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COMPUTATIONAL AND APPLIED MATHEMATICS

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June 1995
CAM Report 95-29

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CAU95-29

PROBABILISTIC ANALYSIS OF GAUSSIAN ELIMINATION WITHOUT PIVOTING*

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Abstract. The numerical instability of Gaussian elimination is proportional to the size of the L and U factors that it produces. The worst case bounds are well known. For the case without pivoting, breakdowns can occur and it is not possible to provide *a priori* bounds for L and U. For the partial pivoting case, the worst case bound is $O(2^m)$, where m is the size of the system. Yet, these worst case bounds are seldom achieved, and in particular Gaussian elimination with partial pivoting is extremely stable in practice. Surprisingly, there has been relatively little theoretical study of the "average" case behaviour. The purpose of our paper is to provide a probabilistic analysis of the case without pivoting. The distribution we use for the entries of A is the normal distribution with mean 0 and unit variance. We first derive the distributions of the entries of L and U. Based on this, we prove that the probability of the occurence of a pivot less than ϵ in magnitude is $O(\epsilon)$. We also prove that the probabilities $Prob(\|U\|_\infty/\|A\|_\infty > m^{2.5})$ and $Prob(\|L\|_\infty > m^3)$ decay algebraically to zero as m tends to infinity. Numerical experiments are presented to support the theoretical results.

Key Words. Gaussian elimination, pivot, growth factor, density function

AMS(MOS) subject classification. 65F05, 65G05

1. Introduction.

Gaussian elimination (GE) is the most common general method for solving an $m \times m$, square, dense, unstructured linear system $Ax = b$. Together with partial pivoting, the method is extremely stable in practice. However, this stability cannot be guaranteed. The worst case examples are well known: without pivoting, breakdowns can occur and even with partial pivoting, the "growth factor" can be as large as $O(2^m)$ (and can occur in practical applications [5]). Obviously, the practical numerical stability of GE can only be explained by an "average case" analysis. Surprisingly, there has been relatively few studies on this topic in the literature. The purpose of our paper is to provide a rather complete analysis for the case without pivoting.

Theoretical studies on the numerical stability of GE have been made since 1940s by a great number of authors, for example, Turing [10], von Neumann and Goldstine [11], [12], Wilkinson [13], [14], and so on. Recently, Trefethen and Schreiber [8] considered the average case analysis. Among their many results, they observed that for many distributions of matrices, the matrix elements after the first few steps of Gaussian elimination with (partial or complete) pivoting are approximately normally distributed. They also found that, for $m \leq 1024$, the average growth factor (normalized by the standard deviation of the initial matrix elements) is within a few percent of $m^{2/3}$ for the partial pivoting case and approximately $m^{1/2}$ for the complete pivoting case.

* Original version: June, 1995. Revised version: October 11, 1995

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Following Trefethen and Schreiber, we study the probability of small pivots and large growth factors in this paper. However, we will only consider the case without pivoting. We are doing so for three reasons. The first is quite obvious: the non-pivoting case is far easier to analyze than the pivoting case. In particular, we are able to derive in close form the density functions of the elements of the LU factors and probabilistic bounds for the occurence of small pivots and the growth factors. The second reason is that, with the advent of parallel computing, there is more incentive to trade off the stability of partial pivoting for the higher performance of simpler but possibly less stable forms of GE, including no pivoting. Finally, we are hoping that our results for GE without pivoting will be useful in the analysis of, as well as providing a basis of comparison for, the partial pivoting case.

Throughout the paper, we suppose $X \in R^{m\times m}$ is a random matrix with independent and identically distributed elements which are $N(0,1)$, the normal distribution with mean 0 and variance 1. This choice is motivated by the empirical results of Trefethen and Schreiber mentioned earlier. Matrices of this type have also been studied by Edelman [2], [3], who derived the expected singular values.

In §2 and §3, we derive the density functions of the entries of L and U respectively, where $X = LU$, the LU factorization of X. In §4, we prove that the probability of the occurence of a pivot less than ϵ in magnitude is $O(\epsilon)$.[1] In §5, we derive bounds on the probabilities of large growth factors. In particular, we prove that the probabilities $Prob(||U||_\infty/||A||_\infty > m^{2.5})$ and $Prob(||L||_\infty > m^3)$ decay algebraically to zero as m tends to infinity. Finally, we present experimental results in §6. We observe that the probabilities $Prob(m \leq ||L||_\infty < m^{1.5})$ and $Prob(m \leq ||U||_\infty/||A||_\infty < m^{1.5})$ tend to one as m goes to infinity. This indicates that our theoretical bounds are not the tightest possible but not too loose either.

2. Density Function of u_{pq}. Let X be an $m \times m$ real matrix with independent and identically distributed elements from $N(0,1)$, to which we simply refer as "$X \sim \mathcal{N}_m(O,I)$". Let $X = LU$, where L is an unit lower triangular matrix and U is an upper triangular matrix, be the LU factorization of X [2]. The (p,q)-th $(p \leq q)$ entry u_{pq} of U and the entries of X have the following relation.

LEMMA 1. *Let $X = LU$ be the LU factorization of X. Then*

$$u_{pq} = x_{pq} - x_{p*}^T X_{p-1}^{-1} x_{*q},$$

where

$$\begin{aligned} x_{p*} &= (x_{p1}, \cdots, x_{pp-1})^T, \\ x_{*q} &= (x_{1q}, \cdots, x_{p-1q})^T, \end{aligned}$$

and X_{p-1} is the $(p-1) \times (p-1)$ leading principal submatrix of X.

[1] We note that Foster [4] has studied the probability of large diagonal elements in the QR factorization of a rectangular matrix A.

[2] Since they just form a set of measure zero, we ignore matrices for which the Gaussian elimination fails.

Proof. Permuting the p-th and q-th columns of X and U simultaneously on both sides of $X = LU$ and then comparing the corresponding blocks, we find

$$\begin{bmatrix} X_{p-1} & x_{*q} \\ x_{p*}^T & x_{pq} \end{bmatrix} = \begin{bmatrix} L_{p-1} & 0 \\ l_{p*}^T & 1 \end{bmatrix} \begin{bmatrix} U_{p-1} & u_{*q} \\ 0 & u_{pq} \end{bmatrix}$$

where

$$\begin{aligned} l_{p*} &= (l_{p1}, \cdots, l_{pp-1})^T, \\ u_{*q} &= (u_{1q}, \cdots, u_{p-1q})^T \end{aligned}$$

and where L_{p-1} and U_{p-1} are the $(p-1) \times (p-1)$ leading principal submatrices of L and U respectively. It follows that

$$X_{p-1} = L_{p-1} U_{p-1}, \qquad l_{p*}^T = x_{p*}^T U_{p-1}^{-1},$$

$$u_{*q} = L_{p-1}^{-1} x_{*q}, \qquad u_{pq} = x_{pq} - l_{p*}^T u_{*q}$$

and these imply the desired equation. □

Let H be an $(p-1) \times (p-1)$ orthogonal matrix, e.g., a Householder matrix, such that

$$x_{p*}^T H = (0, \cdots, 0, s) \equiv \eta^T$$

with $s \geq 0$. Then

$$\begin{aligned} u_{pq} &= x_{pq} - \eta^T \left(X_{p-1} H\right)^{-1} x_{*q} \\ &\equiv x_{pq} - \eta^T Y^{-1} x_{*q}. \end{aligned}$$

It can be shown that the entries s, x_{pq}, x_{iq} and y_{ij}, $i, j = 1, \cdots, p-1$, are mutually independent and all x_{pq}, x_{iq} and y_{ij}, $i, j = 1, \cdots, p-1$, are $N(0,1)$ while s^2 is χ^2_{p-1}. The proof basically follows the approach in [7] and [9]. We now decompose Y as

$$Y = QR$$

where Q is an $(p-1) \times (p-1)$ orthogonal matrix and R an $(p-1) \times (p-1)$ upper triangular matrix with positive diagonal elements. We then further have

$$\begin{aligned} u_{pq} &= x_{pq} - \eta^T R^{-1} Q^T x_{*q} \\ &\equiv x_{pq} - \eta^T R^{-1} \omega \qquad (1) \\ &= x_{pq} - \frac{s w_{p-1}}{r_{p-1p-1}}. \end{aligned}$$

Again, the variables s, x_{pq}, w_i and r_{ij}, $i \leq j$, $i, j = 1, \cdots, p-1$, are independent. s^2 is χ^2_{p-1} and r_{ii}^2 is χ^2_{p-i}, $i = 1, \cdots, p-1$ and all others are $N(0,1)$. The proof basically follows the approach in [7] and [9].

