsymbol{x}\ ) = x_1^{1/2}.$$

Proof: The normal form for generic codimension 1 Legendrian singularities was explained at the end of section 3.1. The normal form for a generic Legendrian blowup may be established along similar lines. First, suppose that the blowup has the form $\psi(\boldsymbol{x}\ ) = x_1^{1/2}$ or $x_1 = \psi^2$, which corresponds to a curve $\mathcal{L}$ in $\tilde{J}^1$ for which $\pi(\mathcal{L})$ has the form $w_1 = 2\psi$. In order to preserve blow-up we only consider perturbations of this curve which continue to pass through the origin $w_1 = \psi = 0$. Under small perturbations of this type, the curve continues to be locally linear to leading order; i.e. of the form

$w_1 = 2(1+\epsilon)\psi$. Thus, the corresponding curve in J^0 will be of the form $x_1 - c = (1+\epsilon)\psi^2$ (c is a constant of integration) which is of the same type as the original curve with a vertical tangent located at $x_1 = c$. Thus, this type of blow-up is stable. On the other hand, any more complicated normal form corresponds to a coincidence of a vertical tangent with a Legendrian singularity. By the first part of this proposition, we may assume that this singularity is equivalent to one of the form $\psi(x) = \frac{3}{2}x^{2/3}$, since otherwise a perturbation will break it up into singularities of this type. However, as was demonstrated in the paragraphs preceding the proposition, further perturbation splits the singularity from the blowup, leaving a blowup point with the stated normal form.

6 Numerical Method

This section presents a numerical method for construction of a particular class of solutions to the system

$$\begin{aligned} \psi_y\rho_x - \psi_x\rho_y &= 0 && (6.1)\\ \psi_y\zeta_x - \psi_x\zeta_y - \rho_x &= 0 && (6.2)\\ -\zeta + \Delta\psi &= 0 && (6.3) \end{aligned}$$

with boundary conditions given by

$$\begin{aligned} \psi(x,\pm 1) &= 0 && (6.4)\\ \psi(x,y) &= \psi(x+2\pi,y) && (6.5)\\ \rho(x,y) &= \rho(x+2\pi,y). && (6.6) \end{aligned}$$

Condition (6.4) states that on the boundary $y = \pm 1$ the normal component of the velocity is 0 and that the total flow in the x direction is 0. After the solution is computed, we then deduce singularity properties (i.e., type and location) via analysis of the Fourier spectrum in x.

6.1 Upper Analytic Solutions

We compute solutions to (6.1),(6.2) and (6.3) which are analytic for $Im(x) \geq 0$, and thus can be expanded in a Fourier series in x as

$$f(x,y) = \sum_{k \geq 0}^{\infty} \hat{f}_k(y) e^{ikx}. \tag{6.7}$$

The aim here is to numerically construct steady Boussinesq solutions having singularities of the type derived above. As we argue below, such singularities are observed in a wide range of numerical solutions, at least for the symmetric case. Thus, the computational results provide numerical affirmation for the genericity of these singularities, which has already been established analytically. They also show that there are no global constraints for Boussinesq solutions that are missing from our local singularity analysis.

Upper analyticity may seem to be an overly restrictive assumption, especially in light of our goal to investigate generic singularities. Note, however, that the upper analyticity of a function is a property of an infinite number of its Taylor coefficients. Singularity type, on the other hand, depends on only a finite number of Taylor coefficients; i.e. it is a more local property than is analyticity. Thus we expect that restriction to upper analyticity does not alter the generic singularity type, although we have no proof of this. This expectation has been partially verified since singularity types (ii) and (iii) have been found numerically, as described below. Type (i) has not yet been observed numerically.

Introducing upper analytic functions of the form (6.7) in (6.1),(6.2) and (6.3) leads to the following equation for the Fourier coefficients:

$$ik\hat{\rho}_k \frac{d}{dy}\psi_0 - ik\hat{\psi}_k \frac{d}{dy}\rho_0 = A_k \tag{6.8}$$

$$ik\hat{\zeta}_k \frac{d}{dy}\psi_0 - ik\hat{\psi}_k \frac{d}{dy}\zeta_0 - ik\hat{\rho}_k = B_k \tag{6.9}$$

$$-\hat{\zeta}_k + \frac{d^2}{dy^2}\hat{\psi}_k - k^2\hat{\psi}_k = 0 \tag{6.10}$$

or equivalently

$$\frac{d^2}{dy^2}\hat{\psi}_k(y) + (\frac{1}{\sigma^2}\rho_0' - k^2)\hat{\psi}_k(y) = C_k \tag{6.11}$$

$$
\begin{aligned}
-\hat{\rho}_k - \frac{i}{\sigma}\hat{\psi}_k\rho_0' &= A_k & (6.12)\\
\hat{\zeta}_k + \frac{i}{\sigma}\hat{\rho}_k &= B_k & (6.13)
\end{aligned}
$$

in which

$$
\begin{aligned}
A_k &= \sum_{l=1}^{k-1} i(k-l)\left[\hat{\psi}_{k-l}\frac{d}{dy}\hat{\rho}_l - \hat{\rho}_{k-l}\frac{d}{dy}\hat{\psi}_l\right] & (6.14)\\
B_k &= \sum_{l=1}^{k-1} i(k-l)\left[\hat{\psi}_{k-l}\frac{d}{dy}\hat{\zeta}_l - \hat{\zeta}_{k-l}\frac{d}{dy}\hat{\psi}_l\right] & (6.15)\\
C_k &= \frac{i}{\sigma^2 k}B_k - \frac{1}{\sigma k}A_k. & (6.16)
\end{aligned}
$$

It should be noted that the A_k and B_k depend on $\hat{\psi}_{k'}$,$\hat{\rho}_{k'}$ and $\hat{\zeta}_{k'}$ only for $k' < k$.

