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ABSTRACT. This paper proposes a stabilization of the classical hierarchical basis (HB) finite element method by modifying the standard nodal basis functions that correspond to the hierarchical complement (in the next finer discretization space) of any successive coarse discretization space using computationally feasible approximate L^2–projections onto the given coarse space. The corresponding multilevel additive and product algorithms give spectrally equivalent preconditioners and one action of such a preconditioner is of almost optimal order. The major results are regularity–free for the continuous problem (second order elliptic) and can be applied to problems with local refinement. Numerical results that illustrate the theory are presented.

1. INTRODUCTION

In this paper we study a modification of the classical hierarchical basis (HB) introduced by Yserentant [19], see also Bank, Dupont and Yserentant [4], for finite element solution of second order elliptic boundary value problems. The method exploits modification of the nodal basis functions at a given discretization level k that correspond to the nodes that are not present at the most recent coarse triangulation (i.e., the triangulation at level $k-1$). The modification of such a basis function ϕ is of the form $(I - Q^a_{k-1})\phi$ where Q^a_{k-1} is sufficiently close to the exact L^2 projection operator onto the level $k-1$ coarse finite element space V_{k-1}. This we call approximate wavelet modification of the HB. Similar approaches, limited for tensor product meshes, were reported recently in Griebel and Oswald [10] and Stevenson [12], [13]. Our approach is general and it applies whenever hierarchical decomposition of the space exists with hierarchical components having nodal basis, including spaces corresponding to highly nonuniform refined meshes.

We prove the following three results. First we show that the block Gauss Seidel (called also multiplicative) preconditioner, resulting from the approximate wavelet modified HB coordinates is spectrally equivalent to the corresponding stiffness matrix. Due to a technical reason we assume here H^2 regularity of the underlined elliptic problem and this is

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the only place we need it. All other results are regularity free. The same spectral equivalence result holds for the block Jacobi (diagonal or additive) preconditioner, again in the approximate wavelet modified HB coordinates. We also show, that actually the stiffness matrix in the approximate wavelet modified HB coordinates is well conditioned. At any rate, however it is always recommended to have some simple preconditioning in order to better scale the resulting stiffness matrix and thus to relax the dependence of the convergence of the iterative method used with respect to the coefficients of the underlined elliptic operator.

The technique we use exploits the now well developed convergence results for the additive multigrid method, see, e.g., the book by Bramble [6], or the survey papers Xu [18] or Yserentant [20]. Using the algebraic analysis for the V cycle hierarchical multilevel methods from Vassilevski [15] (or earlier in [14] and later extended in Axelsson and Vassilevski [2]) we can only prove suboptimal estimates (see Theorem 1 below).

It is interesting to note that the multiplicative preconditioner can be analyzed in a different way than the technique proposed in Bramble, Pasciak, Wang, and Xu [7], see also Wang [17]. This is possible based on the algebraic (i.e., block matrix) approach from [14], [2] or [15], see also the survey [16].

The results of the present paper can be applied to any problem that requires H^1 equivalent basis, for example, elasticity and the Stokes problem.

The remainder of the paper is organized as follows. In Section 2 we consider the case of exact L^2 projections, i.e., modification of the HB using (exact) wavelets. Here the multiplicative preconditioner is analyzed. Section 3 contains the main part of the analysis for the approximate wavelet modified HB (multiplicative) preconditioners. The following section §4 contains some discussion on the computational aspects of the approximate wavelet modified HB preconditioners. In particular, here we show the well conditionedness of the stiffness matrix computed from the approximate wavelet modified HB functions. Finally, in Section 5 some implementation details and numerical illustration of the studied methods is presented.

2. Modified HB Preconditioners using L^2 projections (wavelets)

In this section we introduce the classical hierarchical basis and then its modification with L^2 projections is described. On the basis of the thus modified hierarchical basis the corresponding multilevel product (multiplicative, also called symmetric block Gauss Seidel) preconditioner is formulated. Then, the spectral equivalence relations of this preconditioner with the corresponding discretization elliptic operator is analyzed. We prove a regularity free suboptimal spectrally equivalent result and under H^2 regularity assumption, an optimal spectral equivalence result is then shown. The computationally feasible modification of the HB based on approximate L^2 projections (called in this paper approximate wavelets) is considered in the following section.

Consider the following second order elliptic bilinear form

$$a(u,\phi) = \int_\Omega a\nabla u \cdot \nabla \phi \quad \text{for all } u,\phi \in H_0^1(\Omega), \tag{2.1}$$

where the given bounded domain Ω is either a plane polygon or a 3 D polytope. The coefficient matrix $a = \{a_{r,s}(x)\}$ with bounded and measurable entries in Ω, is assumed symmetric and positive definite in $\overline{\Omega}$. Let $\mathcal{T}_0$ be an initial coarse triangulation of Ω and after a refinement procedure the triangulations $\mathcal{T}_k$ and the corresponding node sets $\mathcal{N}_k$ at discretization level k be generated, $k = 0, 1, \ldots, J$. We will need the set of new node points at level k denoted by $\mathcal{N}_k^{(1)} \equiv N_k \setminus N_{k-1}$. Associated with $\mathcal{T}_k$ are the nested finite element spaces $V_k = \text{span}\ \{\phi_i^{(k)}\ :\ x_i \in \mathcal{N}_k\}$ spanned by the nodal (Lagrangian) basis functions $\phi_i^{(k)}$; i.e., we have $\phi_i^{(k)}(x_j) = \delta_{i,j} = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases}$ the Kronecker symbol. We denote

$$V_k^{(1)} = \text{span}\{\phi_i^{(k)}\ :\ x_i \in \mathcal{N}_k^{(1)}\}. \tag{2.2}$$

The classical hierarchical basis method (see Yserentant [19]) is based on the direct decomposition of $V \equiv V_J$

$$V = V_J^{(1)} + V_{J-1}^{(1)} + \ldots + V_1^{(1)} + V_0. \tag{2.3}$$

The difficulty with this decomposition as well known, is that the interpolation operator $I_k : V \to V_k \subset V$, defined by $I_k v = \sum_{x_i \in \mathcal{N}_k} v(x_i)\phi_i^{(k)}$, is not bounded in the H^1 norm uniformly with respect to the difference $J - k \to \infty$. In this paper we propose to modify the above decomposition (2.3) by using proper approximate projections.

At first however, we consider the following case of exact L^2 projections and then in the following section we consider the case of more computationally feasible approximate L^2 projections giving rise to approximate wavelets.

Let $(.,.)$ be the standard $L^2(\Omega)$ inner product. Consider the L^2 projection operators $Q_k : L^2(\Omega) \to V_k$ defined by

$$(Q_k v, \psi) = (v, \psi) \quad \text{for all } \psi \in V_k. \tag{2.4}$$

Then any function $v \in V$ can be decomposed as follows

$$v = (I - Q_{k-1})v + Q_{k-1}v.$$

Note that $Q_{k-1}v \in V_{k-1}$. The following estimate is well known

$$a(Q_k v, Q_k v) \leq \eta a(v, v) \quad \text{for all } v \in V = V_J, \tag{2.5}$$

for a constant η (≥ 1), independent of J and $k = 1, 2, \ldots, J$. This estimate implies the following corollary (see Vassilevski [15])

$$a((I - Q_k)v, (I - Q_k)v) \leq \eta a(v, v). \tag{2.6}$$

The last estimate used for $k := k - 1$ and $v \in V_k$ implies

$$a((Q_k - Q_{k-1})v, (Q_k - Q_{k-1})v) \leq \eta a(v, v). \tag{2.7}$$

Introduce now the space

$$V_k^1 \equiv (I - Q_{k-1})V_k. \tag{2.8}$$

It is readily seen that

$$V_k^1 = (I - Q_{k-1})V_k^{(1)} = (I - Q_{k-1})(I_k - I_{k-1})V_k,$$

since $(I - Q_{k-1})I_{k-1} = 0$.

The spaces V_k^1 are the modifications of $V_k^{(1)}$ that take part in the hierarchical decomposition (2.2) of V. The modification results from the term $Q_{k-1}V_k^{(1)}$ which is the L^2 projection of the hierarchical component onto the next coarse space V_{k-1}.

We next define the following operators:

- the solution operator $A^{(k)} : V_k \to V_k$,

$$(A^{(k)}v, \psi) = a(v, \psi) \quad \text{for all } v, \psi \in V_k; \tag{2.9a}$$

Denote also λ_k the largest eigenvalue of $A^{(k)}$.

- the solution operator $A_{11}^{(k)} : V_k^1 \to V_k^1$,

$$(A_{11}^{(k)}v, \psi) = a(v, \psi) \quad \text{for all } v, \psi \in V_k^1. \tag{2.9b}$$

Let $\lambda_k^{\max}$ and $\lambda_k^{\min}$ denote the extreme eigenvalues of $A_{11}^{(k)}$.

Similarly,

- the operators $A_{12}^{(k)} : V_{k-1} \to V_k^1$ and $A_{21}^{(k)} : V_k^1 \to V_{k-1}$ are defined by

$$\begin{aligned} (A_{12}^{(k)}\tilde{\psi}, v^1) &= a(v^1, \tilde{\psi}) \quad \text{for all } v^1 \in V_k^1 \text{ and all } \tilde{\psi} \in V_{k-1}, \\ (A_{21}^{(k)}v^1, \tilde{\psi}) &= a(v^1, \tilde{\psi}) \quad \text{for all } v^1 \in V_k^1 \text{ and all } \tilde{\psi} \in V_{k-1}. \end{aligned} \tag{2.9c}$$

Note that $A_{12}^{(k)}$ is the L^2 adjoint of $A_{21}^{(k)}$.

Since the decomposition $v = (I - Q_{k-1})v + Q_{k-1}v$ for any $v \in V_k$ is direct, the solution operator $A^{(k)}$ admits the following two by two block form

$$A^{(k)} = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A^{(k-1)} \end{bmatrix} \begin{matrix} \} & V_k^1 \\ \} & V_{k-1} \end{matrix}. \tag{2.9}$$

That is, for any $v, \psi \in V_k$ decomposed as $v = v^1 + \tilde{v}$, $\psi = \psi^1 + \tilde{\psi}$, where $v^1 = (I - Q_{k-1})v \in V_k^1$, $\psi^1 = (I - Q_{k-1})\psi \in V_k^1$ and $\tilde{v} = Q_{k-1}v \in V_{k-1}$, $\tilde{\psi} = Q_{k-1}\psi \in V_{k-1}$, we have

$$\begin{aligned} (A^{(k)}v, \psi) &= a(v, \psi) \\ &= a(v^1 + \tilde{v}, \psi^1 + \tilde{\psi}) \\ &= a(v^1, \psi^1) + a(\tilde{v}, \psi^1) + a(v^1, \tilde{\psi}) + a(\tilde{v}, \tilde{\psi}) \\ &= (A_{11}^{(k)}v^1, \psi^1) + (A_{12}^{(k)}\tilde{v}, \psi^1) + (A_{21}^{(k)}v^1, \tilde{\psi}) + (A^{(k-1)}\tilde{v}, \tilde{\psi}) \\ &= \left(\begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A^{(k-1)} \end{bmatrix} \begin{bmatrix} v^1 \\ \tilde{v} \end{bmatrix}, \begin{bmatrix} \psi^1 \\ \tilde{\psi} \end{bmatrix} \right). \end{aligned}$$

To proceed with the construction of our wavelet HB preconditioner we assume that there are given approximations $B_{11}^{(k)}$, symmetric and positive definite in V_k^1, to the solution operators $A_{11}^{(k)}$, $k = 1, 2, \ldots, J$. We also assume that the following spectral equivalence estimates hold

$$(A_{11}^{(k)}v^1, v^1) \le (B_{11}^{(k)}v^1, v^1) \le (1 + b_1)(A_{11}^{(k)}v^1, v^1) \quad \text{for all } v^1 \in V_k^1. \tag{2.10}$$

Here $b_1 = \text{Const} \ge 0$ independent of k and J. In the applications $B_{11}^{(k)}$ can simply be a diagonal matrix.

