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ON THE ZERO-VISCOSITY LIMIT OF BURGERS' EQUATION FROM THE PERSPECTIVE OF COMPLEX SINGULARITIES

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Dedicated to my parents, Judy and Paul

Abstract. The zero-viscosity limit of a meromorphic solution to Burgers' equation is found by introducing an integral representation of the Mittag-Leffler expansion of the solution involving a "polar" measure. The weak limit of this Borel measure corresponds to the asymptotic density of poles which describes their condensation on the imaginary axis. The resulting integral representation of the inviscid solution is computed by residues and is shown to match the characteristic solution up to the inviscid shock time t_*. The Mittag-Leffler expansion and the Calogero dynamical system which describes the motion of the poles have a continuum limit consisting of two integro-differential equations which are equivalent to the characteristic equations of the inviscid solution. The analyticity properties of the inviscid solution are also obtained from the large wave number asymptotic expansion of its Fourier transform which is uniformly valid in a neighborhood of the inviscid shock time t_*.

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1. Introduction. In this article we continue the investigation [17] of the spatial analyticity properties of a solution to Burgers' equation

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = \nu\,\frac{\partial^2 u}{\partial x^2}, \qquad x \in \mathbb{R}, t > 0, \nu \geqslant 0. \tag{1.1}$$

Previous work concerning the analyticity properties of Burgers' equation can be found in [3, 4, 12, 13, 18, 20]. We focus on a particular initial value problem for (1.1) which was first studied by Bessis and Fournier in [3, 4], where the initial condition is given by

$$u(x,0) = u_0(x) = 4x^3 - x/t_*, \qquad x \in \mathbb{R}, \tag{1.2}$$

and t_* is a fixed positive parameter. This initial value is chosen for its generic properties which are due to the type of singularity occuring in the inviscid solution at the shock time $t = -\left(\inf_x u_0'(x)\right)^{-1} = t_*$ (cf. [3, 8, 12, 16] for more details). The solution to (1.1)-(1.2) is the meromorphic function

$$u_\nu(x,t) = \frac{x}{t} - \sum_{n=1}^{\infty} \frac{4\nu x}{x^2 + \beta_n^2(t,\nu)}, \tag{1.3}$$

where $\{\pm i\beta_n\}_{n\in\mathbb{N}^*}$, $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$, is a countable set of pure imaginary conjugate poles (the zeros of the Cole-Hopf variable). The motion of these poles on the imaginary axis is governed by a Calogero-type infinite dimensional dynamical system:

$$\forall n \in \mathbb{N}^*, \quad \dot{\beta}_n = \frac{\beta_n}{t} + \frac{\nu}{\beta_n} - 4\nu\beta_n \sum_{\substack{l=1\\ l\neq n}}^{\infty} \frac{1}{\beta_l^2 - \beta_n^2} = \frac{\beta_n}{t} - 2\nu \sum_{\substack{l=-\infty\\ l\neq 0,n}}^{\infty} \frac{1}{\beta_l - \beta_n}. \tag{1.4}$$

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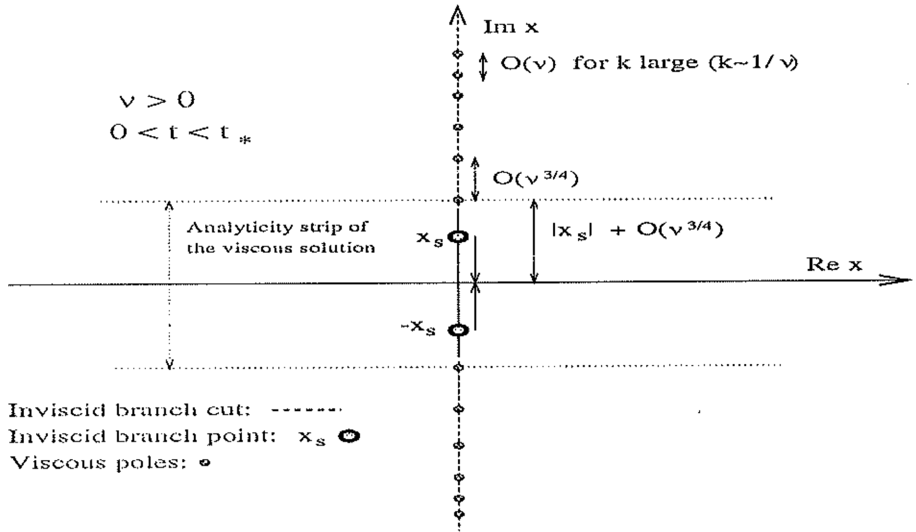

FIG. 1.1. *Inviscid branch points, branch cuts and viscous poles for $\nu > 0$ and $0 < t < t_*$. The poles are located above the inviscid branch point singularities according to the asymptotic formula $\beta_k(t,\nu) = \Im x_s(t) + \mathcal{O}\left((k\nu)^{3/4}\right)$ as $\nu \to 0^+$ for k fixed (or as $k \to \infty$ for ν fixed). The distance separating two successive poles is asymptotically given by $\Delta\beta_k = \mathcal{O}(\nu)$ as $\nu \to 0^+$ for k large ($k \sim 1/\nu$).*

Numerical simulations of the evolution of these poles and the solution itself are described in [17]. For more details on the derivation of (1.3) and (1.4), see the companion article [17, §2].

As $\nu \to 0^+$, these poles condense on the imaginary axis for all $t > 0$. The asymptotic distance between two successive poles as $\nu \to 0^+$ is proportional to ν when the index of these poles grows like $k \sim 1/\nu$ (see Fig. 1.1). This condensation phenomenon is captured by an asymptotic density of poles (also referred to as the limiting pole density in the work of Bessis and Fournier [4]) which, as we show, depends directly on the relevant saddle points of the asymptotic expansion of the solution $u_\nu(i\beta, t)$, $\beta > 0$, on the imaginary axis as $\nu \to 0^+$: We show that the asymptotic density of poles defined by Bessis and Fournier in [3, 4] as

$$\rho(\beta; t) = \lim_{\substack{n\to\infty \\ \nu\to 0^+}} \left.\frac{2\nu}{\Delta\beta_n(t,\nu)}\right|_{\beta_n=\beta},$$

is given by

$$\rho(\beta; t) = \frac{1}{\pi t}\Re z_s^+(\beta; t),$$

where $z_s^+(\beta; t)$ is the relevant saddle point with positive real part in the expansion of $u_\nu(i\beta, t)$ as $\nu \to 0^+$. This density is explicitly calculated for all $t > 0$ using Cardano's formula. The Mittag-Leffler expansion (1.3) has an integral representation which is valid away from the imaginary axis. It can be expressed as the integral of a smooth function against the distributional derivative of a nonnegative regular finite Borel measure

$$\sigma_\nu(\beta; \beta_{\max}, t) = \int_{-\beta_{\max}}^{\beta} 2\nu \sum_{k=1}^{N_\nu} \left[\delta\left(\xi - \beta_k(t,\nu)\right) + \delta\left(\xi + \beta_k(t,\nu)\right)\right] d\xi.$$

The "polar" measure $\sigma_\nu(\beta;\beta_{\max},t)$ is analogous to the spectral measure in the KdV problem (cf. [11, 14]). The zero-viscosity limit of the pole expansion is found by taking the weak limit of $d\sigma_\nu(\beta;\beta_{\max},t)/d\beta$ which approximates the asymptotic density of poles. Thus we show that for $\beta \in [-\beta_{\max},\beta_{\max}]$, $0<\beta_{\max}<+\infty$,

$$\rho(\beta;t) \equiv \text{w-}\lim_{\nu\to 0+} \frac{d\sigma_\nu}{d\beta}(\beta;\beta_{\max},t),$$

where w-$\lim_{\nu\to 0^+}$ denotes the weak limit of measures. The limiting integral representation of the solution given by

$$u(x,t) = \frac{x}{t} - 2x\int_0^\infty \frac{\rho(\beta;t)}{x^2+\beta^2}\,d\beta$$

in which the complex-valued asymptotic density of poles $\rho(\beta;t)$ arises naturally in the integrand, is computed via residues, and the analytic structure of the inviscid solution is explicitly recovered up to t_*.

We also show that the continuum limit of the pair of equations consisting of the pole expansion (1.3) and the dynamical system (1.4), is the system of two integro-differential equations

$$\begin{aligned}\frac{\partial f}{\partial t}(\zeta,t) &= \frac{f(\zeta,t)}{t} - P.V.\int_{-\infty}^{\infty}\frac{d\zeta'}{f(\zeta,t)-f(\zeta',t)}\\ &= \frac{f(\zeta,t)}{t} - f(\zeta,t)\,P.V.\int_{-\infty}^{\infty}\frac{d\zeta'}{f^2(\zeta,t)-f^2(\zeta',t)},\end{aligned}$$

and

$$u(x,t) = \frac{x}{t} - \int_{-\infty}^{\infty}\frac{d\zeta'}{x-f(\zeta',t)} = \frac{x}{t} - x\int_{-\infty}^{\infty}\frac{d\zeta'}{x^2-f^2(\zeta',t)},$$

in which the branch cuts are defined by the condition $x \neq f(\zeta,t)$. This 2×2 system is shown to be equivalent to the characteristic equations of the inviscid Burgers equation. From this analysis, we illustrate the relation between the pole positions and the asymptotic density at $t=t_*$: We show that the pole positions can be recovered from the asymptotic density by choosing the right discretization on the "continuum" curve on which this density lies.

Furthermore, the analyticity properties of the inviscid solution can be analyzed by describing the asymptotic behavior of its Fourier transform (see [12, 20]). We find a uniform asymptotic expansion as $k\to+\infty$ of the Fourier transform of the inviscid solution, clarifying the seemingly discontinuous change of behavior of $\hat{u}(k,t)$ at t_* presented in [12]. This discontinuity in the asymptotic behavior is a direct consequence of the coalescence of the two second order branch points $\pm x_s(t)$ into a third order branch point at the origin $x_s(t_*)=0$. We show that as $k\to+\infty$, $\hat{u}(k,t) = C_0\cdot (tk)^{-4/3}\mathrm{Ai}\left[(-3ikx_s(t)/2)^{2/3}\right](1+O(k^{-1}))$. From the (uniform) asymptotic expansion of the Airy function we find that $\hat{u}(k,t)\sim C_1(t)\cdot(t_*-t)^{-1/4}k^{-3/2}\exp(-k|x_s(t)|)$ for $0<t<t_*$, and $\hat{u}(k,t_*)\sim C_2\cdot(t_*k)^{-4/3}$, where $C_0, C_1(t), C_2$ are appropriate numerical constants.