This system has several important features. First, the non-linear boundary value problem (6.1),(6.2) and (6.3) has been reduced to a series of linear two-point boundary value problems in y for the Fourier coefficients. Second, since the computation of mode k depends on only modes k' with $k' < k$, the system is lower triangular. Therefore no truncation error is introduced by the restriction to a finite set of wavenumbers. Finally, since the calculations are performed entirely in wave space, no aliasing error occurs.

The computation is started by specification of the zero mode $(\rho_0, \psi_0, \zeta_0)$ of the solution. Then the equation for the first mode $(\hat{\rho}_1, \hat{\psi}_1, \hat{\zeta}_1)$ is an eigenfunction problem. Once these two modes are determined, the remaining modes can be computed by solution of equations (6.8), (6.9) and (6.10). These equations are nonsingular as long as no resonance occurs, and none was found in any of the computations.

6.2 Determination of the Zero order Mode

In order to construct solutions to (6.8), (6.9) and (6.10) we need to specify the zero order modes of the Fourier expansion. We select these coefficients in the form

$$
\begin{aligned}
\rho_0 &= \rho_0(y) & (6.17)\\
\psi_0 &= -i\sigma y & (6.18)\\
\zeta_0 &= 0. & (6.19)
\end{aligned}
$$

Equation (6.17) means that the density is stratified in y. Equation (6.18) gives a constant velocity in the x direction.

The equation for $\hat{\psi}_1$ is

$$\begin{aligned} \frac{d^2}{dy^2}\hat{\psi}_1 + (\frac{1}{\sigma^2}\rho_0' - 1)\hat{\psi}_1 &= 0 && (6.20) \\ \hat{\psi}_1(\pm 1) &= 0. && (6.21) \end{aligned}$$

We can solve (6.20) either by specifying ρ_0 and solving the eigen-value problem or by specifying $\hat{\psi}_1$ and solving for ρ_0. The latter is in general much simpler and is the approach that we took.

The specification of $\hat{\psi}_1$ is performed in several steps: First, we determine a preliminary choice $\hat{\psi}_1 = \hat{\psi}_1^0$ for the extreme case in which the density ρ_0 is a Heaviside function. Then, this extreme mode, which is discontinuous, is smoothed out to obtain the usable mode $\hat{\psi}_1$. The smoothing is performed using a function ϕ defined below. Finally, the choice of ϕ, and thus of $\hat{\psi}_1$, is made in two ways. The first, described in Section 6.2.1, is symmetric with respect to y and the second, in Section 6.2.2, is asymmetric.

If the density ρ_0 is a Heaviside function $H(y)$, then equations (6.20) and (6.21) for $\hat{\psi}_1 = \hat{\psi}_1^0$ become

$$\begin{aligned} \frac{d^2}{dy^2}\hat{\psi}_1^0 + (\frac{\delta(y)}{\sigma^2} - 1)\hat{\psi}_1^0 &= 0 && (6.22) \\ \hat{\psi}_1^0(\pm 1) &= 0 \end{aligned}$$

in which $\delta(y) = \frac{dH}{dy}$. Solutions to (6.22) are given by

$$\begin{aligned} \hat{\psi}_1^0(y) &= C\begin{cases} \sinh(y-1) & 0 \le y \le 1 \\ -\sinh(y+1) & -1 \le y \le 1 \end{cases} && (6.23) \\ \sigma &= \frac{\tanh(1)}{2}. \end{aligned}$$

To obtain smooth data, we construct a smooth transition function $\phi = \phi(y)$ such that

$$\begin{aligned} \phi(1) &= 1 \\ \phi(-1) &= -1 && (6.24) \end{aligned}$$

and smooth out $\hat{\psi}_1^0$ as

$$\hat{\psi}_1 = (\frac{\phi+1}{2}) \sinh(y-1) + (\frac{\phi-1}{2}) \sinh(y+1). \tag{6.25}$$

In terms of ϕ, and thus of $\hat{\psi}_1$, we can then solve for ρ_0 from

$$\frac{d}{dy}\rho_0 = \sigma^2(1 - \frac{\hat{\psi}_1''}{\hat{\psi}_1}). \tag{6.26}$$

The conditions

$$\phi'(\pm 1) = \phi''(\pm 1) = 0 \tag{6.27}$$

must be satisfied in order that the limits at ± 1 in (6.26) make sense.

6.2.1 Symmetric

For the symmetric case, we specify a ϕ which maintains the symmetry of the solutions (6.23). We shall use the choice

$$\phi(y) = c_1 \tanh(\frac{y}{\lambda}) + c_2 y + c_3 y^3. \tag{6.28}$$

Choose the c_i so that ϕ satisfies (6.24), (6.27); i.e.

$$c_1 = \frac{3}{d} \tag{6.29}$$

$$c_2 = \frac{3}{2d}(f''(1) - 2f'(1)) \tag{6.30}$$

$$c_3 = \frac{-1}{2d} f''(1) \tag{6.31}$$

in which $f(y) = \tanh(\frac{y}{\lambda})$ and $d = 3f(1) - 3f'(1) + f''(1)$.

The resulting solution has the symmetry

$$\psi(-x, -y) = \psi(x, y) \tag{6.32}$$

which implies that $\psi(-x) = \psi(x)$ for $y = 0$. This is the symmetry condition employed in Theorem 3.2.

