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ADMISSIBLE BOUNDARY CONDITIONS AND STABILITY OF BOUNDARY-LAYERS FOR A HYPERBOLIC RELAXATION SYSTEM

HAILIANG LIU AND WEN-AN YONG

ABSTRACT. This work is concerned with boundary conditions for hyperbolic relaxation systems to have time-asymptotically stable boundary-layers. A new requirement is proposed to characterize a class of boundary conditions for a typical relaxation system. For the corresponding initial-boundary value problems, we prove the global (in time) existence and asymptotic decay of solutions with initial data close to the steady solutions or relaxation boundary-layers.

1. INTRODUCTION

This work is concerned with admissible boundary conditions for the typical hyperbolic relaxation system

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + a^2 u_x &= f(u) - v \end{aligned} \tag{1.1}$$

on the quarter-plane $x, t \geq 0$. Here u, v are unknown scalar functions, a is a positive constant, and $f(u)$ is a given smooth function satisfying the well-known subcharacteristic condition in [9]:

$$|f'(u)| < a \tag{1.2}$$

for all u under consideration. The model (1.1) with (1.2) was introduced in [2] for numerical purposes and serves as a simple example of general hyperbolic relaxation systems [14]. The latters arise in a large number of different physical situations mentioned in [9, 14].

To solve (1.1) on the quarter-plane, we prescribe initial data

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \tag{1.3}$$

at $t = 0$ and appropriate boundary conditions at $x = 0$. Denote by the superscript "*" the transpose operator acting on vectors or matrices. Since the coefficient matrix of $(u_x, v_x)^*$ in (1.1):

$$A = \begin{pmatrix} 0 & 1 \\ a^2 & 0 \end{pmatrix}$$

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has only one positive eigenvalue at $x = 0$, it is well-known (see [3, 5]) that one relation of boundary data:

$$B(u(0,t), v(0,t), t) = 0 \tag{1.4}$$

should be given with $B = B(u, v, t)$ satisfying

$$0 \neq (\partial_u B, \partial_v B) \cdot \begin{pmatrix} 1 \\ a \end{pmatrix} = B_u + aB_v. \tag{1.5}$$

Here $(1, a)^*$ acts as a right eigenvector associated with the positive eigenvalue of the coefficient matrix A.

As well-known, (1.1) with the boundary condition (1.4) can constitute a well-posed IBVP (initial-boundary value problem) under the hyperbolicity requirement $a \neq 0$ and the classical condition (1.5). See [5] and Section 5 below. However, it was pointed out firstly in [14, 15] that an *instability phenomenon* may occur if the *relaxation effect* of the right-hand side in (1.1) is to be taken into account but no further requirement is imposed besides (1.5) and the subcharacteristic condition (1.2). This situation is very similar to that for the IVP (initial value problem), where $a \neq 0$ already gives a well-posed structure but (1.2) was imposed to have controllable relaxation behaviors. See [9, 14] and references cited therein. Thus, like (1.2) for the IVP, a so-called GKC (*generalized Kreiss condition*), stronger than (1.5), was proposed in [14, 15] for general multi-dimensional IBVPs.

The motivation of this work is to show the relevance of the GKC in governing the time-asymptotic stability of steady solutions to hyperbolic relaxation systems together with the boundary conditions. Such steady solutions may be interpreted as boundary-layers for the corresponding systems with a small parameter $\epsilon > 0$, which are for (1.1)

$$\begin{aligned} u_t + v_x &= 0, \\ v_t + a^2 u_x &= (f(u) - v)/\epsilon. \end{aligned} \tag{1.6}$$

Note that the existence of the boundary-layers was established in [15] under the GKC and will be reviewed in Section 2.

The time-asymptotic problem can be precisely formulated as follows. Let $(U, V) = (U(x), V(x))$ be a steady solution to (1.1) with (1.4). Here it is natural to assume that

$$B(u, v, t) \text{ is independent of } t.$$

Our goal is to prove the global (in time) existence and time-asymptotic decay of solutions to (1.1)-(1.4) with initial data close to (U, V). For this purpose, we assume the *0-th consistency condition*

$$B(u_0(0), v_0(0)) = 0 \tag{1.7}$$

to avoid discontinuous solutions.

The reason why to choose the simple system (1.1) is that we want to make our main concerns (boundary conditions) stand out without complicating the presentation below. Furthermore, we consider only the case where

$$B_u \text{ and } B_v \text{ are constants.} \tag{1.8}$$

With these simplifications, we can easily work out a guideline to study the time-asymptotic problem for complicated physical systems.

As a contribution of this work, we single out the following requirement

$$B_u B_v > 0 \tag{1.9}$$

for (1.1) with the boundary condition (1.4). This new requirement is stronger than the GKC and a slightly weaker version of it has been used in [4] to study the zero relaxation limit ($\epsilon \to 0$) for weak entropy solutions (see (1.6)).

Under the new requirement (1.9), we prove the global (in time) existence and time-asymptotic decay of solutions to (1.1)-(1.4) with initial data close to (U, V). The main result is

Theorem 1.1. *Suppose $f = f(u) : \mathbf{R} \to \mathbf{R}$ is continuously differentiable, $f'(u_+) < 0$ with $u_+ \in \mathbf{R}$ fixed, the given data satisfy the two conditions in (1.7)-(1.8), and the subcharacteristic condition (1.2) is fulfilled. Then the followings hold.*

(1). If $B_u \neq 0$ and $B(u_+, f(u_+))$ is small, then (1.1) with (1.4) has a unique smooth steady solution (U, V) converging to $(u_+, f(u_+))$ as $x \to \infty$.

(2). If $B_u B_v > 0$, then there exists a positive constant δ such that if

$$\left\| \int_x^{+\infty} (u_0 - U)(y)dy \right\|_2 + \|v_0 - V\|_1 + |B(u_+, f(u_+))| < \delta,$$

then the IBVP (1.1)-(1.4) has a unique global solution

$$(u, v) \in (U, V) + C(0, \infty; H^1) \cap C^1(0, \infty; H^0)$$

satisfying

$$\lim_{t\to\infty} \sup_{x\in\mathbf{R}_+} |(u, v)(x, t) - (U, V)(x)| = 0.$$

Here and below, H^k stands for the L^2-Sobolev space on $\mathbf{R}_+$ of order k and its norm is denoted with $\|\cdot\|_k$. For $T \in (0, \infty)$, $C(0, T; X)$ and $C^1(0, T; X)$ denote the spaces of continuous and continuously differentiable functions on [0, T] with values in the Banach space X, respectively.