2. Polar measure, integral representation and inviscid limit. In [17], the solution to the initial value problem (1.1)-(1.2) is constructed as follows:

THEOREM 2.1. *Let* $\nu, t > 0$, *then*

$$u_\nu(x,t) = \frac{x}{t} - 2\nu\,\partial_x \log(E_\nu(x,t)),$$
$$E_\nu(x,t) = \int_{-\infty}^{\infty} \exp\left\{\frac{1}{2\nu}\left(\frac{x}{t}y + \alpha y^2 - y^4\right)\right\} dy,$$

where $2\alpha = 1/t_* - 1/t \in \mathbb{R}$. *For fixed* ν, t, $E_\nu(x,t)$ *is an even entire function of* x *of order* $4/3$ *with countably many simple zeros which come in pure imaginary opposite and conjugate pairs. Moreover,* $E_\nu(x,t)$ *has the infinite product representation*

$$E_\nu(x,t) = C_\nu(t)\prod_{n=1}^{\infty}\left(1 + \frac{x^2}{\beta_n^2(t,\nu)}\right), \quad \sum_{n=1}^{\infty}\frac{1}{\beta_n} = +\infty, \quad \sum_{n=1}^{\infty}\frac{1}{\beta_n^2} < +\infty,$$
$$C_\nu(t) = \frac{\sqrt{\alpha}}{2}e^{\frac{\alpha^2}{16\nu}}K_{1/4}\left(\frac{\alpha^2}{16\nu}\right), \quad C_\nu(t_*) = \nu^{1/4}2^{-3/4}\Gamma(1/4),$$

where $K_q(z)$ *is the modified Bessel function of the second kind. Thus the solution* $u_\nu(x,t)$ *has an alternate representation in terms of a Mittag-Leffler (pole) expansion*

$$u_\nu(x,t) = \frac{x}{t} - \sum_{n=1}^{\infty}\frac{4\nu x}{x^2 + \beta_n^2(t,\nu)},$$

which converges uniformly on compact sets for x *away from the poles* $x = \pm i\beta_n$.

In this section, we introduce an integral representation of the solution which enables us to describe the behavior of the solution as the viscosity tends to zero. Using a saddle-point analysis, we have shown in [17, §4.2] and [18] that the dominant behavior of $E_\nu(i\beta, t)$ as $\nu \to 0^+$ or as $\beta \to +\infty$ is

$$\sqrt{\frac{|6z_s(\beta;t)^2 - \alpha|}{2\pi\nu}}\exp\left\{-\frac{1}{2\nu}\Re w(z_s(\beta;t),\beta)\right\}E_\nu(i\beta,t)$$
$$= \cos\left(\frac{\Im w(z_s(\beta;t),\beta)}{2\nu} - \frac{\theta(z_s(\beta;t),t)}{2}\right) + \mathcal{O}\left(\frac{\nu}{\beta^{4/3}}\right),$$

where

$$w(z,\beta) = \int_0^z i\beta/t - \eta/t - u_0(\eta)\,d\eta = i\beta z/t + \alpha z^2 - z^4,$$
$$\theta(z,t) = \arg(\partial_z^2 w) = \arg(6z^2 - \alpha), \quad -\pi < \theta(z,t) < \pi,$$

and $z_s(\beta;t)$ is implicitly defined by

$$0 = w_z(z_s(\beta;t),\beta) = \frac{i\beta}{4t} + \frac{\alpha}{2}z_s - z_s^3. \tag{2.1}$$

For small ν (fixed k) or for large β (large k, fixed ν), the poles β_k are approximated by the roots of the equation

$$\frac{1}{2\nu}\Im\, w(z_0(\beta;t),\beta) - \frac{1}{2}\theta(z_0(\beta;t),t) = \left(k - \frac{1}{2}\right)\pi, \quad k \in \mathbb{N}^*, \tag{2.2}$$

with the convention that $\beta_{-k} \equiv -\beta_k$. Choose a parameter $\beta_{\max} < +\infty$ such that

$$|x_s(t)| < \beta_1(t,\nu) \leqslant \beta_2(t,\nu) \leqslant \cdots \leqslant \beta_k(t,\nu) \leqslant \beta_{N_\nu}(t,\nu) = \beta_{\max} < +\infty, \tag{2.3}$$

for $k \leqslant N_\nu(\beta_{\max})$. For negative indices, the ordering of the β_{-k}'s is the reverse of that given in (2.3). The pair of variables $\{\beta_{\max}, N_\nu(\beta_{\max})\}$ are simultaneously defined by the relation

$$N_\nu(\beta_{\max}) = \frac{\Im w(z_s(\beta_{\max};t),\beta_{\max})}{2\nu\pi} - \frac{1}{2} \equiv \mathrm{Int}\left[\frac{\Im w(z_s(\beta_{\max};t),\beta_{\max})}{2\nu\pi}\right], \tag{2.4}$$

in which Int$[x]$ denotes the integer part of x with half integers rounded down. Let $U_\nu^{\beta_{\max}}(x,t)$ be the $N_\nu(\beta_{\max})$-th partial sum of $U_\nu(x,t)$:

$$\begin{aligned} U_\nu^{\beta_{\max}}(x,t) &= x - t\,u_\nu^{\beta_{\max}}(x,t) = t \cdot \sum_{n=1}^{N_\nu(\beta_{\max})} \frac{4\nu x}{x^2+\beta_n^2} \\ &= t \cdot 2\nu \sum_{n=1}^{N_\nu(\beta_{\max})} \left(\frac{1}{x-i\beta_n} + \frac{1}{x+i\beta_n}\right). \end{aligned} \tag{2.5}$$

Let $U_\nu(x,t)$ be the spatially singular part of the viscous solution defined by

$$U_\nu(x,t) = x - t\,u_\nu(x,t) = t \cdot \sum_{n=1}^{\infty} \frac{4\nu x}{x^2+\beta_n^2(t,\nu)}, \tag{2.6}$$

and let the remainder $R_\nu^{\beta_{\max}}(x,t)$ be defined by

$$R_\nu^{\beta_{\max}}(x,t) = U_\nu(x,t) - U_\nu^{\beta_{\max}}(x,t) = t \cdot \sum_{n=N_\nu+1}^{\infty} \frac{4\nu x}{x^2+\beta_n^2(t,\nu)}.$$

Let $\delta(\beta)$ denote the usual Dirac measure, and define the density $\sigma_\nu(\beta;\beta_{\max},t)$ with support in $[-\beta_{\max},\beta_{\max}]$ by

$$\sigma_\nu(\beta;\beta_{\max},t) = \int_{-\beta_{\max}}^{\beta} 2\nu \sum_{k=1}^{N_\nu} \left[\delta\left(\xi - \beta_k(t,\nu)\right) + \delta\left(\xi + \beta_k(t,\nu)\right)\right] d\xi. \tag{2.7}$$

Since the poles β_n are ordered according to (2.3), this insures that (2.7) is nonnegative and vanishes outside $[-\beta_{\max},\beta_{\max}]$. Moreover, once $\beta_{\max}$ has been chosen, $\sigma_\nu(\beta;\beta_{\max},t)$ is uniformly bounded in t (for $0 < \delta \leqslant t \leqslant t_*$) by $2\nu N_\nu$; i.e. as long as t is bounded below from 0. Thus from the definition (2.4) of N_ν, $\sigma_\nu(\beta;\beta_{\max},t)$ is uniformly bounded in ν, and thus is a regular finite Borel measure. Since

$$\begin{aligned} d\sigma_\nu(\beta;\beta_{\max},t) &= 2\nu \sum_{k=1}^{N_\nu} \left[\delta\left(\beta - \beta_k(t,\nu)\right) + \delta\left(\beta + \beta_k(t,\nu)\right)\right] d\beta \\ &= 2\nu \sum_{\substack{k=-N_\nu \\ k\neq 0}}^{N_\nu} \delta\left(\beta - \beta_k(t,\nu)\right) d\beta, \end{aligned} \tag{2.8}$$

the measure $d\sigma_\nu(\beta;\beta_{\max},t)$ consists of a sum of delta functions with weight (height) 2ν decreasing as $\nu \to 0^+$. The density of these delta functions increases like $1/\Delta\beta_k =$

$\mathcal{O}(1/\nu)$ as $\nu \to 0^+$ for $k \sim 1/\nu$. Using the odd parity of the measure $d\sigma_\nu(-\beta;\beta_{\max},t) = -d\sigma_\nu(\beta;\beta_{\max},t)$, we can represent $U_\nu^{\beta_{\max}}(x,t)$ as

$$U_\nu^{\beta_{\max}}(x,t) = t \cdot \int_{-\infty}^{\infty} \frac{d\sigma_\nu(\beta;\beta_{\max},t)}{x - i\beta} = t \cdot 2x \int_0^{\infty} \frac{d\sigma_\nu(\beta;\beta_{\max},t)}{x^2+\beta^2}, \tag{2.9}$$

where the last integral should be understood as

$$\int_0^{\infty} \frac{d\sigma_\nu(\beta;\beta_{\max},t)}{x^2+\beta^2} \equiv \frac{1}{2}\int_{-\infty}^{\infty} \frac{d\sigma_\nu(\beta;\beta_{\max},t)}{x^2+\beta^2}. \tag{2.10}$$

Since the measure $d\sigma_\nu(\beta;\beta_{\max},t)$ has support in the compact interval $[-\beta_{\max},\beta_{\max}]$, we have proved the following property:

PROPERTY 2.2. *$U_\nu^{\beta_{max}}(x,t)$ has an integral representation for $x \notin [-i\beta_{max}, i\beta_{max}]$ given by*

$$U_\nu^{\beta_{max}}(x,t) = t \cdot \int_{-\beta_{max}}^{\beta_{max}} \frac{d\sigma_\nu(\beta;\beta_{max},t)}{x - i\beta} = t \cdot 2x \int_{-\beta_{max}}^{\beta_{max}} \frac{d\sigma_\nu(\beta;\beta_{max},t)}{x^2+\beta^2}.$$

Now that we have established the validity of the integral representation of Property 2.2, we use this to derive an integral representation of the inviscid solution: We introduce the asymptotic pole density $\rho(\beta;t)$ which also corresponds to the asymptotic distribution of the zeros of E_ν. We define it in relation to the limiting measure $\sigma(\beta;t)$ as follows:

DEFINITION 2.1. *For $\beta \in [-\beta_{max}, \beta_{max}]$,*

$$\sigma(\beta;\beta_{max},t) = \int_0^{\beta} \rho(\zeta;t)\,d\zeta = \frac{1}{\pi}\Im w(z_s(\beta;t),\beta).$$

It follows directly from the definition (2.2) of the zeros $\pm i\beta_k(t,\nu)$ of $E_\nu(x,t)$, and from Definition 2.1 of the asymptotic density of the zeros that

$$\lim_{\nu\to 0^+} \sum_{k=1}^{N_\nu} \frac{2\nu}{x^2+\beta_k^2(t,\nu)} = \int_{-\beta_{\max}}^{\beta_{\max}} \frac{\rho(\beta;t)}{x^2+\beta^2}\,d\beta = \frac{U^{\beta_{\max}}(x,t)}{2tx}. \tag{2.11}$$