Definition 1. *(Wavelet modified HB preconditioner) Let $M^{(0)} = A^{(0)}$. For $k = 1, \dots, J$ define*

$$M^{(k)} = \begin{bmatrix} B_{11}^{(k)} & 0 \\ A_{21}^{(k)} & M^{(k-1)} \end{bmatrix} \begin{bmatrix} I & B_{11}^{(k)^{-1}} A_{12}^{(k)} \\ 0 & I \end{bmatrix}.$$

The operators $B_{11}^{(k)}$ are the given approximations (see (2.10)) to $A_{11}^{(k)}$.

Note that one solution with $M^{(k)}$ requires two solutions with each of the approximations $B_{11}^{(s)}$, $s = 1, 2, \dots, k$, one action with each of $A_{12}^{(s)}$ and $A_{21}^{(s)}$, $s = 1, 2, \dots, k$ and a coarse grid solution with $A^{(0)}$.

The above preconditioner can be analyzed similarly as in Vassilevski [15] (or earlier in [14]).

Consider the difference $M^{(k)} - A^{(k)}$. We have for any $v \in V_k$ decomposed as $v = v^1 + \tilde{v}$, $v^1 \in V_k^1$, $\tilde{v} \in V_{k-1}$,

$$\begin{aligned} &((M^{(k)} - A^{(k)})v, v) \\ &= ((B_{11}^{(k)} - A_{11}^{(k)})v^1, v^1) + ((M^{(k-1)} - A^{(k-1)})\tilde{v}, \tilde{v}) + (B_{11}^{(k)^{-1}} A_{12}^{(k)}\tilde{v}, A_{12}^{(k)}\tilde{v}). \end{aligned} \tag{2.11}$$

Assume now by induction that $M^{(s)} - A^{(s)}$ is positive semi definite on V_s for $s = 0, 1, \dots, k-1$. This holds for $k = 1$ ($M^{(0)} - A^{(0)} = 0$). Since all terms in the identity (2.11) are non negative (by assumption) the positive semi definiteness of $M^{(k)} - A^{(k)}$ immediately follows, and this confirms the induction assumption for $s = k$.

Next, we estimate the same difference $((M^{(k)} - A^{(k)})v, v)$ from above. Using the basic identity (2.11) and the following inequality (which is based on (2.10) and the symmetry and positive definiteness of $A^{(k)}$)

$$\begin{aligned} ((A^{(k-1)} - A_{21}^{(k)} B_{11}^{(k)^{-1}} A_{12}^{(k)})\tilde{v}, \tilde{v}) &\geq ((A^{(k-1)} - A_{21}^{(k)} A_{11}^{(k)^{-1}} A_{12}^{(k)})\tilde{v}, \tilde{v}) \\ &= \inf_{\psi^1 \in V_k^1,\ \psi = \tilde{v} + \psi^1} (A^{(k)}\psi, \psi) \\ &\geq 0, \end{aligned}$$

we get

$$\begin{aligned} ((M^{(k)} - A^{(k)})v, v) &\leq b_1 (A_{11}^{(k)} v^1, v^1) + (A^{(k-1)}\tilde{v}, \tilde{v}) + ((M^{(k-1)} - A^{(k-1)})\tilde{v}, \tilde{v}) \\ &\leq b_1 \sum_{s=1}^{k} (A_{11}^{(s)} v^{(s)^1}, v^{(s)^1}) + \sum_{s=1}^{k} (A^{(s-1)}\tilde{v}^{(s)}, \tilde{v}^{(s)}). \end{aligned} \tag{2.12}$$

Here we have denoted $v^{(k)} = v$ and for $s = k$ down to 1, we first use the decomposition of $v^{(s)} = v^{(s)^1} + \tilde{v}^{(s)}$, where

$$\begin{aligned} v^{(s)^1} &= (I - Q_{s-1})(I_s - I_{s-1})v^{(s)}, \\ \tilde{v}^{(s)} &= Q_{s-1} v^{(s)}, \end{aligned} \tag{2.13}$$

and then define,

$$v^{(s-1)} = \tilde{v}^{(s)} \in V_{s-1}.$$

We also have,

(2.14)
$$v^{(s-1)} = \tilde{v}^{(s)} = Q_{s-1}v^{(s)} = Q_{s-1}Q_s v^{(s+1)} = \ldots = Q_{s-1}Q_s \ldots Q_{k-1}v^{(k)} = Q_{s-1}v,$$
$$v^{(s)^1} = (I - Q_{s-1})v^{(s)} = (Q_s - Q_{s-1})v^{(s)} = (Q_s - Q_{s-1})Q_s v = (Q_s - Q_{s-1})v.$$

Using now the estimates (2.7) and (2.5) we get

$$\begin{aligned}(A_{11}^{(s)}v^{(s)^1}, v^{(s)^1}) &= a((Q_s - Q_{s-1})v, (Q_s - Q_{s-1})v)\\ &\leq \eta a(Q_s v, Q_s v) \leq \eta^2 a(v,v) = \eta^2 (A^{(k)}v, v),\end{aligned}$$

and again by (2.5),

$$(A^{(s)}v, v) = a(Q_s v, Q_s v) \leq \eta a(v,v) = \eta (A^{(k)}v, v).$$

The last two estimates used in (2.12) imply

$$((M^{(k)} - A^{(k)})v, v) \leq k\eta(1 + \eta b_1)(A^{(k)}v, v) \quad \text{for all } v \in V_k.$$

Thus we have proved the first main (suboptimal) result.

Theorem 1. *The following spectral equivalence relations hold*

$$(A^{(k)}v, v) \leq (M^{(k)}v, v) \leq (1 + k\eta(1 + b_1\eta))(A^{(k)}v, v) \quad \textit{for all } v \in V_k,$$

for any $k = 0, 1, \ldots, J$. The constant b_1 is from the estimate (2.10) and η is from the estimate of the norm of the L^2 projection operators Q_s in (2.5).

Remark 1. An optimal spectral equivalence result is possible based on the following well known estimates

$$(2.15)\qquad \begin{aligned}\sum_{s=1}^{k} a((Q_s - Q_{s-1})v, (Q_s - Q_{s-1})v) &\leq \sum_{s=1}^{k} \lambda_s \|(Q_s - Q_{s-1})v\|_0^2\\ &\leq C \sum_{s=1}^{k} h_s^{-2} \|(Q_s - Q_{s-1})v\|_0^2\\ &\leq C h_0^{-2} \sum_{s=1}^{k} 2^{2s} \|(Q_s - Q_{s-1})v\|_0^2\\ &\leq C\|v\|_1^2 \leq Ca(v,v) \quad \text{for all } v \in V_k.\end{aligned}$$

Here $h_s = 2^{-s}h_0$, where h_0 is the characteristic size of the elements of the initial coarse triangulation $\mathcal{T}_0$. The above estimate is shown, e.g., in Oswald [11]. For more detailed derivation of such stability estimates we refer to Bornemann and Yserentant [5].

Another estimate that plays a major role in the analysis is given in the following lemma.

Lemma 1. *Assume that the homogeneous Dirichlet boundary value problem associated with the given bilinear form $a(.,.)$ admits H^2 regular solution for any given L^2 right hand side. Then the following estimate holds*

$$(2.16)\qquad \sum_{s=1}^{k} \lambda_s^{-1} \|A^{(s)}Q_{s-1}v\|_0^2 \leq C \sum_{s=1}^{k} h_s^2 \|A^{(s)}Q_{s-1}v\|_0^2 \leq Ca(v,v), \quad \textit{for any } v \in V_k.$$

Proof. This estimate can be easily derived if H^2 regularity is assumed, i.e., if Ω is convex and the coefficients of the bilinear form (2.1) are smooth enough. Then let P_s be the elliptic projections from $H_0^1(\Omega)$ onto the subspaces V_s with respect to the bilinear form $a(\cdot,\cdot)$ (zero boundary conditions on $\partial\Omega$ are imposed). The H^2 regularity implies that

$$\|(P_j - P_{j-1})v\|_0^2 \le C\lambda_j^{-1} a((P_j - P_{j-1})v, (P_j - P_{j-1})v) \quad \text{for all } v \in V. \tag{2.17}$$

Then we have

$$\begin{aligned}
\sum_{s=1}^{k} \lambda_s^{-1}\|A^{(s)}Q_{s-1}v\|_0^2 &\le 2\sum_{s=1}^{k} \lambda_s^{-1}(\|A^{(s)}(Q_s - Q_{s-1})v\|_0^2 + 2\sum_{s=1}^{k} \lambda_s^{-1}\|A^{(s)}Q_s v\|_0^2 \\
&\le C\sum_{s=1}^{k} \lambda_s\|(Q_s - Q_{s-1})v\|_0^2 + 2\sum_{s=1}^{k} \lambda_s^{-1}\|A^{(s)}Q_s v\|_0^2 \\
&\le Ca(v,v) + 4\sum_{s=1}^{k} \lambda_s^{-1}\|A^{(s)}(Q_s - P_s)v\|_0^2 \\
&\quad +4\sum_{s=1}^{k} \lambda_s^{-1}\|A^{(s)}P_s v\|_0^2 \\
&= Ca(v,v) + 4\sum_{s=1}^{k} \lambda_s^{-1}\|A^{(s)}(Q_s - P_s)v\|_0^2 \\
&\quad +4\sum_{s=1}^{k} \lambda_s^{-1}\|Q_s A v\|_0^2 \\
&\le Ca(v,v) + 4\sum_{s=1}^{k} \lambda_s^{-1}\|A^{(s)}(Q_s - P_s)v\|_0^2.
\end{aligned}$$

Here we have used estimate (2.15), the well known identity $A^{(s)}P_s = Q_sA$ and the uniform convergence of the additive multigrid algorithm (see, e.g., Bramble [6]), i.e., the estimate

$$\sum_{s=1}^{k} \lambda_s^{-1}\|Q_sAv\|_0^2 = \sum_{s=1}^{k} a(T_s v, v) \le Ca(v,v),$$

where $T_s \equiv \lambda_s^{-1}Q_sA$.