6.2.2 Asymmetric

The discontinuous solution (6.23) is symmetric in y. This symmetry may be broken by choosing an asymmetric transition function ϕ of the form

$$\phi(y) = c_5 \tanh(\delta(\frac{y}{\lambda})^2 + (\frac{y}{\lambda})^3) - (c_0 + c_1 y + c_2 y^2 + c_3 y^3 + c_4 y^4), \quad (6.33)$$

in which

$$\begin{aligned} c_0 &= 1 + c_5 f(-1) \\ c_1 &= c_5 f'(-1) \\ c_2 &= \frac{1}{2} c_5 f''(-1) \\ c_3 &= -1 + \frac{c_5}{4}\left[(2f - df')(1) - (2f + 3f' + 2f'')(-1)\right] \\ c_4 &= \frac{1}{48}(f''(1) - f''(-1))c_5 - \frac{1}{4}c_3 \\ c_5 &= 6\left[(3(f - f') + f'')(1) - (3(f + f') + f'')(-1)\right]^{-1} \end{aligned} \quad (6.34)$$

in which $f(y) = \tanh(\delta(\frac{y}{\lambda})^2 + (\frac{y}{\lambda})^3)$. This choice of the c_i's insures that the conditions (6.24), (6.27) on ϕ and its derivatives are satisfied. The parameters δ and λ control the amount of asymmetry and the sharpness of the approximation to the Heaviside function, respectively.

6.3 Description of Numerical Method

The two-point boundary value problems are solved using a centered fourth-order finite difference scheme. The boundary conditions are handled by using an unbalanced fourth order scheme at the endpoints. The resulting system was solved by direct Gaussian elimination at the ends, followed by application of the banded solver DGBSL from SLATEC.

Round-off error grows with the wave number, and could eventually destroy the calculation. Although the mechanism which causes this growth is not fully understood, it is controlled by the use of high precision calculations. We utilized MPFUN, a multi-precision package developed by David Bailey at NASA. Precision levels of 128 and 200 digits were used in the calculations. The computations were run on a Sun SparcServer 1000.

6.4 Numerical Detection of Singularities

The presence and type of singularities was deduced through a numerical analysis of the Fourier coefficients of the computed solution.

6.4.1 Analysis of Fourier Coefficients at Singularity

Consider a function $f : \mathbb{C} \to \mathbb{C}$ that can be represented in a neighborhood of x_* as

$$f(x) = c(x - x_*)^{-(1+a)} \sum_{p=0}^{\infty} a_p (x - x_*)^p. \tag{6.35}$$

The summation is assumed to represent a function that is analytic in some neighborhood of x_*. Under these assumptions, the Fourier coefficients of f are asymptotic to

$$\hat{f}_k \sim bk^{-(\nu+1)} e^{ikx_* + i\frac{\pi}{2}(1+a)} \;\; ; \;\; k \to \infty. \tag{6.36}$$

Details of this result can be found in [7].

Writing out the real and imaginary parts of a, b and x_* as

$$\begin{aligned} a &= a_1 + ia_2 \\ b &= b_1 \exp(ib_2) \\ x_* &= r_2 + ir_1, \end{aligned} \tag{6.37}$$

(6.36)becomes

$$\hat{f}_k \sim b_1 k^{a_1} e^{kr_1} \exp(ikr_2 + ia_2 \log k + ib_2) \;\; ; \;\; k \to \infty \tag{6.38}$$

In the procedure outlined in Section 6.2, $\hat{\psi}_1$ is chosen to be real, which leads to

$$\hat{\psi}_k(y) = i^{k-1} \hat{g}_k(y), \tag{6.39}$$

in which the $\hat{g}_k(y)$ are real. This form of the Fourier coefficients corresponds to the following symmetry in x:

$$\psi(\frac{\pi}{2} + x) = -\psi^*(\frac{\pi}{2} - x^*) \tag{6.40}$$

in which $*$ denotes complex conjugation. This symmetry causes the singularities to occur in pairs. The asymptotics of the Fourier coefficients of ψ are then

$$\hat{\psi}_k \sim b_1 k^{a_1} e^{kr_1} \sin(kr_2 + a_2 \log k + b_2) \ ; \quad k \to \infty. \tag{6.41}$$

The Fourier coefficients for ρ and ζ have a similar form.

6.4.2 Sliding Parameter Fit

To obtain the values of the parameters in (6.41) we perform a sliding six-parameter fit of the Fourier coefficients. Each parameter fit is performed for every fixed value of y as follows:

At each k, we exactly fit the form in (6.41) to the values of the data at $k, k+1, ..., k+5$, using the SLATEC package DNSQE. A fit is deemed successful if the values of the parameters are (approximately) independent of k.

7 Computational Results

The computations were performed for 3200 and 6400 points in y. The precision used was 128 and 200 digits respectively. The parameter λ, which controls the sharpness of the approximation to the Heaviside function, was equal to $\frac{1}{3}$ in both the symmetric and asymmetric cases.

7.1 Symmetric

The computational results and the singularity fit are very precise and clean in the symmetric case. Figure 4 shows contour plots of the real parts of the complex solutions on ψ, ω and ρ for real values of x and y, and a plot of the profile of ρ_0. For each value of y, there is a singularity at some complex value of x. In this solution the closest singularities are at distance $\frac{1}{2}$ from the real axis, and the real parts of the singularity positions x, y are approximately at the centers of the rolls shown in Figure 4. Note, however, that the solution can be given an arbitrary shift in x, so that this could also be a plot for x having imaginary part of $\frac{1}{2}$ of a solution with singularities located on the real axis.

