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Tony F. Chan
Susie Go
Jun Zou

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Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

MULTILEVEL DOMAIN DECOMPOSITION AND MULTIGRID METHODS FOR UNSTRUCTURED MESHES: ALGORITHMS AND THEORY *

TONY F. CHAN †, SUSIE GO † AND JUN ZOU ‡

Abstract. We will summarize some of our recent theoretical and numerical results on domain decomposition and multigrid methods for second order elliptic problems on unstructured meshes. We first present a general framework for convergence analyses applicable to unstructured meshes, which can be viewed as a natural extension of the one formulated by Xu [23] for structured meshes. Then this framework is applied to two level and multilevel Schwarz methods for elliptic problems on unstructured meshes. As we allow general coarse grids whose boundaries may be non-matching to the boundary of the fine grid, special treatments are needed to implement different types of boundary conditions. We will propose a couple of such treatments. Finally, numerical results for domain decomposition and multigrid methods on unstructured meshes are presented to show similar convergence properties as we expect for standard structured meshes.

1. Introduction.

Recently, unstructured finite element meshes have become very popular in scientific computing, cf. Barth [3] and Mavriplis [19], primarily because of their flexibility in adapting to complicated geometries and the resolution of fine scale structures in the solution. Since no natural coarser grids exist as in structured meshes, practical multilevel domain decomposition and multigrid algorithms must allow coarser grids which are non-quasi-uniform and with boundaries and interior elements which are not necessarily matching to that of the fine mesh. Therefore, the traditional solvers have to be modified so that their efficiency will not be adversely affected by the lack of structure.

In this paper, we first propose a general framework for convergence analyses applicable to unstructured meshes, which can be viewed as a natural extension of the one formulated by Xu [23] for structured meshes. Then this framework is applied to two level and multilevel Schwarz methods for elliptic problems on unstructured meshes. Very general meshes and subdomains are allowed: neither the fine mesh nor the coarse mesh need to be quasi-uniform, the subdomains can be of arbitrary shapes and sizes, and the coarse mesh need not be nested to, or cover the same physical domain as the fine mesh. Some existing related works on unstructured meshes can be found in Chan, Smith and Zou[8, 7, 10, 9], Cai [5], Bramble-Pasciak-Xu [4], Bramble-Pasciak-Xu [4], Douglas-Douglas [13], Bank-Xu [1, 2].

We subsequently describe how to create a coarse grid hierarchy by successive coarsening of

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† Dept. of Math., Univ. of Calif. at Los Angeles, Los Angeles, CA 90024-1555 and Dept. of Math. E-mail: chan@math.ucla.edu, sgo@math.ucla.edu.

‡ Dept. of Math., Univ. of Calif. at Los Angeles, CA 90024-1555 and Computing Center, the Chinese Academy of Sciences, Beijing 100080, P. R. China. E-mail: zou@math.ucla.edu.

a fine grid using a maximum independent set approach. As we allow completely non-matching coarse grids to fine grids, special treatments are needed for different types of boundary conditions. We will propose a couple of such treatments to ensure that a proper sequence of coarse subspaces exists for the domain decomposition or multigrid methods. Then we discuss how to implement interpolation operators from coarse grids to finest grids.

Finally, numerical experiments on domain decomposition and multigrid methods on unstructured meshes will be presented to demonstrate similar convergence properties as we expect for standard structured meshes.

The paper is arranged as follows: In Section 2, we formulate the general framework for the convergence analysis for the additive type domain decomposition methods, and a perturbation extension of the result will be presented afterwards for the application to non self-adjoint parabolic problems. In Section 3 we introduce the fine and coarse finite element spaces, domain decompositions, the L^2-optimal approximation and H^1-stability of the standard finite element interpolant and the Clément interpolant. Section 4 will be devoted to the application of the abstract convergence theory developed in Section 2 to second order elliptic problems. Section 5 discusses how to generate a coarse grid sequence and to implement boundary conditions for Neumann boundary part, and Section 6 shows some numerical experiments using domain decomposition and multigrid methods. The results in Sections 2-4 are a summary of some of the main results in Chan-Zou [10].

2. Convergence theory for additive preconditioners. Let V, and V^k, $0 \le k \le p$ be finite dimensional vector spaces with inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_k$, resp. The spaces V^k are not necessarily subspaces of V. The space V^0 is special, usually referring to the coarse grid space.

Given a symmetric positive definite (SPD) operator A on V and $f \in V$, we are interested in solving the equation $Au = f$ on V, which arises from the discretization of elliptic or parabolic problems by using finite element methods. As A is ill-conditioned, our goal is to find a good preconditioner M for A such that MA is better conditioned than A, and the action of M is inexpensive to calculate. Then one can use iterative methods, like *Conjugate Gradient* method, for $MAu = Mf$ instead of $Au = f$.

We will study preconditioners of the following additive type:

$$M = \sum_{k=0}^{p} I_k R_k Q_k, \tag{2.1}$$

where the "interpolation" operators $I_k : V^k \to V$ are linear, and the "projection" operators Q_k are the adjoints of I_k defined by

$$(Q_k u, v_k)_k = (u, I_k v_k),\ \forall\, u \in V, \quad v_k \in V^k, \tag{2.2}$$

and $R_k : V^k \to V^k$ are given SPD operators, approximating the inverses of the restrictions of A on V^k in some sense. It is easy to verify that M is an SPD operator on V.

We remark that the preconditioner form (2.1) is a natural extension of the one introduced by Xu [23] with nested subspaces. The awareness of this general form (2.1) was due to [15], see also [7], [22] and [18].