Thus we have that

$$|U^{\beta_{\max}}(x,t) - U_\nu^{\beta_{\max}}(x,t)| = t\,|2x| \cdot \left| \int_{-\beta_{\max}}^{\beta_{\max}} \frac{d\sigma - d\sigma_\nu}{x^2+\beta^2} \right| < \epsilon/3 \tag{2.12}$$

for ν small enough on compact sets for x and t away from the branch cuts defined by $(-i\infty, -i|x_s|] \cup [i|x_s|, +i\infty)$ (for a similar argument see for example [11]). Thus the convergence of the measure $d\sigma_\nu$ to $d\sigma$ is described in the following way:

PROPERTY 2.3. *For $\beta \in [-\beta_{max}, \beta_{max}]$ and $0 < \delta \leqslant t \leqslant t_*$, the sequence of distributions $d\sigma_\nu(\beta;\beta_{max} t)$ converges weakly to $d\sigma(\beta;\beta_{max}, t)$:*

$$w\text{-}\lim_{\nu\to 0+} d\sigma_\nu(\beta;\beta_{max},t) = d\sigma(\beta;\beta_{max},t) = \rho(\beta;t)\,d\beta = \frac{1}{\pi}\Im dw(z_s(\beta;t),\beta).$$

This measure is analogous to the spectral measure introduced in [11, 14]. Here $\text{w-lim}_{\nu\to 0+}$ stands for a limit in the sense of weak convergence of measures, that is $\text{w-lim}_{\nu\to 0+} d\mu_\nu(\beta) = d\mu(\beta)$ if

$$\lim_{\nu\to 0+} (\phi, d\mu_\nu) = (\phi, d\mu) = \int_{-\beta_{\max}}^{\beta_{\max}} \phi(\beta)\,d\mu(\beta).$$

for every continuous function ϕ in $[-\beta_{\max}, \beta_{\max}]$. Note that we suspect the convergence

$$\lim_{\nu \to 0^+} U_\nu^{\beta_{\max}}(x,t) = U^{\beta_{\max}}(x,t) \tag{2.13}$$

to hold uniformly over compact sets for t and x away from the branch cuts.

From the definition of the limiting function $U(x,t)$

$$U(x,t) = t \cdot 2x \int_{-\infty}^{\infty} \frac{\rho(\beta;t)}{x^2+\beta^2}\, d\beta,$$

the remainder $R^{\beta_{\max}}(x,t)$, defined for $x \notin (-i\infty, -i\beta_{\max}] \cup [i\beta_{\max}, i\infty)$ as

$$R^{\beta_{\max}}(x,t) = U^{\beta_{\max}}(x,t) - U(x,t) = t \cdot 2x \cdot \int_{|\beta| \geqslant \beta_{\max}} \frac{\rho(\beta;t)}{x^2+\beta^2}\, d\beta, \tag{2.14}$$

can be shown to go to zero as $\beta_{\max} \to +\infty$ independently of ν: Fix an $R > 0$ such that $|x| \leqslant R < \beta_{\max}$, then $|x^2+\beta^2| \geqslant \beta^2 - R^2$. Let $\theta > 1$ be a fixed parameter, then for $\beta > \beta_{\max} > \sqrt{\theta/(\theta-1)}R$, we have $1/(\beta^2 - R^2) < \theta/\beta^2$. Then since $\rho(\beta;t) = \mathcal{O}(\beta^{1/3})$ as $\beta \to +\infty$ (see Theorem 3.1), we can estimate (2.14) as $\beta_{\max} \to +\infty$ as follows:

$$\begin{aligned}\left| \int_{|\beta| \geqslant \beta_{\max}} \frac{\rho(\beta;t)}{x^2+\beta^2}\, d\beta \right| &\leqslant \int_{|\beta| \geqslant \beta_{\max}} \frac{\beta^{1/3}}{\beta^2 - R^2}\, d\beta \\ &\leqslant \theta \int_{|\beta| \geqslant \beta_{\max}} \frac{d\beta}{\beta^{5/3}} = \mathcal{O}(\beta_{\max}^{-2/3}).\end{aligned}$$

Therefore, on compact sets for x and t

$$|U^{\beta_{\max}}(x,t) - U(x,t)| < \epsilon/3 \tag{2.15}$$

for $\beta_{\max}$ large enough independently of ν.

The last estimate concerns

$$\begin{aligned}|R_\nu^{\beta_{\max}}(x,t)| &= |U_\nu^{\beta_{\max}}(x,t) - U_\nu(x,t)| \\ &= t\,|2x| \left| \sum_{n=N_\nu+1}^{\infty} \frac{2\nu}{x^2 + \beta_n^2(t,\nu)} \right|.\end{aligned} \tag{2.16}$$

In [17, §4.2], it is shown that $\beta_n(t,\nu) \sim \Im x_s(t) + \mathcal{O}\left((n\nu)^{3/4}\right)$ as $n \to +\infty$ for fixed $\nu > 0$. Therefore let $y_n^\nu = |x_s|^{4/3} + Cn\nu$ where C is an appropriate asymptotic constant which depends on t (see (5.1) for such a representation). This assumption can also be justified by combining (5.1) and the fact that the order $\lambda = 4/3$ of the entire function E_ν is also the order of convergence of its zeros (see [17, §2.1]). Then since $N_\nu(\beta_{\max}) = \mathrm{Int}[\Im w(\beta_{\max})/2\nu]$, we may estimate (2.16) as

$$\begin{aligned}|U_\nu^{\beta_{\max}}(x,t) - U_\nu(x,t)| &\leqslant t\,|2x| \left| \sum_{n=N_\nu+1}^{\infty} \frac{2\nu}{x^2 + y_n^{\nu\,3/2}} \right| \\ &\leqslant t\,|2x| \left| \int_{\beta_{\max}}^{+\infty} \frac{2C\, dy}{x^2 + y^{3/2}} \right| = \mathcal{O}(\beta_{\max}^{-1/2})\end{aligned}$$

as $\beta_{\max} \to +\infty$, uniformly in ν on compact sets for t and x away from the branch cuts. Choosing $\beta_{\max}$ large enough so that $|U(x,t) - U^{\beta_{\max}}(x,t)| < \epsilon/3$ and $|U_\nu^{\beta_{\max}}(x,t) -$

$U_\nu(x,t)| < \epsilon/3$ independently of ν, and then choosing ν small enough in such a way that $|U^{\beta_{\max}}(x,t) - U_\nu^{\beta_{\max}}(x,t)| < \epsilon/3$, we finally have

$$|U_\nu(x,t) - U(x,t)| \leq |U(x,t) - U^{\beta_{\max}}(x,t)| \\ + |U^{\beta_{\max}}(x,t) - U_\nu^{\beta_{\max}}(x,t)| + |U_\nu^{\beta_{\max}}(x,t) - U_\nu(x,t)| < \epsilon.$$

Using the fact that $\rho(\beta;t) = 0$ for $|x| < |x_s(t)|$ when $0 < t \leqslant t_*$, we have proved

THEOREM 2.4. *For $t \in (0,t_*]$, on compact sets for x away from the branch cuts defined by $(-i\infty, -i|x_s|] \cup [i|x_s|, +i\infty)$, we have*

$$\lim_{\nu\to 0+} U_\nu(x,t) = t \cdot 2x \int_{|x_s(t)|}^{\infty} \frac{\rho(\beta;t)}{x^2+\beta^2}\, d\beta.$$

3. Asymptotic density of poles. Now that we have defined the asymptotic density of poles, we proceed with its explicit computation for the different time intervals $(0,t_*)$, $t = t_*$ and $(t_*, +\infty)$. As in § 2, let w-$\lim_{\nu\to 0+}$ denote a weak limit in the sense of weak convergence of measures. Then we prove the following:

THEOREM 3.1. *For $0 < \beta_{max} < +\infty$, the asymptotic density of poles $\rho(\beta;t)$: $[-\beta_{max}, \beta_{max}] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a positive even function of β defined by*

$$\rho(\beta;t) \equiv w\text{-}\lim_{\nu\to 0+} \frac{d\sigma_\nu}{d\beta}(\beta;\beta_{max},t) = \frac{1}{\pi t}\Re z_s^+(\beta;t),$$

where $z_s^+(\beta;t)$ is the saddle point with positive real part (that is z_0 if $t < t_*$ and z_1 if $t > t_*$) which is relevant to the asymptotic expansion of $E_\nu(i\beta,t)$ as $\nu \to 0^+$. This saddle point is determined by the implicit equation

$$\frac{\partial w}{\partial z}(z_s(\beta;t),\beta) = \frac{i\beta}{t} - \frac{z_s(\beta;t)}{t} - u_0(z_s(\beta;t)) = 0.$$

Let $\pm x_s(t) = \pm i\,(3t_*)^{-3/2}(t_*-t)^{3/2}t^{-1/2}$ be the second order branch points of the inviscid solution arising from the initial data $u_0(x) = 4x^3 - x/t_*$. For $t > t_*$ and for $t < t_*$, $|\beta| > |x_s|$,

$$\rho(\beta;t) = \frac{2^{2/3}\sqrt{3}}{\pi}(4t)^{-4/3}\left\{\sqrt[3]{|\beta| + \sqrt{\beta^2+x_s^2}} - \sqrt[3]{|\beta| - \sqrt{\beta^2+x_s^2}}\right\}.$$

For $t < t_*$, $|\beta| < |x_s|$,

$$\rho(\beta;t) = 0.$$

For $t = t_*$,

$$\rho(\beta;t_*) = \frac{2\sqrt{3}}{\pi}(4t_*)^{-4/3}|\beta|^{1/3}.$$

For $t > t_*$, $\beta = 0$,

$$\rho(0;t) = \lim_{\substack{\beta\to 0\\ t>t_*}} \rho(\beta;t) = \frac{1}{2\pi}(t-t_*)^{1/2}t^{-3/2}t_*^{-1/2}.$$