Figure 5 shows plots of the parameter fits for the stream function with discretizations of 3200 and 6400 points. The parameter a_1 is of main interest because it determines singularity type. Figures (6),(7) show plots of the a_1 parameter fits for the density and vorticity with discretizations of 3200 and 6400 points. The results are as follows:

Stream Function

$$a_1 = -\tfrac{5}{3}; \quad \psi(x) \sim x^{\frac{2}{3}}$$

Vorticity

$$a_1 = \tfrac{1}{3}; \quad \zeta(x) \sim x^{-\frac{4}{3}}$$

Density

$$a_1 = -\tfrac{1}{3}; \quad \rho(x) \sim x^{-\frac{2}{3}}.$$

This is a singularity of type (iii). Note that the vorticity ζ, the density ρ and the velocity $v = \psi_x$ all blow up at the singularity, and that $\rho = O(v^2)$.

While these results were for $\lambda = 1/3$, other values of λ gave similar results with the same values of a_1, i.e. with the same singularity type. This is numerical evidence for the genericity of singularity type (iii) in the symmetric case.

7.2 Asymmetric

The numerical detection of singularity properties in the asymmetric case is not as straightforward as in the symmetric case. Although the full reasons for this difficulty are not very clear, we believe that it is partly because singularities come in pairs that are very close together, for small asymmetry parameter δ, since they come from the perturbation of a symmetric singularity, as in Figure 3. For such nearby singularities, it is difficult to separate the different exponential rates of decay, due to slightly different imaginary components r_1. In addition, small differences in the real part r_2 may result in "beating" of the oscillatory part of (6.38). Since we do not have direct control over the location of the singularities, a number of computational trials were performed with different values of δ in order to obtain one with a clean fit to the singularity parameters. The results presented here are for $\delta = -3.25$.

Figure 8 shows contour plots of the real parts of the complex solutions ψ, ω, ρ for real values of x, y, and a plot of the profile of ρ_0. As in the symmetric case, the singularities are at distance $\frac{1}{2}$ from the real axis. Figure 9 shows plots of the parameter fits for the stream function with discretizations of 3200 and 6400 points. Figures 6 and 7) show plots of the a_1 parameter fits for the density and vorticity with discretizations of 3200 and 6400 points. The results can be outlined as follows:

Stream Function

$$a_1 = -\tfrac{3}{2}; \quad \psi(x) \sim x^{\frac{1}{2}}$$

Vorticity

$$a_1 = \tfrac{1}{2}; \quad \zeta(x) \sim x^{-\frac{3}{2}}$$

Density

$$a_1 = 0; \quad \rho(x) \sim x^{-1}.$$

This is a singularity of type (ii), and is evidence for the genericity of this singularity type. We emphasize again, however, that good fits to the singularity parameters were not obtained for general values of λ and δ.

8 Conclusions

By a combination of analysis and numerical calculation, we have derived and verified the generic types of singularities of codimension 1 for the steady Boussinesq equations of two-dimensional, stratified flow. The singularity types are

(i) $\psi \approx x^{3/2}, u \approx x^{1/2}, \zeta \approx x^{-1/2}$
(ii) $\psi \approx x^{1/2}, u \approx x^{-1/2}, \zeta \approx x^{-3/2}$.

If the solution has the symmetry $\psi(-x, 0) = \psi(x, 0)$, the generic type is

(iii) $\psi \approx x^{2/3}, u \approx x^{-1/3}, \zeta \approx x^{-4/3}$.

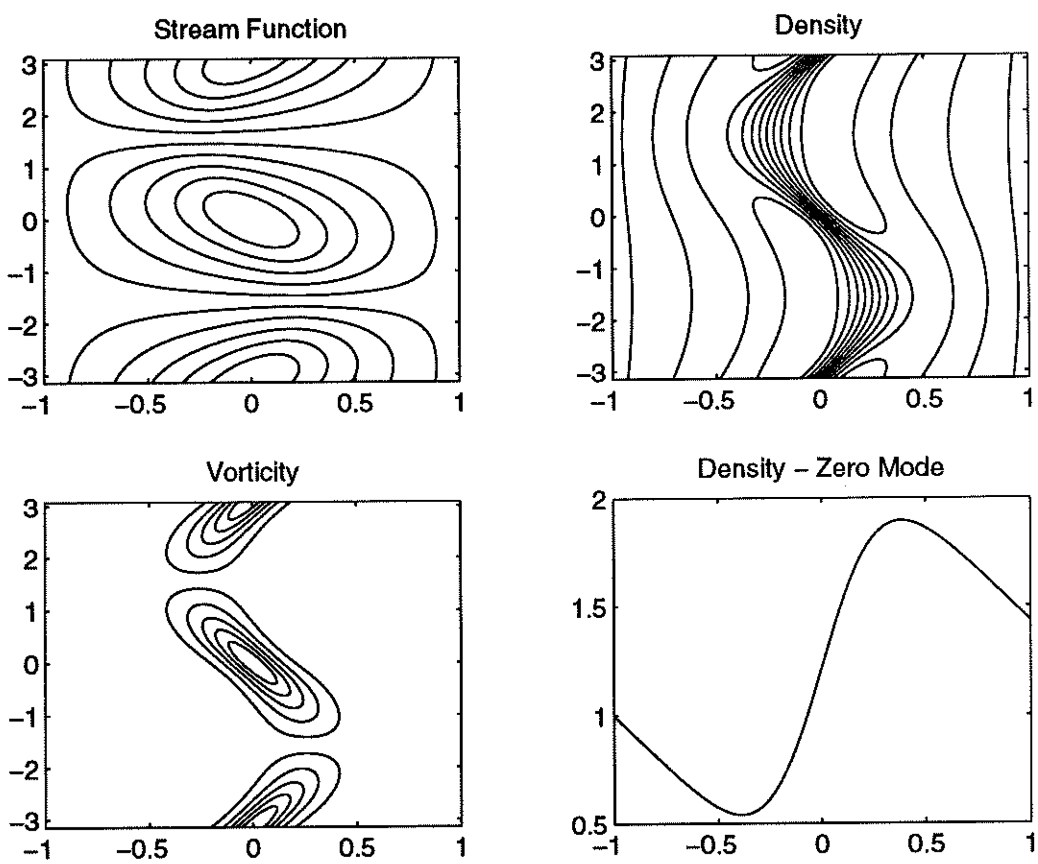

Figure 4: Symmetric Case: Real Part of Solutions.