Since the variables on the right-hand-side of (1) are independent and their density functions are known, it is straightforward to determine the density function of u_{pq}.

THEOREM 1. *Suppose $X \sim \mathcal{N}_m(O, I)$ and let $X = LU$ be the LU factorization of X. Then the density function of the (p,q)-th entry of U is*

$$(2)\; f_{u_{pq}}(t) = \frac{\sqrt{2}}{\pi}\frac{\Gamma(p/2)}{\Gamma((p-1)/2)}\left(\sum_{i=0}^{\lfloor\frac{p-3}{2}\rfloor}\xi_{i,p}\,t^{-2i-2} + (-1)^{\lfloor\frac{p-1}{2}\rfloor}\zeta_p\,t^{-p+1}exp(-\frac{1}{2}t^2)\,\phi_p(t)\right)$$

where

$$\xi_{i,p} = \begin{cases} (-1)^i\prod_{j=0}^{i-1}(p-2j-3) & i > 0 \\ 1 & i = 0\,, \end{cases}$$

$$\zeta_p = \begin{cases} (p-3)!! & p > 3 \\ 1 & p = 2,\ 3\,, \end{cases}$$

$$\phi_p(t) = \left(\int_0^t exp\left(\frac{1}{2}x^2\right)dx\right)^{p-1-2\lfloor(p-1)/2\rfloor}$$

and where $-\infty < t < \infty$, $2 \le p \le q$.

Proof. Since the variables r^2_{p-1p-1} ($r_{p-1p-1} \ge 0$), s^2 ($s \ge 0$), w_{p-1} and x_{pq} in (1) are χ^2_1, χ^2_{p-1} and $N(0,1)$ respectively, the density functions of r_{p-1p-1}, s, w_{p-1} and x_{pq} are given as follows,

$$f_{r_{p-1p-1}}(t) = \begin{cases} \sqrt{\frac{2}{\pi}}exp\,(-t^2/2) & t > 0 \\ 0 & t \le 0\,, \end{cases}$$

$$f_s(t) = \begin{cases} \frac{1}{2^{(p-3)/2}\Gamma((p-1)/2)}t^{p-2}exp\,(-t^2/2) & t > 0 \\ 0 & t \le 0\,, \end{cases}$$

$$f_{w_{p-1}}(t) = \frac{1}{\sqrt{2\pi}}exp\,(-t^2/2)$$

and

$$f_{x_{pq}}(t) = \frac{1}{\sqrt{2\pi}}exp\,(-t^2/2)\,.$$

Since r_{p-1p-1}, s, w_{p-1} and x_{pq} are independent, their joint density function is given by

$$f(r,s,w,x) = f_{r_{p-1p-1}}(r)\, f_s(s)\, f_{w_{p-1}}(w)\, f_{x_{pq}}(x)$$
$$= \begin{cases} \tilde{c}\, s^{p-2} exp\left(-\frac{1}{2}(s^2+r^2+w^2+x^2)\right) & r,\ s > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\tilde{c} = \frac{1}{\pi^{3/2}2^{(p-2)/2}\Gamma((p-1)/2)}$. Thus, the distribution function $F_{u_{pq}}(\alpha)$ of u_{pq} is

$$F_{u_{pq}}(\alpha) = \tilde{c} \iiiint_{u_{pq}\le\alpha} s^{p-2} exp\left(-\frac{1}{2}(s^2+r_{p-1p-1}^2+w_{p-1}^2+x_{pq}^2)\right) ds\, dr_{p-1p-1} dw_{p-1} dx_{pq}$$

$$= \tilde{c} \iiiint_{x_{pq}-\frac{s\, w_{p-1}}{r_{p-1p-1}}\le\alpha} s^{p-2} exp\left(-\frac{1}{2}(s^2+r_{p-1p-1}^2+w_{p-1}^2+x_{pq}^2)\right) \times$$

$$ds\, dr_{p-1p-1} dw_{p-1} dx_{pq}\,.$$

Using Lemma 3 in Appendix, we can show that

$$f_{u_{pq}}(t) = \tilde{c}\int_0^\infty dx \int_{-\infty}^\infty \frac{x^{p-1}}{x^2+y^2} exp\left(-\frac{1}{2}\left(x^2+(y+t)^2\right)\right) dy \tag{3}$$

which can be further reduced to (2) by Lemma 4 in Appendix. □

3. Density Function of l_{pq}. Similar to the derivation of the density function of u_{pq}, we first establish a relation between l_{pq} and the entries of X and then simplify it. Let $X = LU$ and $X^T = \tilde{L}\tilde{U}$ be the LU factorizations of X and X^T respectively. Set $\tilde{D} = \text{diag}(\tilde{u}_{11}, \cdots, \tilde{u}_{mm})$. Thus, $X^T = \tilde{L}\tilde{D}\tilde{D}^{-1}\tilde{U}$. So $X = \left(\tilde{D}^{-1}\tilde{U}\right)^T\left(\tilde{L}\tilde{D}\right)^T$. Note that $\left(\tilde{D}^{-1}\tilde{U}\right)^T$ is unit lower triangular and $\left(\tilde{L}\tilde{D}\right)^T$ upper triangular. By the uniqueness of the LU factorization of X, we have

$$L = \left(\tilde{D}^{-1}\tilde{U}\right)^T.$$

Hence

$$l_{pq} = \tilde{u}_{qp}/\tilde{u}_{qq}$$

for $1 \le q < p \le m$. By Lemma 1,

$$\tilde{u}_{qp} = x_{pq} - x_{*q}^T X_{q-1}^{-T} x_{p*}$$

and

$$\tilde{u}_{qq} = x_{qq} - x_{*q}^T X_{q-1}^{-T} x_{q*}$$

where

$$\begin{aligned} x_{p*} &= (x_{p1}, \cdots, x_{pq-1})^T, \\ x_{q*} &= (x_{q1}, \cdots, x_{qq-1})^T, \\ x_{*q} &= (x_{1q}, \cdots, x_{q-1q})^T \end{aligned}$$

and X_{q-1} is the $(q-1) \times (q-1)$ leading principal submatrix of X. We now let H be an $(q-1) \times (q-1)$ orthogonal matrix such that

$$x_{*q}^T H = (0, \cdots, 0, s) \equiv \eta^T$$

with $s \geq 0$. Then

$$\begin{aligned} l_{pq} &= \frac{x_{pq} - \eta^T \left(X_{q-1}^T H\right)^{-1} x_{p*}}{x_{qq} - \eta^T \left(X_{q-1}^T H\right)^{-1} x_{q*}} \\ &\equiv \frac{x_{pq} - \eta^T Y^{-1} x_{p*}}{x_{qq} - \eta^T Y^{-1} x_{q*}}. \end{aligned}$$

As in the case of u_{pq} in §2, all the entries in the above expression are mutually independent and s^2 is χ^2_{q-1} while others are $N(0,1)$. Let

$$Y = QR$$

be the QR factorization of Y where R has positive diagonal elements. Then the expression can be reduced to

$$\begin{aligned} l_{pq} &= \frac{x_{pq} - \eta^T R^{-1} Q^T x_{p*}}{x_{qq} - \eta^T R^{-1} Q^T x_{q*}} \\ &\equiv \frac{x_{pq} - \eta^T R^{-1} \omega}{x_{qq} - \eta^T R^{-1} \mu} \qquad (4) \\ &= \frac{r_{q-1q-1} x_{pq} - s\omega_{q-1}}{r_{q-1q-1} x_{qq} - s\mu_{q-1}}. \end{aligned}$$

The entries x_{pq}, x_{qq}, ω_i, μ_i and $r_{ij} (i < j)$ are $N(0,1)$ while s^2 is χ^2_{q-1} and r^2_{ii} is χ^2_{q-i}, where $i = 1, \cdots, q-1$, $j = 2, \cdots, q-1$. They are all independent.