It remains then to show the estimate

$$\sum_{s=1}^{k} \lambda_s^{-1}\|A^{(s)}(Q_s - P_s)v\|_0^2 \le \sum_{s=1}^{k} \lambda_s\|(Q_s - P_s)v\|_0^2 \le Ca(v,v). \tag{2.18}$$

Let $\|.\|$ mean the energy norm $a(.,.)^{\frac{1}{2}}$. We have, for any $v \in V_k$, using the decomposition $(I - P_s)v = \sum_{j=s+1}^{k} (P_j - P_{j-1})v$, estimate (2.17), and the $\|.\|$ orthogonality of the elliptic projections P_j, i.e., that $\|(P_j - P_{j-1})v\|^2 = \|P_j v\|^2 - \|P_{j-1}v\|^2$,

$$
\begin{aligned}
\sum_{s=1}^{k} \lambda_s \|(Q_s - P_s)v\|_0^2 &= \sum_{s=1}^{k} \lambda_s ((Q_s - P_s)v, (I - P_s)v) \\
&= \sum_{s=1}^{k} \sum_{j=s+1}^{k} \lambda_s ((Q_s - P_s)v, (P_j - P_{j-1})v) \\
&\leq \sum_{s=1}^{k} \sum_{j=s+1}^{k} \lambda_s \|(Q_s - P_s)v\|_0 \|(P_j - P_{j-1})v\|_0 \\
&\leq C \sum_{s=1}^{k} \sum_{j=s+1}^{k} \lambda_s \lambda_j^{-\frac{1}{2}} \|(Q_s - P_s)v\|_0 \|(P_j - P_{j-1})v\| \\
&\leq C \sum_{s=1}^{k} \sum_{j=s+1}^{k} \lambda_s^{\frac{1}{2}} 2^s 2^{-j} \|(Q_s - P_s)v\|_0 \|(P_j - P_{j-1})v\| \\
&\leq C \left[\sum_{s=1}^{k} \sum_{j=s+1}^{k} \lambda_s \left(\tfrac{1}{2}\right)^{j-s} \|(Q_s - P_s)v\|_0^2 \right]^{\frac{1}{2}} \\
&\quad \times \left[\sum_{s=1}^{k} \sum_{j=s+1}^{k} \left(\tfrac{1}{2}\right)^{j-s} \|(P_j - P_{j-1})v\|^2 \right]^{\frac{1}{2}} \\
&\leq C \left[\sum_{s=1}^{k} \lambda_s \|(Q_s - P_s)v\|_0^2 \right]^{\frac{1}{2}} \left[\sum_{j=1}^{k} \|(P_j - P_{j-1})v\|^2 \right]^{\frac{1}{2}} \\
&= C \left[\sum_{s=1}^{k} \lambda_s \|(Q_s - P_s)v\|_0^2 \right]^{\frac{1}{2}} \left[\sum_{j=1}^{k} \left(\|P_j v\|^2 - \|P_{j-1}v\|^2 \right) \right]^{\frac{1}{2}} \\
&\leq C \left[\sum_{s=1}^{k} \lambda_s \|(Q_s - P_s)v\|_0^2 \right]^{\frac{1}{2}} [a(v,v)]^{\frac{1}{2}},
\end{aligned}
$$

which implies the desired inequality (2.18). Thus estimate (2.16) has been verified. □

We will prove in a following section the following lemma.

Lemma 2. *The following bounds for the extreme eigenvalues of $A_{11}^{(k)}$ hold*

$$C_1 h_k^{-2} \leq \lambda_k^{\min} \leq \lambda_k^{\max} \leq C_2 h_k^{-2},$$

for some positive constants C_1 and C_2 independent of k and J. This in particular shows that the operators $A_{11}^{(k)}$ are well conditioned.

Then (2.16) and Lemma 2 imply the estimate

$$
\begin{aligned}
\sum_{s=1}^{k} \left(A_{11}^{(s)^{-1}} A_{12}^{(s)} \tilde{v}^{(s)}, A_{12}^{(s)} \tilde{v}^{(s)}\right) &\leq \sum_{s=1}^{k} \left(\lambda_s^{\min}\right)^{-1} \|(Q_s - Q_{s-1}) A^{(s)} Q_{s-1} v\|_0^2 \\
&\leq \sum_{s=1}^{k} \left(\lambda_s^{\min}\right)^{-1} \|A^{(s)} Q_{s-1} v\|_0^2 \\
&\leq C_1^{-1} \sum_{s=1}^{k} h_s^2 \|A^{(s)} Q_{s-1} v\|_0^2 \\
&\leq C a(v,v), \quad \text{for any } v \in V_k.
\end{aligned}
\tag{2.19}
$$

Here we have used the representation $A_{12}^{(s)} = (Q_s - Q_{s-1})A^{(s)}Q_{s-1}$.

Next we note that a sharper inequality than (2.12) holds; namely, we have

(2.20)
$$\begin{aligned}((M^{(k)} - A^{(k)})v, v) &\leq b_1(A_{11}^{(k)}v^1, v^1) + (A_{11}^{(k)^{-1}}A_{12}^{(k)}\tilde{v}, A_{12}^{(k)}\tilde{v}) + ((M^{(k-1)} - A^{(k-1)})\tilde{v}, \tilde{v}) \\ &\leq b_1 \sum_{s=1}^{k} (A_{11}^{(s)}v^{(s)^1}, v^{(s)^1}) + \sum_{s=1}^{k} (A_{11}^{(s)^{-1}}A_{12}^{(s)}\tilde{v}^{(s)}, A_{12}^{(s)}\tilde{v}^{(s)}).\end{aligned}$$

Using now estimates (2.15) and (2.19) the factor k in the right hand estimate in Theorem 1 can be removed. We have

$$\sum_{s=1}^{k}(A_{11}^{(s)}v^{(s)^1}, v^{(s)^1}) = \sum_{s=1}^{k} a((Q_s - Q_{s-1})v, (Q_s - Q_{s-1})v) \leq C(A^{(k)}v, v),$$

and

$$\sum_{s=1}^{k}(A_{11}^{(s)^{-1}}A_{12}^{(s)}\tilde{v}^{(s)}, A_{12}^{(s)}\tilde{v}^{(s)}) \leq C(A^{(k)}v, v).$$

The last two estimates and (2.20) show the following optimal spectral equivalence result.

Theorem 1'. *The following spectral equivalence relations hold, under the H^2 regularity assumption,*

$$(A^{(k)}v, v) \leq (M^{(k)}v, v) \leq C(A^{(k)}v, v) \quad \text{for all } v \in V_k,$$

for any $k = 0, 1, \ldots, J$. The constant C depends only on the constant b_1 from the estimate (2.10), on the constant η which is from the estimate of the norm of the L^2 projection operators Q_s in (2.5) and on the constants involved in the estimates (2.15), (2.16) and constant C_1 from Lemma 2.

Remark 2. Since in any of the spaces V_k^1, for the finite element application we are interested in, there is no basis of locally supported functions, the multilevel preconditioner defined above in Definition 1, based on the operator blocks $A_{11}^{(k)}, A_{12}^{(k)}$ and $A_{21}^{(k)}$, does not lead to computationally feasible sparse matrix computations. This gives rise to the (non local) wavelet bases for finite element spaces. This difficulty can be removed by appropriate approximations Q_k^a to Q_k. This is possible since the actions of Q_k require solutions of mass matrix problems which are well conditioned and for problems with a source right hand sides the corresponding solutions have good decay rate, hence the inverses of the mass matrices can be well approximated by sparse matrices.

3. Multilevel HB Preconditioners based on Approximate Wavelets

In this section we consider the case of more computationally feasible modification of the HB functions based on approximate L^2 projections (called in this paper approximate wavelets). On the basis of the thus modified hierarchical basis the corresponding multilevel product (multiplicative) preconditioners, assuming H^2 regularity property of the underlined second order elliptic problem, are spectrally equivalent to the corresponding discretization finite element elliptic operators. If we do not assume H^2 regularity then a suboptimal result can be proved. The multilevel additive (block Jacobi, or block diagonal) preconditioner is formulated and analyzed in the following section.

Let Q_k^a be an approximation to the exact L^2 projection Q_k. We assume that for sufficiently small $\tau > 0$ there holds

$$\|(Q_k^a - Q_k)v)\|_0 \le \tau \|Q_k v\|_0 \quad \text{for any } v \in V. \tag{3.1}$$

Examples of such approximations will be given in the following section.

Now we repeat the procedure applied in the previous section, replacing everywhere the exact projection operators Q_k by their accurate approximations Q_k^a. The analysis then of the resulting approximate wavelet HB preconditioner will only be a perturbation of that carried out before.

Consider first the spaces

$$V_k^1 \equiv (I - Q_{k-1}^a)V_k^{(1)} = (I - Q_{k-1}^a)(I_k - I_{k-1})V_k,$$

and introduce then the operators $A_{11}^{(k)} : V_k^1 \to V_k^1$, $A_{12}^{(k)} : V_{k-1} \to V_k^1$, and $A_{21}^{(k)} : V_k^1 \to V_{k-1}$ by the same formulae (2.9a) (2.9c), as in Section 2. Then $A^{(k)}$ admits a two by two block form (2.9) (with now a different space V_k^1).

In the same manner as in Definition 1, we define the multilevel, now approximate, wavelet HB preconditioner $M^{(k)}$, for given, symmetric and positive definite, approximations $B_{11}^{(k)}$ to the solution operators $A_{11}^{(k)}$. We also assume that estimates (2.10) hold for these approximations with a constant $b_1 \ge 0$, independent of J and k.

The analysis of the spectral equivalence relations between $M^{(k)}$ and $A^{(k)}$ proceeds in the same manner as in Section 2, including the estimate (2.12) now with the following function decompositions. Starting with $v^{(k)} = v$, for $s = k$ down to 1 one first decomposes $v^{(s)}$ as

$$v^{(s)} = v^{(s)^1} + \tilde{v}^{(s)},$$

where

$$\begin{aligned} v^{(s)^1} &= (I - Q_{s-1}^a)(I_s - I_{s-1})v^{(s)} \in V_s^1 = (I - Q_{s-1}^a)V_s^{(1)}, \\ \tilde{v}^{(s)} &= Q_{s-1}^a v^{(s)} + (I - Q_{s-1}^a)I_{s-1}v^{(s)} \in V_{s-1}, \end{aligned}$$

and then defines

$$v^{(s-1)} = \tilde{v}^{(s)}.$$

One easily verifies that $v^{(s)} = v^{(s)^1} + v^{(s-1)}$.