Following the framework of Xu [23] for structured meshes in which all V^k are subspaces of V, one can bound the condition number of MA for the present unstructured cases in terms of three parameters K_0, ω_0 and α_0 defined as follows:

(P1) For any $u \in V$, there exist $u_k \in V^k$ $(0 \le k \le p)$ such that $u = \sum_{k=0}^{p} I_k u_k$ and

$$\sum_{k=0}^{p} (R_k^{-1} u_k, u_k)_k \le K_0 (Au, u).$$

(P2) For any $u_k \in V^k$, $k = 0, 1, \cdots, p$,

$$(AI_k u_k, I_k u_k) \le \omega_0 (R_k^{-1} u_k, u_k)_k.$$

(P3) For any $u \in V$ and $u_k \in V^k$ $(1 \le k \le p)$,

$$\sum_{k=1}^{p} (Au, I_k u_k) \le \alpha_0^{\frac{1}{2}} (Au, u)^{\frac{1}{2}} \Big(\sum_{k=1}^{p} (AI_k u_k, I_k u_k) \Big)^{\frac{1}{2}}.$$

We have the following bound for the condition number of $\kappa(MA)$ (see [10] for the proof):
THEOREM 2.1. *Under the assumptions **(P1)** - **(P3)**,*

$$\kappa(MA) \le \omega_0 (\alpha_0 + 1) K_0.$$

REMARK 2.1. ***(P1)** and **(P2)** are natural extensions of assumptions from the theory for structured meshes by Xu [23], where all the spaces V^k, $0 \le k \le p$, are assumed to be subspaces of V. **(P1)** means that any function in V can be decomposed into a sum of functions in spaces V^k and this partition is stable with the perturbed "energy" norm in some sense. **(P2)** is equivalent to $\lambda_{\max}(R_k A_k) \le \omega_0$ where $A_k = Q_k A I_k$ is the "restriction" of A on V^k, it means that the approximation of R_k to the inverse of A_k cannot be "too bad". **(P3)** is a condition on the "local" properties of V^k $(1 \le k \le p)$ in some sense, i.e. the image spaces $I_k V_k$ of V^k under the mapping I_k cannot overlap one another too much in the fine space V.*

*Note that our **(P3)** is not the extension of the corresponding assumption used in [23]. It might be replaced by the extension of the so-called strengthened Cauchy-Schwarz inequality in [23] for nested subspaces with identity operators I_i $(1 \le i \le p)$:*
***(P3*)** Let $\varepsilon_{ij} \in (0,1]$ be the smallest constants satisfying that*

$$(AI_i u_i, I_j u_j) \le \varepsilon_{ij} (AI_i u_i, I_i u_i)^{\frac{1}{2}} (AI_j u_j, I_j u_j)^{\frac{1}{2}}, \quad \forall u_i \in V^i, u_j \in V^j, i, j = 1, \cdots, p.$$

*It is easy to verify that **(P1)**, **(P2)** and **(P3*)** imply **(P3)**. Thus **(P3)** is a weaker assumption than **(P3*)**. We prefer **(P3)** to **(P3*)** as **(P3)** is more convenient to check than **(P3*)** for the non-nested subspaces.*

2.1. Multilevel additive preconditioners for SPD operators. Let V and V^l $(0 \le l \le L)$ be defined as in Section 2, and furthermore, we assume that for each $l : 1 \le l \le L$, the space V^l can be decomposed into a sum of subspaces V_k^l $(1 \le k \le N_l)$. Then the multilevel additive preconditioners for the given SPD operator A is defined as follows

$$M = \sum_{l=0}^{L} \sum_{k=1}^{N_l} I_k^l R_k^l Q_k^l \tag{2.3}$$

where the "interpolation" operators $I_k^l : V_k^l \to V$ are linear, and the "projection" operators Q_k^l are the adjoints of I_k^l defined by

$$(Q_k^l u, v_k^l)_l = (u, I_k^l v_k^l), \ \forall\, u \in V, \quad v_k^l \in V_k^l, \tag{2.4}$$

and $R_k^l : V_k^l \to V_k^l$ are given SPD operators, approximating the inverses of the restrictions of A on V_k^l in some sense. It is easy to verify that M is an SPD operator on V. Note that for $l = 0$, we adopt the notation

$$N_0 = 1, \quad I_k^0 = I^0, \quad Q_k^0 = Q^0, R_k^0 = R^0.$$

As in the last section, the condition number of MA can be bounded in terms of three parameters K_0, ω_0 and α_0 defined as follows:

(P1′) For any $u \in V$, there exist $u_k^l \in V_k^l$ $(0 \le l \le L,\ 1 \le k \le N_l)$ such that $u = \sum_{l=0}^{L} \sum_{k=1}^{N_l} I_k^l u_k^l$ and

$$\sum_{l=0}^{L} \sum_{k=1}^{N_l} ((R_k^l)^{-1} u_k^l, u_k^l)_k \le K_0(Au, u).$$

(P2′) For any $u_k^l \in V_k^l$, $0 \le l \le L$, $1 \le k \le N_l$,

$$(AI_k^l u_k^l, I_k^l u_k^l) \le \omega_0((R_k^l)^{-1} u_k^l, u_k^l)_l.$$

(P3′) For any $u \in V$ and $u_k^l \in V_k^l$, $0 \le l \le L$, $1 \le k \le N_l$,

$$\sum_{l=0}^{L} \sum_{k=1}^{N_l} (Au, I_k^l u_k^l) \le \alpha_0^{\frac{1}{2}} (Au, u)^{\frac{1}{2}} \Big(\sum_{l=0}^{L} \sum_{k=1}^{N_l} (AI_k^l u_k^l, I_k^l u_k^l) \Big)^{\frac{1}{2}}.$$