THEOREM 2. *Suppose $X \sim \mathcal{N}_m(O, I)$ and let $X = LU$ be the LU factorization of X. Then the density function of the (p,q)-th entry of L is*

$$f_{l_{pq}}(t) = \frac{1}{\pi} \frac{1}{1+t^2} \qquad (5)$$

where $-\infty < t < \infty$ and $1 \leq q < p \leq m$.

Proof. Suppose $q > 1$ and let $F_{l_{pq}}(\alpha)$ be the distribution function of l_{pq}. Since the joint density function of r_{q-1q-1}, x_{pq}, x_{qq}, ω_{q-1}, μ_{q-1} and s is

$$f(r_{q-1q-1}, x_{pq}, x_{qq}, \omega_{q-1}, \mu_{q-1}, s)$$

$$= \begin{cases} \dfrac{1}{2^{q/2}\pi^{5/2}\Gamma((q-1)/2)} s^{q-2} \times \\ exp\left(-\dfrac{1}{2}(r^2_{q-1q-1} + x^2_{pq} + x^2_{qq} + \omega^2_{q-1} + \mu^2_{q-1} + s^2)\right) & r_{q-1q-1}, s > 0 \\ 0 & \text{otherwise} \end{cases},$$

and since

$$F_{l_{pq}}(\alpha) = \int\cdots\int_{l_{pq}\le\alpha} f(r_{q-1q-1}, x_{pq}, x_{qq}, \omega_{q-1}, \mu_{q-1}, s)dr_{q-1q-1}dx_{pq}dx_{qq}d\omega_{q-1}d\mu_{q-1}ds,$$

(5) holds from (4) and Lemmas 5 and 6 in Appendix. The case where $q = 1$ is quite trivial if we notice that l_{p1} is the division of two $N(0, 1)$ variables x_{p1} and x_{11}. □

4. Probability of Small Pivot. In practice, if one of the pivot elements u_{pp} is zero or smaller in magnitude than a preset tolerance ϵ, Gaussian elimination will fail. In this section, we describe the probability of the occurrence of such a situation. First, we give a bound on the density function $f_{u_{pq}}(t)$ of u_{pq}.

LEMMA 2.

$$\frac{1}{\pi\sqrt{2}}\frac{\Gamma(p/2)}{\Gamma((p+1)/2)} exp\left(-\frac{t^2}{2}\right) \le f_{u_{pq}}(t) \le \frac{1}{\pi}\frac{\Gamma(p/2)}{\Gamma((p+1)/2)} exp\left(\frac{t^2}{2}\right)$$

for $-\infty < t < \infty$ *and* $p \ge 2$.

Proof. From (3), we have

$$f_{u_{pq}}(t) = \tilde{c}\int_0^\infty dx\int_0^\infty \frac{x^{p-1}}{x^2+y^2}\left(exp\left(-\frac{1}{2}\left(x^2+(y+t)^2\right)\right) + exp\left(-\frac{1}{2}\left(x^2+(y-t)^2\right)\right)\right)dy.$$

Letting $y = xz$, this can be written as

$$f_{u_{pq}}(t) = \tilde{c}\int_0^\infty dx\int_0^\infty \frac{x^{p-2}}{1+z^2}\left(exp\left(-\frac{1}{2}\left(x^2+(xz+t)^2\right)\right) + exp\left(-\frac{1}{2}\left(x^2+(xz-t)^2\right)\right)\right)dz$$

$$= \tilde{c}\int_0^\infty dx\int_0^\infty \frac{x^{p-2}}{1+z^2} exp\left(-\frac{1}{2}((1+z^2)x^2+t^2)\right)(exp(-xzt) + exp(xzt))\,dz.$$

(6)

Since $exp(\xi) + exp(-\xi) \ge 2$, we have

$$f_{u_{pq}}(t) \ge 2\tilde{c}\int_0^\infty dx\int_0^\infty \frac{x^{p-2}}{1+z^2} exp\left(-\frac{1}{2}((1+z^2)x^2+t^2)\right)dz$$

$$= 2\tilde{c}\, exp\left(-\frac{1}{2}t^2\right)\int_0^\infty dz\int_0^\infty \frac{x^{p-2}}{1+z^2} exp\left(-\frac{1}{2}(1+z^2)x^2\right)dx.$$

Let $w = \frac{1}{2}(1+z^2)x^2$. Then

$$\begin{aligned}
f_{u_{pq}}(t) &\geq \frac{\sqrt{2}}{\pi^{3/2}\Gamma((p-1)/2)}\, exp\left(-\frac{1}{2}t^2\right)\int_0^\infty dz \int_0^\infty (1+z^2)^{-(p+1)/2} w^{(p-3)/2} exp\,(-w)\,dw \\
&= \frac{1}{\pi\sqrt{2}}\,\frac{\Gamma(p/2)}{\Gamma((p+1)/2)}\, exp\left(-\frac{1}{2}t^2\right).
\end{aligned}$$

Moreover, from (6) we have

$$\begin{aligned}
f_{u_{pq}}(t) &\leq 2\tilde{c}\int_0^\infty dx \int_0^\infty \frac{x^{p-2}}{1+z^2} exp\left(-\frac{1}{2}\left((1+z^2)x^2+t^2\right)\right) exp\,(xz|t|)\,dz \\
&\leq 2\tilde{c}\int_0^\infty dx \int_0^\infty \frac{x^{p-2}}{1+z^2} exp\left(-\frac{1}{2}\left((1+z^2)x^2+t^2\right)\right) exp\left(\frac{1}{2}\left(\frac{1}{2}(xz)^2+2t^2\right)\right) dz \\
&= 2\tilde{c}\, exp\left(\frac{1}{2}t^2\right)\int_0^\infty dz \int_0^\infty \frac{x^{p-2}}{1+z^2} exp\left(-\frac{1}{2}\left(1+\frac{1}{2}z^2\right)x^2\right) dx.
\end{aligned}$$

Letting $u = \frac{1}{2}\left(1+\frac{1}{2}z^2\right)x^2$, we finally have

$$\begin{aligned}
f_{u_{pq}}(t) &\leq \frac{\sqrt{2}}{\pi^{3/2}\Gamma((p-1)/2)}\, exp\left(\frac{1}{2}t^2\right)\int_0^\infty dz \int_0^\infty (1+z^2)^{-1}\left(1+\frac{1}{2}z^2\right)^{-(p-1)/2} \times \\
&\quad u^{(p-3)/2} exp\,(-u)\,du \\
&= \frac{\sqrt{2}}{\pi^{3/2}}\, exp\left(\frac{1}{2}t^2\right)\int_0^\infty (1+z^2)^{-1}\left(1+\frac{1}{2}z^2\right)^{-(p-1)/2} dz \\
&\leq \frac{\sqrt{2}}{\pi^{3/2}}\, exp\left(\frac{1}{2}t^2\right)\int_0^\infty \left(1+\frac{1}{2}z^2\right)^{-(p+1)/2} dz \\
&= \frac{2}{\pi^{3/2}}\, exp\left(\frac{1}{2}t^2\right)\int_0^\infty (1+z^2)^{-(p+1)/2} dz \\
&= \frac{1}{\pi}\,\frac{\Gamma(p/2)}{\Gamma((p+1)/2)}\, exp\left(\frac{1}{2}t^2\right). \quad \square
\end{aligned}$$

To make the statements below neatly, we use a shorthand notation here. For given $\epsilon > 0$ and $1 \leq p \leq m$, we define

$$E_{p,\epsilon} = \{X \in R^{m\times m} \,|\, |u_{pp}| < \epsilon\}.$$

Then the event that at least one u_{pp} has $|u_{pp}| < \epsilon$ is naturally denoted by $\bigcup_{p=1}^{m} E_{p,\epsilon}$.