For the deviation $e_{s-1} \equiv v^{(s-1)} - Q_{s-1}v$ the following representation holds

$$\begin{aligned} e_{s-1} &= v^{(s-1)} - Q_{s-1}v \\ &= (Q_{s-1} - Q_{s-1}^a)I_{s-1}v^{(s)} + Q_{s-1}^a v^{(s)} - Q_{s-1}v \\ &= (Q_{s-1} - Q_{s-1}^a)I_{s-1}(v^{(s)} - Q_s v) \\ &\quad + Q_{s-1}^a(v^{(s)} - Q_s v) \\ &\quad + (Q_{s-1} - Q_{s-1}^a)I_{s-1}Q_s v + Q_{s-1}^a Q_s v - Q_{s-1}Q_s v \\ &= (Q_{s-1} - Q_{s-1}^a)I_{s-1}e_s + Q_{s-1}^a I_s e_s \\ &\quad + Q_{s-1}(I_{s-1}Q_s v - Q_s v) \\ &\quad - Q_{s-1}^a(I_{s-1}Q_s v - Q_s v) \\ &= (Q_{s-1} - Q_{s-1}^a)(I_{s-1} - I_s)e_s \\ &\quad + (Q_{s-1} - Q_{s-1}^a)e_s + Q_{s-1}^a e_s \\ &\quad + (Q_{s-1} - Q_{s-1}^a)(I_{s-1} - I_s)Q_s v \\ &= \left[Q_{s-1} + (Q_{s-1} - Q_{s-1}^a)(I_{s-1} - I_s)\right] e_s \\ &\quad + (Q_{s-1} - Q_{s-1}^a)(I_{s-1} - I_s)Q_s v. \end{aligned}$$

Hence the following recurrence holds (note that $(I_{s-1} - I_s)Q_{s-1} = 0$)

$$e_{s-1} = [Q_{s-1} + R_{s-1}]e_s + R_{s-1}(Q_s - Q_{s-1})v, \tag{3.2}$$

where $R_{s-1} = (Q_{s-1} - Q^a_{s-1})(I_{s-1} - I_s)$. Note that based on (3.1) and on the L^2 boundedness of the nodal interpolation operators $I_{s-1} : V_s \to V_{s-1}$ we have

$$\|R_{s-1}v\|_0 \leq C\tau\|v\|_0 \quad \text{for any } v \in V_s. \tag{3.3}$$

The above recurrence (3.2) implies then the following L^2 estimate

$$\|e_{s-1}\|_0 \leq (1 + C\tau)\|e_s\|_0 + C\tau\|(Q_s - Q_{s-1})v\|_0,$$

where we have also used the estimate (3.3).

Using the fact that $e_k = 0$ by simple recurrence we obtain

$$\|e_{s-1}\|_0 \leq C\tau C_0 \sum_{j=s}^{k} \|(Q_j - Q_{j-1})v\|_0, \tag{3.4}$$

where $C_0 = (1 + C\tau)^J \leq e^{C\tau J}$, which is bounded if $\tau \leq CJ^{-1}$, i.e., for τ sufficiently small. From now on we assume that

$$\tau \leq CJ^{-1}. \tag{3.5}$$

Proceeding further, we get from $v^{(s)^1} = v^{(s)} - v^{(s-1)} = e_s + Q_s v - e_{s-1} - Q_{s-1}v$ and using estimate (3.4)

$$\begin{aligned} \|v^{(s)^1}\|_0 &\leq \|(Q_s - Q_{s-1})v\|_0 + \|e_s\|_0 + \|e_{s-1}\|_0 \\ &\leq \|(Q_s - Q_{s-1})v\|_0 + 2C\tau C_0 \sum_{j=s}^{k} \|(Q_j - Q_{j-1})v\|_0. \end{aligned} \tag{3.6}$$

To complete the proof we need to estimate the two sums in (2.20). Recall that λ_s is the largest eigenvalue of the solution operator $A^{(s)}$ and that $\lambda_s^{\min}$ is the minimal eigenvalue of the solution operator $A_{11}^{(s)}$. The following lemma will be proved in the following section.

Lemma 2'. *The extreme eigenvalues of the solution operator $A_{11}^{(k)}$ defined on the space $V_k^1 = (I - Q_k^a)(I_k - I_{k-1})V_k$ are of order h_k^{-2}; i.e., the following estimates hold*

$$C_1 h_k^{-2} \leq \lambda_k^{\min} \leq \lambda_k^{\max} \leq C_2 h_k^{-2},$$

for some positive constants C_1 and C_2 independent of k and J. This in particular shows that the operators $A_{11}^{(k)}$ are well conditioned.

We have, using (3.6), that

(3.7)

$$\begin{aligned} \sum_{s=1}^{k} (A_{11}^{(s)} v^{(s)^1}, v^{(s)^1}) &\leq \sum_{s=1}^{k} \lambda_s \|v^{(s)^1}\|_0^2 \\ &\leq \sum_{s=1}^{k} \lambda_s \left(\|(Q_s - Q_{s-1})v\|_0 + 2C\tau C_0 \sum_{j=s}^{k} \|(Q_j - Q_{j-1})v\|_0 \right)^2 \\ &\leq 2 \sum_{s=1}^{k} \lambda_s [\|(Q_s - Q_{s-1})v\|_0^2 \\ &\quad + (k - s + 1)(2C\tau C_0)^2 \sum_{j=s}^{k} \|(Q_j - Q_{j-1})v\|_0^2] \\ &\leq 2(1 + 4C^2 C_0^2 (\tau J)^2) \sum_{s=1}^{k} \lambda_s \|(Q_s - Q_{s-1})v\|_0^2. \end{aligned}$$

To estimate the second sum in (2.20) we first note that for any $w \in V_{s-1}$

$$\|A_{12}^{(s)} w\|_0^2 = a(w, A_{12}^{(s)} w) = (A^{(s)} w, A_{12}^{(s)} w) \leq \|A^{(s)} w\|_0 \|A_{12}^{(s)} w\|_0.$$

Hence

$$\|A_{12}^{(s)} w\|_0 \leq \|A^{(s)} w\|_0.$$

The second sum in (2.20) is then estimated as follows using Lemma 2', $\tilde{v}^{(s)} = e_{s-1} + Q_{s-1} v$ and the above estimate:

$$
\begin{aligned}
\sum_{s=1}^{k} ({A_{11}^{(s)}}^{-1} A_{12}^{(s)} \tilde{v}^{(s)}, A_{12}^{(s)} \tilde{v}^{(s)}) &\leq \sum_{s=1}^{k} \left(\lambda_s^{\min}\right)^{-1} \|A_{12}^{(s)} \tilde{v}^{(s)}\|_0^2 \\
&\leq C_1^{-1} \sum_{s=1}^{k} h_s^2 \|A^{(s)} \tilde{v}^{(s)}\|_0^2 \\
&\leq 2C_1^{-1} \sum_{s=1}^{k} h_s^2 \left(\|A^{(s)} e_{s-1}\|_0^2 + \|A^{(s)} Q_{s-1} v\|_0^2\right) \\
&\leq C \sum_{s=1}^{k} \lambda_s \|e_{s-1}\|_0^2 + 2C_1^{-1} \sum_{s=1}^{k} h_s^2 \|A^{(s)} Q_{s-1} v\|_0^2.
\end{aligned}
\tag{3.8}
$$

The first sum in (3.8) is estimated as follows using (3.4)

$$
\begin{aligned}
\sum_{s=1}^{k} \lambda_s \|e_{s-1}\|_0^2 &\leq C\tau^2 C_0^2 \sum_{s=1}^{k} \lambda_s \left(\sum_{j=s}^{k} \|(Q_j - Q_{j-1}) v\|_0 \right)^2 \\
&\leq C\tau^2 C_0^2 \sum_{s=1}^{k} \lambda_s (k - s + 1) \sum_{j=s}^{k} \|(Q_j - Q_{j-1}) v\|_0^2 \\
&\leq C\tau^2 C_0^2 \sum_{s=1}^{k} (k - s + 1) \sum_{j=s}^{k} \lambda_j \|(Q_j - Q_{j-1}) v\|_0^2 \\
&\leq C(\tau J)^2 \sum_{s=1}^{k} 2^{2s} \|(Q_s - Q_{s-1}) v\|_0^2 \\
&\leq C(\tau J)^2 \|v\|_1^2 \\
&\leq C(\tau J)^2 a(v, v).
\end{aligned}
\tag{3.9}
$$

Since we have already assumed that $\tau = O(J^{-1})$ (this is (3.5)), based on the last estimate and estimate (2.16) used in (3.8), and having in mind that $\lambda_s = O(h_s^{-2})$ in (3.7), we proved the following main result.

Theorem 2. *The approximate wavelet HB preconditioner $M^{(k)}$ defined by Definition 1 on the basis of the spaces $V_k^1 = (I - Q_{k-1}^a)(I_k - I_{k-1}) V_k$, is spectrally equivalent to the solution operator $A^{(k)}$ if the approximate L^2 projection operators Q_k^a are sufficiently close to the exact ones, such that the estimate (3.1) holds for $\tau = O(J^{-1})$ and assuming H^2 regularity of the underlined elliptic problem. If the H^2 regularity does not hold the approximate wavelet HB preconditioner is only nearly spectrally equivalent to the corresponding solution operator.*

Proof. The suboptimal result is proved in the same way as demonstrated above noting only that the estimate (2.16) trivially holds with Ck instead of just a constant C. □

4. Computational Aspects

In this section we first prove Lemma 2 and Lemma 2', formulated in the previous two sections concerning the condition number of the major operator blocks $A_{11}^{(k)}$ of the solution operators $A^{(k)}$. We also comment on the additive or block diagonal version of the wavelet modified HB preconditioners for $A^{(k)}$. As a consequence we get that actually the matrices obtained using the wavelet modified HB functions are well conditioned. Hence we can simply use the CG method for the thus transformed matrix. Of course, it is always recommended to have some simple preconditioning in order to better scale the resulting matrices thus relaxing the dependence of the convergence of the CG method with respect to the coefficients involved in the elliptic bilinear form (2.1). We stress that the results in this section are regularity free.

Lemma 3. *Consider the standard nodal basis of* $V_k^{(1)} = (I_k - I_{k-1})V_k$, $\{\phi_i^{(k)} \;:\; x_i \in \mathcal{N}_k^{(1)} \equiv \mathcal{N}_k \setminus \mathcal{N}_{k-1}\}$. *Then* $\{(I - Q_{k-1}^a)\phi_i^{(k)} \;:\; x_i \in \mathcal{N}_k^{(1)}\}$ *forms a basis of* $V_k^1 = (I - Q_{k-1}^a)(I_k - I_{k-1})V_k$.