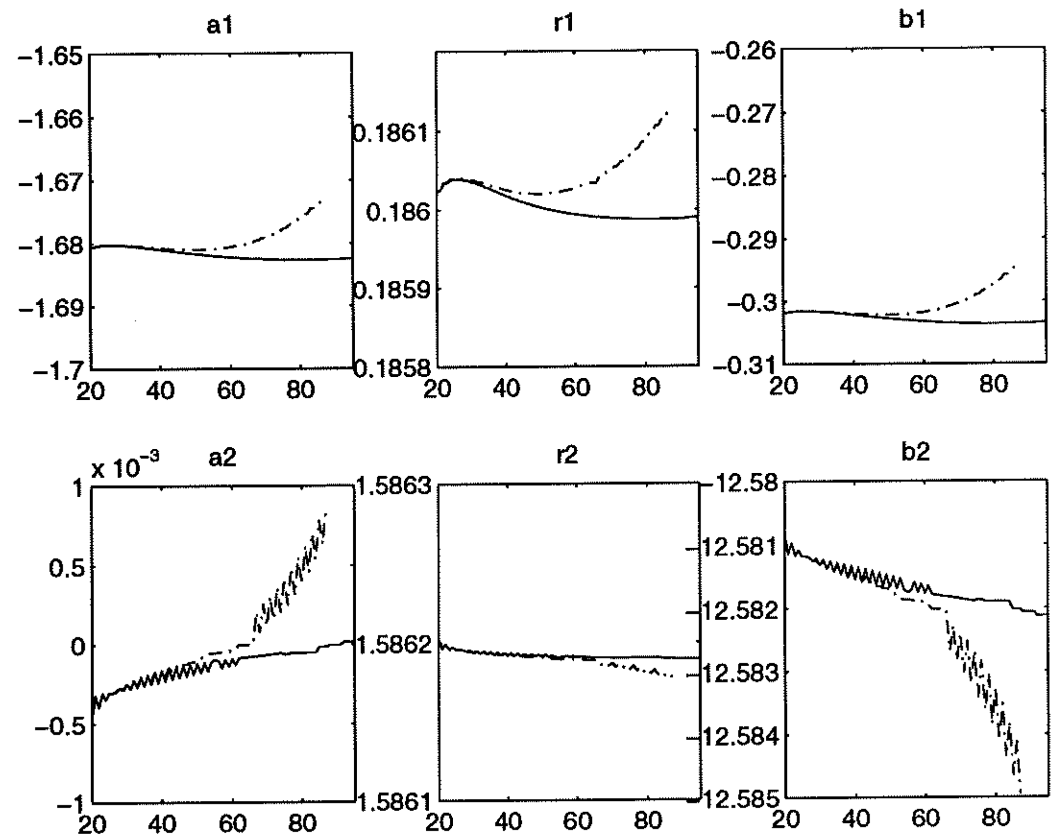

Figure 5: Parameters vs. k for stream function in the symmetric case with 3200 and 6400 points. Note that the scale is very fine for each of the plots, so that each of the parameters is nearly constant.