COROLLARY 1. *Suppose $X \sim \mathcal{N}_m(O, I)$ and let $X = LU$ be the LU factorization of X. Given $\epsilon > 0$ and $1 \le p \le m$. Then*

$$Prob(E_{p,\epsilon}) = \alpha_{p,\epsilon} \frac{\Gamma(p/2)}{\Gamma((p+1)/2)}$$

where $\dfrac{\sqrt{2}}{\pi} \displaystyle\int_0^{\epsilon} exp\left(-\frac{1}{2}t^2\right) dt \le \alpha_{p,\epsilon} \le \frac{2}{\pi} \int_0^{\epsilon} exp\left(\frac{1}{2}t^2\right) dt.$

Proof. For the case where $p = 1$, it is sufficient to note that

$$Prob(E_{1,\epsilon}) = Prob(|x_{11}| < \epsilon) = \frac{1}{\sqrt{2\pi}} \int_{-\epsilon}^{\epsilon} exp\left(-\frac{1}{2}t^2\right) dt.$$

Other cases are just the direct results of Lemma 2. $\square$

THEOREM 3. *Suppose $X \sim \mathcal{N}_m(O, I)$ and let $X = LU$ be the LU factorization of X. Then*

$$Prob\left(\bigcup_{p=1}^{m} E_{p,\epsilon}\right) \le c(m)\, \epsilon\, exp\left(\frac{1}{2}\epsilon^2\right) \tag{7}$$

where $c(m) = \dfrac{2}{\pi} \displaystyle\sum_{p=1}^{m} \frac{\Gamma(p/2)}{\Gamma((p+1)/2)}$.

Proof. Since $Prob(\bigcup_{p=1}^{m} E_{p,\epsilon}) \le \sum_{p=1}^{m} Prob(E_{p,\epsilon})$, (7) follows by Corollary 1. $\square$

The coefficient $c(m)$ of $\epsilon\, exp\left(\frac{1}{2}\epsilon^2\right)$ is a rather slow-growing function of m. In fact, it is about 1800 even when $m = 10^6$. So, if ϵ is small enough, (7) will certainly give a satisfying bound for the desirable probability. Moreover, the right hand side of (7) is approximately linear with ϵ for small ϵ.

5. Probability of Large Growth Factor. When Gaussian elimination is performed on an $m \times m$ matrix A in floating point arithmetic, the computed LU factors $\hat{L}$ and $\hat{U}$ are produced. Then, by solving two corresponding triangular systems, we obtain the solution $\hat{x}$ to $Ax = b$. The computed solution $\hat{x}$ satisfies

$$(A + E)\,\hat{x} = b$$

with

$$|E| \le m\boldsymbol{u}\left(3\,|A| + 5\,|\hat{L}|\,|\hat{U}|\right) + O(\boldsymbol{u}^2)$$

where $\boldsymbol{u}$ is the unit roundoff and where, for any matrix M, we use $|M|$ to denote the matrix obtained by taking the absolute value of the elements of M, see, for instance, [6, Theorem 3.3.2]. From this, it follows that

$$\|E\|_\infty \le m\boldsymbol{u}\|A\|_\infty \left(3 + 5\|\hat{L}\|_\infty \frac{\|\hat{U}\|_\infty}{\|A\|_\infty}\right) + O(\boldsymbol{u}^2).$$

We define the growth factors ρ_L and ρ_U to be

$$\rho_L = \|L\|_\infty, \qquad \rho_U = \|U\|_\infty / \|A\|_\infty .$$

It is possible that ρ_L and ρ_U can be very large because small pivots can appear. The following Theorem gives probabilistic bounds on the sizes of ρ_L and ρ_U.

THEOREM 4. *Suppose* $X \sim \mathcal{N}_m(O, I)$ *and let* $X = LU$ *be the LU factorization of* X. *Then there exist numbers* $1 > b > 0$ *and* $c > 0$, *independent of* m, *such that*

$$Prob(\rho_U > r) \;\leq\; \frac{c}{r} m^{5/2} + \min\left(\frac{c}{r} m^{7/2}, \frac{1}{m}\right) + b^m$$

and

$$Prob(\rho_L > r) \;\leq\; \frac{c}{r} m^3$$

for any $r \geq 1$.

Proof. We first claim that there exists a $c_1 > 0$, independent of m, such that

$$Prob(\|U\|_\infty > r) \;\leq\; \frac{c_1}{r} m^{7/2}. \tag{8}$$

In fact, by (3), we have

$$\begin{aligned}
f_{u_{pq}}(t) &= \tilde{c}\int_0^\infty dx \int_{|y+t|\geq|t|/2} \frac{x^{p-1}}{x^2+y^2} exp\left(-\frac{1}{2}\left(x^2+(y+t)^2\right)\right) dy + \\
&\quad \tilde{c}\int_0^\infty dx \int_{|y+t|<|t|/2} \frac{x^{p-1}}{x^2+y^2} exp\left(-\frac{1}{2}\left(x^2+(y+t)^2\right)\right) dy \\
&\leq \tilde{c}\int_0^\infty dx \int_{|y+t|\geq|t|/2} \frac{x^{p-1}}{x^2+y^2} exp\left(-\frac{1}{2}(x^2+\frac{1}{4}t^2)\right) dy + \\
&\quad \tilde{c}\int_0^\infty dx \int_{|y+t|<|t|/2} \frac{x^{p-1}}{x^2+t^2/4} exp\left(-\frac{1}{2}\left(x^2+(y+t)^2\right)\right) dy \\
&\leq \tilde{c}\, exp\left(-\frac{1}{8}t^2\right)\int_0^\infty dx \int_{-\infty}^\infty \frac{x^{p-1}}{x^2+y^2} exp\left(-\frac{1}{2}x^2\right) dy + \\
&\quad \frac{4\tilde{c}}{t^2}\int_0^\infty dx \int_{-\infty}^\infty x^{p-1} exp\left(-\frac{1}{2}\left(x^2+(y+t)^2\right)\right) dy \\
&= \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{8}t^2\right) + \frac{4\sqrt{2}}{\pi}\frac{\Gamma(p/2)}{\Gamma((p-1)/2)}\frac{1}{t^2} \\
&\leq \left(\frac{4}{\sqrt{p}} + \frac{4\sqrt{2}}{\pi\sqrt{p}}\frac{\Gamma(p/2)}{\Gamma((p-1)/2)}\right)\frac{\sqrt{p}}{t^2}.
\end{aligned}$$

Since

$$\lim_{k\to+\infty}\left(\frac{4}{\sqrt{k}}+\frac{4\sqrt{2}}{\pi\sqrt{k}}\frac{\Gamma(k/2)}{\Gamma((k-1)/2)}\right)$$

exists by Stirling's formula

$$\lim_{x\to+\infty}\frac{\Gamma(x+1)}{x^x exp\,(-x)\sqrt{2\pi x}} \;=\; 1,$$

we can find a c_2 such that

$$\frac{4}{\sqrt{k}}+\frac{4\sqrt{2}}{\pi\sqrt{k}}\frac{\Gamma(k/2)}{\Gamma((k-1)/2)} \;\le\; c_2$$

for all k. Hence

$$f_{u_{pq}}(t) \;\le\; c_2\sqrt{p}/t^2\,.$$

Therefore

$$\begin{aligned}
Prob(\|U\|_\infty > r) &\le \sum_{p=1}^{m}\sum_{q=p}^{m} P(|u_{pq}| > r/m) \\
&= \sum_{p=1}^{m}\sum_{q=p}^{m}\int_{|t|>r/m} f_{pq}(t)\,dt \\
&\le \sum_{p=1}^{m}\sum_{q=p}^{m}\int_{|t|>r/m}\frac{c_2\sqrt{p}}{t^2}dt \\
&\le \frac{c_2c_3}{r}m^{7/2}
\end{aligned}$$

for some $c_3>0$, independent of m. The existence of c_3 is due to the existence of the limit

$$\lim_{k\to+\infty}\frac{1}{k^{5/2}}\sum_{p=1}^{k}(k-p+1)\sqrt{p} \;=\; \int_0^1(1-t)\sqrt{t}\,dt\,.$$