Proof. This is seen as follows. Assume that the following linear combination is zero, i.e.,

$$\sum_{x_i \in \mathcal{N}_k^{(1)}} c_i (I - Q_{k-1}^a)\phi_i^{(k)} = 0.$$

Then $\sum_{x_i \in \mathcal{N}_k^{(1)}} c_i \phi_i^{(k)} =$ function in V_{k-1}. Hence

$$0 = (I_k - I_{k-1})\,(\text{function in } V_{k-1}) = (I_k - I_{k-1}) \sum_{x_i \in \mathcal{N}_k^{(1)}} c_i \phi_i^{(k)} = \sum_{x_i \in \mathcal{N}_k^{(1)}} c_i \phi_i^{(k)}.$$

This is only possible if $c_i = 0$ for all i. □

Remark 3. Lemma 3 shows that we can explicitly construct a basis of V_k^1 by approximately projecting any $\phi_i^{(k)}$, $x_i \in \mathcal{N}_k^{(1)}$ onto V_{k-1} and then forming $(I - Q_{k-1}^a)\phi_i^{(k)}$, expanding it, e.g., in terms of the nodal basis of V_k. This will be computationally feasible if the support of any $(I - Q_{k-1}^a)\phi_i^{(k)}$ is not too large.

We recall now the following strengthened Cauchy inequality for the standard two level hierarchical decomposition of $V_k = V_k^{(1)} + V_{k-1}$, see, e.g., Bank and Dupont [3] or Axelsson and Gustafsson [1].

Lemma 4. *There exists a constant* $\gamma \in (0,1)$, *independent of the mesh size (or equivalently of the discretization level index* k*) such that the following strengthened Cauchy inequality holds,*

$$a(\phi^1, \tilde{\phi}) \le \gamma [a(\phi^1, \phi^1)]^{\frac{1}{2}} [a(\tilde{\phi}, \tilde{\phi})]^{\frac{1}{2}}, \quad \textit{for all } \phi^1 \in V_k^{(1)} \textit{ and all } \tilde{\phi} \in V_{k-1}.$$

We will be actually needing the following corollary of the above strengthened Cauchy inequality,

$$a(\phi^1 + \tilde{\phi}, \phi^1 + \tilde{\phi}) \ge (1 - \gamma^2) a(\phi^1, \phi^1), \quad \textit{for all } \phi^1 \in V_k^{(1)} \textit{ and all } \tilde{\phi} \in V_{k-1}. \tag{4.1}$$

Proof of Lemma 2 and Lemma 2'. Based on inequality (4.1), applied for any $\phi^1 \in V_k^{(1)}$ and $\tilde{\phi} = Q_{k-1}^a \phi^1$ we get

$$
\begin{aligned}
(A_{11}^{(k)}(\phi^1 - Q_{k-1}^a\phi^1), \phi^1 - Q_{k-1}^a\phi^1) &= a(\phi^1 - Q_{k-1}^a\phi^1, \phi^1 - Q_{k-1}^a\phi^1) \\
&\geq (1-\gamma^2) a(\phi^1, \phi^1) \\
&\geq C h_k^{-2} \|\phi^1\|_0^2 \\
&= C h_k^{-2} \left(\|Q_{k-1}\phi^1\|_0^2 + \|(I - Q_{k-1})\phi^1\|_0^2 \right) \\
&\geq C h_k^{-2} \|(I - Q_{k-1})\phi^1\|_0^2.
\end{aligned}
\tag{4.2}
$$

Here we have also used the estimate

$$
\|\phi^1\|_0 = \|(I_k - I_{k-1})\phi^1\|_0 \leq C h_{k-1} \|\phi^1\|_1 \leq C h_k [a(\phi^1, \phi^1)]^{\frac{1}{2}}. \tag{4.3}
$$

Using finally the following sequence of inequalities based on the triangle inequality, the estimates (3.1), (4.3), (4.1) and an inverse inequality, we get

$$
\begin{aligned}
\|\phi^1 - Q_{k-1}^a\phi^1\|_0 &\leq \|\phi^1 - Q_{k-1}\phi^1\|_0 + \|(Q_{k-1} - Q_{k-1}^a)\phi^1\|_0 \\
&\leq \|\phi^1 - Q_{k-1}\phi^1\|_0 + \tau \|\phi^1\|_0 \\
&\leq \|\phi^1 - Q_{k-1}\phi^1\|_0 + \tau C h_{k-1} [a(\phi^1, \phi^1)]^{\frac{1}{2}} \\
&\leq \|\phi^1 - Q_{k-1}\phi^1\|_0 + \tau C h_{k-1} [\tfrac{1}{1-\gamma^2} a(\phi^1 - Q_{k-1}^a\phi^1, \phi^1 - Q_{k-1}^a\phi^1)]^{\frac{1}{2}} \\
&\leq \|\phi^1 - Q_{k-1}\phi^1\|_0 + C\tau \|\phi^1 - Q_{k-1}^a\phi^1\|_0,
\end{aligned}
$$

i.e.,

$$
(1 - C\tau) \|\phi^1 - Q_{k-1}^a\phi^1\|_0 \leq \|\phi^1 - Q_{k-1}\phi^1\|_0. \tag{4.4}
$$

The lower bound is positive if τ is sufficiently small, which we have already assumed (see (3.5)).

The last estimate used in (4.2) shows

$$
(A_{11}^{(k)}(\phi^1 - Q_{k-1}^a\phi^1), \phi^1 - Q_{k-1}^a\phi^1) \geq C(1 - C\tau) h_k^{-2} \|\phi^1 - Q_{k-1}^a\phi^1\|_0^2.
$$

This estimate, together with the inverse inequality

$$
(A_{11}^{(k)}(\phi^1 - Q_{k-1}^a\phi^1), \phi^1 - Q_{k-1}^a\phi^1) = a(\phi^1 - Q_{k-1}^a\phi^1, \phi^1 - Q_{k-1}^a\phi^1) \leq C h_k^{-2} \|\phi^1 - Q_{k-1}^a\phi^1\|_0^2,
$$

show Lemmas 2 and 2', since any element in $V_k^1 = (I - Q_{k-1}^a)(I_k - I_{k-1})V_k$ has the form $(I - Q_{k-1}^a)\phi^1$, $\phi^1 \in V_k^{(1)} = (I_k - I_{k-1})V_k$. □

Now we comment on the additive version of the wavelet modified HB preconditioner further denoted by D.

Definition 2. *(Additive (or block Jacobi) wavelet modified HB preconditioner). It is defined by the following quadratic form*

$$
(Dv, v) \equiv \sum_{s=1}^{J} (B_{11}^{(s)} v^{(s)^1}, v^{(s)^1}) + (A^{(0)} v^{(0)}, v^{(0)}),
$$

based on the symmetric and positive definite approximations $B_{11}^{(s)}$ to the solution operators $A_{11}^{(s)}$ (see (2.10)) and on the direct decomposition of any $v \in V$

$$v = \sum_{s=1}^{J} v^{(s)^1} + v^{(0)}, \tag{4.5}$$

where starting with $v = v^{(J)}$ for $s = J$ down to 1 the component $v^{(s)^1} \in V_s^1$ of v and $v^{(s-1)} \in V_{s-1}$ are defined as follows,

$$\begin{aligned} v^{(s)^1} &= (I - Q_{s-1}^a)(I_s - I_{s-1})v^{(s)} \in V_s^1 = (I - Q_{s-1}^a)V_s^{(1)}, \\ v^{(s-1)} &= Q_{s-1}^a v^{(s)} + (I - Q_{s-1}^a)I_{s-1}v^{(s)} \in V_{s-1}. \end{aligned} \tag{4.6}$$

Note that one action of D^{-1} requires one solution with each $B_{11}^{(s)}$, $s = 1, 2, \ldots, J$ and a coarse grid solution with $A^{(0)}$. This is because D admits a block diagonal stiffness matrix structure in the approximate wavelet modified HB coordinates.

It is obvious that D is spectrally equivalent to the preconditioner with the exact blocks, i.e., $B_{11}^{(k)}$ replaced with $A_{11}^{(k)}$. It is then straightforward to show the following spectral bound,

$$\begin{aligned} a(v,v) \le C\|v\|_1^2 &\le C \sum_{s=1}^{J} \lambda_s^{\min} \|(Q_s - Q_{s-1})v\|_0^2 + Ca(Q_0 v, Q_0 v) \\ &= C \sum_{s=1}^{J} \lambda_s^{\min} \|v^{(s)^1} + e_{s-1} - e_s\|_0^2 + Ca(v^{(0)}, v^{(0)}) + Ca(e_0, e_0) \\ &\le 2C \sum_{s=1}^{J} \lambda_s^{\min} \left[\|v^{(s)^1}\|_0^2 + \|e_s - e_{s-1}\|_0^2 \right] + Ch_0^{-2}\|e_0\|_0^2 + Ca(v^{(0)}, v^{(0)}) \\ &\le C \left[\sum_{s=1}^{J} (A_{11}^{(s)} v^{(s)^1}, v^{(s)^1}) + a(v^{(0)}, v^{(0)}) \right] + C(\tau J)^2 a(v,v) \\ &= CD(v,v) + C(\tau J)^2 a(v,v), \end{aligned}$$

where we have used estimates (3.9). Now assuming that (3.5) holds with sufficiently small constant C we get that

$$(1 - C(\tau J)^2) a(v,v) \le C(Dv, v). \tag{4.7}$$

The reverse inequality is given by estimates (3.7), (2.15), (2.5) and (3.9), i.e., we have that

$$\begin{aligned} (Dv, v) &\le C(1 + C(\tau J)^2) a(v,v) + a(v^{(0)}, v^{(0)}) \\ &\le C(1 + C(\tau J)^2) a(v,v) + a(Q_0 v, Q_0 v) + a(e_0, e_0) \\ &\le C(1 + C(\tau J)^2) a(v,v) + \eta a(v,v) + Ch_0^{-2}\|e_0\|_0^2 \\ &\le C(1 + C(\tau J)^2) a(v,v). \end{aligned} \tag{4.8}$$

The estimates (4.7) and (4.8) imply the following theorem.

Theorem 3. *The additive, or block Jacobi, preconditioner D defined in Definition 2 is spectrally equivalent to the solution operator $A = A^{(J)}$ if estimate (3.5) holds with sufficiently small constant C.*

A simple and practical choice for the approximations $B_{11}^{(k)}$ is the diagonal matrices $\{a(\phi_i^{(k)} - Q_{k-1}^a \phi_i^{(k)}, \phi_i^{(k)} - Q_{k-1}^a \phi_i^{(k)}) \; : \; x_i \in \mathcal{N}_k^{(1)}\}$. For the multiplicative (i.e., Gauss Seidel) preconditioner from Definition 1, they have to be properly scaled in order to ensure estimate (2.10). The latter can be achieved by the Lanczos method for example.

Now, we are ready to prove that the approximate wavelet modified HB decomposition of $v \in V$ gives optimal condition number of the resulting stiffness matrix $A = A^{(J)}$.