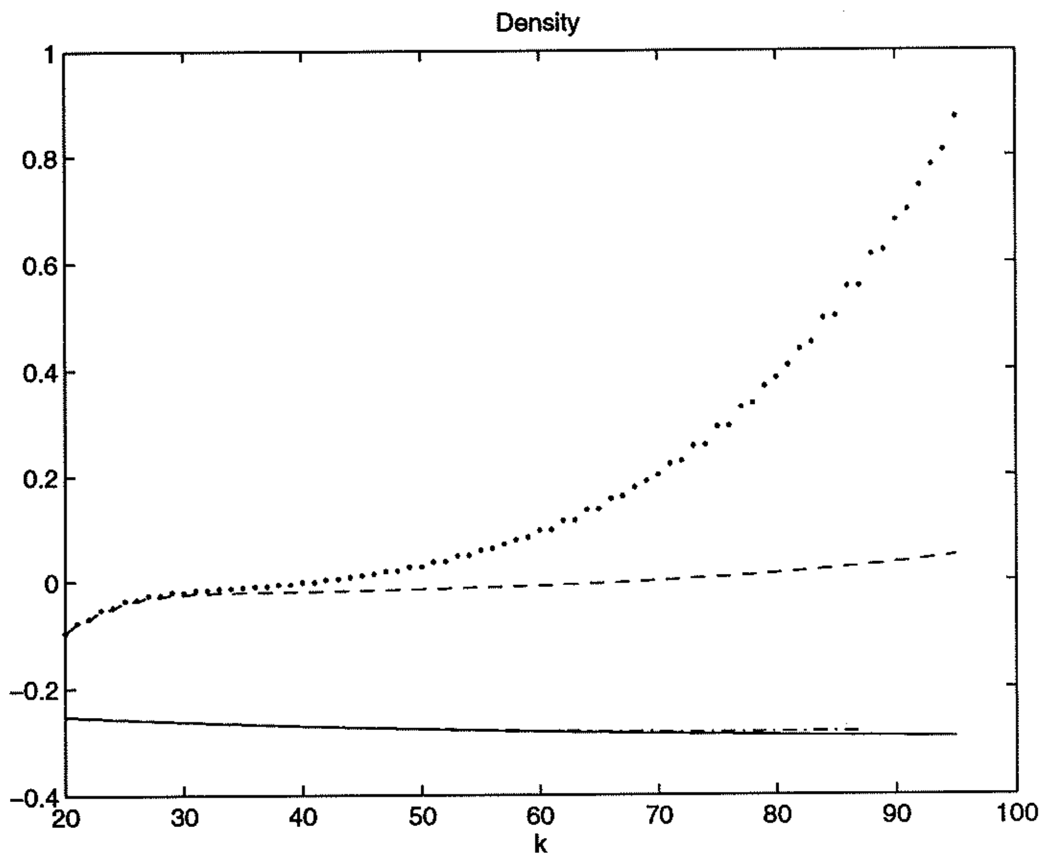

Figure 6: a_1 vs. k for ρ. Symmetric: –· 3200 points, – 6400 points. Asymmetric case: · 3200 points, –– 6400 points.

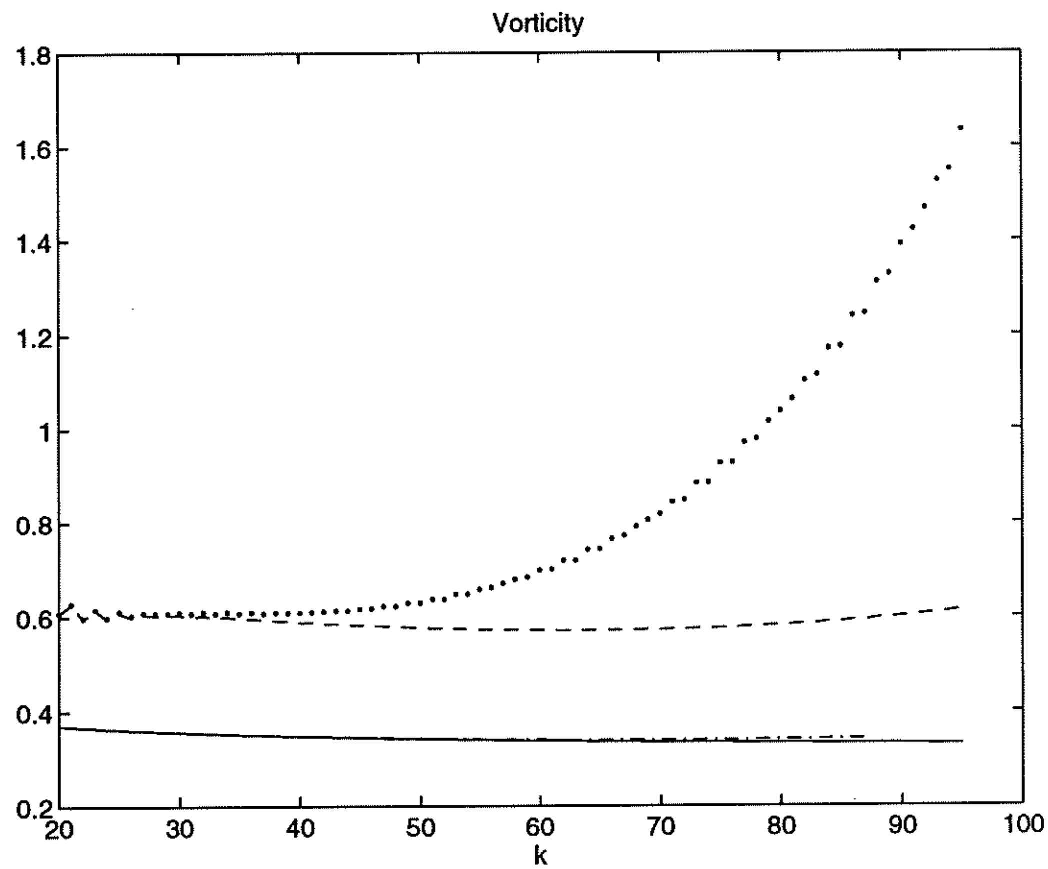

Figure 7: a_1 vs. k for ζ. Symmetric: $-\cdot$ 3200 points, $-$ 6400 points. Asymmetric: $\cdot$ 3200 points, $--$ 6400 points.