We set $c_1=c_2c_3$ and then (8) is proven. For proving the first inequality in the theorem, we note that the expected value μ and the variance σ^2 of the variable $x_1\equiv\sum_{q=1}^{m}|x_{1q}|$ are

$$\mu=m\sqrt{\frac{2}{\pi}}\quad,\quad \sigma^2=\left(1-\frac{2}{\pi}\right)m\,.$$

Setting $\varepsilon=m\sqrt{1-\frac{2}{\pi}}$ in Chebyshev's inequality [1, p.183]

$$Prob\,(|x_1-\mu|\ge\varepsilon)\le\frac{\sigma^2}{\varepsilon^2}\,,$$

we have

$$(9) \qquad Prob\,(x_1 < mc_4) \;\leq\; \frac{1}{m},$$

where $c_4 = \sqrt{\frac{2}{\pi}} - \sqrt{1 - \frac{2}{\pi}}$. Combining (8) and (9) we find

$$\begin{aligned}
Prob\,(\rho_U > r) &= Prob\,(\|U\|_\infty > r\|A\|_\infty) \\
&\leq Prob\,(\|U\|_\infty > rx_1) \\
&= Prob\,(\|U\|_\infty > rx_1\,,\, x_1 \geq mc_4) + Prob\,(\|U\|_\infty > rx_1\,,\, mc_4 > x_1 > 1) + \\
&\quad Prob\,(\|U\|_\infty > rx_1\,,\, x_1 \leq 1) \\
&\leq Prob\,(\|U\|_\infty > mrc_4) + \min\left(Prob\,(\|U\|_\infty > r)\;,\; Prob\,(x_1 < mc_4)\right) + \\
&\quad Prob\left(|x_{1q}| \leq 1, \forall\, 1 \leq q \leq m\right) \\
&\leq \frac{c_1}{c_4 r} m^{5/2} + \min\left(\frac{c_1}{r} m^{7/2}\,,\, \frac{1}{m}\right) + \left(\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} exp\left(-\frac{1}{2}t^2\right) dt\right)^m .
\end{aligned}$$

Finally,

$$\begin{aligned}
Prob\,(\rho_L > r) &\leq \sum_{p=2}^{m} \sum_{q=1}^{p-1} Prob\left(|l_{pq}| > \frac{r-1}{m-1}\right) \\
&= \frac{1}{\pi} \sum_{p=2}^{m} \sum_{q=1}^{p-1} \int_{|t| > \frac{r-1}{m-1}} \frac{1}{1+t^2} dt \\
&\leq \frac{c_5}{r} m^3
\end{aligned}$$

for some c_5. $\square$

6. Numerical Experiments. In this section, we present numerical results to support Theorems 1 - 4. All our calculations have been carried out in MATLAB 4.2c on SUN workstations.

In our first experiment, 595000 matrices of dimension $m = 31$ were selected at random from the class $\mathcal{N}_{31}(O, I)$. Then Gaussian elimination was applied to each of the matrices and then statistics on the elements $l_{13,12}$, $l_{30,29}$, $u_{12,12}$ and $u_{31,31}$ were accumulated. The data are plotted in Figures 1 - 4 together with the corresponding functions indicated in Theorems 1 and 2. In order to make clearer the difference between Figure 1(b) and 2(b), we present them together in Figure 5(a).

The purpose of our second experiment is to test formula (7). Matrices of several dimensions m were selected at random from $\mathcal{N}_m(O, I)$, with the sample size varying. A

m	ϵ	*Sample Size*	*Frequency*	*Empirical probability*	*Theoretical bound*
25	10^{-5}	10^5	5	5×10^{-5}	8.2853×10^{-5}
50	10^{-3}	10^4	90	0.009	0.012
50	10^{-4}	10^4	8	8×10^{-4}	0.0012
50	10^{-5}	10^4	0	0	1.2014×10^{-4}
50	10^{-5}	10^5	9	9×10^{-5}	1.2014×10^{-4}
75	10^{-3}	10^4	89	0.0089	0.0149
75	10^{-4}	10^4	8	8×10^{-4}	0.0015
75	10^{-5}	10^4	0	0	1.4876×10^{-4}
100	10^{-3}	10^4	115	0.0115	0.0173

TABLE 1
Probabilities of small pivot.

few tolerances ϵ were used. The results are outlined in Table 1. The frequency column of the table provides the numbers of matrices which, in their LU factors, have at least one u_{pp} less than ϵ in magnitude. By comparing with the empirical probabilities, we conclude that the bound given in (7) is a fairly tight one.

Finally, if we set $r = m^\alpha, \alpha > 2.5$ for ρ_U and $\alpha > 3$ for ρ_L, in Theorem 4, then we can see that the probabilities $Prob(\rho_L > m^\alpha)$ and $Prob(\rho_U > m^\alpha)$ decrease with m increasing. In fact, empirically this is true even for smaller α, say, $\alpha > 1.5$ for both ρ_L and ρ_U, as illustrated in Figures 5(b) and 6. In this experiment, we chose sample sizes to be $968500, 365500$ and 98000 for $m = 25, 50$ and 100 respectively. In each sample, we calculated ρ_L and ρ_U for each matrix X. Then the data of ρ_L and ρ_U were grouped into ten classes respectively. In the case of ρ_L, for example, the first class consists of matrices X with $m^0 \leq \rho_L < m^{0.5}$ and the second class with $m^{0.5} \leq \rho_L < m^1$, the third one with $m^1 \leq \rho_L < m^{1.5}$ and so on. The number of matrices in each class was then divided by the corresponding sample size to get the percentage frequency to the class. The distributions have been plotted in the form of histograms. Empirically, there is a tendency that $Prob(m \leq \rho_L < m^{1.5})$ and $Prob(m \leq \rho_U < m^{1.5})$ tend to one as m goes to infinity.

7. Appendix

LEMMA 3.

$$\iiiint_\Omega w^{p-1} exp\left(-\frac{1}{2}\left(x^2+y^2+z^2+w^2\right)\right) dx\, dy\, dz\, dw$$

$$= \int_{-\infty}^{\alpha} dt \int_{-\infty}^{\infty} dx \int_0^\infty \frac{w^p}{w^2+x^2} exp\left(-\frac{1}{2}\left((x+t)^2+w^2\right)\right) dw$$

where $\Omega = \{(x,y,z,w) \mid x - yw/z \leq \alpha, w > 0, z > 0\}$ *and* $1 \leq p$.

Proof.

$$
\begin{aligned}
F(\alpha) &\equiv \iiiint_{\Omega} w^{p-1} exp\left(-\frac{1}{2}\left(x^2+y^2+z^2+w^2\right)\right) dx\, dy\, dz\, dw \\
&= \int_{-\infty}^{\infty} dx \int_0^{\infty} dw \int_0^{\infty} dz \int_{(x-\alpha)z/w}^{\infty} w^{p-1} exp\left(-\frac{1}{2}\left(x^2+y^2+z^2+w^2\right)\right) dy
\end{aligned}
$$

Letting $y = uz$ we find

$$
\begin{aligned}
F(\alpha) &= \int_{-\infty}^{\infty} dx \int_0^{\infty} dw \int_0^{\infty} dz \int_{(x-\alpha)/w}^{\infty} z w^{p-1} exp\left(-\frac{1}{2}\left(x^2+u^2z^2+z^2+w^2\right)\right) du \\
&= \int_{-\infty}^{\infty} dx \int_0^{\infty} dw \int_{(x-\alpha)/w}^{\infty} du \int_0^{\infty} z w^{p-1} exp\left(-\frac{1}{2}\left(x^2+u^2z^2+z^2+w^2\right)\right) dz \\
&= \int_{-\infty}^{\infty} dx \int_0^{\infty} dw \int_{(x-\alpha)/w}^{\infty} \frac{w^{p-1}}{1+u^2} exp\left(-\frac{1}{2}(x^2+w^2)\right) du.
\end{aligned}
$$