Similar to Theorem 2.1, we have

THEOREM 2.2. *Under the assumptions **(P1′) - (P3′)**,*

$$\kappa(MA) \le \omega_0(\alpha_0 + 1)K_0.$$

2.2. Additive preconditioners for small perturbations of SPD operators. The results of this section are applicable to general non-symmetric parabolic problems, cf. [9, 10]. Let V be a finite dimensional space with the scalar product $(\cdot,\cdot)$, and E a non-symmetric operator on V which is a small perturbation of the SPD operator A, that is, $E = A + B$, and we solve the equation

$$Eu \equiv (A + B)u = f$$

on V. Our goal is to find a good preconditioner M for the non-symmetric operator E. Then we can use iterative methods, like *GMRES* or *BiCGSTAB*, to solve

$$MEu = Mf$$

instead of $Eu = f$. Let us consider the *GMRES* method. It is known (cf. [14]) that the convergence rate of *GMRES* depends on the following two parameters:

$$\beta_1 = \min_{u \neq 0} \frac{(u, MEu)_A}{(u, u)_A}, \quad \beta_2 = \max_{u \neq 0} \frac{\|MEu\|_A}{\|u\|_A}. \tag{2.5}$$

If $\beta_1 > 0$, *GMRES* converges, and at the mth iteration the residual is bounded as (cf. [14])

$$\|Mf - MEu^m\|_A \le \left(1 - \frac{\beta_1^2}{\beta_2^2}\right)^{m/2} \|Mf - MEu^0\|_A.$$

Let the spaces V^k $(1 \le k \le p)$, the scalar products $(\cdot,\cdot)_k$, linear operators $I_k : V^k \to V$, the adjoints Q_k of I_k and the SPD operators $R_k : V^k \to V^k$ be defined as in Section 2. Then we define the preconditioner M as in (2.1) by $M = \sum_{k=0}^{p} I_k R_k Q_k$ for operator E. Note that we still use an SPD preconditioner M even though E is non-symmetric.

We introduce two assumptions for the perturbation operator B:

(P4) For any $u \in V$ and $u_k \in V^k$, $1 \le k \le p$,

$$\sum_{k=1}^{p}(Bu, I_k u_k) \le \alpha_1^{\frac{1}{2}}(Au,u)^{\frac{1}{2}}\Big(\sum_{k=1}^{p}(AI_k u_k, I_k u_k)\Big)^{\frac{1}{2}}.$$

(P5) There exists a constant $\mu_1 \in (0,1)$ such that for any $u, v \in V$,

$$|(Bu,v)| \le \mu_1 \,\|u\|_A \|v\|_A.$$

We have the following estimates about two parameters β_1 and β_2 which determine the convergence rate of *GMRES* iteration for solving $MEu = Mf$ (see [10] for the proof):

THEOREM 2.3. *If in addition to **(P1) - (P5)**, we assume further that*

$$\mu_1^2 + \alpha_1 \le \frac{(1-\mu_1)^2}{2\omega_0 K_0}, \tag{2.6}$$

then we have

$$\beta_1 = \min_{u\neq 0}\frac{(u,ME)_A}{(u,u)_A} \ge \frac{(1-\mu_1)^2}{4K_0}, \quad \beta_2 = \max_{u\neq 0}\frac{\|MEu\|_A}{\|u\|_A} \le 2\omega_0\alpha_0^{\frac{1}{2}}(1+\alpha_0+\alpha_1+\mu_1^2)^{\frac{1}{2}}.$$

3. Finite elements and domain decompositions. For an open bounded domain Ω in $\subset R^d$ $(d = 2,3)$, suppose we are given a family of shape regular (not necessarily quasi-uniform) triangulations $\{\mathcal{T}^h\}$ on Ω, consisting of simplices. We will not discuss the effects of approximating Ω but always assume that the triangulations $\{\mathcal{T}^h\}$ of Ω are exact, i.e., Ω is either a polygon or a polyhedron, and

$$\Omega = \Omega^h \equiv \cup_{\tau^h \in \mathcal{T}^h} \tau^h.$$

Let V^h be a piecewise linear finite element subspace of $H_0^1(\Omega)$ defined on $\mathcal{T}^h$.

Decompose the domain Ω into p non-overlapping subdomains $\tilde{\Omega}^k$ $(1 \le k \le p)$, then extend each $\tilde{\Omega}^k$ to a larger one Ω^k such that the distance between $\partial\Omega^k$ and $\partial\tilde{\Omega}^k$ is bounded from below by $\delta_k > 0$. We allow each Ω^k to be of quite different size and of quite different shape from other subdomains. Then we define the subspaces $\{V^k\}_{k=1}^p$ of V^h corresponding to the subdomains $\{\Omega^k\}_{k=1}^p$ by

$$V^k = \{v \in V^h;\ \ v = 0 \ \ on \ \ \Omega \setminus \Omega^k\}. \tag{3.7}$$

We introduce also a coarse grid $\mathcal{T}^H$ which forms a σ_0-shape regular triangulation of Ω, but has nothing to do with $\mathcal{T}^h$, i.e., none of the nodes of $\mathcal{T}^H$ need to be nodes of $\mathcal{T}^h$. Let Ω^0 be the coarse grid domain, i.e. $\Omega^0 = \cup_{\tau^H \in \mathcal{T}^H} \tau^H$.

Denote by V^0 (resp. $\hat{V}^0$) the subspace of $H_0^1(\Omega^0)$ (resp. $H^1(\Omega^0)$) consisting of piecewise linear functions defined on $\mathcal{T}^H$. Note that Ω^0 usually does not match with Ω, and $V^0 \not\subset V^h$.

In addition, we need to impose a few reasonable assumptions on the coarse grid Ω^0. Roughly speaking, we assume that for each coarse element τ^H, all its neighboring fine elements having non-empty intersections with τ^H form a subregion whose measure can be bounded by a constant times the one of τ^H; the coarse grid part outside the fine grid is of the fine element sizes while the fine grid part outside the coarse grid is of the coarse element sizes. See detailed assumptions in [10].