Concerning Theorem 1.1, we make three remarks. (1). Implied by the GKC but not the classical condition (1.5), $B_u \neq 0$ is necessary to have a unique steady solution. (2). We do not assume the mass of the initial perturbations in the u-component to be zero. Indeed, unlike for IVPs [8], the total mass for IBVPs is not conserved but changing with the boundary flow. (3). The semilinearity of (1.1) makes it possible to work with H^1-solutions under the 0-th consistency condition (1.7) merely!

The proof of Theorem 1.1 involves weighted energy estimates similar to that in [7, 8] for IVPs and new efforts are needed to treat the boundary conditions. It is worthwhile to mention that the weighted function enables us to overcome the troubles caused by not only the possible non-convexity of f but also the possible non-negativeness of $f'(U(0))$. In addition, the local existence theory in [5] seems not to meet our needs and thus a different theory is included in this paper.

We should mention that the above time-asymptotic stability problem was previuosly considered by several authors for some special boundary conditions, say, $B_v = 0$. See [13]. Unlike in those works, our analysis is straightforward and quite independent of the knowledges about shock profiles. In addition, the closely related viscosity boundary-layer problems were studied in [10, 1] respectively in the L^2-norm and L^1-norm. Another related is the zero relaxation limit problems with boundaries and the interested reader is referred to [4, 12] and references cited therein.

This paper is organized as follows. In Section 2 we explain the GKC for the simple system (1.1) with (1.4). Section 3 is devoted to the stability of boundary-layers. The required a priori estimates are derived in Section 4. The final section presents a self-contained proof of a local existence result.

2. Admissible Boundary Conditions

In this section we introduce admissible boundary conditions of the form (1.4)-(1.5) for the semilinear system in (1.1)-(1.2) to have stable boundary-layers.

To begin with, we show the existence of boundary-layers or (1) of Theorem 1.1.

Theorem 2.1. *Suppose $f'(u_+) < 0$, B_u and B_v are constants, and $B_u \neq 0$. Then there is a $\delta_0 > 0$, depending only on f and u_+, such that if $|B(u_+, f(u_+))| < \delta_0 |B_u|$, then the system (1.1) has a unique smooth steady solution $(U(x), V(x))$ satisfying $B(U(0), V(0)) = 0$ and decaying exponentially to $(u_+, f(u_+))$ as x goes to infinity.*

Note that the smallness of $B(u_+, f(u_+))$ can be relaxed but we won't do it here.

Proof. As a steady solution to (1.1), V satisfies $V_x = 0$ and thereby $V(x) = V(0)$. To fulfil the decay property, $V(x)$ should be equal to $f(u_+)$. Thus, U solves

$$a^2 U_x = f(U) - f(u_+). \tag{2.1}$$

Since $f'(u_+) < 0$, it is well-known that there is a $\delta_0 > 0$, depending only on f and u_+, such that if $|U(0) - u_+| < \delta_0$, then (2.1) has a unique global smooth solution $U(x)$ decaying exponentially to u_+ as x goes to infinity. On the other hand, since $B(u, v)$ is affine with respec to (u, v) and $B_u \neq 0$, $U(0)$ can be uniquely determined from $B(U(0), V(0)) = 0$ and satisfies

$$B_u(U(0) - u_+) = B(U(0), V(0)) - B(u_+, V(0)) = -B(u_+, f(u_+)).$$

Thus, $|U(0) - u_+| < \delta_0$ follows from $|B(u_+, f(u_+))| < \delta_0 |B_u|$. Consequently, $U(x)$ is uniquely determined. □

The above proof indicates that there are infinitely many steady solutions or boundary-layers if $B_u = 0$ (and $B(u_+, f(u_+)) = 0$). The latter does not contradict the classical condition (1.5). This supports the observation in [14, 15] that the well-known uniform Kreiss condition is inadequate to have controllable relaxation behaviors induced by the right-hand side in (1.1). Because of this, the GKC (*generalized Kreiss condition*) was proposed in [14, 15] as a stronger requirement for general multi-dimensional IBVPs. Like the uniform Kreiss condition, the GKC is difficult to be checked even for one-dimensional problems due to an additional parameter η. However, for special systems, easily-checked formulations of it could always be expected.

Referring to [14, 15], we write down the GKC for (1.1) to have an easily-checked formulation. Let A and Q_U be the coefficient matrix of $(u_x, v_x)^*$ and the Jacobian (with respect to $(u,v)^*$) of the right-hand side in (1.1), respectively. Define

$$M(\eta,\xi_0) := A^{-1}(\eta Q_U - \xi_0 I)$$

for $\eta \geq 0$ and ξ_0 complex with $\mathrm{Re}\xi_0 > 0$. Under the subcharacteristic condition $|f'(u)| \leq a$, it can be verified directly (or see Lemma 2.3 in [15]) that $M(\eta,\xi_0)$ has an eigenvalue $\kappa_- = \kappa_-(\eta,\xi_0)$ with $\mathrm{Re}\kappa_- < 0$ and the real part of another eigenvalue κ_+ is positive for $\eta \geq 0$ and $\mathrm{Re}\xi_0 > 0$. Thus, a *right-S matrix* $R_M^S(\eta,\xi_0)$ of $M(\eta,\xi_0)$ is an eigenvector associated with the eigenvalue κ_- and the GKC in [14, 15] is that *there is a constant $c_K > 0$ such that*

$$|(B_u, B_v)\cdot R_M^S(\eta,\xi_0)| \geq c_K |R_M^S(\eta,\xi_0)| \tag{2.2}$$

for all $\eta \geq 0$ and all complex ξ_0 with $\mathrm{Re}\xi_0 > 0$.

To describe (2.2) more explicitly, we compute

$$A = \begin{pmatrix} 0 & 1 \\ a^2 & 0 \end{pmatrix}, \qquad Q_U = \begin{pmatrix} 0 & 0 \\ f'(u) & -1 \end{pmatrix},$$

$$M(\eta,\xi_0) = \begin{pmatrix} 0 & a^{-2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -\xi_0 & 0 \\ a\eta\lambda & -\eta-\xi_0 \end{pmatrix} = \begin{pmatrix} \eta\lambda a^{-1} & -(\eta+\xi_0)a^{-2} \\ -\xi_0 & 0 \end{pmatrix}$$

with $\lambda = a^{-1}f'(u)$, and the characteristic equation of $M(\eta,\xi_0)$:

$$a^2\kappa^2 - \eta\lambda a\kappa - (\eta+\xi_0)\xi_0 = 0. \tag{2.3}$$

It is not difficult to see that $\begin{pmatrix} 1 \\ a^2(\eta+\xi_0)^{-1}\kappa_+ \end{pmatrix}$ is an eigenvector associated with κ_- and therefore can be taken as a right-S matrix of $M(\eta,\xi_0)$. Namely,

$$R_M^S(\eta,\xi_0) = \begin{pmatrix} 1 \\ a\theta \end{pmatrix} \qquad \text{with} \qquad \theta = \frac{a\kappa_+(\eta,\xi_0)}{\eta+\xi_0}.$$