Letting $u = v/w$ this can then be written

$$
\begin{aligned}
F(\alpha) &= \int_{-\infty}^{\infty} dx \int_0^{\infty} dw \int_{x-\alpha}^{\infty} \frac{w^p}{w^2+v^2} exp\left(-\frac{1}{2}(x^2+w^2)\right) dv \\
&= \int_{-\infty}^{\infty} dx \int_{x-\alpha}^{\infty} dv \int_0^{\infty} \frac{w^p}{w^2+v^2} exp\left(-\frac{1}{2}(x^2+w^2)\right) dw.
\end{aligned}
$$

Finally, letting $v = x - t$ we have

$$
\begin{aligned}
F(\alpha) &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\alpha} dt \int_0^{\infty} \frac{w^p}{w^2+(x-t)^2} exp\left(-\frac{1}{2}(x^2+w^2)\right) dw \\
&= \int_{-\infty}^{\alpha} dt \int_{-\infty}^{\infty} dx \int_0^{\infty} \frac{w^p}{w^2+(x-t)^2} exp\left(-\frac{1}{2}(x^2+w^2)\right) dw \\
&= \int_{-\infty}^{\alpha} dt \int_{-\infty}^{\infty} dx \int_0^{\infty} \frac{w^p}{w^2+x^2} exp\left(-\frac{1}{2}\left((x+t)^2+w^2\right)\right) dw. \quad \square
\end{aligned}
$$

LEMMA 4.

$$
\int_0^{\infty} dx \int_{-\infty}^{\infty} \frac{x^{p-1}}{x^2+y^2} exp\left(-\frac{1}{2}\left(x^2+(y+t)^2\right)\right) dy
$$

$$
= 2^{(p-1)/2}\sqrt{\pi}\,\Gamma\left(\frac{p}{2}\right)\left(\sum_{i=0}^{\lfloor \frac{p-3}{2} \rfloor} \xi_{i,p}\, t^{-2i-2} + (-1)^{\lfloor (p-1)/2 \rfloor} \zeta_p\, t^{-p+1} exp(-\frac{1}{2}t^2)\, \phi_p(t)\right)
$$

where

$$\xi_{i,p} = \begin{cases} (-1)^i \prod_{j=0}^{i-1} (p-2j-3) & i > 0 \\ 1 & i = 0\,, \end{cases}$$

$$\zeta_p = \begin{cases} (p-3)!! & p > 3 \\ 1 & p = 2,\ 3\ , \end{cases}$$

$$\phi_p(t) = \left(\int_0^t exp\left(\frac{1}{2}x^2\right) dx \right)^{p-1-2\lfloor (p-1)/2 \rfloor}$$

and where $-\infty < t < \infty$, $2 \le p$.

Proof.

$$\begin{aligned}
f(t) &\equiv \int_0^\infty dx \int_{-\infty}^\infty \frac{x^{p-1}}{x^2+y^2} exp\left(-\frac{1}{2}\left(x^2+(y+t)^2\right)\right) dy \\
&= exp\left(-\frac{1}{2}t^2\right) \int_0^\infty dx \int_{-\infty}^\infty \frac{x^{p-1}}{x^2+y^2} exp\left(-\frac{1}{2}\left(x^2+y^2\right)\right) exp\left(-yt\right) dy \\
&= exp\left(-\frac{1}{2}t^2\right) \int_0^\infty dx \int_{-\infty}^\infty \frac{x^{p-1}}{x^2+y^2} exp\left(-\frac{1}{2}\left(x^2+y^2\right)\right) \sum_{n=0}^\infty \frac{(-yt)^n}{n!} dy \\
&= exp\left(-\frac{1}{2}t^2\right) \sum_{n=0}^\infty \frac{(-t)^n}{n!} \int_0^\infty dx \int_{-\infty}^\infty \frac{x^{p-1}y^n}{x^2+y^2} exp\left(-\frac{1}{2}\left(x^2+y^2\right)\right) dy \\
&= 2\, exp\left(-\frac{1}{2}t^2\right) \sum_{n=0}^\infty \frac{t^{2n}}{(2n)!} \int_0^\infty dx \int_0^\infty \frac{x^{p-1}y^{2n}}{x^2+y^2} exp\left(-\frac{1}{2}\left(x^2+y^2\right)\right) dy \\
&= 2\, exp\left(-\frac{1}{2}t^2\right) \sum_{n=0}^\infty \frac{t^{2n}}{(2n)!} \int_0^\infty dz \int_0^\infty \frac{z^{2n}}{1+z^2}\, x^{2n+p-2} exp\left(-\frac{1}{2}x^2\left(1+z^2\right)\right) dx\,.
\end{aligned}$$

where $y = xz$. Let $w = x^2\left(1+z^2\right)/2$. Then

$$\begin{aligned}
f(t) &= 2^{(p-1)/2} exp\left(-\frac{1}{2}t^2\right) \sum_{n=0}^\infty \frac{2^n t^{2n}}{(2n)!} \int_0^\infty dz \int_0^\infty \frac{z^{2n}}{\left(1+z^2\right)^{n+(p+1)/2}}\, w^{n+(p-3)/2} exp\left(-w\right) dw \\
&= 2^{(p-3)/2} exp\left(-\frac{1}{2}t^2\right) \sum_{n=0}^\infty \frac{2^n t^{2n}}{(2n)!} \Gamma\left(n+\frac{p-1}{2}\right) B\left(\frac{p}{2}, n+\frac{1}{2}\right) \\
&= 2^{(p-3)/2} exp\left(-\frac{1}{2}t^2\right) \sum_{n=0}^\infty \frac{2^n t^{2n}}{(2n)!} \Gamma\left(n+\frac{p-1}{2}\right) \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{p+1}{2}\right)}
\end{aligned}$$

$$
\begin{aligned}
&= 2^{(p-1)/2}\sqrt{\pi}\,\Gamma\left(\frac{p}{2}\right) exp\left(-\frac{1}{2}t^2\right)\sum_{n=0}^{\infty}\frac{1}{2n+p-1}\frac{(2n-1)!!}{(2n)!}t^{2n} \\
&= 2^{(p-1)/2}\sqrt{\pi}\,\Gamma\left(\frac{p}{2}\right) exp\left(-\frac{1}{2}t^2\right)\sum_{n=0}^{\infty}\frac{1}{n!\,2^n}\frac{1}{2n+p-1}t^{2n} \\
&= 2^{(p-1)/2}\sqrt{\pi}\,\Gamma\left(\frac{p}{2}\right) exp\left(-\frac{1}{2}t^2\right)t^{-p+1}\sum_{n=0}^{\infty}\frac{1}{n!\,2^n}\frac{1}{2n+p-1}t^{2n+p-1} \\
&\equiv 2^{(p-1)/2}\sqrt{\pi}\,\Gamma\left(\frac{p}{2}\right) exp\left(-\frac{1}{2}t^2\right)t^{-p+1}g(t)\,,
\end{aligned}
$$

where and below we define $0! = 0!! = (-1)!! = 1$. Since

$$
\frac{d}{dt}g(t) \;=\; \sum_{n=0}^{\infty}\frac{1}{n!\,2^n}t^{2n+p-2} \;=\; t^{p-2}exp\left(\frac{1}{2}t^2\right),
$$

we find

$$
\begin{aligned}
g(t) &= \int_0^t x^{p-2}exp\left(\frac{1}{2}x^2\right)dx \\
&= exp\left(\frac{1}{2}t^2\right)\sum_{i=0}^{\lfloor\frac{p-3}{2}\rfloor}(-1)^i\prod_{j=0}^{i-1}(p-2j-3)\,t^{p-2i-3} + \\
&\quad (-1)^{\lfloor(p-1)/2\rfloor}(p-3)!!\left(\int_0^t exp\left(\frac{1}{2}x^2\right)dx\right)^{p-1-2\lfloor(p-1)/2\rfloor}
\end{aligned}
$$

by integration by parts and then the desired result follows. □

LEMMA 5.

$$
\int\cdots\int_{\Omega} x_1^{q-2}exp\left(-\frac{1}{2}\sum_{i=1}^{6}x_i^2\right)\prod_{i=1}^{6}dx_i
$$

$$
= \pi^{1/2}2^{(q-2)/2}\Gamma((q+1)/2)\int_{-\infty}^{\alpha}dy_1\int_{-\infty}^{\infty}dy_2\int_{-\infty}^{\infty}dy_3\int_{-\infty}^{\infty}\frac{|y_3|}{\left(1+y_2^2+y_4^2\right)^{(q+1)/2}}\times
$$

$$
\frac{1}{\left(1+(y_1y_3+y_2)^2+(y_3+y_4)^2\right)^{3/2}}dy_4
$$

where $q \geq 2$ *and* $\Omega = \{(x_1,\cdots,x_6) \mid (x_1x_3-x_2x_4)/(x_1x_5-x_2x_6) \leq \alpha,\ x_1>0,\ x_2>0\}$.