Proof. In [3, 4], Bessis and Fournier introduced a limiting density of poles which characterizes the process of condensation of poles on the imaginary axis as the viscosity $\nu \to 0^+$. They defined it in [3] in the following way:

$$\rho(\beta;t) \equiv \lim_{\substack{n\to\infty\\ \nu\to 0^+}} \left.\frac{2\nu}{\Delta\beta_n(t,\nu)}\right|_{\beta_n=\beta} : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+,$$

with $\Delta\beta_n(t,\nu) = \beta_{n+1}(t,\nu) - \beta_n(t,\nu) > 0$, where $n \in \mathbb{Z}\backslash\{0\} = \{\pm1, \pm2, \cdots\}$, with the convention that $\beta_{-n} = -\beta_n$. Since

$$\frac{\partial w}{\partial \beta}(z_s(\beta;t),\beta) = \frac{i}{t}\, z_s(\beta;t), \tag{3.1a}$$

$$\frac{\partial w}{\partial z}(z_s(\beta;t),\beta) = \frac{i\beta}{t} - \frac{z_s(\beta;t)}{t} - u_0(z_s(\beta;t)) = 0, \tag{3.1b}$$

from Property 2.3 we find that the density is given by

$$\rho(\beta;t) = \Im\frac{dw}{d\beta}(z_s(\beta;t),\beta) = \frac{1}{\pi t}\,\Re z_s^+(\beta;t), \tag{3.2}$$

where $z_s^+(\beta;t)$ is the relevant saddle point with positive real part. Since $z_s(-\beta;t) = -z_s(\beta;t)$, in order to have $\rho(-\beta;t) = \rho(\beta;t) > 0$, we must take in both cases $\beta > 0$ and $\beta < 0$ the saddle point with positive real part. That is, z_0 for $\beta > 0$ and z_1 for $\beta < 0$ since they are related by $z_0(-\beta;t) = -z_0(\beta;t) = \overline{z_1}(\beta;t)$, and $z_1(-\beta;t) = -z_1(\beta;t) = \overline{z_0}(\beta;t)$ (see [17, §4.2]). With this choice of saddle points, the asymptotic density defined in (3.2) is positive whether $\beta > 0$ or $\beta < 0$. Note that if we let $x = i\beta$, $x_0 = x_0(x,t) = z_s(\beta;t)$, then the inviscid solution is $u(x,t) = u_0(x_0(x,t),t)$ where $x - x_0 - t\,u_0(x_0) = 0$ (see Appendix A). Solving (2.1), (3.1b) using Cardano's formula (see [17, Appendix B]), we find that

$$\begin{cases} z_0 = \frac{\sqrt{3}}{2}(\mathcal{A}-\mathcal{B}) + \frac{i}{2}(\mathcal{A}+\mathcal{B}) \\ z_1 = -\overline{z_0} = \frac{\sqrt{3}}{2}(\mathcal{B}-\mathcal{A}) + \frac{i}{2}(\mathcal{A}+\mathcal{B}) \\ z_2 = -i(\mathcal{A}+\mathcal{B}) \end{cases} \tag{3.3}$$

where for $\beta > |x_s(t)|$,

$$\begin{cases} \mathcal{A}(\beta;t) = (8t)^{-1/3}\,\sqrt[3]{\beta + \sqrt{\beta^2 + x_s^2}} > 0 \\ \mathcal{B}(\beta;t) = (8t)^{-1/3}\,\sqrt[3]{\beta - \sqrt{\beta^2 + x_s^2}} > 0 \end{cases} \tag{3.4}$$

For $\beta < -|x_s(t)|$, $\mathcal{A}$ and $\mathcal{B}$ are defined by the odd parity condition $\mathcal{A}(-\beta;t) = -\mathcal{A}(\beta;t)$ and $\mathcal{B}(-\beta;t) = -\mathcal{B}(\beta;t)$, so that

$$z_s(-\beta;t) = -z_s(\beta;t). \tag{3.5}$$

We can now describe the various cases $t = t_*$, $0 < t < t_*$, $t > t_*$, and $x = 0, t > t_*$:

(*i*) $\underline{t = t_*}$: From (3.2), (3.3) and (3.4), we find

$$\rho(\beta;t_*) = \frac{1}{\pi t_*}\Re z_s(\beta;t_*) = \frac{\sqrt{3}}{2\pi t_*}\left(\frac{\beta}{4t_*}\right)^{1/3} = \frac{2\sqrt{3}}{\pi}(4t_*)^{-4/3}|\beta|^{1/3}. \tag{3.6}$$

In the last step of (3.6), we replace $\beta^{1/3}$ by $|\beta|^{1/3}$ to allow for both $\beta > 0$ and $\beta < 0$. Note that we can obtain (3.6) by taking the limit as $t \to t_*$ in (3.7) or (3.8).

(*ii*) $\underline{0 < t < t_*}$: The density is zero for $|\beta| \leqslant |x_s|$ and for $|\beta| > |x_s|$, using (3.4), (3.3) and (3.2), we find

$$\rho(\beta;t) = \frac{2^{2/3}\sqrt{3}}{\pi}(4t)^{-4/3}\left\{\sqrt[3]{|\beta| + \sqrt{\beta^2 + x_s^2}} - \sqrt[3]{|\beta| - \sqrt{\beta^2 + x_s^2}}\right\}. \tag{3.7}$$

The behavior of $\rho(\beta;t)$ in a neighborhood of $\beta = \pm|x_s|$, ($|\beta| > |x_s|$) is

$$\rho(\beta;t) = \frac{t^{-4/3}}{\sqrt{6}\pi}\frac{\sqrt{|\beta| - |x_s|}}{|x_s|^{1/6}} + \mathcal{O}\Big(\big(|\beta| - |x_s|\big)^{3/2}\Big),$$

as mentioned in [3].

(*ii*) $\underline{t > t_*}$: From (3.2) and (3.3) we find $\forall\beta \in \mathbb{R}$,

$$\rho(\beta;t) = \frac{2^{2/3}\sqrt{3}}{\pi}(4t)^{-4/3}\left\{\sqrt[3]{\beta + \sqrt{\beta^2 + x_s^2}} + \sqrt[3]{-\beta + \sqrt{\beta^2 + x_s^2}}\right\}. \tag{3.8}$$

(*iii*) $\underline{x = 0, t \geqslant t_*}$: An interesting case occurs at the origin for $t > t_*$, as was pointed out by Bessis and Fournier in [3, 4]. The inviscid solution $u(0,t)$ at the shock is given by the asymptotic density of poles $\rho(0;t)$ (see (4.4)). If we look at the solution at the origin ($\beta \to 0$), $z_s(0;t)$ is the solution to

$$-\frac{z_s(0;t)}{t} = u_0(z_s(0;t)),$$

When $u_0(x) = 4x^3 - x/t_*$,

$$-\frac{z_s(0;t)}{t} = 4z_s(0;t)^3 - \frac{z_s(0;t)}{t_*}.$$

The non-zero pair of opposite saddle points are therefore

$$z_s^{\pm}(0;t) = \pm\frac{1}{2}\sqrt{\frac{t - t_*}{tt_*}} = \pm\sqrt{\frac{\alpha}{2}} \geqslant 0 \qquad \text{when} \quad t \geqslant t_*.$$

The corresponding density is easily found to be

$$\rho(0;t) = \frac{1}{\pi t}\Re z_s^+(0;t) = \begin{cases} \frac{1}{2\pi}(t - t_*)^{1/2}t^{-3/2}t_*^{-1/2} & t > t_* \\ 0 & t \leqslant t_* \end{cases} \tag{3.9}$$

It makes sense that the density $\rho(0;t)$ is null when $t < t_*$ for then all the poles β_n are located above the inviscid branch points x_s on the imaginary axis, and $|x_s| > 0$. This could have been found by letting $\beta \to 0$ in (3.8). □.

4. Residue computation of the integral representation of the inviscid limit. In this section, we compute via residues the integral representation of the inviscid solution found in § 2 using the explicit values of the asymptotic pole densities computed in § 3. We prove that it matches the inviscid solution found by the method of characteristics:

THEOREM 4.1. *The analytic structure of the inviscid solution can be obtained from the asymptotic pole density and the integral representation of the inviscid solution for*

$0 < t \leqslant t_*$: *Let* $\beta \in \mathbb{C}$, *and consider* $\rho(\beta; t) : \mathbb{C}\backslash \{(-\infty, -|x_s|] \cup [|x_s|, +\infty)\} \times (0, t_*) \to \mathbb{C}$ *defined by*

$$\rho(\beta;t) = \begin{cases} \frac{2^{2/3}\sqrt{3}}{\pi}(4t)^{-4/3}\left\{\sqrt[3]{\beta+\sqrt{\beta^2+x_s^2}}+\sqrt[3]{\beta-\sqrt{\beta^2+x_s^2}}\right\} & 0<t<t_* \\ \frac{2\sqrt{3}}{\pi}(4t_*)^{-4/3}\sqrt[3]{\beta} & t=t_* \end{cases}$$

$\forall x \notin (-i\infty, -i|x_s|] \cup [i|x_s|, +i\infty)$ *for* $0 < t < t_*$, *and* $\forall x \notin i\mathbb{R}_+$ *for* $t = t_*$, *we have*

$$t \cdot 2x \int_{|x_s(t)|}^{\infty} \frac{\rho(\beta;t)}{x^2+\beta^2}\, d\beta = U(x,t),$$

where $U(x,t)$ *is the spatially singular part of the inviscid solution defined by*

$$U(x,t) = \begin{cases} (8t)^{-1/3}\left\{\sqrt[3]{x+\sqrt{x^2-x_s^2}}+\sqrt[3]{x-\sqrt{x^2-x_s^2}}\right\} & 0<t<t_* \\ \sqrt[3]{\frac{x}{4t_*}} & t=t_* \end{cases}$$

with branch cuts above mentioned.

Proof. In [4], Bessis and Fournier claim that expressing the limiting pole expansion (2.1) as a Cauchy integral allows us to recover the inviscid solution from the density of poles. We now verify this claim for $0 < t \leqslant t_*$. We must no longer use the real valued definition of the density $\rho(\beta; t)$ in the integral in Theorem 2.2, but the complex-valued one (see (3.3)) which we denote by $\rho(\beta; t)$:

$$\rho(\beta;t) = \frac{\mathcal{A}(\beta;t)+\mathcal{B}(\beta;t)}{\pi t}$$

where $\mathcal{A}(\beta;t)$, $\mathcal{B}(\beta;t)$ are given by $(8t)^{-1/3}\sqrt[3]{\beta \pm \sqrt{\beta^2+x_s^2}}$. Thus define

$$\rho(\beta;t) \equiv \frac{2^{2/3}\sqrt{3}}{\pi}(4t)^{-4/3}\left\{\sqrt[3]{\beta+\sqrt{\beta^2+x_s^2}}+\sqrt[3]{\beta-\sqrt{\beta^2+x_s^2}}\right\}. \tag{4.1}$$

We choose the branches of the complex-valued function $\rho(\beta; t)$ in (4.1) in such a way that $\rho(-\beta; t) = -\rho(\beta; t)$, and whose branch cuts defined by the imaginary interval $(-i\infty, -i|x_s|] \cup [i|x_s|, +i\infty)$ for $0 < t < t_*$, and the ray $i\mathbb{R}_+$ when $t = t_*$ are displayed on Figs. 4.1 and 4.2; thus we have the equality

$$\int_{|x_s(t)|}^{\infty} \frac{\rho(\beta;t)}{x^2+\beta^2}\, d\beta = \int_{-|x_s(t)|}^{-\infty} \frac{\rho(\beta;t)}{x^2+\beta^2}\, d\beta.$$