In view of $\kappa_+(\sigma\eta,\sigma\xi_0) = \sigma\kappa_+(\eta,\xi_0)$ for any $\sigma > 0$ and $\mathrm{Re}\xi_0 > 0$, we rescale the parameters by dividing $\mathrm{Re}\xi_0$ to write

$$\theta = \frac{a\kappa_+(\eta, 1+i\xi)}{\eta+1+i\xi} \equiv \theta(\eta,\xi). \tag{2.4}$$

Thus the inequality (2.2) becomes

$$|B_u + a\theta(\eta,\xi)B_v| \geq c_K\sqrt{1+a^2|\theta(\eta,\xi)|^2} \tag{2.5}$$

for all $\eta \geq 0$ and all real ξ. Note that the GKC is stronger than the well-known condition (1.5), since $\theta(0,\xi) = 1$ follows from (2.4) and (2.3).

Concerning the function $\theta(\eta,\xi)$, we have

Proposition 2.2. *(1).* $\lim_{\eta\to+\infty}\theta(\eta,0) = \max(\lambda,0)$. *(2). Under the subcharacteristic condition, $|\theta(\eta,\xi)| < (1+\sqrt{5})/2$ for all $\eta \geq 0$ and ξ. (3). $\mathrm{Re}\theta(\eta,\xi) > 0$ for all $\eta \geq 0$ and ξ.*

Note that further discussions of $\theta(\eta,\xi)$ are possible but won't be given here.

Proof. ¿From (2.4) and the characteristic equation (2.3), it follows

$$\theta(\eta, 0) = \frac{\eta\lambda + \sqrt{\eta^2\lambda^2 + 4(\eta+1)}}{2(\eta+1)}.$$

Thus (1) is clear.

For (2), it follows from (2.3) that $\theta = \theta(\eta, \xi)$ satisfies

$$\theta^2 - (\eta + 1 + i\xi)^{-1}\eta\lambda\theta = (1 + i\xi)(\eta + 1 + i\xi)^{-1}.$$

Thus, $|\theta|\big||\theta| - |(\eta + 1 + i\xi)^{-1}\eta\lambda|\big| \le 1$ due to $\eta \ge 0$. Because $|\lambda| \le 1$ (the subcharacteristic condition), it is not difficult to see that $|\theta| < (1+\sqrt{5})/2$.

To show (3), we set $a\kappa_+(\eta, 1 + i\xi) = \alpha + i\beta$. Then $\alpha > 0$ and

$$\mathrm{Re}\theta = \frac{(\eta+1)\alpha + \xi\beta}{(\eta+1)^2 + \xi^2}$$

due to (2.4). On the other hand, we deduce from (2.3) that

$$\alpha - \eta\lambda = -a\mathrm{Re}\kappa_-(\eta, 1 + i\xi) > 0 \qquad \text{and} \qquad 2\alpha\beta - \eta\lambda\beta = (\eta+2)\xi.$$

Thus,

$$\xi\beta = \frac{(\eta+2)\xi^2}{\alpha + \alpha - \eta\lambda} \ge 0$$

and therefore $\mathrm{Re}\theta > 0$. This completes the proof. □

Thanks to (2) of Proposition 2.2, *the GKC is equivalent to that there is a constant* $c_K > 0$ *such that*

$$|B_u + aB_v\theta(\eta, \xi)| \ge c_K \qquad \textit{for } \eta \ge 0 \textit{ and real } \xi.$$

Moreover, the GKC holds if

$$B_u \ne 0 \qquad \text{and} \qquad aB_v/B_u \in \Big(\frac{1-\sqrt{5}}{2}, +\infty\Big).$$

In fact, if $|aB_v/B_u| < (\sqrt{5}-1)/2$, then it follows from (2) that

$$|B_u + aB_v\theta| \ge |B_u|(1 - |aB_v\theta/B_u|) \ge |B_u|(1 - (\sqrt{5}+1)|aB_v/B_u|/2)(> 0).$$

On the other hand, if $aB_v/B_u > 0$, then it follows from (3) that

$$|B_u + aB_v\theta| \ge |B_u|(1 + (aB_v/B_u)Re\theta) \ge |B_u|(> 0).$$

Because of (1) of Proposition 2.2, *the GKC is violated if* $B_u = 0$ *and* $\lambda = a^{-1}f'(u) < 0$ *is concerned.* This again shows the importance of the requirement $B_u \ne 0$ in Theorem 2.1. In the following sections, we will analyse the stability of the boundary-layers under the subcharacteristic condition and the requirement $aB_v/B_u > 0$. The latter is equivalent to that in (1.9).

3. Stability of boundary-layers

This section is devoted to proving (2) of Theorem 1.1 or the stability of boundary-layers (U, V) constructed in Theorem 2.1.

First of all, we normalize the unknowns by introducing

$$\bar{\phi}(x,t) = u(x,t) - U(x) \qquad \text{and} \qquad \bar{\psi}(x,t) = v(x,t) - V(x).$$

Then the system (1.1) can be rewritten as

$$\begin{aligned} &\bar{\phi}_t + \bar{\psi}_x = 0, \\ &\bar{\psi}_t + a^2\bar{\phi}_x = f(U+\bar{\phi}) - f(U) - \bar{\psi}. \end{aligned} \tag{3.1}$$

Initial data are

$$(\bar{\phi}(x,0), \bar{\psi}(x,0)) = (u_0(x) - U(x), v_0(x) - V(x)) \tag{3.2}$$

and the boundary condition (1.4) becomes

$$B_u\bar{\phi}(0,t) + B_v\bar{\psi}(0,t) = 0. \tag{3.3}$$

Here we have used the assumption (1.8) that B_u and B_v are constants. Moreover, since $B(U(0), V(0)) = 0$, the 0-th consistency condition (1.7) becomes

$$B_u\bar{\phi}(0,0) + B_v\bar{\psi}(0,0) = 0. \tag{3.4}$$

Thus, our task is reduced to showing the global existence and time-asymptotic behaviors of solutions $(\bar{\phi}, \bar{\psi})$ to the IBVP (3.1)-(3.3). To do this, our starting point is the following local existence result.