Proof. Let

$$x_3 = (y_1 + y_2)x_2 \quad , \quad x_4 = x_1 y_2 \quad ,$$

$$x_5 = (y_3 + y_4)x_2 \quad , \quad x_6 = x_1 y_4 \quad .$$

Then

$$\begin{aligned}
F(\alpha) &\equiv \int \cdots \int_{\Omega} x_1^{q-2} exp\left(-\frac{1}{2}\sum_{i=1}^{6} x_i^2\right) \prod_{i=1}^{6} dx_i \\
&= \int \cdots \int_{y_1/y_3 \leq \alpha} x_1^q x_2^2 exp\left(-\frac{1}{2}\left(x_1^2\left(1 + y_2^2 + y_4^2\right) + x_2^2\left(1 + (y_1+y_2)^2 + (y_3+y_4)^2\right)\right)\right) \times \\
&\quad dx_1\, dx_2 \prod_{i=1}^{4} dy_i \\
&= \iiiint_{y_1/y_3 \leq \alpha} \left(\int_0^\infty x_1^q exp\left(-\frac{1}{2}x_1^2(1 + y_2^2 + y_4^2)\right) dx_1\right) \times \\
&\quad \left(\int_0^\infty x_2^2 exp\left(-\frac{1}{2}x_2^2\left(1 + (y_1+y_2)^2 + (y_3+y_4)^2\right)\right) dx_2\right) \prod_{i=1}^{4} dy_i \\
&= \pi^{1/2} 2^{(q-2)/2} \Gamma((q+1)/2) \iiiint_{y_1/y_3 \leq \alpha} \frac{1}{\left(1 + (y_1+y_2)^2 + (y_3+y_4)^2\right)^{3/2}} \times \\
&\quad \frac{1}{\left(1 + y_2^2 + y_4^2\right)^{(q+1)/2}} \prod_{i=1}^{4} dy_i \\
&= \pi^{1/2} 2^{(q-2)/2} \Gamma((q+1)/2) \iint_{y_1/y_3 \leq \alpha} dy_1\, dy_3 \int_{-\infty}^{\infty} dy_2 \int_{-\infty}^{\infty} \frac{1}{\left(1 + y_2^2 + y_4^2\right)^{(q+1)/2}} \times \\
&\quad \frac{1}{\left(1 + (y_1+y_2)^2 + (y_3+y_4)^2\right)^{3/2}}\, dy_4 \,.
\end{aligned}$$

Since

$$\iint_{x/y \leq \alpha} f(x,y)\, dx\, dy = \int_{-\infty}^{\alpha} dx \int_{-\infty}^{\infty} f(xy, y)|y|\, dy \,,$$

we have

$$F(\alpha) = \pi^{1/2} 2^{(q-2)/2} \Gamma((q+1)/2) \int_{-\infty}^{\alpha} dy_1 \int_{-\infty}^{\infty} dy_3 \int_{-\infty}^{\infty} dy_2 \int_{-\infty}^{\infty} \frac{|y_3|}{\left(1 + y_2^2 + y_4^2\right)^{(q+1)/2}} \times$$

$$\frac{1}{\left(1+(y_1y_3+y_2)^2+(y_3+y_4)^2\right)^{3/2}}dy_4 . \quad \square$$

LEMMA 6. *Let*

$$f(t) \;=\; \frac{q-1}{4\pi^2}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\int_{-\infty}^{\infty}\frac{|x_1|}{\left(1+(x_2+x_1t)^2+(x_3+x_1)^2\right)^{3/2}}\times$$

$$\frac{1}{\left(1+x_2^2+x_3^2\right)^{(q+1)/2}}dx_3$$

where $2 \le q$ *and* $-\infty < t < \infty$. *Then*

$$f(t) \;=\; \frac{1}{\pi}\frac{1}{1+t^2} .$$

Proof. We rewrite the expression of $f(t)$ as

$$f(t) \;=\; \frac{q-1}{4\pi^2}\frac{1}{1+t^2}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dx_2\int_{-\infty}^{\infty}\frac{|x_1|}{\left(1+(x_2+cx_1)^2+(x_3+sx_1)^2\right)^{3/2}}\times$$

$$\frac{1}{\left(1+x_2^2+x_3^2\right)^{(q+1)/2}}dx_3$$

where $c=t/\sqrt{1+t^2}$ and $s=1/\sqrt{1+t^2}$. Let

$$x_1=y_1 , \qquad x_2=y_1(cy_2-sy_3) , \qquad x_3=y_1(sy_2+cy_3) .$$

Then

$$f(t) \;=\; \frac{q-1}{4\pi^2}\frac{1}{1+t^2}\int_{-\infty}^{\infty}dy_1\int_{-\infty}^{\infty}dy_2\int_{-\infty}^{\infty}\frac{y_1^2|y_1|}{\left(1+y_1^2\left((1+y_2)^2+y_3^2\right)\right)^{3/2}}\times$$

$$\frac{1}{\left(1+y_1^2\left(y_2^2+y_3^2\right)\right)^{(q+1)/2}}dy_3$$

$$\equiv\; \frac{1}{\pi}\frac{1}{1+t^2}\xi .$$

Since

$$\int_{-\infty}^{\infty}f(t)dt \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+t^2}dt \;=\; 1 ,$$

we have $\xi = 1$ and therefore the lemma follows. □

Acknowledgement. The authors wish to thank Professor Alan Edelman for providing us with several important references and suggestions on an earlier draft which improves the proof in Lemma 6 and the derivations of equations (1) and (4). In addition, the authors would like to thank Professors Gene Golub and Nick Trefethen for their valuable comments and suggestions on the earlier draft.

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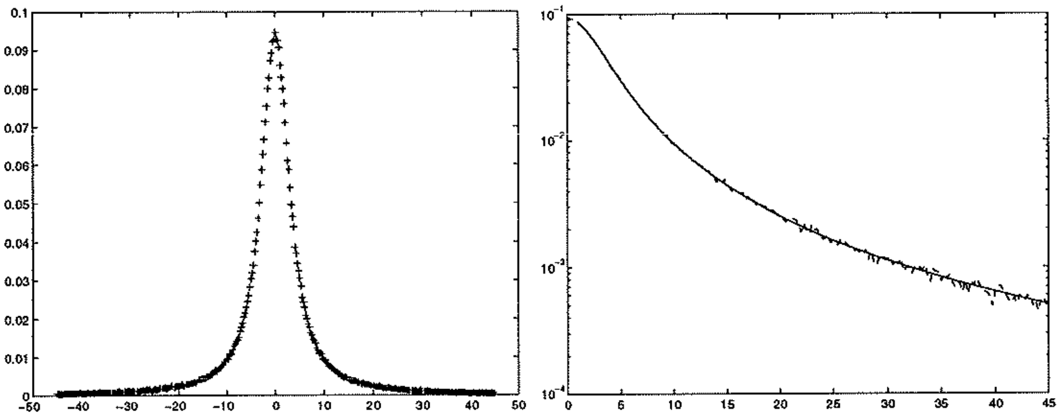

FIG. 1. *(a) Distribution of* $u_{12,12}$. *(b) Observed density function (dashed) of* $u_{12,12}$ *and its predicted function (solid).*