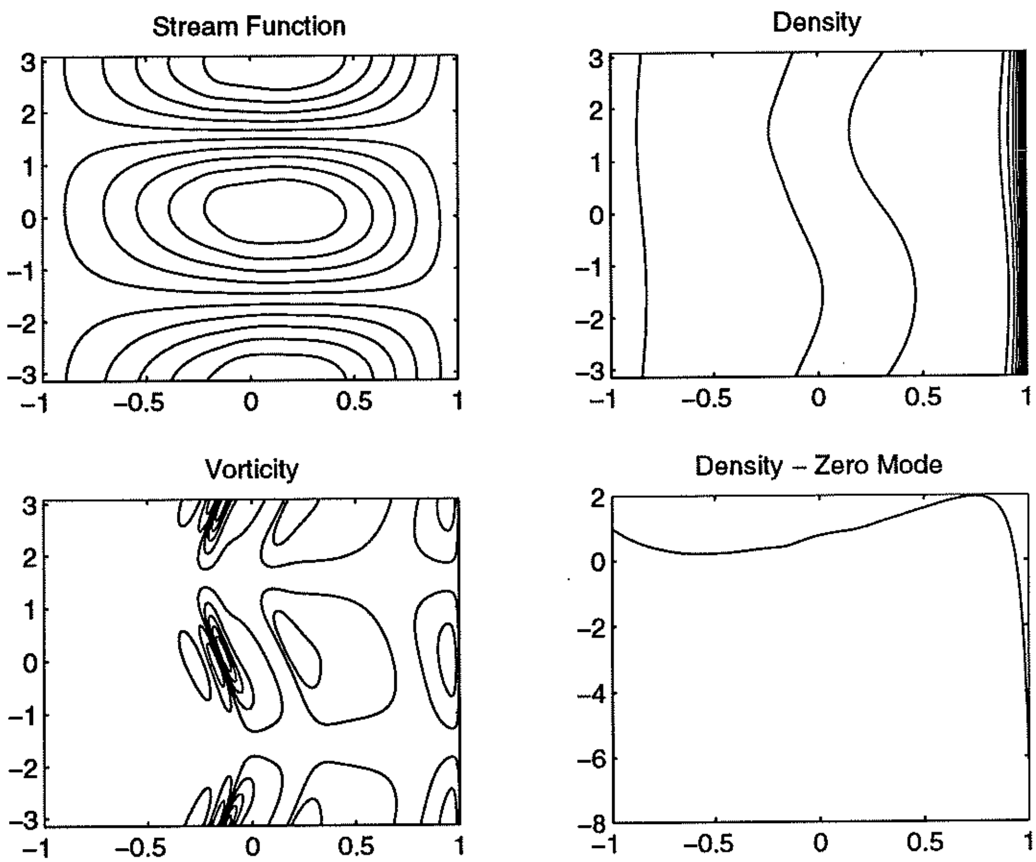

Figure 8: Asymmetric Case: Real Part of Solutions.

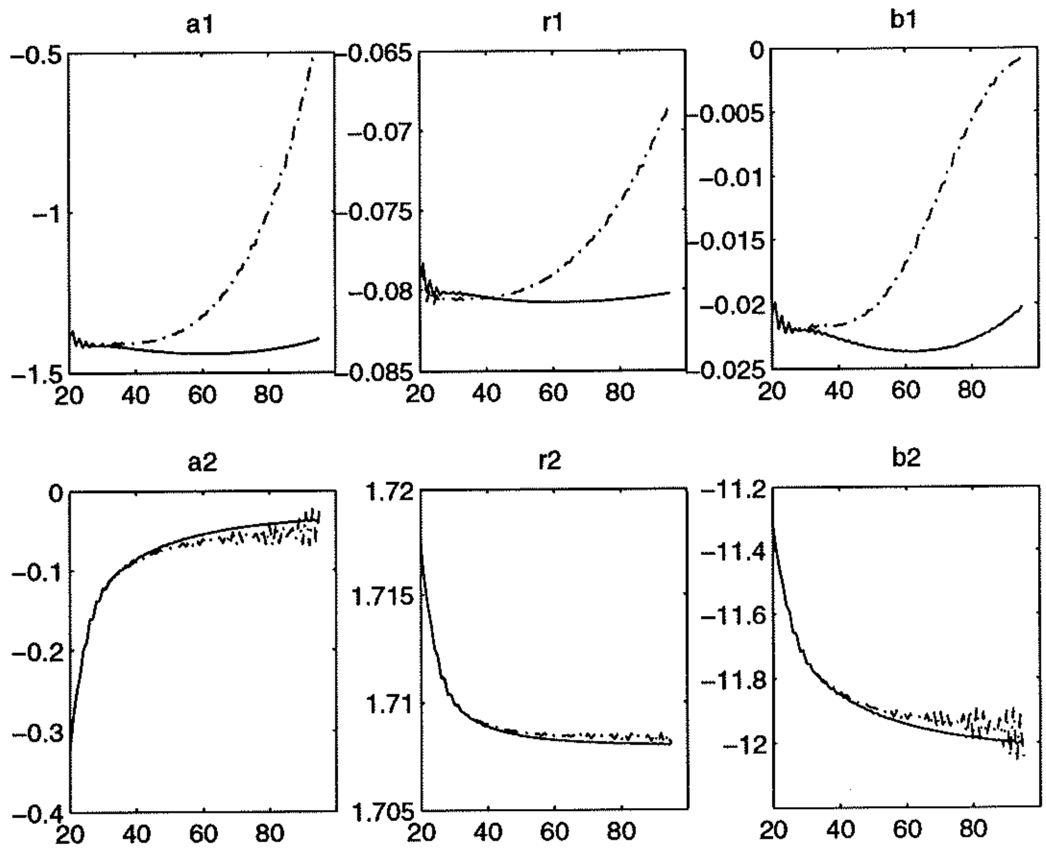

Figure 9: Parameters vs. k for stream function in the asymmetric case with 3200 and 6400 points.

In the latter two cases, the velocity v blows up, and also the density ρ blows up like v^2. Extension of these results to the Euler equations for axi-symmetric flow with swirl is straightforward.

The significance of these results is, first, that they demonstrate the effectiveness of the geometric approach for analyzing singularities of PDEs. In particular, we have succeeded here in analyzing problems in which the dependent variables have blowup.

Second, we expect that these results can be a guide in the search for time dependent singularities for the Boussinesq and Euler equations. In particular, we expect that such singularities may propagate in the complex space plane ($(x, y) \in \mathbb{C}^2$) and that a complex conjugate pair may hit the real plane at some finite time. Moreover, the generic singularity types for the steady Boussinesq system are also generic among traveling wave solutions, as shown in Appendix A. Therefore, as long as the singularities remain away from the real plane, we conjecture that the form of the singularity may be the same (or very similar to) the generic steady singularities described here. When the collision of singularities occurs on the real plane, the singularity type will necessarily change, but still the generic types in the complex plane will influence the possible types of singularities at the collision.