3.1. H^1-stability and L^2-optimal approximation of linear interpolants. As the coarse space V^0 is non-nested to the fine space V^h for our interest, the convergence proof for the domain decomposition methods requires the existence of an operator I_0 mapping the non-nested coarse space V^0 to a nested subspace of V^h satisfying the following H^1-stability and L^2-optimal approximation properties: for any coarse grid function $u \in V^0$,

$$|I_0 u|_{1,\Omega} \leq C|u|_{1,\Omega^0}, \quad \|u - I_0 u\|_{0,\Omega} \leq Ch|u|_{1,\Omega^0}$$

where Ω is the coarse grid domain. More specifically, we require these two properties to hold locally in order to deal with general unstructured meshes.

There exist a lot of options for such grid-transfer operators. But as this grid-transfer operator I_0 enters the algorithm, we want it to be as simple as possible.

Standard and modified finite element interpolants. The simplest one is obviously the standard finite element interpolant Π_h corresponding to the fine space V^h. To make $I_0 V^0$ a subspace of the fine space V^h, we need some special treatment on the part of the coarse grid boundary close to the fine grid boundary part which assumes Neumann boundary conditions. If one has pure Dirichlet boundary conditions, then the zero extension operator outside of the coarse grid domain should be the most natural and also effective option, cf. [11, 7]. If one has other type of boundary conditions, one choice is to require that the coarse grid covers the fine grid Neumann boundary part, then Π_h is well-defined everywhere in the fine grid domain, cf. [7, 10, 9]. It is shown that this interpolant has the required two properties.

Another choice is to define the operator I_0 in the interior part of the coarse grid domain by the standard interpolant Π_h but outside of the coarse grid domain by other simple linear interpolation, see Section 5 for more details.

Clément's interpolant. For the proof of the convergence of domain decomposition methods, or more specifically for the verification of the partition assumption **(P1)** in Theorem 2.1, we need another grid-transfer operator R_H mapping the fine space V^h to the coarse space V^0 and R_H must also possess the two properties of the H^1-stability and L^2-optimal approximation: for any fine grid function $u \in V^h$,

$$|R_H u|_{1,\Omega^0} \leq C|u|_{1,\Omega}, \quad \|u - R_H u\|_{0,\Omega^0} \leq C\,H\,|u|_{1,\Omega}.$$

However, as we are dealing with unstructured meshes which can be non-quasi-uniform, R_H should be defined completely locally. We know the standard finite element interpolant corresponding to the coarse space V^0 is defined locally, but it does not possess the required two properties. Two satisfactory operators are Clément interpolant [12] and Scott-Zhang interpolant [21], cf. [7, 10, 9].

4. Two level and multilevel additive Schwarz method for elliptic problems. In this section, we apply the general theory of Section 2 to the following second order elliptic problems:

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j}) + b\,u = f, \quad in \;\; \Omega \tag{4.1}$$

with some mixed boundary conditions on $\partial\Omega$. Here $\Omega \subset R^d$ $(d = 2, 3)$ as described in Section 3, $(a_{ij}(x))$ is symmetric, uniformly positive definite, and $b(x) \geq 0$ in Ω.

The weak formulation of the above problem is: Find $u \in H_0^1(\Omega)$ such that

$$A_\Omega(u, v) = (f, v), \; \forall\, v \in H_0^1(\Omega)$$

with

$$A_\Omega(u,v) = \int_\Omega \Big(\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + b\, uv \Big) dx.$$

The finite element problem is: Find $u \in V^h$ such that

$$A_\Omega(u,v) = (f,v),\ \forall v \in V^h. \tag{4.2}$$

4.1. Two level additive Schwarz method. Based on the local finite element spaces V^k $(1 \le k \le p)$ and the coarse space $V^0 = V^H$ defined in Section 3, Schwarz methods are preconditioning for the linear system (4.2) that are built using local and coarse grid solves.

We define scalar products $(\cdot,\cdot)_k = (\cdot,\cdot)_{0,\Omega^k}$ on V^k for $1 \le k \le p$ and $(\cdot,\cdot) = (\cdot,\cdot)_{0,\Omega}$ on V^h, and then define an SPD operator A on V^h and a coarse operator A_0 by

$$(Au,v) = A_\Omega(u,v),\ \forall u,v \in V^h; \quad (A_0 u, v)_0 = A_{\Omega^0}(u,v),\ \forall u,v \in V^0$$

and local operators A_k, $1 \le k \le p$ by

$$(A_k u, v)_k = A_{\Omega^k}(u,v),\ \forall u,v \in V^k.$$

Since $V^k \subset V^h$, $1 \le k \le p$, we define $I_k : V^k \to V^h$ to be the natural injection operator. Note that $V^0 \not\subset V^h$. Define $I_0 : V^0 \to V^h$ to be the standard interpolant Π_h discussed in Section 3.1. One may also use other choices of I_0, e.g., the Clément interpolant R_h^0. Choose the local solvers R_k, $0 \le k \le p$ to be exact solvers, i.e. $R_k^{-1} = A_k$. Then the preconditioner M in (2.1) for A is:

$$M = \sum_{k=0}^{p} I_k A_k^{-1} Q_k.$$

The following theorem gives the bound of the condition number $\kappa(MA)$. These results were proved in our previous work [8, 7], but only global bounds were obtained there.

THEOREM 4.1. *Under the assumptions (A1) - (A6), we have*

$$\kappa(MA) \le C \max_{1 \le k \le p} \frac{H_k^2}{\delta_k^2}.$$

Outline of the proof. By Theorem 2.1, it suffices to verify the three assumptions **(P1)**, **(P2)** and **(P3)**. **(P2)** is a direct consequence of the Cauchy-Schwarz inequality and the two properties of H^1-stability and the L^2-optimal approximation for the interpolants I_k discussed in Section 3.1, which gives the bound $\omega_0 = O(1)$. **(P3)** follows easily from the Cauchy-Schwarz inequality and the assumption that any point in Ω belongs to only a fixed number of subdomains, which gives us a bound $\alpha_0 = O(1)$. For **(P1)**, we use the Clément interpolant R_H and the partition of unity associated with the subdomains to define the required partition. Then **(P1)** can be proved using the two properties of H^1-stability and the L^2-optimal approximation for the interpolants I_k and the Clément interpolant R_H, giving a bound $K_0 = \max_k H_k^2/\delta_k^2$.