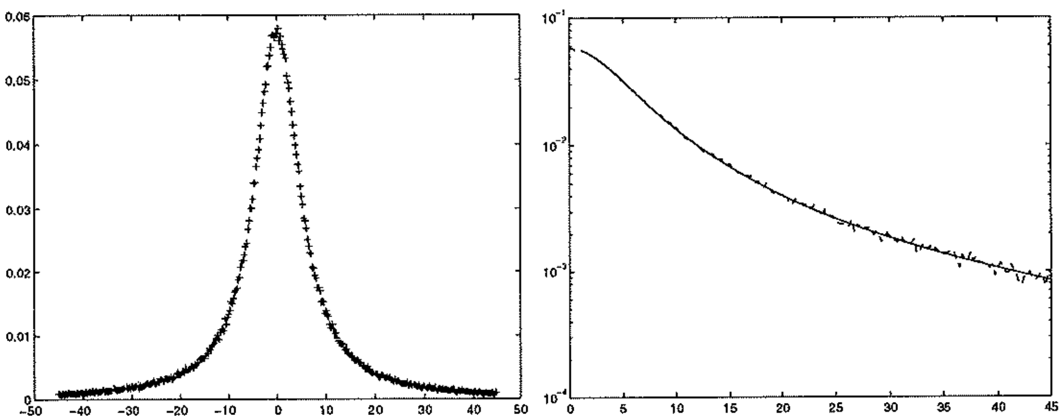

FIG. 2. *(a) Distribution of* $u_{31,31}$. *(b) Observed density function (dashed) of* $u_{31,31}$ *and its predicted function (solid).*

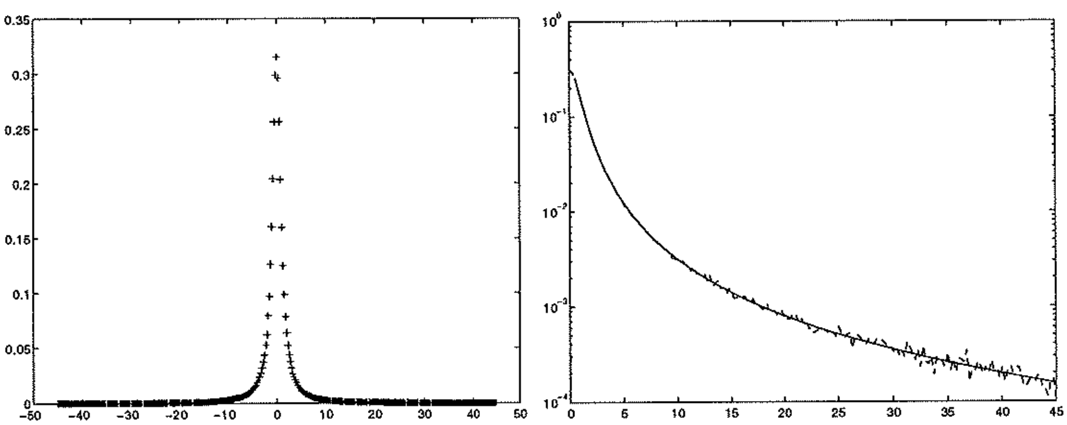

FIG. 3. *(a) Distribution of* $l_{13,12}$. *(b) Observed density function (dashed) of* $l_{13,12}$ *and its predicted function (solid).*

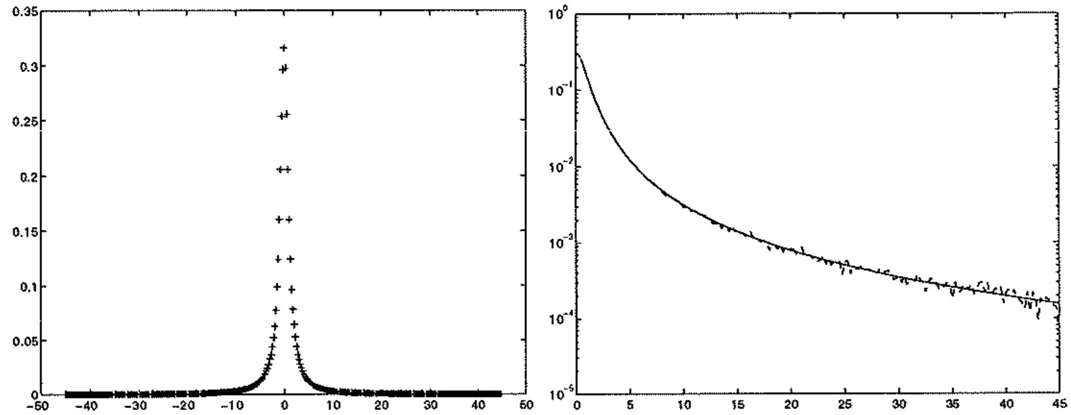

FIG. 4. *(a) Distribution of* $l_{30,29}$. *(b) Observed density function (dashed) of* $l_{30,29}$ *and its predicted function (solid).*

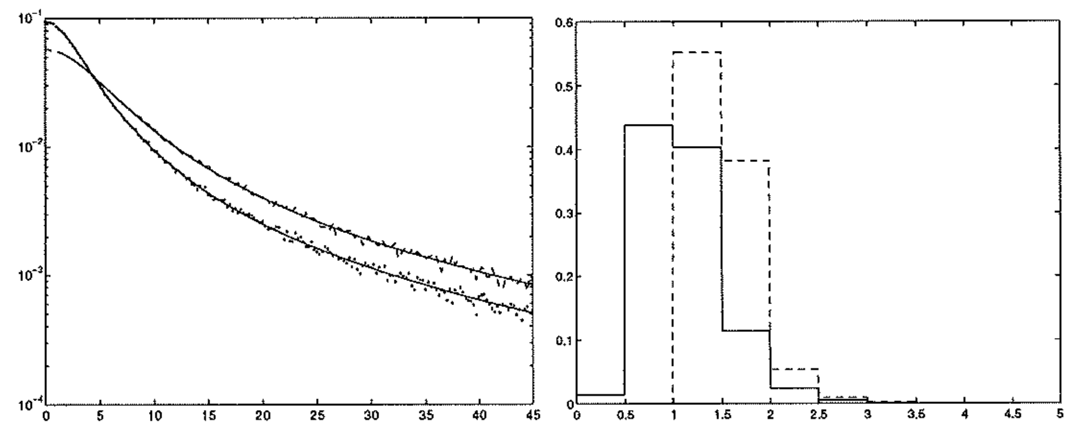

FIG. 5. *(a) Overlap of Figure 1(b) and 2(b). (b) Percentage frequency distributions of* ρ_L *(dashed) and* ρ_U *(solid).* $m = 25$.

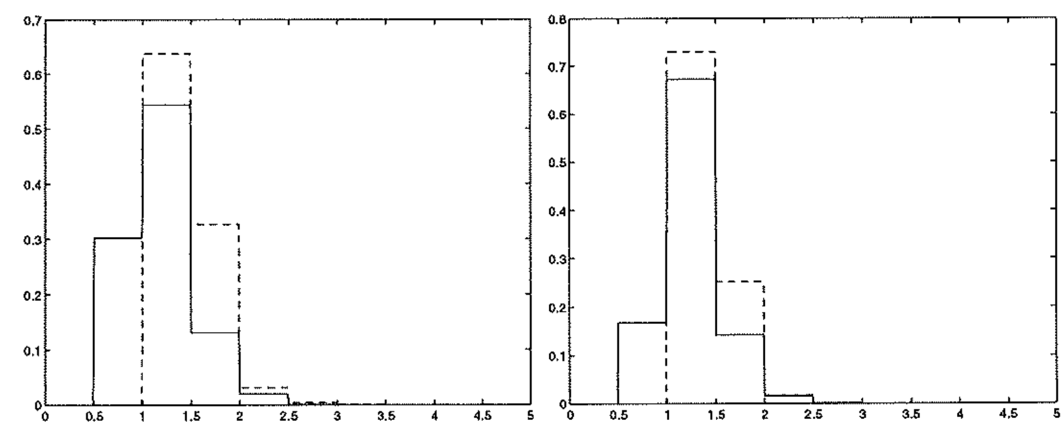

FIG. 6. *Percentage frequency distributions of* ρ_L *(dashed) and* ρ_U *(solid). (a)* $m = 50$. *(b)* $m = 100$.