Lemma 3.1. *Suppose $f = f(u)$ is continuously differentiable, $(u_0, v_0) \in (U,V) + H^1$, and the 0-th consistency condition (3.4) holds.*

Then there exists a positive constant T_* such that (3.1)-(3.3) has a unique solution $(\bar{\phi}, \bar{\psi}) \in C(0, T_*; H^1)$ with $(\bar{\phi}, \bar{\psi})(0,t) \in H^1(0, T_*)$. Moreover, the solution satisfies the following estimate

$$\sup_{0\le t\le T_*} \|(\bar{\phi}, \bar{\psi})(\tau)\|_1 \le 2\|(u_0 - U, v_0 - V)\|_1$$

Here T_* depends only on the range of $U(x)$ and any upper bound of $\|(u_0-U, v_0-V)\|_1$.

Note that the new notation $H^k(\Omega)$ stands for the L^2-Sobolev space on Ω of order k and its norm will be denoted with $\|\cdot\|_{H^k(\Omega)}$.

Remark 3.1. Lemma 3.1 is different from the local existence theory due to Li and Yu in [5]. In fact, by Li-Yu's theory, the precondtions of Lemma 3.1 only guarantee that $(\bar{\phi}, \bar{\psi}) \in C(\mathbf{R}_+ \times [0, T_*])$ with T_* having similar dependence. However, Lemma 3.1 claims more regularities of the solutions.

A self-contained proof of Lemma 3.1 is given in Section 5 and we continue the argument. In order to get the global existence, we follow the well-known "partial integration" approach to reformulate the above problem with

$$\phi(x,t) = -\int_x^\infty \bar{\phi}(y,t)dy \qquad \text{and} \qquad \psi = \bar{\psi}.$$

Notice that $\psi(+\infty, t) = 0$ and $\bar{\phi} = \phi_x$. By integrating the first equation in (3.1) from x to $+\infty$ we obtain $\psi = -\phi_t$ and thereby the second equation in (3.1) can be rewritten as

$$\psi_t + a^2\phi_{xx} = f(U + \phi_x) - f(U) - \psi.$$

Namely, the equations (3.1) can be reset as

$$\psi = -\phi_t,$$

$$\phi_t + f'(U)\phi_x - \psi_t - a^2\phi_{xx} = F(U, \phi_x) \tag{3.5}$$

with

$$F = f(U) + f'(U)\phi_x - f(U + \phi_x) \equiv \phi_x^2 K(U, \phi_x) \tag{3.6}$$

and

$$K(U, \phi_x) = -\int_0^1 \int_0^1 f''(U + \xi\eta\phi_x) d\xi d\eta.$$

The initial and boundary conditions become

$$(\phi(x,0), \psi(x,0)) = (\phi_0, \psi_0)(x) \equiv \Big(-\int_x^\infty (u_0 - U)(y)dy,\ v_0(x) - V(x) \Big) \tag{3.7}$$

and

$$B_u\phi_x(0,t) + B_v\psi(0,t) = 0. \tag{3.8}$$

For this reformulated problem, we follow [16] and revise Lemma 3.1 to obtain the following local existence result.

Lemma 3.2. *Suppose $f = f(u)$ is continuously differentiable, $(\phi_0, \psi_0) \in H^2 \times H^1$, and the 0-th consistency condition holds.*

Then there exists $T_0 > 0$, depending only on any upper bound of $\|(\phi_{0x}, \psi_0)\|_1$, such that (3.5)-(3.8) has a unique solution $(\phi, \psi) \in C(0, T_0; H^2) \times C(0, T_0; H^1)$ with $(\phi_x, \psi)|_{x=0} \in H^1(0, T_0)$. Moreover, the solution satisfies the following estimate

$$\sup_{0 \le t \le T_0} \|(\phi, \psi, \phi_x)(t)\|_1 \le 4\|(\phi_0, \psi_0, \phi_{0x})\|_1.$$

Proof. Let $(\bar{\phi}, \bar{\psi}) \in C(0, T_*; H^1)$ be the solution to (3.1)-(3.3). Define $\psi = \bar{\psi}$ and

$$\phi(x,t) = \int_0^x \bar{\phi}(\zeta, t) d\zeta - \int_0^t \bar{\psi}(0, \tau) d\tau + \phi_0(0).$$

It is easy to verify that (ϕ, ψ) is a solution to (3.5)-(3.8). To see $\phi \in C(0, T_*; H^2)$ and the desired estimate, we use the first equation in (3.5) to obtain

$$\phi(x,t) = \phi_0(x) - \int_0^t \psi(x, \tau) d\tau.$$

This shows that $\phi \in C(0, T_*; H^1)$ and

$$\sup_{0 \le \tau \le t} \|\phi(\tau)\|_1 \le \|\phi_0\|_1 + t \sup_{0 \le \tau \le t} \|\psi(\tau)\|_1.$$

It follows from $\phi_x = \bar{\phi} \in C(0, T_*; H^1)$ that $\phi \in C(0, T_*; H^2)$. With $T_0 = \min\{1, T_*\}$, the desired estimate follows from the last one and that in Lemma 3.1.

The uniqueness can easily be shown again via the hyperbolic system (3.1). This completes the proof. □

In view of Lemma 3.2, we introduce a *solution space* for (3.5)-(3.7):

$$X(0,T) = \Big\{(\phi,\psi) \in C(0,T;H^2)\times C(0,T;H^1) : (\phi_x,\psi)|_{x=0} \in H^1(0,T)\Big\}.$$

Set $N(t) := \sup_{0\le\tau\le t}\|(\phi,\psi,\phi_x)(\tau)\|_1$. We will prove in the next section the following *a priori* estimate.

Lemma 3.3. *Suppose the subcharacteristic condition holds, $B_uB_v>0$, $f'(u_+)<0$, and $(\phi,\psi)\in X(0,T)$ is a solution to (3.5)-(3.8). Then there exist two positive constants δ_1 and C, independent of t, such that if*

$$N(T) + |B(u_+, f(u_+))| < \delta_1,$$

then

$$N^2(t) + \int_0^t \|(\psi,\phi_x)(\tau)\|_1^2 d\tau + \int_0^t \big[\phi^2(0,\tau)+\phi_{xt}^2(0,\tau)\big]d\tau \le CN^2(0) \tag{3.9}$$

for all $t\in[0,T]$.

Having the above local existence lemma and this *a priori* estimate, we follow the standard continuation argument (see, e.g., [8]) to conclude

Theorem 3.4. *Suppose $(\phi_0,\psi_0)\in H^2\times H^1$, the subcharacteristic condition and the 0-th consistency condition hold, $B_uB_v>0$ and $f'(u_+)<0$. Then there exists a $\delta_2>0$ such that if*

$$\|\phi_0\|_2 + \|\psi_0\|_1 + |B(u_+, f(u_+))| < \delta_2,$$

then (3.5)-(3.8) has a unique global solution $(\phi,\psi)\in X(0,\infty)$ satisfying the estimate in (3.9) for all $t>0$.