Finally, it is important to point out that the question of singularity formation in finite time is somewhat artificial, in that nearly singular behavior is indistinguishable physically and numerically from an actual singularity for many applications. Therefore this theory will be equally effective if it can be used in the description of singularities that are not real, but lie in the complex plane close to the real plane.

9 Appendix A. - Traveling Waves

Consider solutions to the time dependent problem

$$\rho_t + \psi_y \rho_x - \psi_x \rho_y = 0 \tag{9.1}$$

$$\zeta_t + \psi_y \zeta_x - \psi_x \zeta_y + \rho_x = 0 \tag{9.2}$$

$$\zeta + \Delta \psi = 0 \tag{9.3}$$

which are traveling waves in x with constant speed $i\sigma$. This corresponds to solutions in which information (i.e. a singularity) propagates in the complex plane at speed σ in the imaginary direction. It follows that any singularities in the complex plane will reach the real axis in finite time. A second interpretation of this ansatz is that the Fourier expansion in x consists of purely growing modes.

Combining the traveling wave ansatz with upper analyticity gives solutions of the following form.

$$f(x,y,t) = f_0(y) + \sum_{k>0}^{\infty} \hat{f}_k(y) e^{ikx+\sigma t}. \tag{9.4}$$

In this context, the zero order mode is a steady state, the $k = 1$ term is an unstable linear mode added on to this steady state, and the remaining terms are added to make the sum a nonlinear solution. Physically, we can take a steady state in which the density varies smoothly in y and in which there is no fluid motion. This type of steady state is described here by

$$\begin{aligned} \rho_0 &= \rho_0(y) && (9.5) \\ \psi_0 &= 0 && (9.6) \\ \zeta_0 &= 0. && (9.7) \end{aligned}$$

The steady solutions can be recovered from the traveling wave solutions via Galilean invariance; i.e.

$$\begin{aligned} \psi_{steady} &= \psi_{traveling} - i\sigma y && (9.8) \\ \zeta_{steady} &= \zeta_{traveling} && (9.9) \\ \rho_{steady} &= \rho_{traveling}. && (9.10) \end{aligned}$$

The zero order modes will be adjusted accordingly to give the form (6.18). Consideration of the traveling wave ansatz was the starting point of the numerical investigation of this problem.

This shows that the class of traveling wave solutions is equivalent to the class of steady solutions for the Boussinesq equations.

References

[1] Claude Bardos and S. Benachour. Domaine d'analycite des solutions de l'equation d'Euler dans un ouvert de R^n. *Annali della Scuola Normale Superiore di Pisa*, IV,4:647–687, 1977.

[2] J.T. Beale, T. Kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Communications on Mathematical Physics*, 94:61–66, 1984.

[3] R.E. Caflisch. Singularity formation for complex solutions of the 3D incompressible Euler equations. *Physica D*, 67:1–18, 1993.

[4] R.E. Caflisch and N. Ercolani. Unfolding partial differential equations and their singularities.

[5] R.E. Caflisch, N. Ercolani, T.Y. Hou, and Y. Landis. Multi-valued solutions and branch point singularities for nonlinear hyperbolic or elliptic systems. *Communications on Pure and Applied Mathematics*, 46:453–499, 1993.

[6] R.E. Caflisch, Isaac Klapper, and Gregory Steele. Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD. 1995.

[7] George F. Carrier, Max Krook, and Carl E. Pearson. *Functions of a Complex variable; theory and technique.* McGraw-Hill, 1966.

[8] S. Chern. *Complex Manifolds without Potential Theory, ed. 2.* Springer-Verlag, New York, 1979.

[9] S. Childress, G.R. Ierley, E.A. Spiegel, and W.R. Young. Blow-up of unsteady two-dimensional Euler and Navier-Stokes solutions having stagnation point form. *Journal of Fluid Mechanics*, 203:1–22, 1989.

[10] Peter Constantin, Weinan E, and Edriss S. Titi. Onsager's conjecture on energy conservation for solutions of Euler's equation. *Communications on Mathematical Physics*, 165:207–209, 1994.

[11] Weinan E and C.W. Shu. Numerical study of the small scale structures in Boussinesq convection. *Physics of Fluids A*, 6:49–58, 1994.

[12] Gregory Eyink. Energy dissipation without viscosity in ideal hydrodynamics. 1. Fourier analysis and local energy transfer. *Physica D*, 78:222–240, 1994.

[13] J. Milnor. *Morse Theory.* Princeton University Press, 1969.

[14] L. Nirenberg. An abstract form of the nonlinear Cauchy-Kowalewski theorem. *J. Diff. Geom.*, 6:561–576, 1972.

[15] L. Onsager. Statistical hydrodynamics. *Nuovo Cimento (Supplemento)*, 6:279, 1949.

[16] Alain Pumir and Eric D. Siggia. Development of singular solutions to the axisymmetric Euler equations. *Physics of Fluids*, A4:1472–1491, 1992.

[17] P. Griffiths R.Bryant and L. Hsu. Hyperbolic exterior differential systems and their conservation laws. *to appear in Mathematica Selecta*, 1994.

[18] P. Griffiths R.Bryant and L. Hsu. Toward a geometry of differential equations. *to appear in the Proceedings of the 1993 JDG Geometry and Topology Conference (dedicated to Raoul Bott)*, 1994.

[19] M.V. Safonov. The abstract Cauchy-Kovalevskaya theorem in a weighted Banach space. *Communications on Pure and Applied Mathematics*, to appear, 1995.

[20] John Stilwell. *Classical Topology and Combinatorial Group Theory.* Springer-Verlag, New York, 1993.

[21] S.M. Gusein-Zade V.I. Arnold and A.N. Varchenko. *Singularities of Differentiable Mappings Volume 1.* Birkhäuser, 1985.