(*i*) <u>$t = t_*$</u>: Since $x_s(t_*) = 0$, using the contour of integration displayed in Fig. 4.1, we find

$$\int_0^{\infty} \frac{\beta^{1/3}}{x^2+\beta^2}\, d\beta = \frac{\pi}{\sqrt{3}x^{2/3}}. \tag{4.2}$$

Combining (3.6), (2.2) and (4.2), we recover the inviscid solution at $t = t_*$:

$$u(x,t_*) = \lim_{\nu\to 0+} u_\nu(x,t_*) = \frac{x}{t_*} - \left(\frac{x}{4t_*^4}\right)^{1/3} = \frac{x}{t_*} - \frac{U(x,t_*)}{t_*}. \tag{4.3}$$

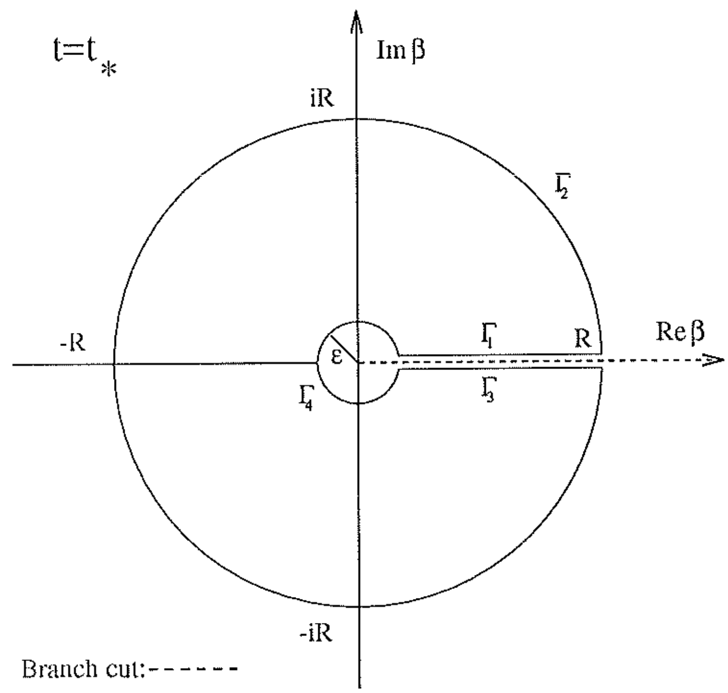

FIG. 4.1. *Contour of integration for the inviscid limit at $t = t_*$*

(ii) $0 < t < t_*$: In order to recover the inviscid solution via (2.2), we use the contour of integration displayed in Fig. 4.2 to evaluate

$$I = \int_{|x_s|}^{\infty} \frac{\rho(\beta;t)}{x^2+\beta^2}\,d\beta = \frac{2^{2/3}\sqrt{3}}{\pi}(4t)^{-4/3}\int_{|x_s|}^{\infty} f(\beta;t)\,d\beta,$$

where $(x^2+\beta^2)\cdot f(\beta;t) = \sqrt[3]{\beta+\sqrt{\beta^2+x_s^2}}+\sqrt[3]{\beta-\sqrt{\beta^2+x_s^2}}$, with branch cuts described in Fig. 4.2. By Cauchy's theorem

$$\int_\gamma f(\beta;t)\,d\beta = 2\pi i\left\{\mathrm{Res}[f(\beta;t),\beta=ix] + \mathrm{Res}[f(\beta;t),\beta=-ix]\right\}$$

where $\gamma = \sum_{i=1}^{8}\Gamma_i$. Letting $J = \int_{|x_s|}^{\infty} f(\beta;t)\,d\beta$, we find

$$\int_{\Gamma_1} f(\beta;t)\,d\beta \xrightarrow[\epsilon\to 0+]{R\to+\infty} J, \qquad \int_{\Gamma_3} f(\beta;t)\,d\beta \xrightarrow[\epsilon\to 0+]{R\to+\infty} -J$$

$$\int_{\Gamma_5} f(\beta;t)\,d\beta \xrightarrow[\epsilon\to 0+]{R\to+\infty} J, \qquad \int_{\Gamma_7} f(\beta;t)\,d\beta \xrightarrow[\epsilon\to 0+]{R\to+\infty} -\omega\,J = e^{-\frac{i\pi}{3}}\,J$$

$$\int_{\Gamma_4} f(\beta;t)\,d\beta = \int_{\Gamma_8} f(\beta;t)\,d\beta \xrightarrow{\epsilon\to 0^+} 0$$

$$\int_{\Gamma_2} f(\beta;t)\,d\beta = \int_{\Gamma_6} f(\beta;t)\,d\beta \xrightarrow{R\to+\infty} 0$$

Since $\beta = \pm ix$ are simple poles of f, we find

$$\left(1+e^{-\frac{i\pi}{3}}\right)J = 2\pi i\left(\frac{g(ix;t)}{2ix} - \frac{g(-ix;t)}{2ix}\right) \Longrightarrow J = \frac{\pi}{x}\frac{e^{\frac{i\pi}{6}}}{\sqrt{3}}(g(ix;t)-g(-ix;t))$$

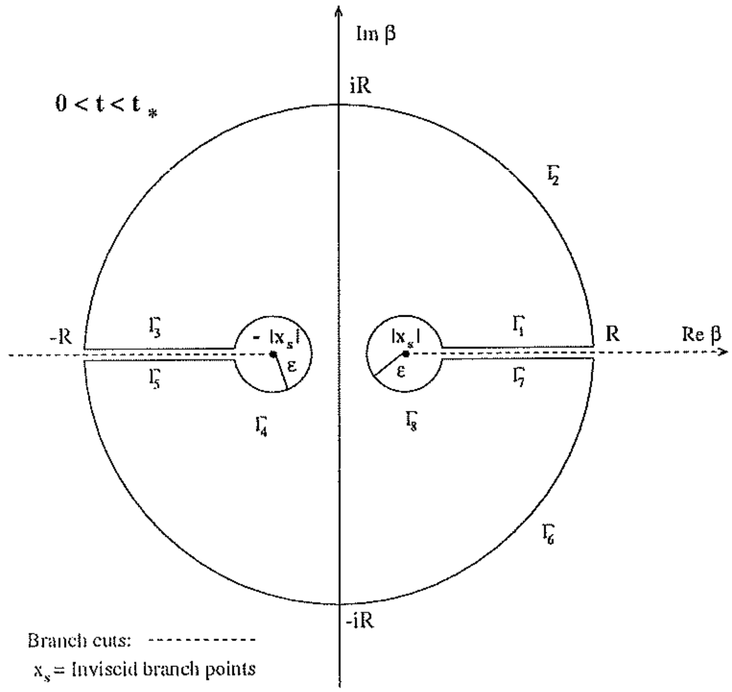

FIG. 4.2. *Contour of integration for the inviscid limit for $t < t_*$*

where $g(\beta;t) = \sqrt[3]{\beta + \sqrt{\beta^2 + x_s^2}} + \sqrt[3]{\beta - \sqrt{\beta^2 + x_s^2}}$. Since

$$I = \frac{2^{2/3}\sqrt{3}}{\pi}(4t)^{-4/3} J,$$

this yields

$$t \cdot 2x \int_{|x_s|}^{\infty} \frac{\rho(\beta;t)}{x^2+\beta^2}\, d\beta = (8t)^{-1/3} e^{\frac{i\pi}{6}} \left(g(ix;t) - g(-ix;t)\right).$$

We find that

$$e^{\frac{i\pi}{6}} \left(g(ix;t) - g(-ix;t)\right) = \left(e^{\frac{i\pi}{3}} - e^{\frac{2\pi i}{3}}\right) \left\{ \sqrt[3]{x + \sqrt{x^2 - x_s^2}} + \sqrt[3]{x - \sqrt{x^2 - x_s^2}} \right\}.$$

Since $e^{\frac{i\pi}{3}} - e^{\frac{2\pi i}{3}} = 1$, we finally have

$$t \cdot 2x \int_{|x_s|}^{\infty} \frac{\rho(\beta;t)}{x^2+\beta^2}\, d\beta = (8t)^{-1/3} \left\{ \sqrt[3]{x + \sqrt{x^2 - x_s^2}} + \sqrt[3]{x - \sqrt{x^2 - x_s^2}} \right\} = U(x,t),$$

where $U(x,t)$ is the spatially singular part of the inviscid solution defined in (A.8). Thus we have found that the inviscid limit is recovered in the sense that

$$\lim_{\nu \to 0+} u_\nu(x,t) = \frac{x}{t} - \frac{U(x,t)}{t} = u(x,t), \quad \text{for } 0 < t \leqslant t_*.$$

(iii) $\underline{x = 0, t \geqslant t_*}$: Since $z_s(0;t) \in \mathbb{R}$ for $t \geqslant t_*$, $\rho(0;t) = z_s(0;t)/\pi t$. Thus the inviscid solution at the origin (shock) for $t > t_*$ is given by (see (4.8b))

$$u(0,t) = \frac{u(0^-,t) - u(0^+,t)}{2} = \pi\rho(0;t) = \frac{1}{2}(t-t_*)^{1/2} t^{-3/2} t_*^{-1/2}, \tag{4.4}$$

i.e. the solution at the shock satisfies the jump condition (see (A.9) and [3, 15]):

$$u(0^{\mp},t) = \pm\frac{1}{2}(t-t_*)^{1/2}t^{-3/2}t_*^{-1/2}.$$

Although this result was stated by Bessis and Fournier and can be derived by taking $\lim_{\beta\to 0}\rho(\beta;t)$, it is mentioned here to verify the formula $\rho(\beta;t) = (\pi t)^{-1}\Re z_s^+(\beta;t)$ where $z_s^+(\beta;t)$ is the saddle point relevant to the asymptotic expansion with positive real part.

As a final remark, we would like to point out that this procedure which consists in recovering the analytic structure of the inviscid solution via the limiting pole density and the pole expansion is no longer possible when $t > t_*$ and $x \neq 0$. Indeed in this case we are faced with the same (apparent) paradox that is present in the asymptotic expansion of the (spatial) Fourier transform of the inviscid solution (see § 6 and also [3]). Thus the only way to recover the inviscid solution for $t > t_*$ and $x \neq 0$ using the limiting pole density is by extending the solution obtained for $t < t_*$ to $t > t_*$.