4.2. Multilevel additive Schwarz method for elliptic problems. Let $\Omega \subset R^d$ $(d = 2,3)$ be a convex polygonal or polyhedral domain. Consider the same elliptic problem as defined in (4.1) and its finite element discretization (4.2). We will construct multilevel additive type preconditioners for the finite element system.

Let $\{\mathcal{T}^l\}_{l=0}^L$ be a not necessarily nested sequence of shape regular triangulations on Ω with h_l the maximum diameter of all elements in $\mathcal{T}^l$. $\mathcal{T}^L = \mathcal{T}^h$ is the finest triangulation on which the finite element space V^h and in return the finite element problem are defined. Denote the coarser domains corresponding to the coarser triangulations $\mathcal{T}^l$, $0 \le l \le L-1$ by Ω^l.

For $0 \le l < L$, let $V^l \subset H^1(\Omega^l)$ be the piecewise linear finite element space defined on $\mathcal{T}^l$ with proper boundary conditions imposed.

Assume that for each level $l = 0, 1, \cdots, L$, $\{\Omega_k^l\}_{k=1}^{N_l}$ is an overlapping domain decomposition of Ω^l, obtained by extending a given non-overlapping subdomain covering $\{\tilde{\Omega}_k^l\}_{k=1}^{N_l}$ of Ω^l such that $dist(\partial\tilde{\Omega}_k^l, \partial\Omega_k^l \cap \Omega^l) \ge \delta_l > 0$, $1 \le k \le N_l$. δ_l is called the l-th level overlapping ratio. Here the boundaries of the subdomains Ω_k^l are required to align with the boundaries of the l-th level elements in $\mathcal{T}^l$. We also impose some assumptions on each coarse grid similar to the ones discussed in Section 3, see [10] for details.

For each coarser space V^l $(1 \le l \le p)$, we define a mapping $I_l : V^l \to V^h$ to be the standard finite element interpolant Π_h with proper modifications on boundary nodes, and for each subdomain Ω_k^l on l-th level, define a local subspace by

$$V_k^l = \{v \in V^l;\ \ v = 0 \ \ on \ \ \partial\Omega_k^l \cap \Omega^l\} \subset V^l$$

and a prolongation operator $I_k^l : V_k^l \to V^h$ to be I_l, but I_k^l's adjoint $Q_k^l : V^h \to V_k^l$ by

$$(Q_k^l u, v_k^l)_l = (u, I_k^l v_k^l),\ \forall\, u \in V^h,\ v_k^h \in V_k^l$$

where $(\cdot,\cdot)_l = (\cdot,\cdot)_{0,\Omega^l}$ is the scalar product in $L^2(\Omega^l)$.

Furthermore, we define local operators $A_k^l : V_k^l \to V_k^l$ by

$$(A_k^l u, v)_l = A_{\Omega_k^l}(u, v),\ \forall\, u, v \in V_k^l,$$

and let $R_k^l = (A_k^l)^{-1}$ for simplicity of exposition, then we may construct the additive Schwarz preconditioner as in (2.3) by

$$M = \sum_{l=0}^{L}\sum_{k=1}^{N_l} I_k^l R_k^l Q_k^l \equiv \sum_{l=0}^{L}\sum_{k=1}^{N_l} I_k^l (A_k^l)^{-1} Q_k^l.$$

For the condition number $\kappa(MA)$, we have

THEOREM 4.2. *Under the assumptions* **(H1)** *-* **(H4)**,

$$\kappa(MA) \lesssim \rho^2 L^2,$$

where $\rho = \max_{1\le l \le L}(h_l + h_{l-1})/\delta_l$.

Outline of the proof. By Theorem 2.2, it suffices to verify the three assumptions **(P1′)**, **(P2′)** and **(P3′)**. To this aim, we need the two properties of the H^1-stability and L^2-optimal approximation for the interpolants I_k and the Clément interpolant discussed in Section 3.1; and moreover, we need also the H^1-stability and L^2-stability of the interpolants I_k.

To define the proper partition in **(P1′)**, we use the orthogonal projections $P^l : H^1(\Omega) \to I_l V^l$ defined by

$$A_\Omega(P^l u, v) = A_\Omega(u, v),\ \forall\, u \in H^1(\Omega), v \in I_l V^l,$$

and the following property of projections P^l which can be proved by using the Aubin-Nitsche trick, Sobolev extension theorem and Clément's interpolant: for $0 \le l < L$,

$$\|v - P^l v\|_{0,\Omega} \lesssim h_l\, \|v\|_{1,\Omega},\ \forall\, v \in V^h.$$

Using these results, we can show that $K_0 = O(\rho^2 L)$, $\omega_0 = O(1)$ and $\alpha_0 = O(L)$.

REMARK 4.1. *The condition number bound given in Theorem 4.2 grows like L^2. It is known that in the structured case, one can remove this dependence on L and obtain optimal condition number (cf. Zhang [24] and Oswald [20]). At this point, we do not know how to obtain a similar optimal bound for our unstructured case.*

4.3. Additive Schwarz method for non-symmetric parabolic problems. The abstract framework of Section 2.2 can be applied to general non-symmetric second order parabolic problems to obtain similar optimal convergence results as in structured meshes. We remark that symmetric positive definite solvers can be used both for local subproblems and for the global coarse problem. We refer to [9, 10] for more details.