Finally, Theorem 1.1 can be deduced from Theorem 3.4 by following the standard argument (see, e.g., [8]).

4. *A Priori* Estimates

We prove Lemma 3.3 in this section and start with a basic weighted L^2-estimate.

Lemma 4.1. *Under the conditions of Lemma 3.3, there exists a positive constant C such that*

$$\begin{aligned}
&\|(\phi,\psi,\phi_x)(t)\|^2 + \int_0^t \|(\psi,\phi_x)(\tau)\|^2 d\tau + \int_0^t \|\sqrt{|U_x|}\phi(\tau)\|^2 d\tau \\
&\quad + \int_0^t \big(\phi^2(0,\tau)+\psi^2(0,\tau)\big)d\tau \\
&\le C\Big\{\|(\phi,\psi,\phi_x)(0)\|^2 + \int_0^t\int_0^{+\infty} |(\phi-2\psi)F|dxd\tau\Big\}
\end{aligned} \tag{4.1}$$

for all $t\in[0,T]$. Here $\|\cdot\|\equiv\|\cdot\|_0$.

Proof. First of all, we choose a weighted function w to overcome the difficulties caused by the possible positiveness of $f'(U(0))$ or the possible nonconvexity of f (see [11, 6] for this trick). To this end, we notice that the continuous function

$$H(u) := -1 - |u - U(0)|$$

is strictly increasing (resp. decreasing) on $[u_+, U(0)]$ (resp. $[U(0), u_+]$) and satisfying $H(U(0)) < 0$. With this $H(u)$, we define

$$w(u) = \frac{\int_{u_+}^{u} H(s)ds}{f(u) - f(u_+)}$$

for u between u_+ and $U(0)$. If f' has the aforesaid properties of $H(u)$, we simply take $w = 1$. For such a $w(u)$, $w(x) = w(U(x))$ has a positive lower bound. Moreover, $w(x)$ has an upper bound if and only if $f'(u_+) < 0$. Note that

$$H(u) = w'(u)\big(f(u) - f(u_+)\big) + w(u)f'(u). \tag{4.2}$$

Given the weighted function $w = w(x)$, we follow [8] and multiply the second equation in (3.5) with $(\phi - 2\psi)w$ to obtain

$$w(\phi - 2\psi)\big(\phi_t + f'(U)\phi_x - \psi_t - a^2\phi_{xx}\big) = w(\phi - 2\psi)F.$$

By using $\psi = -\phi_t$ due to (3.5), the left-hand side can be rewritten as

$$\begin{aligned}
&w\big(\phi\phi_t + f'(U)\phi\phi_x - \phi\psi_t - a^2\phi\phi_{xx} - 2\psi\phi_t - 2f'(U)\psi\phi_x + 2\psi\psi_t + 2a^2\psi\phi_{xx}\big)\\
=&w\big(\phi^2/2 - \phi\psi + \psi^2\big)_t + w\psi^2 - 2wf'(U)\psi\phi_x + \big(wf'(U)\phi^2/2\big)_x\\
&- \big(wf'(U)\big)_x\phi^2/2 - \big(a^2w\phi\phi_x\big)_x + a^2w\phi_x^2 + (a^2w_x\phi^2/2)_x\\
&- a^2w_{xx}\phi^2/2 + \big(2a^2w\phi_x\psi\big)_x - 2a^2w_x\phi_x\psi + 2a^2w\phi_x\phi_{xt}\\
=&w\big(\psi^2 - \phi\psi + \phi^2/2 + a^2\phi_x^2\big)_t + w\psi^2 - 2(a^2w_x + wf'(U))\psi\phi_x\\
&+ a^2w\phi_x^2 + D\phi^2/2 - B_{1x}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&w\big(\psi^2 - \phi\psi + \phi^2/2 + a^2\phi_x^2\big)_t + D\phi^2/2 - B_{1x}\\
&\quad + w\psi^2 - 2(a^2w_x + wf'(U))\psi\phi_x + a^2w\phi_x^2 = w(\phi - 2\psi)F.
\end{aligned} \tag{4.3}$$

Here

$$\begin{aligned}
D =& - a^2w_{xx} - \big(wf'(U)\big)_x = -\big(a^2w_x + wf'(U)\big)_x,\\
B_1 =&B_1(x,t) = a^2w\phi_x(\phi - 2\psi) - (a^2w_x + wf'(U))\phi^2/2.
\end{aligned} \tag{4.4}$$

Next we analyze the terms in (4.3). It is clear that

$$\psi^2 - \phi\psi + \phi^2/2 + a^2\phi_x^2 \sim |(\phi, \psi, \phi_x)|^2. \tag{4.5}$$

¿From the equation for $U = U(x)$ and the relation (4.2), we deduce that

$$\begin{aligned}
D =& - \big(a^2w'(U)U_x + w(U)f'(U)\big)_x\\
=& - \big(w'(U)\big(f(U) - f(u_+)\big) + w(U)f'(U)\big)_x\\
=& - (H(U))_x = -H'(U)U_x \geq |U_x|.
\end{aligned} \tag{4.6}$$

Here we have used the monotonicity of $H(u)$ and $U(x)$. Moreover, thanks to the subcharacteristic condition, it is clear that there is a constant $c > 0$ such that

$$\psi^2 - 2f'(U)\phi_x\psi + a^2\phi_x^2 \geq c|(\psi, \phi_x)|^2.$$

Thus, if $a^2 w_x/w$ is sufficiently small, then it is immediate that

$$w\big(\psi^2 - 2(a^2 w_x/w + f'(U))\phi_x\psi + a^2\phi_x^2\big) \geq cw|(\psi, \phi_x)|^2 \tag{4.7}$$

for another constant $c > 0$. Note that

$$a^2 w_x/w = a^2 w'(U)U_x/w = w'(U)\big(f(U) - f(u_+)\big)/w$$

is small provided that w is a constant or $|U(0) - u_+| \equiv |B_u^{-1}B(u_+, v_+)|$ is small. Having (4.5)-(4.7), we integrate (4.3) over $\mathbf{R}_+ \times [0, t]$ to obtain

$$\begin{aligned} &|(\phi, \psi, \phi_x)(t)|_w^2 + c\int_0^t |(\psi, \phi_x)(\tau)|_w^2 d\tau \\ &\quad + 2^{-1}\int_0^t \|\sqrt{|U_x|}\phi(\tau)\|^2 d\tau + \int_0^t B_1(0, \tau)d\tau \\ \leq& |(\phi, \psi, \phi_x)(0)|_w^2 + \int_0^t\int_0^{+\infty} w|(\phi - 2\psi)F|dxd\tau. \end{aligned} \tag{4.8}$$

Here $|\cdot|_w^2 := \int_0^\infty w(x)|\cdot(x)|^2 dx$.