4.1. Analytic extension of the integral representation of the inviscid solution on the imaginary axis. Let

$$u(x,t) = \frac{x}{t} - 2x\int_0^\infty \frac{\rho(\beta;t)}{x^2+\beta^2}\,d\beta = \frac{x}{t} - \int_{-\infty}^\infty \frac{\rho(\beta;t)}{x-i\beta}\,d\beta, \tag{4.5a}$$

$$\tilde{u}(y,t) = \frac{y}{t} + 2y\,P.V.\int_0^\infty \frac{\rho(\beta;t)}{y^2-\beta^2}\,d\beta = \frac{y}{t} + P.V.\int_{-\infty}^\infty \frac{\rho(\beta;t)}{y-\beta}\,d\beta. \tag{4.5b}$$

Then one can show that $\rho(y;t)$ is a density function which satisfies the conservation equation

$$\rho_t + (\rho\,\tilde{u})_y = 0. \tag{4.6}$$

Indeed one only needs to verify that $u(x,t)$ defined by (4.5a) satisfies the inviscid Burgers equation $u_t + uu_x = 0$ under the assumption that (4.6) holds. Since u has branch cuts on the imaginary axis for $t < t_*$, $\tilde{u}$ has branch cuts for y real (i.e. also on the imaginary axis), one can analytically continue u on the imaginary axis using the paths displayed in Fig. 4.3. Define the solution on the left (u_-) and right (u_+) of the imaginary axis by

$$u_\pm(iy,t) = \lim_{x\to 0^\pm} u(z = x+iy, t). \tag{4.7}$$

The discontinuity at $x = 0$ characterizes the shock solution. Thus we find that

$$\begin{aligned} u_+(iy,t) &= \frac{iy}{t} - P.V.\int_{-\infty}^\infty \frac{\rho(\beta;t)}{iy-i\beta}\,d\beta - \frac{\rho(y;t)}{i}\overbrace{\left(-\frac{1}{2}\oint_{|y-\beta|=\epsilon}\frac{d\beta}{y-\beta}\right)}^{\pi i} \\ &= i\left\{\frac{y}{t} + \int_{-\infty}^\infty \frac{\rho(\beta;t)}{y-\beta}\,d\beta + i\pi\rho(y;t)\right\}. \end{aligned}$$

Similarly we have that

$$u_-(iy,t) = i\left\{\frac{y}{t} + \int_{-\infty}^\infty \frac{\rho(\beta;t)}{y-\beta}\,d\beta - i\pi\rho(y;t)\right\}.$$

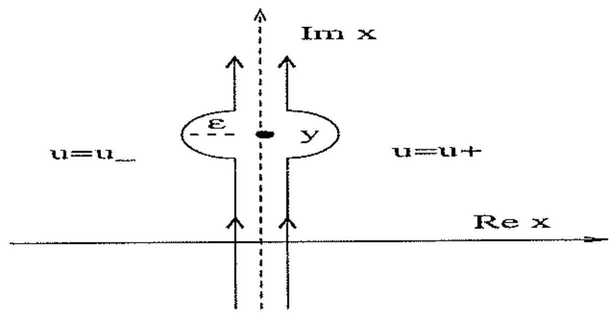

FIG. 4.3. *Analytic continuation of the integral representation of the inviscid solution on the imaginary axis*

Therefore

$$\tilde{u}(y,t) = \frac{1}{2i}(u_+(iy,t) + u_-(iy,t)), \tag{4.8a}$$

$$\rho(y;t) = \frac{1}{2\pi}(u_-(iy,t) - u_+(iy,t)). \tag{4.8b}$$

Since $u_\pm$ are real on the real axis, it is clear that they satisfy the symmetry relations

$$u_\pm(\overline{x},t) = \overline{u_\pm(x,t)},$$
$$u_-(-x,t) = -u_+(x,t),$$

and therefore

$$u_\pm(iy,t) = \overline{u_\pm(-iy,t)} = -\overline{u_\mp(iy,t)}.$$

From this we have that

$$\tilde{u}(y,t) = \Im u_+(iy,t), \tag{4.9a}$$

$$\rho(y;t) = -\frac{1}{\pi}\Re u_+(iy,t) = \frac{1}{\pi}\Re u_-(iy,t), \tag{4.9b}$$

and the symmetry relations

$$\tilde{u}(-y,t) = -\tilde{u}(y,t),$$
$$\rho(-y;t) = \rho(y;t).$$

Notice finally the similarity between the definitions (3.2) and (4.9b); they show that on the imaginary axis we have (see § 3)

$$u_\mp(iy,t) = \pm z_s^+(y;t)/t. \tag{4.10}$$

5. Continuum limit of the pole expansion and the Calogero dynamical system. It is shown in [17] that the poles satisfy the following property:

PROPERTY 5.1. *The imaginary part $\beta_n = \beta_n(t,\nu) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ of the simple poles $x = \pm i\beta_n$ of $u_\nu(x,t)$ satisfy the Calogero-type infinite dimensional dynamical system*

$$\forall n \in \mathbb{N}^*, \quad \dot{\beta}_n = \frac{\beta_n}{t} - 2\nu \sum_{\substack{l=-\infty \\ l\neq 0,n}}^{\infty} \frac{1}{\beta_l - \beta_n}$$

Let $a_n(t,\nu) = i\beta_n(t,\nu)$, and define the complex map $\mathcal{F}(\zeta,\nu,t)$ as

$$a_n(t,\nu) = \mathcal{F}(\zeta_n^\nu = \nu n, \nu, t) : \mathbb{Z}^* \times \mathbb{R}_+^2 \to i\mathbb{R}_+, \qquad a_{-n} = -a_n. \tag{5.1}$$

Then, re-formulating in terms of a_n the Mittag-Leffler expansion of u_ν in Theorem 2.1 and the ODE in Property 5.1, we find

$$\dot{a}_n = \frac{a_n}{t} - 2\nu \sum_{\substack{l=-\infty \\ l\neq 0,n}}^{\infty} \frac{1}{a_n - a_l}, \tag{5.2a}$$

$$u_\nu(x,t) = \frac{x}{t} - 2\nu \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{1}{x - a_l}, \tag{5.2b}$$

where both of these symmetric sums should be understood as

$$\sum_{\substack{l=-\infty \\ l\neq 0,n}}^{\infty} \frac{1}{a_n - a_l} = \frac{1}{2a_n} + 2a_n \sum_{\substack{l=1 \\ l\neq n}}^{\infty} \frac{1}{a_n^2 - a_l^2} = \frac{1}{2a_n} + a_n \sum_{\substack{l=-\infty \\ l\neq 0,\pm n}}^{\infty} \frac{1}{a_n^2 - a_l^2},$$

and

$$\sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{1}{x - a_l} = 2x \sum_{l=1}^{\infty} \frac{1}{x^2 - a_l^2} = x \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{1}{x^2 - a_l^2}.$$

At t_*, we have (cf. [17, §4.1])

$$\begin{aligned} a_n(t_*,\nu) &= \mathcal{F}(\zeta_n^\nu = \nu n, \nu, t_*) = i \cdot 4t_* \left(2\nu\mu_n\right)^{3/4} \\ &= i \cdot 4t_* \left(2\nu(c_{-1}n + c_0 + c_1/n + \cdots)\right)^{3/4} \\ &= i \cdot 4t_* \left(c_{-1}(2\nu n) + c_0 2\nu + c_1(2\nu)^2/(2\nu n) + \cdots)\right)^{3/4}. \end{aligned}$$

Introduce the map

$$f(\zeta,t) = \mathcal{F}(\zeta,0,t) : \mathbb{R} \times \mathbb{R}_+ \to i\mathbb{R}_+, \qquad f(-\zeta,t) = -f(\zeta,t), \tag{5.3}$$

where the continuous variable ζ corresponds to a position on the real axis which can be thought of as a variable obtained by simultaneously letting $\nu \to 0^+$ and $n \to +\infty$. Assume that

$$a_n(t,\nu) = \mathcal{F}(n\nu,\nu,t) = f(n\nu,t) + e_n(\nu,t) \tag{5.4}$$

in which $e_n(\nu,t)$ is a small error term that goes to 0 as $\nu \to 0^+$. Thus, formally we have

$$\begin{aligned} 2\nu \sum_{\ell\neq n} \frac{1}{a_n(t,\nu) - a_\ell(t,\nu)} &\simeq 2\nu \sum_{\ell\neq n} \frac{1}{f(n\nu,t) - f(\ell\nu,t)} \\ &\xrightarrow{\nu\to 0^+} 2P.V. \int_{-\infty}^{\infty} \frac{d\zeta'}{f(\zeta,t) - f(\zeta',t)}. \end{aligned} \tag{5.5}$$

Moreover, this approximation shows that the representation (5.4) is valid for all time if it is true at $t = t_*$. A rigorous analysis of the approximation (5.5) has been performed in the context of vortex sheets in [9]. It is then clear that the pair of equations (5.2a) and (5.2b) satisfy the following:

PROPERTY 5.2. *The continuum limit of the Calogero dynamical system and the pole expansion is the system of integro-differential equations defined for any x such that $\forall \zeta \in \mathbb{R}$, $x \neq f(\zeta, t)$, by*

$$\frac{\partial f}{\partial t}(\zeta, t) = \frac{f(\zeta,t)}{t} - P.V. \int_{-\infty}^{\infty} \frac{d\zeta'}{f(\zeta,t) - f(\zeta',t)},$$
$$u(x,t) = \frac{x}{t} - \int_{-\infty}^{\infty} \frac{d\zeta'}{x - f(\zeta',t)}.$$

This property can also be expressed as

$$\begin{aligned}\frac{\partial f}{\partial t}(\zeta, t) &= \frac{f(\zeta,t)}{t} - 2f(\zeta,t) \int_{0}^{\infty} \frac{d\zeta'}{f^2(\zeta,t) - f^2(\zeta',t)} \\ &= \frac{f(\zeta,t)}{t} - f(\zeta,t)\, P.V. \int_{-\infty}^{\infty} \frac{d\zeta'}{f^2(\zeta,t) - f^2(\zeta',t)},\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}u(x,t) &= \frac{x}{t} - 2x \int_{0}^{\infty} \frac{d\zeta'}{x^2 - f^2(\zeta',t)} \\ &= \frac{x}{t} - x \int_{-\infty}^{\infty} \frac{d\zeta'}{x^2 - f^2(\zeta',t)}, \qquad x \neq f(\zeta,t).\end{aligned} \tag{5.7}$$

Equation (5.7) defines the branch cuts of the inviscid solution as the set of complex x-points for which $x = f(\zeta,t)$, while equation (5.6) defines the dynamics of these branch cuts.