5. Treatment of Neumann boundary conditions. Practical multilevel algorithms using unstructured grids require some method to produce the coarse grid hierarchy (since it not naturally obtained from the fine grid problem), along with the associated interpolation and restriction operators. In [11], we generate a sequence of coarse grids by recursively coarsening a fine, unstructured grid using a maximal independent set approach [17]. We observed that the performance of multilevel methods using grids generated by this method performed as well as standard multilevel methods on a structured mesh, but the performance of the methods deteriorated considerably when a mixed boundary condition was used instead of a purely Dirichlet boundary condition.

The convergence rate proofs in [7] for domain decomposition methods with non-matching grids using interpolations with zero extension required the assumptions that the coarse grid covers all of the Neumann boundary and that no coarse grid element lies completely outside the fine grid. The motivation for this was that with zero extension outside the coarse grid, corrections were being improperly made for the parts of the fine grid which lie outside the coarse domains. While this would not have a serious effect for problems with Dirichlet boundary conditions, it would slow down the method for problems with Neumann boundary conditions. Since the maximal independent set approach for grid coarsening generally creates coarse grid domains which are subsets of the fine grid, this suggests that the observed deterioration in the rate of convergence for mixed boundary condition problems may be remedied by either creating coarse grid domains which completely cover the fine grid domain or improving the transfer matrices used in the multilevel methods to get from one level to the next. We will discuss these two approaches next.

5.1. Modifying coarse grid boundaries. A sequence of coarse grids is generated by finding the maximal independent set of boundary nodes by eliminating every other boundary node and then finding the maximal independent set of the interior nodes. The resulting vertex set is then triangulated using a triangulation algorithm. Since this approach for grid coarsening generally creates coarse grid domains which do not cover the fine grid domain, we modify the coarse grids so that they do not violate this condition. In our implementation, we physically move the boundary nodes of the coarse grids in a systematic way so that if a fine grid node to be eliminated is exterior to the coarse grid boundary, the positions of one or more nearby coarse boundary nodes are adjusted so that the fine grid node will be interior to the new coarse grid boundary (see Figure 1). For our purposes, the boundary adjustment algorithm need only be applied to the edges where a Neumann boundary condition occurs and is not necessary for edges with a Dirichlet boundary condition. For practical purposes however, we applied the boundary adjustment algorithm to all edges regardless of the boundary condition.

5.2. Approximate interpolation/restriction operators. An alternative to modifying the coarse grid domains is to instead improve the interpolation and restriction operators

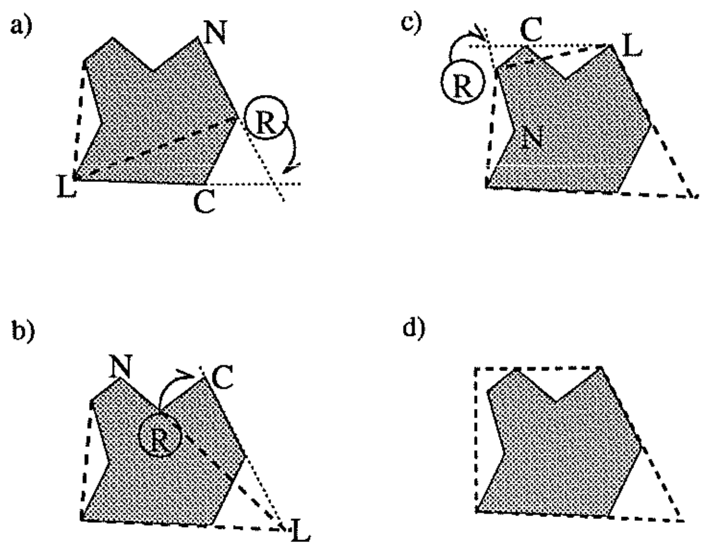

FIG. 1. *Modifying the coarse grid boundaries. Shaded region is the fine grid domain, dashed line is the coarse grid boundary, dotted lines show coarse grid boundary adjustment. L and R are coarse boundary nodes, while C is the fine grid boundary node to be eliminated. Modify coarse edges by moving node R.*

used to transfer information between different levels. The interpolation matrices used in [11] were formed by taking each fine node and searching for the coarse grid element in which it lies, then interpolating with the coarse nodes which make up that element. If no such coarse grid element can be found, then zero weights were set for all coarse nodes. This results in a zero extension for all fine nodes exterior to the coarse domain. The restriction matrices were taken to be the transposes of these interpolation matrices.

Instead of zero extension for fine nodes exterior to the coarse grid domain, we have modified the current interpolation matrices for these points by selecting the nearest coarse boundary edge and interpolating with the two coarse nodes which make up that edge. Values at these exterior fine nodes were approximated with the value at the nearest point on this coarse boundary edge (see Figure 2). This idea was motivated by Bank and Xu's [2] coarsening algorithm for hierarchical basis methods.

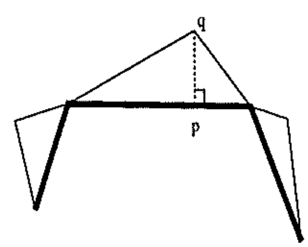

FIG. 2. *Values on a fine grid point outside the coarse domain, q, are approximated with the nearest point, p, on the nearest coarse boundary edge (thick line).*

6. Numerical results. In this section, we provide some numerical results of domain decomposition and multigrid experiments on unstructured grids for the Poisson equation with the airfoil mesh (from T. Barth and D. Jesperson of NASA Ames) shown in Figure 3 as our fine grid domain. All numerical experiments were performed using the Portable, Extensible Toolkit for Scientific Computation (PETSc) of Gropp and Smith [16], running on a Sun SPARC 20. Piecewise linear finite elements were used for the discretizations and the resulting linear system was solved using either two-level overlapping Schwarz or multigrid preconditioning with full *GMRES* as an outer accelerator. We compared two different triangulation algorithms, a Cavendish [6] and Baker (from Timothy Baker of Princeton University) algorithm. The coarse grid hierarchy of the airfoil mesh triangulated with Baker's algorithm is shown in Figure 4 where G^2 refers to the first coarsening of the fine grid, G^1 is the coarsening of G^2, and G^0 is the coarsening of the G^1.