It remains to estimate the boundary term $B_1(0, \tau)$. Since $B_u \neq 0$ and $H(U(0)) = -1$, we deduce from (4.4) and the boundary condition (3.8) that

$$\begin{aligned} B_1(0, t) &= a^2 w(U(0))\big[\phi_x(\phi - 2\psi)\big](0, t) - H(U(0))\phi^2(0, t)/2 \\ &= a^2 w B_u^{-1}B_v\big[\phi\phi_t + 2\psi^2\big](0, t) + \phi^2(0, t)/2 \\ &= \big(a^2 w B_u^{-1}B_v\phi^2(0, t)/2\big)_t + 2a^2 w B_u^{-1}B_v\psi^2(0, t) + \phi^2(0, t)/2. \end{aligned}$$

By substituting this into (4.8) and using the condition $B_u B_v > 0$ and the elementary inequality $|\phi| \leq \|\phi\|_1$, (4.1) follows from the boundedness of w. This completes the proof. □

Next we estimate the derivatives. Differentiating the equations (3.5) with respect to x gives

$$\begin{aligned} (\phi_x)_t + f'(U)(\phi_x)_x - (\psi_x)_t - a^2(\phi_x)_{xx} &= F_x + [f'(U), \partial_x]\phi_x, \\ \psi_x &= -(\phi_x)_t, \end{aligned} \tag{4.9}$$

where

$$[f'(U), \partial_x]\phi_x = f'(U)\partial_x\phi_x - \partial_x\big(f'(U)\phi_x\big) \tag{4.10}$$

is a commute term. As in obtainning (4.8), we take $w = 1$ and put the corresponding term "$D\phi^2/2$" in the right-hand side of the inequality to obtain

$$\begin{aligned}
&\|(\phi_x, \psi_x, \phi_{xx})(t)\|^2 + \int_0^t \|(\psi_x, \phi_{xx})(\tau)\|^2 d\tau + \int_0^t B_2(0,\tau)d\tau \\
&\le C\Big\{\|(\phi_x, \psi_x, \phi_{xx})(0)\|^2 + \int_0^t \|\sqrt{|U_x|}\phi_x(\tau)\|^2 d\tau \\
&\quad + \int_0^t \int_0^{+\infty} \big|(\phi_x - 2\psi_x)\big(F_x + [f'(U), \partial_x]\phi_x\big)\big| dx d\tau\Big\}.
\end{aligned} \tag{4.11}$$

Here $B_2 = B_2(x,t) = a^2\phi_{xx}(\phi_x - 2\psi_x) - f'(U)\phi_x^2/2$.

Now we turn to deal with the boundary term $B_2(0,\tau)$. From the boundary condition (3.8) and $(\phi_x, \psi)|_{x=0} \in H^1(0,T)$, it follows that $B_u\phi_{xt}(0,t) = -B_v\psi_t(0,t)$. Combining this with the equations (3.5) gives

$$\begin{aligned}
B_2(0,t) =& \big(\phi_t + f'(U)\phi_x - \psi_t - F\big)(\phi_x - 2\psi_x) - f'(U)\phi_x^2/2 \\
=& \big(-\psi + f'(U)\phi_x - F\big)(\phi_x - 2\psi_x) - f'(U)\phi_x^2/2 - \psi_t(\phi_x - 2\psi_x) \\
\ge& - C\big(\phi_x^2 + \psi^2 + F^2\big) - B_v^{-1}B_u\psi_x^2 + B_v^{-1}B_u\phi_{xt}(\phi_x + 2\phi_{xt}) \\
\ge& - C\big(\phi_x^2 + \psi^2 + F^2\big) - B_v^{-1}B_u\phi_{xt}^2 + B_v^{-1}B_u\phi_{xt}(\phi_x + 2\phi_{xt}) \\
\ge& - C\big(\phi_x^2(0,t) + \psi^2(0,t) + F^2(0,t)\big) + B_v^{-1}B_u\phi_{xt}^2(0,t)/2
\end{aligned} \tag{4.12}$$

for some constant C. Furthermore, since $F(\infty,t) = 0 = \phi_x(\infty,t)$, F^2 and ϕ_x^2 can be bounded as

$$\begin{aligned}
F^2(0,t) =& -\int_0^{+\infty} \frac{\partial F^2}{\partial x}(x,t)dx = -2\int_0^{+\infty} F(x,t)F_x(x,t)dx \\
\le& 2\int_0^{+\infty} |F(x,t)F_x(x,t)|dx, \\
C\phi_x^2(0,t) \le& \frac{1}{2}\|\phi_{xx}(t)\|^2 + C\|\phi_x(t)\|^2.
\end{aligned} \tag{4.13}$$

Substituting the last two inequalities for $B_2(0,t)$ into (4.11) yields

Lemma 4.2. *The assumptions of Lemma 3.3 further imply that*

$$\begin{aligned}
&\|(\phi_x, \psi_x, \phi_{xx})(t)\|^2 + \int_0^t \|(\psi_x, \phi_{xx})(\tau)\|^2 d\tau + \int_0^t \phi_{xt}^2(0,\tau)d\tau \\
&\le C\Big\{\|(\phi_x, \psi_x, \phi_{xx})(0)\|^2 + \int_0^t \|\phi_x(\tau)\|^2 d\tau + \int_0^t \psi^2(0,\tau)d\tau \\
&+ \int_0^t \int_0^{+\infty} \big[|(\phi_x - 2\psi_x)\big(F_x + [f'(U), \partial_x]\phi_x\big)| + |F(x,\tau)F_x(x,\tau)|\big] dx d\tau\Big\}
\end{aligned}$$

for $t \in [0,T]$.