Theorem 4. *Let*

$$v = \sum_{x_i \in \mathcal{N}_0} c_{0,i}\phi_i^{(0)} + \sum_{k=1}^{J} \sum_{x_i \in \mathcal{N}_k^{(1)}} c_{k,i}(I - Q_{k-1}^a)\phi_i^{(k)}$$

be the expansion of v with respect to the approximate wavelet modified HB functions of V. Then the following estimates hold

$$C^{-1}(1 - C(\tau J)^2) \sum_{k=0}^{J} h_k^{-2} \sum_{x_i \in \mathcal{N}_k^{(1)}} d_{k,i}c_{k,i}^2 \leq a(v,v) \leq C \sum_{k=0}^{J} h_k^{-2} \sum_{x_i \in \mathcal{N}_k^{(1)}} d_{k,i}c_{k,i}^2.$$

Here $d_{k,i} = \|(I - Q_{k-1}^a)\phi_i^{(k)}\|_0^2$ are just positive coefficients.

Proof. The estimates (4.7) and (4.8), for $B_{11}^{(k)}$ the diagonal part of $A_{11}^{(k)}$, and Lemma 2' imply the desired result. □

Corollary 1. Theorem 4 implies that for τ satisfying (3.5) with sufficiently small constant C, the approximate wavelet modified HB functions give well conditioned stiffness matrices.

5. Implementation and Numerical Examples

In this section we consider some possible implementation of the presented methods and in particular we outline algorithms how to compute the actions of Q_{k-1}^a on functions $v \in V_k^{(1)}$, i.e., from the hierarchical complement of V_{k-1} in V_k. We also formulate the multiplicative and additive version of the studied approximate wavelet modified HB method in matrix vector form which we find more suitable for actual implementation.

5.1. Computing actions of Q_{k-1}^a. We begin with describing algorithms for computing the actions of Q_{k-1}^a. Given $v \in V_k^{(1)}$ let $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ 0 \end{bmatrix} \begin{matrix} \}\mathcal{N}_k \setminus \mathcal{N}_{k-1} \\ \}\mathcal{N}_{k-1} \end{matrix}$, be its coefficient vector with respect to the standard nodal basis of V_k. The second block component of $\mathbf{v}$ is zero since v vanishes on $\mathcal{N}_{k-1}$. The actions of Q_{k-1}^a are computed by approximate solving the following equation,

$$(Q_{k-1}v, w) = (v, w) \quad \text{for all } w \in V_{k-1}. \tag{5.1}$$

Let $I_{k-1}^k = \begin{bmatrix} J_{12} \\ I \end{bmatrix} \begin{matrix} \}\mathcal{N}_k \setminus \mathcal{N}_{k-1} \\ \}\mathcal{N}_{k-1} \end{matrix}$ $(J_{12} = J_{12}^{(k)})$ and $I_k^{k-1} = I_{k-1}^{k^T}$ be the natural coarse to fine, and respectively, the fine to coarse transformation matrices. For example, if the nodal basis coefficient vector of a function $v_2 \in V_{k-1}$ in terms of the nodal basis of V_{k-1} is $\mathbf{v}_2$ then its coefficient vector with respect to the nodal basis of V_k (note that $v_2 \in V_{k-1} \subset V_k$) will be $I_{k-1}^k \mathbf{v}_2 = \begin{bmatrix} J_{12}\mathbf{v}_2 \\ \mathbf{v}_2 \end{bmatrix} \begin{matrix} \}\mathcal{N}_k \setminus \mathcal{N}_{k-1} \\ \}\mathcal{N}_{k-1} \end{matrix}$.

Denote now by $G_k = \{(\phi_j^{(k)}, \phi_i^{(k)})\}_{x_j, x_i \in \mathcal{N}_k}$ the kth level space mass (or Gramm) matrices. Then (5.1) admits the following matrix vector form:

$$\mathbf{w}_2^T G_{k-1} \mathbf{v}_2 = (I_{k-1}^k \mathbf{w}_2)^T G_k \mathbf{v} \quad \text{for all } \mathbf{w}_2.$$

Here $\mathbf{v}_2$ is the $(k-1)$th level space nodal coefficient vector of $Q_{k-1}v$ and $\mathbf{w}_2$ is also the $(k-1)$th level space nodal coefficient vector of $w \in V_{k-1}$. Therefore we get the following mass matrix problem to solve:

$$G_{k-1}\mathbf{v}_2 = I_k^{k-1} G_k \mathbf{v}. \tag{5.2}$$

This shows that the $(k-1)$th level space nodal coefficient vector of $Q_{k-1}v$ equals

$$G_{k-1}^{-1} I_k^{k-1} G_k \mathbf{v}.$$

Hence

$$\|Q_{k-1}v\|_0^2 = \left(G_{k-1}^{-1} I_k^{k-1} G_k \mathbf{v}\right)^T G_{k-1} \left(G_{k-1}^{-1} I_k^{k-1} G_k \mathbf{v}\right) = \|G_{k-1}^{-\frac{1}{2}} I_k^{k-1} G_k \mathbf{v}\|^2. \tag{5.3}$$

Above and in what follows we use the notation $\|\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{x}$.

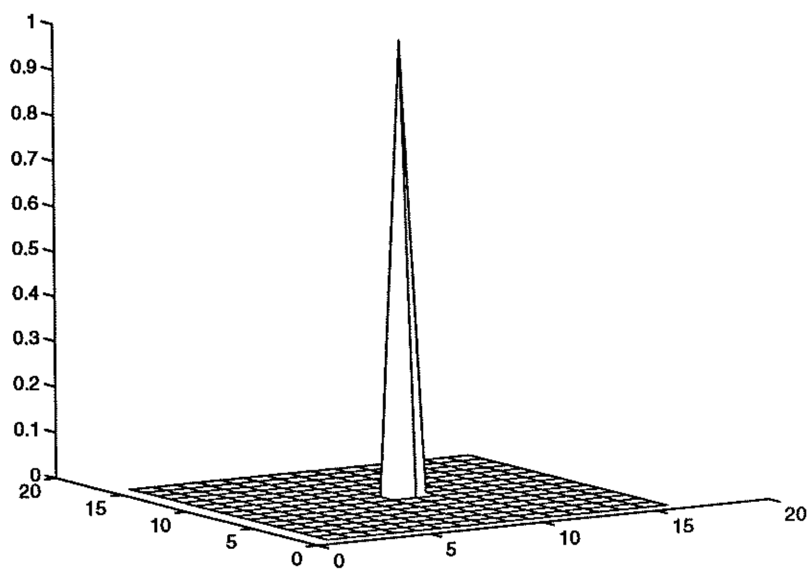

Figure 1. Plot of a HB function (no modification)

To be computationally feasible, we have to replace G_{k-1}^{-1} by any cheaper approximation $\tilde{G}_{k-1}^{-1}$ which action can be computed by some simple iterative method applied to (5.2). Such iterative methods lead to the following polynomial approximation to G_{k-1}^{-1},

$$\tilde{G}_{k-1}^{-1} = \left[I - \pi_m\left(G_{k-1}\right)\right] G_{k-1}^{-1},$$

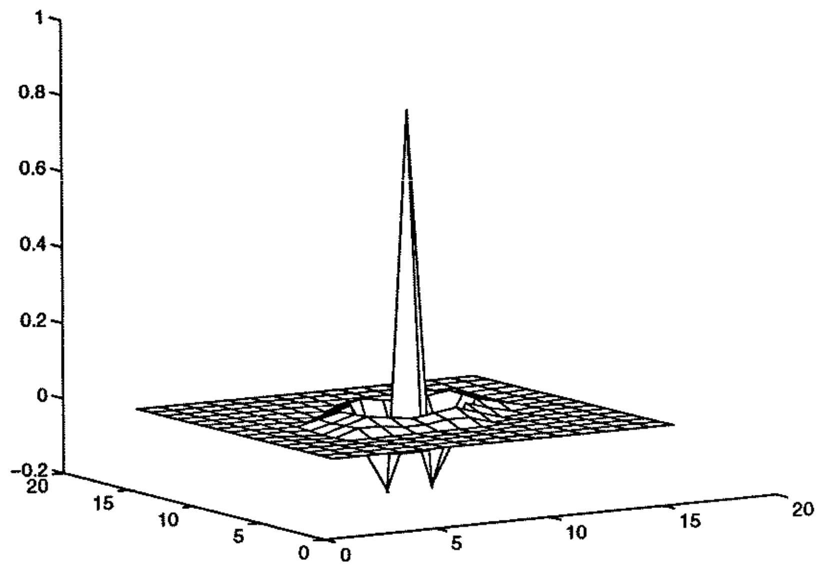

Figure 2. Plot of a wavelet modified HB function; $m = 2$

where π_m is a polynomial of some small degree $m \geq 1$. The polynomial π_m also satisfies $\pi_m(0) = 1$ and $0 \leq \pi_m(t) < 1$ for $t \in [\alpha, \beta]$, where the latter interval contains the spectrum of the mass matrix G_{k-1}. Since G_{k-1} is well conditioned, one can choose the interval $[\alpha, \beta]$ independent of k (i.e., mesh independent) and hence m, the polynomial degree, can be chosen mesh independent such that a given prescribed accuracy $\tau > 0$ is guaranteed. More precisely, given a tolerance $\tau > 0$, one can choose $m = m(\tau)$ such that the following holds:

$$\begin{aligned} \|Q^a_{k-1}v - Q_{k-1}v\|_0 &= \|G^{\frac{1}{2}}_{k-1}\left(G^{-1}_{k-1} - \tilde{G}^{-1}_{k-1}\right) I^{k-1}_k G_k \mathbf{v}\| \\ &= \|G^{\frac{1}{2}}_{k-1}\pi_m\left(G_{k-1}\right) G^{-1}_{k-1} I^{k-1}_k G_k \mathbf{v}\| \\ &\leq \max_{t\in[\alpha,\beta]} \pi_m(t) \|G^{-\frac{1}{2}}_{k-1} I^{k-1}_k G_k \mathbf{v}\| \\ &= \max_{t\in[\alpha,\beta]} \pi_m(t) \|Q_{k-1}v\|_0. \end{aligned}$$

Here we have used identity (5.3) and the properties of π_m. The latter estimate implies (3.1) with

$$\tau \geq \max_{t\in[\alpha,\beta]} \pi_m(t).$$

Since typically, $\max_{t\in[\alpha,\beta]} \pi_m(t) \leq Cq^m$ for some constants $C > 0$ and $q \in (0, 1)$, both independent of k (i.e., mesh independent), we see that to guarantee the prescribed accuracy τ, we need to choose $m = O(\log \tau^{-1})$. Since our restriction for τ in (3.1) was $\tau = O(J^{-1})$, we obtain that m has to satisfy asymptotically the estimate

$$m = O(\log J). \tag{5.4}$$

Inequality (5.4), of course, imposes very mild restrictions on m, and in practice, it is expected to be able to choose reasonably small m, e.g., $m = 1, 2$. This is confirmed by our numerical tests presented a little further in this section.

We show in Fig. 1 a typical plot of a nodal basis function of $V_k^{(1)}$ and its approximate wavelet modification for $m = 2$ in Fig. 2 and for $m = 4$ in Fig. 3. Its cross section, for $m = 4$, is shown in Fig. 4. We have used m steps of the conjugate gradient method to provide approximations to the solution of the mass matrix problem (5.2).