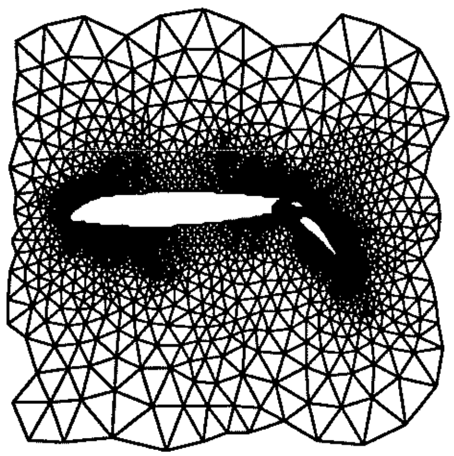

FIG. 3. *The airfoil grid with 4253 unknowns.*

In our first experiment, we solve a mildly varying coefficient problem:

$$\frac{\partial}{\partial x}((1+xy)\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}((1+\sin(4x+4y)\frac{\partial u}{\partial y}) = x^2\sin(3y)$$

with either a purely Dirichlet boundary condition or a mixed boundary condition: Dirichlet for $x \leq 0.2$ and homogeneous Neumann for $x > 0.2$. For this problem, the Dirichlet condition was $u = x^2\sin(4y)$.

We solved this problem using two-level additive and multiplicative Schwarz preconditioning with the fine grid domain partitioned into 32 subdomains using the Recursive Spectral Bisection method as in [11]. The initial iterate is set to be zero and the iteration is stopped when the discrete norm of the residual is reduced by a factor of 10^{-5}.

The results are summarized in Tables 1–2. For these results, the coarse grids used were triangulated using Cavendish's algorithm. The column labeled "unmodified boundaries" shows the number of iterations until convergence for coarse grid domains which do not cover the fine domain with both non-zero extension of fine nodes in $G^{k+1}\backslash G^k$, and zero extension (in parentheses). The column labeled "modified boundaries" shows the results for coarsening with coarse grid domains covering fine grid domains.

The second experiments show the results for the Poisson equation using multigrid preconditioning. The same two kinds of boundary conditions were used, but with a homogeneous

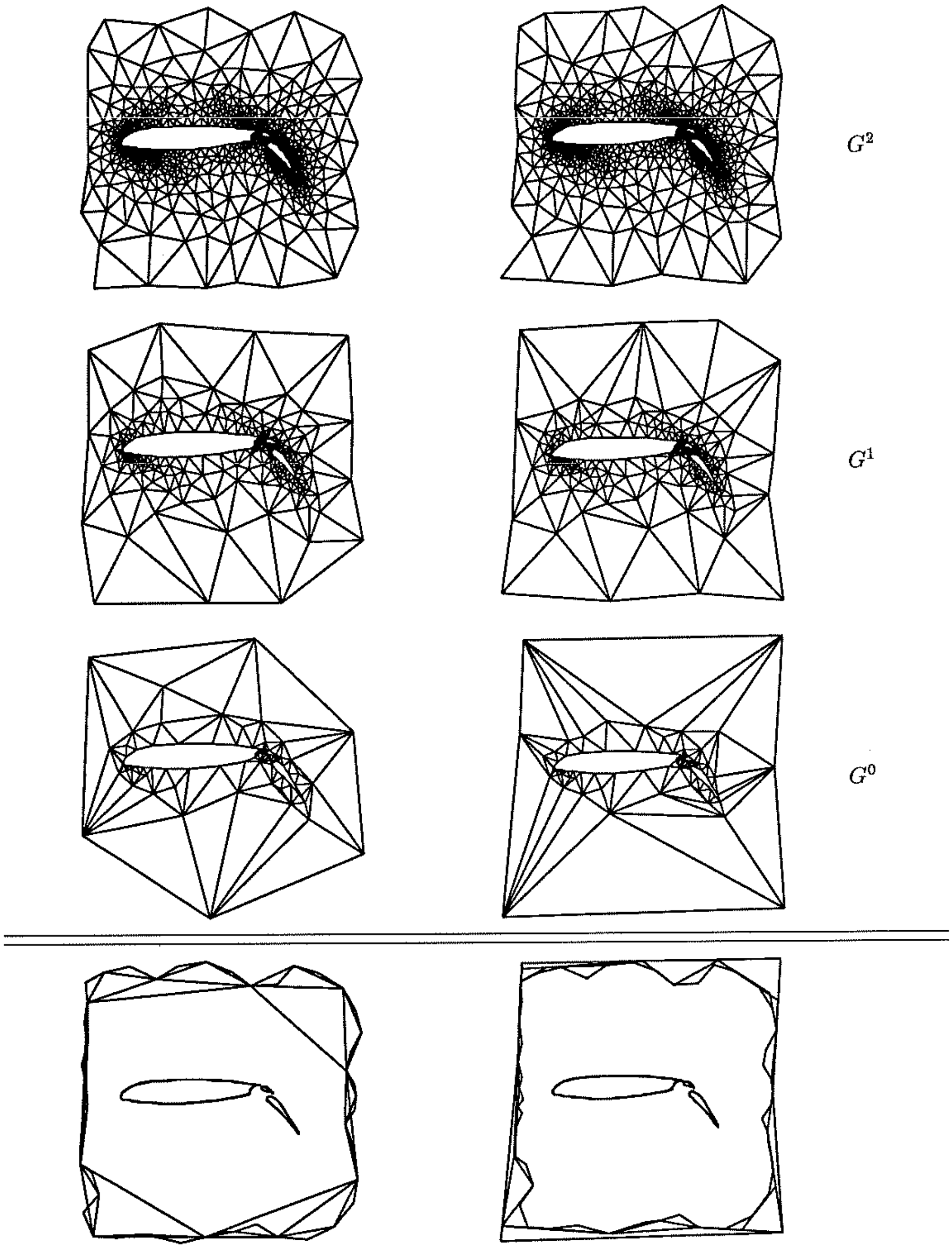

FIG. 4. *Airfoil grid hierarchy with unmodified boundaries (left) and modified boundaries (right).*