Finally, we deduce from (3.6) and (4.10) that $F = \phi_x^2 K(U, \phi_x)$ and $[f'(U), \partial_x]\phi_x = -f'(U)_x \phi_x$. It is not difficult to see that

$$\int_0^t \int_0^{+\infty} |(\phi - 2\psi)F| dx d\tau \leq C_N N(t) \int_0^t \|\phi_x(\tau)\|^2 d\tau,$$

$$\begin{aligned}
&\int_0^t \int_0^{+\infty} \big\{ |(\phi_x - 2\psi_x)\big(F_x + [f'(U), \partial_x]\phi_x\big)| + |F(x,\tau)F_x(x,\tau)| \big\} dx d\tau \\
\leq & \frac{1}{2C} \int_0^t \|\psi_x(\tau)\|^2 d\tau + C \int_0^t \|\phi_x(\tau)\|^2 d\tau + C \int_0^t \int_0^\infty (F^2 + F_x^2) dx d\tau \\
\leq & \frac{1}{2C} \int_0^t \|\psi_x(\tau)\|^2 d\tau + C_N \int_0^t \|\phi_x(\tau)\|^2 d\tau + C_N N(t) \int_0^t \|\phi_{xx}(\tau)\|^2 d\tau.
\end{aligned}$$

Here C_N is a generic constant depending on $N(t)$ and remains bounded when $N(t)$ goes to zero. Having the last inequalities, we deduce from lemmas 4.1 and 4.2 that

$$\begin{aligned}
&\|(\phi, \psi, \phi_x)(t)\|_1^2 + (1 - C_N N(T)) \int_0^t \|(\psi, \phi_x)(\tau)\|_1^2 d\tau \\
&+ \int_0^t [\phi^2(0,\tau) + \phi_{xt}^2(0,\tau)] d\tau \leq C \|(\phi, \psi, \phi_x)(0)\|_1^2.
\end{aligned}$$

Thus, if $N(T)$ is sufficiently small so that $C_N N(T) < 1/2$, then Lemma 3.3 follows immediately.

5. Local Existence

This section presents a self-contained proof of the local existence result in Lemma 3.1. Consider the following slightly more general semilinear system

$$\begin{aligned}
w_t - a w_x &= f_1(w, z), \\
z_t + a z_x &= f_2(w, z)
\end{aligned} \tag{5.1}$$

on $\Omega_T \equiv \mathbf{R}_+ \times [0, T] \ni (x, t)$ with initial data $(w_0(x), z_0(x))$ and boundary conditions of the form

$$z(0, t) = b(w(0, t), t). \tag{5.2}$$

Here $(f_1, f_2) \in C^1(G)$ with G an open set in $\mathbf{R}^2$, T is a given positive constant, and $b(w, t)$ is a given smooth function of $(w, t) \in \mathbf{R} \times \mathbf{R}_+$.

To begin with, we consider the following linear system

$$\begin{aligned}
w_t - a w_x &= f_1(x, t), \\
z_t + a z_x &= f_2(x, t).
\end{aligned} \tag{5.3}$$

Its solutions can be explicitly given as

$$\begin{aligned}
w(x,t) &= w_0(x + at) + \int_0^t f_1(x + at - a\tau, \tau) d\tau, \\
z(x,t) &= \begin{cases} z_0(x - at) + \int_0^t f_2(x - at + a\tau, \tau) d\tau, & x \geq at \\ z(0, t - x/a) + \int_{t-x/a}^t f_2(x - at + a\tau, \tau) d\tau, & x < at \end{cases}.
\end{aligned} \tag{5.4}$$

Thus, if $w_0 \in C^k(\bar{\mathbf{R}}_+)$ and $f_1 \in C^k(\bar{\Omega}_T)$ for some integer $k \geq 0$, then it is clear that $w \in C^k(\bar{\Omega}_T)$. Similarly, we have $z \in C^k\{(x,t) \in \bar{\Omega}_T : x \neq at\}$ if $z_0 \in C^k(\bar{\mathbf{R}}_+)$, $f_2 \in C^k(\bar{\Omega}_T)$ and $z(0,\cdot) \in C^k[0,T]$.

Furthermore, under the *0-th consistency condition*

$$z(0,0) = z_0(0),$$

it is clear that $z \in C(\bar{\Omega}_T)$. For the boundary condition (5.2), this consistency condition reads as

$$b(w_0(0),0) = z_0(0), \tag{5.5}$$

which is a constraint only on the given data. Thus, we can state

Lemma 5.1. *If f_1, f_2, w_0, z_0 and b are all continuously differentiable and satisfy the 0-th consistency condition (5.5), then (5.3)-(5.2) has a unique global continuous solution (w,z) given in (5.4), which is piecewise continuously differentiable and the trace $(w(0,t), z(0,t))$ is continuously differentiable with respect to t. Moreover, if $(f_1, f_2) \in C(0,T;H^1)$ and $(w_0, z_0) \in H^1$, then $(w,z) \in C(0,T;H^1)$.*

Proof. In view of the discussions before this lemma, we only need to show that $(w,z) \in C(0,T;H^1)$. To this end, we extend the data as

$$\tilde{w}_0(x) = w_0(|x|), \quad \tilde{z}_0(x) = z_0(|x|), \quad \tilde{f}_1(x,t) = f_1(|x|,t), \quad \tilde{f}_2(x,t) = f_2(|x|,t)$$

for $x \in \mathbf{R}$. Since $(f_1, f_2) \in C(0,T;H^1)$ and $(w_0,z_0) \in H^1$, we have $(\tilde{f}_1, \tilde{f}_2) \in C(0,T;H^1(\mathbf{R}))$ and $(\tilde{w}_0, \tilde{z}_0) \in H^1(\mathbf{R})$.

Having these extended data, we consider the following IVP

$$\begin{aligned} \tilde{w}_t - a\tilde{w}_x =& \tilde{f}_1(x,t), \\ \tilde{z}_t + a\tilde{z}_x =& \tilde{f}_2(x,t), \\ (w(x,0), z(x,0)) =& (\tilde{w}_0(x), \tilde{z}_0(x)), \end{aligned}$$

where the equations are exactly those in (5.3) with f_1 and f_2 replaced by $\tilde{f}_1$ and $\tilde{f}_2$, respectively. According to the standard existence theory (see, e.g., [14]) for linear symmetrizable hyperbolic systems, this IVP has a unique solution $(\tilde{w}, \tilde{z})$ in $C(0,T;H^1(\mathbf{R}))$.

Thanks to the above extension, we have $w(x,t) = \tilde{w}(x,t)$ for $x > 0$ and $z(x,t) = \tilde{z}(x,t)$ for $x > at$. Thus, it is clear that $w \in C(0,T;H^1)$ and $z(\cdot,t) \in H^1(at,+\infty)$ for each $t \geq 0$. On the other hand, since z_0, f_2, b and w are continuously differentiable, it is direct to deduce from the expression in (5.4) that both $z_x(x,t)$ and $z(x,t)$ are bounded on any bounded subset of Ω_T. In particular, we have $z(\cdot,t) \in H^1(0, at+a)$ and therefore $z(\cdot,t) \in H^1$ by combining the fact shown above.