Let $\zeta' = (f^{-1}(i\beta))_t$ be the spatial projection of the inverse map of f, that is

$$f\left((f^{-1}(i\beta))_t, t\right) = i\beta. \tag{5.8}$$

We can introduce this change of variable in Property 5.2 to obtain

$$\frac{\partial f}{\partial t}(\zeta,t) = \frac{f(\zeta,t)}{t} - \int_{-\infty}^{\infty} \frac{d\zeta'/d\beta}{f(\zeta,t) - i\beta}\, d\beta = \frac{f(\zeta,t)}{t} - 2f(\zeta,t) \int_{0}^{\infty} \frac{\rho(\beta;t)}{f(\zeta,t)^2 + \beta^2}\, d\beta,$$

and

$$u(x,t) = \frac{x}{t} - \int_{-\infty}^{\infty} \frac{d\zeta'/d\beta}{x - i\beta}\, d\beta = \frac{x}{t} - 2x \int_{0}^{\infty} \frac{\rho(\beta;t)}{x^2 + \beta^2}\, d\beta.$$

Thus we have recovered the limiting integral representation of the inviscid solution (cf. § 2). It is then easy to read the previous system as the characteristic solution emanating from the point $f(\zeta,t)$ since we can write it as

$$U(x,t) = t \cdot 2x \int_{0}^{\infty} \frac{\rho(\beta;t)}{x^2 + \beta^2}\, d\beta, \tag{5.9a}$$

and

$$\frac{\partial f}{\partial t}(\zeta,t) = \frac{f(\zeta,t)}{t} - \frac{U(f(\zeta,t),t)}{t} = u(f(\zeta,t),t). \tag{5.9b}$$

Moreover we have that

$$\frac{d\zeta}{d\beta} = \frac{d\left(f^{-1}(i\beta)\right)_t}{d\beta} = \rho(\beta;t), \tag{5.10}$$

from which we find that

$$\zeta = \left(f^{-1}(i\beta)\right)_t = \int_0^\beta \rho(\xi;t)\,d\xi, \tag{5.11}$$

and in particular

$$\begin{aligned}\zeta_* &= \left(f^{-1}(i\beta)\right)_{t_*} = \int_0^{\beta_*} \rho(\xi;t_*)\,d\xi \\ &= \int_0^{\beta_*} \frac{2\sqrt{3}}{\pi}(4t_*)^{-4/3}\xi^{1/3}\,d\xi = \left(\frac{2\pi}{3\sqrt{3}}\right)^{-1}\left(\frac{\beta_*}{4t_*}\right)^{4/3}.\end{aligned}$$

Inverting this relation in terms of β_* we find that

$$\beta_* = 4t_*\left(\frac{2\pi}{3\sqrt{3}}\zeta_*\right)^{3/4}. \tag{5.12}$$

Similar computations can be found in [21]. In order to recover the correct discretization, it suffices to choose

$$\zeta_* = \zeta_n(t_*,\nu) = 2\nu\mu_n\left(\frac{2\pi}{3\sqrt{3}}\right)^{-1} \Rightarrow \beta_* = \beta_n(t_*,\nu) = 4t_*\left(2\nu\mu_n\right)^{3/4}, \tag{5.13}$$

where $\mu_n = \frac{2\pi}{3\sqrt{3}}(n-\frac{1}{3}) + \mathcal{O}(1/n)$ as $n \to +\infty$ (see [17, §4.1]).

6. Uniform asymptotic expansion in a neighborhood of t_* of the spatial Fourier transform of the inviscid solution $\hat{u}(k,t)$ as $k \to +\infty$. The analyticity properties of the inviscid solution can also be analyzed by describing the asymptotic behavior of its Fourier transform (see [12, 20]). We find a uniform asymptotic expansion as $k \to +\infty$ of the Fourier transform of the inviscid solution in a neighborhood of $t = t_*$ where two second order branch points $\pm x_s(t)$ coalesce into a third order branch point at the origin $x_s(t_*) = 0$. Thus we clarify the seemingly discontinuous change of behavior of $\hat{u}(k,t)$ at t_* presented in [12]. This result is resumed in the following theorem:

THEOREM 6.1. *The uniform asymptotic expansion of the Fourier transform of the inviscid solution for $0 < t \leqslant t_*$ is*

$$\hat{u}(k,t) = C_0 \cdot (tk)^{-4/3} Ai\left[\left(-3ikx_s(t)/2\right)^{2/3}\right]\left(1 + \mathcal{O}\left(k^{-1}\right)\right) \quad \text{as } k \to +\infty.$$

Thus from the asymptotic property of the Airy function and its value at the origin Ai(0) we have

$$\hat{u}(k,t) \sim \begin{cases} C_1(t)\cdot(t_*-t)^{-1/4}k^{-3/2}e^{-k|x_s(t)|} & 0 < t < t_* \\ C_2\cdot(t_*k)^{-4/3} & t = t_* \end{cases} \quad \text{as } k \to +\infty.$$

Proof. In [12], Fournier and Frisch derive the asymptotic behavior of the inviscid solution via the so-called Fourier-Lagrangian (F-L) representation which is valid up to the time where the relation $x(x_0,t) = x_0 + tu_0(x_0)$ is invertible, i.e. up to $t_* = -(\inf_{x_0} u_0'(x_0))^{-1}$. There is a discontinuous change in the behavior of $\hat{u}(k,t)$ in

$k^{-3/2}\exp(-k|x_s(t)|)$ before t_* to $(t_*k)^{-4/3}$ at t_* which arises from the fact that the two saddle points of multiplicity 1 for $0 < t < t_*$ coalesce at the origin to form a saddle point of multiplicity 2 at $t = t_*$. The Fourier-Lagrangian representation is found by changing variables from the Eulerian coordinate to the Lagrangian coordinate, followed by an integration by parts:

$$\begin{aligned}\hat{u}(k,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}u(x,t)\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx(x_0,t)}u_0(x_0)\frac{\partial x}{\partial x_0}dx_0 \\ &= \frac{1}{\sqrt{2\pi}ik}\int_{-\infty}^{\infty} e^{-ikx(x_0,t)}u_0'(x_0)\,dx_0, \qquad (k \neq 0).\end{aligned}$$

For $u_0(x) = 4x^3 - x/t_*$, $x(x_0,t) = 4tx_0^3 + x_0(1 - t/t_*)$, we find

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}ik}\int_{-\infty}^{\infty} \exp\left\{-12ikt\left(\frac{x_0^3}{3} - \frac{\alpha}{6}x_0\right)\right\} u_0'(x_0)\,dx_0,$$

where $2\alpha = 1/t_* - 1/t$. Let $\lambda = -12ikt$, then we are interested in finding the behavior of $\hat{u}(k,t)$ as $k \to +\infty$, that is the behavior as $\lambda \to \infty$ of the integral

$$\int_{-\infty}^{\infty} \exp\left\{\lambda\left(\frac{x_0^3}{3} - \frac{\alpha}{6}x_0\right)\right\} u_0'(x_0)\,dx_0. \tag{6.1}$$

The saddle points of the integrand occur when $\partial x/\partial x_0 = 0$, so that (see (A.7))

$$x_0 = x_0^{\pm}(t) = \pm\sqrt{\frac{\alpha}{6}} = \pm\sqrt[3]{\frac{x_s(t)}{8t}} \Rightarrow x\left(x_0^{\pm}(t)\right) = \pm x_s(t).$$

At $t = t_*$, $x_0^{\pm}(t_*) = 0$, and the two saddle points of multiplicity 1 have coalesced into a saddle point of multiplicity 2 at the origin. Let

$$f(x_0) = \frac{x(x_0,t)}{12t} = \frac{x_0^3}{3} - \frac{\alpha}{6}x_0,$$

and recall that $x_s(t) = t(2\alpha/3)^{3/2}$ (see (A.6)), then

$$f(x_0^{\pm}(t)) = -\frac{2}{3}x_0^{\pm}(t)^3 = \mp\frac{x_s(t)}{12t}.$$

We introduce the coefficients

$$\begin{aligned}\zeta^{3/2} &= \frac{3}{4}\left(f(x_0^-(t)) - f(x_0^+(t))\right) = \frac{3}{2}f(x_0^-(t)) = \frac{x_s(t)}{8t}, \\ \eta &= \frac{1}{2}\left(f(x_0^-(t)) + f(x_0^+(t))\right) = 0,\end{aligned}$$

which arise in the construction of a uniform asymptotic expansion of an integral with two coalescing saddle points. The integral defined in (6.1) is already in a format appropriate for such a derivation. Indeed it is an integral of the form

$$I(\lambda;\zeta,\eta) = \int_C \exp\left\{\lambda\left(u^3/3 - \zeta u + \eta\right)\right\}\phi_0(u)\,du,$$

where $\lambda \to \infty$. Thus the 1-1 analytic transformation $x_0 \leftrightarrow u$ given by the equation $f(x_0) = u^3/3 - \zeta u + \eta$ is simply the identity $x_0 \equiv u$. Therefore the time-uniform asymptotic expansion of the spatial Fourier transform of the solution $\hat{u}(k,t)$ is immediately found in terms of the Airy function and its derivative (cf. [10] and [23, VII-4]): Let

$$a_0 = \frac{1}{2}\left[u_0'(\zeta^{1/2}) + u_0'(-\zeta^{1/2})\right], \qquad b_0 = \frac{1}{2\zeta^{1/2}}\left[u_0'(\zeta^{1/2}) - u_0'(-\zeta^{1/2})\right],$$

then

$$\hat{u}(k,t) = \frac{e^{-\lambda\eta}}{\sqrt{2\pi ik}} \cdot 2\pi i \left[\frac{\mathrm{Ai}\left[\lambda^{2/3}\zeta\right]}{\lambda^{1/3}}(a_0 + \mathcal{O}(1/\lambda)) + \frac{\mathrm{Ai}'\left[\lambda^{2/3}\zeta\right]}{\lambda^{2/3}}(b_0 + \mathcal{O}(1/\lambda))\right],$$

as $\lambda(k) = -12ikt \to \infty$. Since $\eta = 0$, $\zeta^{3/2} = x_s(t)/8t$, $\zeta^{1/2} = x_0^+(t)$, $u_0'(x_0) = 12x_0^2 - 1/t_*$, $a_0 = u_0'(x_0^\pm(t)) = 1/t$, $b_0 = 0$, we obtain the asymptotic behavior of $\hat{u}(k,t)$ as $k \to +\infty$ uniform in a compact interval containing $t = t_*$:

$$\hat{u}(k,t) = C_0 \cdot (tk)^{-4/3}\mathrm{Ai}\left[(-3ikx_s(t)/2)^{2/3}\right]\left(1 + \mathcal{O}\left(k^{-1}\right)\right) \quad \text{as } k \to +\infty, \tag{6.2}$$

where C_0 is an appropriate numerical constant. Note that (6.2) can be obtained without recourse to this method by a classical asymptotic analysis in which one would express the expansion in terms of the Airy function. We choose this derivation due to the simplicity of its construction. For $0 < t < t_*$, using the fact that

$$\mathrm{Ai}(z) = \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}}\left(1 + \mathcal{O}\left(z^{-3/2}\right)\right) \quad \text{as } z \to \infty \text{ in } |\arg z| < \pi,$$

and for $t = t_*$, evaluating $\mathrm{Ai}(0) = 3^{-2/3}/\Gamma(2/3)$, we obtain the asymptotic behavior of $\hat{u}(k,t)$ as $k \to +\infty$ for $0 < t \leqslant t_*$ described in the second part of Theorem 6.1 in which $C_1(t)$ is a constant depending on t, and C_2 is a numerical constant. These expansions are consistent with the fact that the Fourier transform of an analytic function with a branch point singularity at $z_0 = x_0 + iy_0$ of the form (cf. [20]):

$$v(z) \sim (z - z_0)^\mu, \qquad \mu \notin \mathbb{Z},$$

has an asymptotic behavior of the form

$$\hat{v}(k) \sim k^{-(\mu+1)}e^{-ky_0}e^{ix_0k} \quad \text{as } k \to +\infty.$$

Note that the expansion for $t > t_*$ obtained from (6.2) yields the incorrect behavior $|\hat{u}(k,t)| \sim C_1(t)\cdot(t-t_*)^{-1/4}k^{-3/2}$ which is valid only for moderate wave numbers of the form $1 \ll k \leqslant 1/|x_s|$. This is due to the fact that the formal F-L representation is no longer valid beyond t_*. The correct behavior after t_* of the form $|\hat{u}_I(k,t)| \sim C_3(t)\cdot(t-t_*)^{1/2}k^{-1}$ for $k > 1/|x_s|$ which reflects the presence of a shock must then be obtained by following the work of Fournier and Frisch in [12]. The two expansions agree when $k \simeq 1/|x_s|$ giving a behavior of the form $|\hat{u}(k,t)|, |\hat{u}_I(k,t)| \sim C_4(t)\cdot(t-t_*)^2$ for t close to t_* (see [12]).