Overlap (# elements)	Coarse grid	Unmodified Boundaries		Modified Boundaries	
		Dir BC	Mixed BC	Dir BC	Mixed BC
0	None	43 (*)	40 (*)	*	*
	G^2	17 (18)	23 (74)	18	24
	G^1	31 (33)	43 (86)	31	42
	G^0	54 (56)	46 (84)	51	52
1	None	31 (*)	96 (*)	*	*
	G^2	15 (16)	22 (64)	15	22
	G^1	21 (22)	29 (69)	21	29
	G^0	35 (36)	40 (70)	34	42
2	None	26 (*)	74 (*)	*	*
	G^2	15 (16)	21 (60)	15	22
	G^1	19 (19)	26 (61)	19	25
	G^0	29 (30)	36 (63)	28	36

TABLE 1

*Additive DD iterations for the airfoil grid with 4253 unknowns. * indicates identical results since no coarse grid was used.*

Dirichlet condition instead. A V-cycle multigrid method with pointwise Gauss-Seidel smoothing and 2 pre and 2 post smoothings per level was used with the same initial iterate and the stopping criterion was reduced to 10^{-6}. In addition to the airfoil domain, we ran experiments on an annulus domain [11] for comparison.

The results of the multigrid experiments on the airfoil grid are summarized in Tables 3–4. Table 3 shows results with Cavendish's triangulations, while Table 4 shows results with Baker's triangulations. Multigrid results on the annulus grid are summarized in Table 5–7.

The numerical results show the significance of the assumption that when interpolations with zero extension are used, the coarse grid must cover the Neumann boundary of the fine grid problem; when the coarse grid domains do not cover the Neumann boundary, the convergence rates deteriorate noticeably.

The method of triangulation seemed to have little effect on the convergence rates. The multigrid experiments on the annulus grid show that in both approaches used to treat Neumann boundary conditions, we obtained methods which were nearly mesh-size independent.

The zero extension transfer operator is the simplest to implement and effective for Dirichlet boundary conditions. If there are Neumann boundaries, then the improved interpolation approach seems to work well and is not too difficult to implement. It can also be used with independently generated coarse grids. The modified boundary approach is equally effective but requires a re-triangulation of the coarse grid.

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Overlap (# elements)	Coarse grid	Unmodified Boundaries		Modified Boundaries	
		Dir BC	Mixed BC	Dir BC	Mixed BC
0	None	23 (*)	73 (*)	*	*
	G^2	9 (9)	9 (17)	9	10
	G^1	15 (12)	17 (24)	15	17
	G^0	20 (20)	21 (33)	19	24
1	None	13 (*)	39 (*)	*	*
	G^2	6 (6)	7 (9)	7	8
	G^1	9 (9)	10 (16)	10	10
	G^0	12 (12)	13 (19)	12	15
2	None	10 (*)	30 (*)	*	*
	G^2	5 (5)	6 (9)	6	7
	G^1	8 (7)	8 (12)	8	8
	G^0	9 (9)	11 (15)	9	13

TABLE 2

*Multiplicative DD iterations for the airfoil grid with 4253 unknowns. * indicates identical results since no coarse grid was used.*

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Cavendish Triangulation Results

MG levels	Unmodified Boundaries			Modified Boundaries		
	nodes	Dir BC	Mixed BC	nodes	Dir BC	Mixed BC
2	1170	5 (5)	6 (8)	1170	5	6
3	338	5 (5)	6 (9)	342	5	7
4	95	5 (5)	7 (9)	98	5	8

TABLE 3
MG iterations for the airfoil grid with 4253 unknowns.

Baker Triangulation Results

MG levels	Unmodified Boundaries			Modified Boundaries		
	nodes	Dir BC	Mixed BC	nodes	Dir BC	Mixed BC
2	1170	4 (4)	5 (8)	1170	4	6
3	336	4 (4)	6 (9)	333	5	7
4	98	5 (5)	7 (9)	98	5	7

TABLE 4
MG iterations for the airfoil grid with 4253 unknowns.

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Baker Triangulation Results for Annulus

MG levels	Unmodified Boundaries			Modified Boundaries		
	nodes	Dir BC	Mixed BC	nodes	Dir BC	Mixed BC
2	2175	5 (5)	8 (48)	2175	5	8
3	574	5 (5)	8 (58)	572	6	8
4	158	5 (5)	8 (50)	156	6	8

TABLE 5

MG iterations for the annulus grid with 8448 unknowns.

MG levels	Unmodified Boundaries			Modified Boundaries		
	nodes	Dir BC	Mixed BC	nodes	Dir BC	Mixed BC
2	575	5 (5)	7 (20)	575	5	7
3	159	5 (5)	7 (20)	159	5	7
4	47	5 (5)	7 (20)	47	5	7

TABLE 6

MG iterations for the annulus grid with 2176 unknowns.

MG levels	Unmodified Boundaries			Modified Boundaries		
	nodes	Dir BC	Mixed BC	nodes	Dir BC	Mixed BC
2	159	4 (4)	6 (11)	159	4	7
3	47	4 (4)	6 (12)	47	4	7
4	15	4 (4)	6 (12)	15	4	7

TABLE 7

MG iterations for the annulus grid with 576 unknowns.