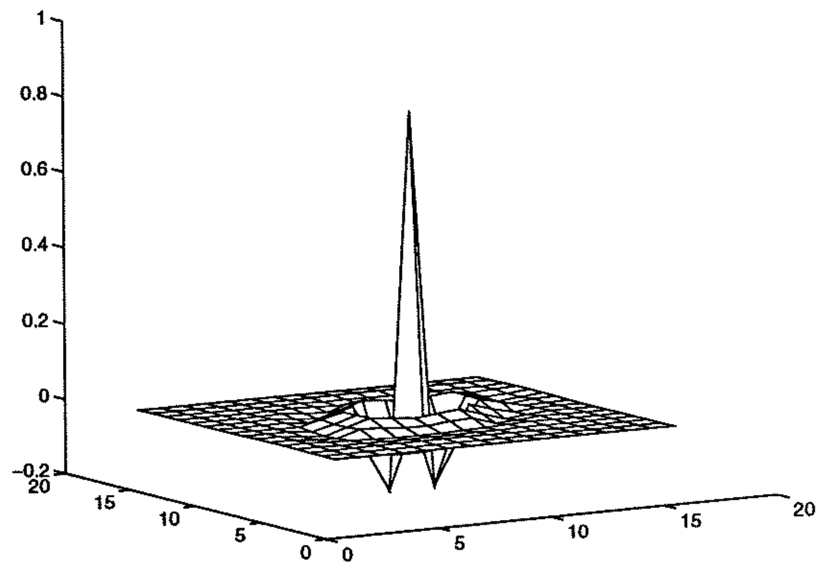

Figure 3. Plot of a wavelet modified HB function; $m = 4$

5.2. Matrix–vector representation of the AWM–HB methods. We now turn to the description of the multiplicative and additive variants of the studied approximate wavelet modified HB (AWM HB) methods in matrix vector form.

We first derive matrix representations of the operators $A_{11}^{(k)}$, $A_{12}^{(k)}$ and $A_{21}^{(k)}$ introduced in Section 2 (see (2.9b), (2.9c) and (2.9)). We keep the same notation for the matrix representation as for the operators.

Given a $v \in V_k$ and its nodal coefficient vector $\mathbf{v}$ we decompose it as follows:

$$v = (I - Q_{k-1}^a)(I_k - I_{k-1})v + w_2,$$

where $w_2 \in V_{k-1}$ is uniquely determined as $w_2 = I_{k-1}v + Q_{k-1}^a(I_k - I_{k-1})v$. Our goal is to find a vector representation of the components of v. Since the above decomposition is direct, it is clear that for some vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ the following holds:

$$\mathbf{v} = (I - I_{k-1}^k \tilde{G}_{k-1}^{-1} I_k^{k-1} G_k) \begin{bmatrix} \hat{\mathbf{v}}_1 \\ 0 \end{bmatrix} \begin{matrix} \}\mathcal{N}_k \setminus N_{k-1} \\ \}\mathcal{N}_{k-1} \end{matrix} + I_{k-1}^k \hat{\mathbf{v}}_2. \tag{5.5}$$

Here $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ represent the two components of our wavelet modified two level hierarchical basis coefficient vector $\hat{\mathbf{v}} = \begin{bmatrix} \hat{\mathbf{v}}_1 \\ \hat{\mathbf{v}}_2 \end{bmatrix}$ of v.

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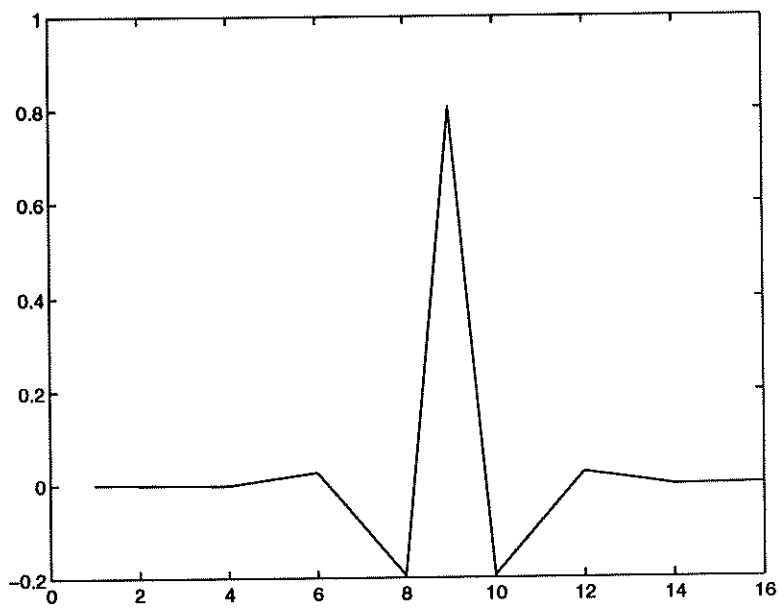

Figure 4. Cross section plot of a wavelet modified HB function, $m = 4$

Now, given the following problem in the standard nodal basis matrix vector form,

$$A\mathbf{v} = \mathbf{d}, \tag{5.6}$$

we can transform it into the approximate wavelet modified two level HB by testing (5.6) with the two components $(I - I^k_{k-1}\tilde{G}^{-1}_{k-1}I^{k-1}_k G_k)\left[\begin{smallmatrix}\mathbf{w}_1\\ 0\end{smallmatrix}\right]$ and $I^k_{k-1}\mathbf{w}_2$ for arbitrary $\mathbf{w}_1$ and $\mathbf{w}_2$. We get the following two by two block system for the approximate wavelet modified two level HB components of $\hat{\mathbf{v}}$ (denoted by $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$),

$$\begin{bmatrix} A^{(k)}_{11} & A^{(k)}_{12} \\ A^{(k)}_{21} & A^{(k)}_{22} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{v}}_1 \\ \hat{\mathbf{v}}_2 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{d}}_1 \\ \hat{\mathbf{d}}_2 \end{bmatrix}, \tag{5.7}$$

where

$$\begin{aligned} A^{(k)}_{11} &= [I \quad 0]\left(I - G_k I^k_{k-1}\tilde{G}^{-1}_{k-1}I^{k-1}_k\right) A^{(k)} \left(I - I^k_{k-1}\tilde{G}^{-1}_{k-1}I^{k-1}_k G_k\right)\begin{bmatrix} I \\ 0 \end{bmatrix}; \\ A^{(k)}_{12} &= [I \quad 0]\left(I - G_k I^k_{k-1}\tilde{G}^{-1}_{k-1}I^{k-1}_k\right) A^{(k)} I^k_{k-1}; \\ A^{(k)}_{21} &= I^{k-1}_k A^{(k)} \left(I - I^k_{k-1}\tilde{G}^{-1}_{k-1}I^{k-1}_k G_k\right)\begin{bmatrix} I \\ 0 \end{bmatrix}; \\ A^{(k)}_{22} &= I^{k-1}_k A^{(k)} I^k_{k-1} \\ &= A^{(k-1)}. \end{aligned} \tag{5.8}$$

The transformed right hand side vectors of (5.7) read similarly,

$$\begin{aligned}\hat{\mathbf{d}}_1 &= [I \quad 0]\left(I - G_k I^k_{k-1} \tilde{G}^{-1}_{k-1} I^{k-1}_k\right)\mathbf{d},\\ \hat{\mathbf{d}}_2 &= I^{k-1}_k \mathbf{d}\cdot\end{aligned}$$

Then the multiplicative AWM HB preconditioner $M^{(k)}$ from Definition 1, starting with

$$M^{(0)} = A^{(0)},$$

for $k = 1, 2, \ldots, J$, takes the following block matrix factored form :

$$M^{(k)} = \begin{bmatrix} B^{(k)}_{11} & 0 \\ A^{(k)}_{21} & M^{(k-1)} \end{bmatrix} \begin{bmatrix} I & B^{(k)^{-1}}_{11} A^{(k)}_{12} \\ 0 & I \end{bmatrix}. \tag{5.9}$$

Note that (5.9) has precisely the same form as the algebraic multilevel method studied in Vassilevski [14] (see also Axelsson and Vassilevski [2] and Vassilevski [15]).

In (5.9), $B^{(k)}_{11}$ are some appropriately scaled approximations to $A^{(k)}_{11}$. Recall that $A^{(k)}_{11}$ are well conditioned (by Lemma 2). Hence some simple polynomial approximations $B^{(k)}_{11}$ to $A^{(k)}_{11}$ are possible. However, in order to take the coefficients of the differential operator into account it would be better to compute, for example, the diagonal part of $A^{(k)}_{11}$. This will be computationally feasible since the basis functions of $V^1_k = (I - Q^a_{k-1})V^{(1)}_k$ (given in Lemma 3) will have reasonably narrow support if m is not too large, which should be the case in practice. We, however, used in our tests below the CG method to compute (fairly accurate) approximate actions of $A^{(k)^{-1}}_{11}$.

Now we formulate one preconditioning solution step with the multiplicative AWM HB preconditioner $M = M^{(J)}$.

Algorithm AWM–HB. *(Multiplicative version.) Given the problem*

$$M\mathbf{v} = \mathbf{d}\cdot$$

Initiate:

$$\mathbf{d}^{(J)} = \mathbf{d}\cdot$$

(I) **Forward recurrence.** *For $k = J$ down to 1 perform:*

(1) Compute:

$$\mathbf{d}^{(k)}_1 = [I \quad 0]\left(I - G_k I^k_{k-1} \tilde{G}^{-1}_{k-1} I^{k-1}_k\right)\mathbf{d}^{(k)};$$

(2) Solve:

$$A^{(k)}_{11}\mathbf{w}_1 = \mathbf{d}^{(k)}_1;$$

(3) Transform basis:

$$\mathbf{w} = \left(I - I^k_{k-1}\tilde{G}^{-1}_{k-1} I^{k-1}_k G_k\right)\begin{bmatrix}\mathbf{w}_1\\ 0\end{bmatrix} \begin{matrix}\}\mathcal{N}_k \setminus \mathcal{N}_{k-1}\\ \}\mathcal{N}_{k-1}\end{matrix};$$

(4) *Coarse grid defect restriction:*

$$\begin{aligned}\mathbf{d}^{(k-1)} &= I_k^{k-1}\mathbf{d}^{(k)} - A_{21}^{(k)}\mathbf{w}_1 \\ &= I_k^{k-1}(\mathbf{d}^{(k)} - A^{(k)}\mathbf{w});\end{aligned}$$

(5) *Set $k = k-1$. If $k > 0$ go to (1), else:*
(6) *Solve on the* **coarsest level**:

$$A^{(0)}\mathbf{x}^{(0)} = \mathbf{d}^{(0)};$$

(II) **Backward recurrence.**
(7) *Interpolate result: Set $k := k+1$ and compute*

$$\mathbf{x}^{(k)} = I_{k-1}^k \mathbf{x}^{(k-1)};$$

(8) *Update fine grid residual:*

$$\begin{aligned}\mathbf{d}_1^{(k)} &:= \mathbf{d}_1^{(k)} - A_{12}^{(k)}\mathbf{x}^{(k-1)} \\ &= \mathbf{d}_1^{(k)} - [I \quad 0](I - G_k I_{k-1}^k \tilde{G}_{k-1}^{-1} I_k^{k-1})A^{(k)}\mathbf{x}^{(k)} \\ &= [I \quad 0](I - G_k I_{k-1}^k \tilde{G}_{k-1}^{-1} I_k^{k-1})(\mathbf{d}^{(k)} - A^{(k)}\mathbf{x}^{(k)});\end{aligned}$$

(9) *Solve:*

$$A_{11}^{(k)}\mathbf{w}_1 = \mathbf{d}_1^{(k)};$$

(10) *Change the basis:*

$$\mathbf{w} = (I - I_{k-1}^k \tilde{G}_{k-1}^{-1} I_k^{k-1} G_k)\begin{bmatrix}\mathbf{w}_1 \\ 0\end{bmatrix};$$

(11) *Finally set:*

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k)} + \mathbf{w}.$$

(12) *Set $k := k+1$. If $k < J$ go to (7), else* set

$$\mathbf{v} = \mathbf{x}^{(J)}.$$

END (Algorithm AWM HB multiplicative version)

One preconditioning solution step with $D = D^{(J)}$, the additive version of the AWM HB method, takes similarly the following form.