A. Inviscid solution ($\nu = 0$). The inviscid Burger's equation ($\nu = 0$)

$$\begin{cases} u_t + u\,u_x = 0 \\ u(x,0) = u_0(x) = 4x^3 - x/t_* \end{cases} \tag{A.1}$$

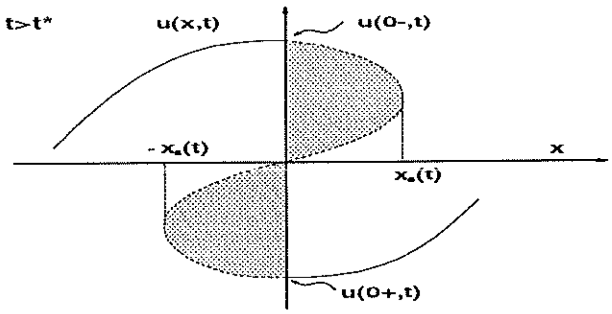

FIG. A.1. *Shock, multi-valuedness, branch points and Maxwell's equal area rule for $t > t_*$.*

states that the velocity of a fluid particle is conserved along certain trajectories, namely the characteristic lines

$$\dot{x} = \frac{dx}{dt}(t) = u(x(t), t) = u_0(x_0(x,t)) \tag{A.2}$$

in the (x,t) plane. The implicit solution obtained by the method of characteristics reflects the conservation of the velocity along these special curves:

$$\begin{cases} u = u(x,t) = u_0(x_0(x,t)) \\ x = x_0 + t\,u_0(x_0(x,t)) \end{cases} \tag{A.3}$$

A fluid particle originally at a (Lagrangian) position x_0 in space will be at a new (Eulerian) position x after a certain time t with the same velocity along this line. We can express (A.3) as

$$\begin{cases} u(x,t) = x/t - U(x,t)/t \\ U(x,t) = x_0(x,t) \end{cases} \tag{A.4}$$

Substituting $u_0(x)$ in (A.3), we find that U satisfies the cubic equation

$$U^3 - \frac{\alpha}{2}U - \frac{x}{4t} = 0, \qquad \alpha = \frac{t - t_*}{2tt_*}. \tag{A.5}$$

This defines a three-sheeted Riemann surface for the solution with a third order branch point at infinity and two opposite second order branch points at $\pm x_s(t)$ defined by

$$x_s(t) = \begin{cases} i\,(3t_*)^{-3/2}(t_* - t)^{3/2}t^{-1/2} & \in i\mathbb{R} \quad 0 < t \leqslant t_* \\ (3t_*)^{-3/2}(t - t_*)^{3/2}t^{-1/2} & \in \mathbb{R} \quad\quad t \geqslant t_*. \end{cases} \tag{A.6}$$

The envelope of the characteristic lines is given by the branch point:

$$\begin{aligned} 0 = \frac{\partial x}{\partial x_0} = \frac{\partial x}{\partial U} \Rightarrow x_0 = x_0^{\pm}(t) = \pm\sqrt{\frac{t - t_*}{12tt_*}} = \pm\sqrt{\frac{\alpha}{6}} = \pm\sqrt[3]{\frac{x_s}{8t}} \\ \Rightarrow x\left(x_0^{\pm}(t)\right) = x_0^{\pm}(t) + t\,u_0\left(x_0^{\pm}(t)\right) = \pm x_s(t). \end{aligned} \tag{A.7}$$

and the solution is

$$U(x,t) = \begin{cases} (8t)^{-1/3}\left\{\sqrt[3]{x + \sqrt{x^2 - x_s^2}} + \sqrt[3]{x - \sqrt{x^2 - x_s^2}}\right\} & t \neq t_* \\ \sqrt[3]{\frac{x}{4t_*}} & t = t_* \end{cases} \tag{A.8}$$

Note the particular (real) values of $u(x,t)$ at the shock at $x=0$:

$$u(0^{\pm},t) = -U(0^{\pm},t)/t = \begin{cases} \mp\frac{1}{2}(t-t_*)^{1/2}t^{-3/2}t_*^{-1/2} & t \geqslant t_* \\ 0 & t < t_* \end{cases} \tag{A.9}$$

The topology of the three-sheeted Riemann surface given by (A.8) and the interpretation of the shock as the permutation of two Riemann sheets has been fully explained by Bessis and Fournier in [3].

B. Generalization of the initial data to $u_0(x) = 2nx^{2n-1} - x/t_*$, $n \geqslant 2$. Although the inviscid singularity resulting from a polynomial of arbitrary odd order of the form $u_0(x) = 2nx^{2n-1} - x/t_*$, $n \in \mathbb{N}$, $n \geqslant 2$, is no longer generic (see [12, p.707]), it is still interesting to describe the behavior of the inviscid solution and the related asymptotic density of poles. We first describe the inviscid solution and its branch points: substituting $u_0(x)$ in (A.3) and using (A.4), we find that U is a root of the polynomial of degree $2n-1$: Let

$$\alpha_n(t) = \frac{t-t_*}{ntt_*}, \quad \text{and} \quad P_n(U) = U^{2n-1} - \frac{\alpha_n(t)}{2}U - \frac{x}{2nt} = 0.$$

Let $U_s(t)$ satisfy

$$0 = \frac{\partial x}{\partial U}(U_s(t)) = P_n'(U_s(t)) \Longrightarrow U_s(t) = \left(\frac{\alpha_n(t)/2}{2n-1}\right)^{1/(2n-2)}.$$

The $2n-2$ branch points of the inviscid solution are then given by

$$x_s(t) = x(U_s(t)) = 2ntU_s\left(U_s^{2n-2} - \alpha_n(t)/2\right) = C_n \cdot (t-t_*)^{\frac{2n-1}{2n-2}} \cdot t_*^{-\frac{2n-1}{2n-2}} \cdot t^{-\frac{1}{2n-2}},$$

where $C_n = -(2n-2)(2n-1)^{\frac{2n-1}{2n-2}}(2n)^{-\frac{1}{2n-2}}$. Notice that the $2n-2$ branch points coalesce at the origin at t_*. Since $U(0^{\pm},t) = \pm\pi t\rho(0;t) = \pm(\alpha_n(t)/2)^{\frac{1}{2n-2}}$ (cf. (3.9)), the (real) value of the inviscid solution at the origin (shock) is given by

$$u(0^{\pm},t) = -U(0^{\pm},t)/t = \mp\pi\rho(0;t) = \begin{cases} \mp(t-t_*)^{\frac{1}{2n-2}}(2nt_*)^{\frac{-1}{2n-2}}t^{-\frac{2n-1}{2n-2}} & t \geqslant t_* \\ 0 & t < t_* \end{cases}$$

The limiting pole density and inviscid limit are obtained using results from [16] where it is shown that the relevant saddle points in the asymptotic analysis of $E_\nu(i\beta,t_*)$ are

$$z_0(\beta;t_*) = \exp\left(\frac{i\pi}{4n-2}\right)\left(\frac{\beta}{2nt_*}\right)^{\frac{1}{2n-1}}, \quad z_1 = -\overline{z_0}.$$

Since

$$\rho(\beta;t_*) = \frac{\Re z_0(\beta;t_*)}{\pi t_*},$$

and appealing to further results in [16] concerning the asymptotic behavior of the zeros $\mu_{k,n}$ of $\mathcal{F}_n(\mu) = \int_{-\infty}^{\infty} e^{\mu(2niz - z^{2n})}dz$, we have the following result:

THEOREM B.1. *For any integer $n \geqslant 2$, the density of poles at the shock time t_* arising from the initial data $u_0(x) = 2nx^{2n-1} - x/t_*$ is*

$$\rho(\beta;t_*) = \frac{1}{\pi}\cos\left(\frac{\pi}{4n-2}\right)\left(\frac{\beta}{2nt_*^{2n}}\right)^{\frac{1}{2n-1}}.$$

The density at the origin ($\beta = 0$) is

$$\rho(0;t) = \begin{cases} \frac{1}{\pi t}\left(\frac{t-t_*}{2nt t_*}\right)^{\frac{1}{2n-2}} & t > t_* \\ 0 & t \leqslant t_* \end{cases}$$

Moreover at t_*, the $k-th$ ordered pole of the solution for $\nu > 0$ is located at

$$a_{k,n}(t_*, \nu) = i \cdot 2nt_* \left(2\nu\mu_{k,n}\right)^{\frac{2n-1}{2n}}$$

where the positive coefficients $\mu_{k,n}$ are asymptotically given by

$$\mu_{k,n} = \frac{\pi}{4n-2}\sec\left(\frac{\pi}{4n-2}\right)\left(\frac{n-1}{2n-1} + 1 + 2k\right) + \mathcal{O}\left(\frac{1}{k}\right) \quad \textit{as } k \to +\infty.$$

Higher order approximations of $\mu_{k,n}$ are provided in [16]. At $t = t_*$ the inviscid solution can be found via the complex-valued limiting pole density $\rho(\beta; t_*)$ and the pole expansion as in (4.3), or via the characteristic equation as in appendix A. The resulting singularity is a branch point of order $2n-1$ which arises from the coalescence of $n-1$ pair(s) of conjugate branch points of order $2n-2$ (see [12, p.707]):

$$u(x, t_*) = \frac{x}{t_*} - \left(\frac{x}{2nt_*^{2n}}\right)^{\frac{1}{2n-1}}.$$

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