Algorithm AWM–HB. *(Additive version.) Given the problem*

$$D\mathbf{v} = \mathbf{d}\cdot$$

Initiate:

$$\mathbf{d}^{(J)} = \mathbf{d}\cdot$$

(I) **Forward recurrence.** *For $k = J$ down to 1 perform:*

(1) Compute:

$$\mathbf{d}_1^{(k)} = [I \quad 0]\left(I - G_k I_{k-1}^k \tilde{G}_{k-1}^{-1} I_k^{k-1}\right)\mathbf{d}^{(k)};$$

(2) Solve:

$$A_{11}^{(k)}\mathbf{w}_1 = \mathbf{d}_1^{(k)};$$

(3) Transform basis:

$$\mathbf{x}^{(k)} = \left(I - I_{k-1}^k \tilde{G}_{k-1}^{-1} I_k^{k-1} G_k\right)\begin{bmatrix}\mathbf{w}_1\\ 0\end{bmatrix}\begin{matrix}\}\mathcal{N}_k\setminus\mathcal{N}_{k-1}\\ \}\mathcal{N}_{k-1}\end{matrix};$$

(4) Coarse grid defect restriction:

$$\mathbf{d}^{(k-1)} = I_k^{k-1}\mathbf{d}^{(k)};$$

(5) Set $k = k - 1$. If $k > 0$ go to (1), else :
(6) Solve on the **coarsest level***:*

$$A^{(0)}\mathbf{x}^{(0)} = \mathbf{d}^{(0)};$$

(II) **Backward recurrence.**
(7) Interpolate result: Set $k := k + 1$ and compute

$$\mathbf{w} = I_{k-1}^k \mathbf{x}^{(k-1)};$$

(8) Update at level k:

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k)} + \mathbf{w};$$

(9) Set $k := k + 1$. If $k < J$ go to (7), else set*:*

$$\mathbf{v} = \mathbf{x}^{(J)}.$$

END (Algorithm AWM HB additive version)

For both algorithms it is readily seen that the above implementations require only actions of the stiffness matrices $A^{(k)}$, the mass matrices $M^{(k)}$ and the transformation matrices I_{k-1}^k and I_k^{k-1}. The approximate inverse actions of $A_{11}^{(k)}$ are needed via some inner iteration algorithm. Similarly, the action of $\tilde{G}_{k-1}^{-1}$ can be computed by approximate solution of the corresponding mass matrix problems using m steps of some simple inner iteration method. It is clear then, that at every discretization level k, one performs a number of arithmetic operations proportional to the degrees of freedom at that level denoted by N_k. If we have locally refined meshes the corresponding operations involve the stiffness and mass matrices computed for the subdomains where local refinement is present. Hence even in the case of locally refined meshes the cost of the algorithms AWM HB is proportional to $N = N_J$. The proportionality constant, however depends linearly on $m = O(\log J) = O(\log\log N)$. This dependence however, is not really seen in practice.

It is clear that the solution vector $\mathbf{v}$ in the additive algorithm is given by an expression of the form

$$\mathbf{v} = R_0^T {A^{(0)}}^{-1} R_0 \mathbf{d} + \sum_{k=1}^{J} R_k^T {A_{11}^{(k)}}^{-1} R_k \mathbf{d}, \tag{5.10}$$

where the matrices R_k are given accordingly as follows:

$$\begin{aligned} R_0^T &= I_0^J, \quad \text{for } k = 0, \\ R_k^T &= I_k^J \left[I - I_{k-1}^k \tilde{G}_{k-1}^{-1} I_k^{k-1} G_k \right] \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \text{for } k \geq 1. \end{aligned} \tag{5.11}$$

Here I_k^J stands for the natural coarse to fine transfer matrix from level k to the finest level.

It is clear from (5.10), (5.11) that the additive version of the algorithm can be implemented in parallel but this will require $O\left(N \log N \log \tau^{-1}\right)$ operations versus $O\left(N \log \tau^{-1}\right)$ operations in the consecutive implementation which is due to the more expensive simultaneous transfer of the same data from the finest grid directly to all the kth level grids for $k = J-1, \ldots, 0$ and vice versa.

5.3. Numerical experiments. Finally, we present some numerical results of the performance of the AWM HB algorithms, additive and multiplicative, as described in § 5.2.

Throughout this subsection Ω will be taken to be the unit square $(0,1)^2$. Furthermore, the spaces V_k will be the piecewise linear conforming spaces of functions that vanish on $\Gamma_D \equiv \{(x,0) \,:\, 0 < x < 1\} \cup \{(0,y) \,:\, 0 < y < 1\}$. The spaces V_k correspond to uniform triangulations of Ω of isosceles rightangled triangles of size $h_k = 2^{-k}$, $k = 0, 1, 2, \ldots, J$. The mass matrix problems involved in both algorithms AWM HB are solved by $m \geq 0$ steps of the CG method (no preconditioning). The problems with $A_{11}^{(k)}$ are solved by the CG method until high relative residual tolerance is reached. That is, one may assume that the actions of ${A_{11}^{(k)}}^{-1}$ are practically exact. The diffusion coefficient $a = a(x,y)$ of the elliptic bilinear form (2.1) was chosen:

$$a = 1 + x^2 + y^2.$$

In the test we varied the number of inner iterations $m = 0, 2, 4$ for solving the mass matrix problems in order to compute the actions of Q_{k-1}^a needed in the approximate wavelet modification of the HB. The multiplicative method with $m = 0$ corresponds to the method of Vassilevski [14], which in this case (i.e., exact blocks $A_{11}^{(k)}$) coincides with the HB MG method of Bank, Dupont and Yserentant [4]. The additive method with $m = 0$ is then a variant of the HB method of Yserentant [19].

In the tables below we show the number of iterations, *iter*, in the preconditioned conjugate gradient method applied to solve a problem

$$A\mathbf{x} = \mathbf{b},$$

$A = A^{(J)}$ for $J = 3, 4, 5, 6, 7$ (i.e., meshsize $h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$) with the same m for a given table; $m = 0, 2, 4$, and a right hand side vector $\mathbf{b}$ corresponding to a given prescribed solution $u(x,y)$.

The stopping criterion used is

$$\mathbf{r}^T W^{-1} \mathbf{r} \leq 10^{-18} \mathbf{r}_0^T W^{-1} \mathbf{r}_0,$$

where W is the preconditioner (i.e., M or D), $\mathbf{r}$ is the current residual and $\mathbf{r}_0 = (I - AW^{-1})\mathbf{b}$ is the initial residual. We also show in the tables the following average convergence rate factor $\rho = \left[\sqrt{\frac{\mathbf{r}^T W^{-1}\mathbf{r}}{\mathbf{r}_0^T W^{-1}\mathbf{r}_0}}\right]^{\frac{1}{\text{iter}}}$.

We also show approximations of the minimum ($\lambda_{\min}$) and maximum ($\lambda_{\max}$) eigenvalues of ${A^{(k)}}^{-1}M^{(k)}$ and ${A^{(k)}}^{-1}D^{(k)}$, $k = 3, \ldots, J$, computed by the Lanczos method.

It is noticed, In Tables 2 3, that the number of iterations (as well as the estimated extreme eigenvalues) tend to stabilize. This very well seen for the multiplicative AWM HB in Table 3, $m = 4$). That is, the choice of m is very little influenced by J. For example, $m = 2$ (see Table 2) gives also weakly sensitive values of the number of iterations (as well as eigenvalues) with J varying in the range $2 \leq J \leq 7$ for both additive and multiplicative AWH HB preconditioners. The improvement in spectral properties as well as in number of iterations over the pure HB method (see Table 1) is also clearly demonstrated. All this well illustrates the theory presented in the present paper.

TABLE 1. HB Multilevel Preconditioners; ($m = 0$)

levels	*Additive*				*Multiplicative*			
J	λ_{min}	λ_{max}	ρ	*iter*	λ_{min}	λ_{max}	ρ	*iter*
3	0.462	5.167	0.435	25	1.000	2.677	0.127	10
4	0.396	7.674	0.566	38	1.000	3.459	0.234	14
5	0.358	10.52	0.640	48	1.000	4.433	0.298	17
6	0.333	13.26	0.690	59	1.000	5.522	0.347	19
7	0.316	16.09	0.726	69	1.000	6.732	0.383	22

TABLE 2. AWM HB Multilevel Preconditioners; $m = 2$

levels	*Additive*				*Multiplicative*			
J	λ_{min}	λ_{max}	ρ	*iter*	λ_{min}	λ_{max}	ρ	*iter*
3	0.542	2.846	0.375	21	0.972	1.577	0.118	10
4	0.481	3.395	0.466	28	0.990	1.711	0.143	11
5	0.443	3.564	0.486	30	0.990	1.798	0.156	11
6	0.418	3.674	0.499	31	0.989	1.832	0.157	11
7	0.401	3.698	0.505	32	0.989	1.877	0.156	12

TABLE 3. AWM HB Multilevel Preconditioners; $m = 4$

levels	*Additive*				*Multiplicative*			
J	λ_{min}	λ_{max}	ρ	*iter*	λ_{min}	λ_{max}	ρ	*iter*
3	0.544	2.862	0.364	21	0.997	1.572	0.098	9
4	0.481	3.393	0.447	26	0.999	1.724	0.130	10
5	0.442	3.633	0.668	28	0.998	1.808	0.143	11
6	0.417	3.722	0.484	30	0.999	1.856	0.147	11
7	0.399	3.769	0.498	32	0.999	1.905	0.147	11

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