Furthermore, we recall that $z(x,t)$ is continuous and piecewise continuously differentiable. Then $z_x(x,t)$ and $z(x,t)$ converge respectively to $z_x(x,t_0)$ and $z(x,t_0)$ for almost every $x \in [0, at_0 + a]$ as t tends to $t_0 \in [0,T]$. Thus, it follows from the bounded convergence theorem that

$$\lim_{t \to t_0} \|z(\cdot,t) - z(\cdot,t_0)\|_{H^1(0,at_0+a)} = 0.$$

Consequently, we see that

$$\begin{aligned}&\|z(\cdot,t)-z(\cdot,t_0)\|_{H^1}\\ =&\|z(\cdot,t)-z(\cdot,t_0)\|_{H^1(0,at_0+a)}+\|z(\cdot,t)-z(\cdot,t_0)\|_{H^1(at_0+a,+\infty)}\\ =&\|z(\cdot,t)-z(\cdot,t_0)\|_{H^1(0,at_0+a)}+\|\tilde{z}(\cdot,t)-\tilde{z}(\cdot,t_0)\|_{H^1(at_0+a,+\infty)}\end{aligned}$$

tends to zero as t goes to t_0. Hence $z\in C(0,T;H^1)$ and the proof is complete. □

Next we use a density argument to remove the precondition in Lemma 5.1 that w_0, z_0, f_1 and f_2 are continuously differentiable. For simplicity, we assume below that $b_w(w,t)$ is constant.

Lemma 5.2. *Suppose $(f_1,f_2)\in C(0,T;H^1), (w_0,z_0)\in H^1$, b_w is constant, and the 0-th consistency condition holds. Then (5.3)-(5.2) has a unique global solution*

$$(w,z)\in C_{tr}(0,T;H^1):=\{(w,z)\in C(0,T;H^1):(w,z)|_{x=0}\in H^1(0,T)\}.$$

Moreover, let $(\hat{w},\hat{z})\in H^2$ satisfy $\hat{z}(0)=b(\hat{w}(0),0)$ (consistency condition). Then the solution satisfies the estimate

$$\begin{aligned}\|\big(\kappa(w(t)-\hat{w}),\,z(t)-\hat{z}\big)\|_1^2\le e^t\|\big(\kappa(w_0-\hat{w}),z_0-\hat{z}\big)\|_1^2+\\ +(1+5a^{-1})\int_0^t e^{t-\tau}\|\big(\kappa(f_1(\tau)+a\hat{w}_x),f_2(\tau)-a\hat{z}_x\big)\|_1^2d\tau\end{aligned}$$

with $\kappa\ge\sqrt{5}|b_w|$.

Proof. For $n=1,2,\cdots$, choose $w_0^n,z_0^n\in C^1(\bar{\mathbf{R}}_+)$ and $f_1^n,f_2^n\in C^1(\bar{\Omega}_T)\cap C(0,T;H^1)$ so that

$$\|(w_0^n-w_0,z_0^n-z_0)\|_1,\quad \max_{t\in[0,T]}\|\big(f_1^n(t)-f_1(t),f_2^n(t)-f_2(t)\big)\|_1\le 2^{-n}. \tag{5.6}$$

Moreover, we may assume that the perturbed data satisfy the 0-th consistency condition

$$z_0^n(0)=b(w_0^n(0),0),$$

since so does $(w_0,z_0)=\lim_{n\to\infty}(w_0^n,z_0^n)$ in H^1. According to Lemma 5.1, there exists $(w^n,z^n)\in C(0,T;H^1)$ with $(w,z)(0,\cdot)\in C^1[0,T]$ so that

$$\begin{aligned}w_t^n-aw_x^n=&f_1^n(x,t),\\ z_t^n+az_x^n=&f_2^n(x,t)\end{aligned}$$

with initial data $(w_0^n(x),z_0^n(x))$ and boundary conditions

$$z^n(0,t)=b(w^n(0,t),t).$$

We show below that (w^n,z^n) is a Cauchy sequence in the Banach space $C_{tr}(0,T;H^1)$ with the norm

$$\|(w,z)\|_*:=\max_{t\in[0,T]}\|\big(w(t),z(t)\big)\|_1+\|(w,z)|_{x=0}\|_{H^1(0,T)}.$$

To this end, set $(\tilde{w}^n,\tilde{z}^n)=(w^{n+1},z^{n+1})-(w^n,z^n)$. Then $(\tilde{w}^n,\tilde{z}^n)$ satisfies

$$\begin{aligned}\tilde{w}_t^n-a\tilde{w}_x^n=&\tilde{f}_1^n\equiv f_1^{n+1}-f_1^n,\\ \tilde{z}_t^n+a\tilde{z}_x^n=&\tilde{f}_2^n\equiv f_2^{n+1}-f_2^n\end{aligned}\tag{5.7}$$

with initial data $(\tilde{w}_0^n, \tilde{z}_0^n)$ and boundary conditions

$$\tilde{z}^n(0,t) = b_w \tilde{w}^n(0,t). \tag{5.8}$$

Moreover, differentiating the equations (5.7) with respect to x gives

$$\begin{aligned} \tilde{w}_{xt}^n - a\tilde{w}_{xx}^n =& \tilde{f}_{1x}^n, \\ \tilde{z}_{xt}^n + a\tilde{z}_{xx}^n =& \tilde{f}_{2x}^n. \end{aligned} \tag{5.9}$$

Thanks to the constancy of b_w, it follows from the boundary condition (5.8) that

$$\tilde{z}_t^n(0,t) = b_w \tilde{w}_t^n(0,t). \tag{5.10}$$

Using the equations (5.7) we obtain a boundary condition for $(\tilde{w}_x, \tilde{z}_x)$:

$$\tilde{z}_x^n(0,t) = -b_w \tilde{w}_x^n(0,t) + a^{-1}\big(\tilde{f}_2^n(0,t) - b_w \tilde{f}_1^n(0,t)\big). \tag{5.11}$$

Let $\kappa \geq \sqrt{5}|b_w|$ be constant and define

$$E_\kappa(t) = \|\big(\kappa\tilde{w}^n(t), \tilde{z}^n(t)\big)\|_1.$$

By a simple calculation based on the equations in (5.7) and (5.9), we get

$$\begin{aligned} &\frac{dE_\kappa^2(t)}{dt} + a(|\kappa\tilde{w}^n(x,0)|^2 - |\tilde{z}^n(x,0)|^2) + a(|\kappa\tilde{w}_x^n(x,0)|^2 - |\tilde{z}_x^n(x,0)|^2) \\ \leq& E_\kappa^2(t) + \|\big(\kappa\tilde{f}_1^n(t), \tilde{f}_2^n(t)\big)\|_1^2. \end{aligned} \tag{5.12}$$

Set $\epsilon = a^{-2}/2$. By using (5.8), (5.11), (5.10) and (5.7), the boundary terms can be estimated as

$$\begin{aligned} &2(|\kappa\tilde{w}^n(0,t)|^2 - |\tilde{z}^n(0,t)|^2) + 2(|\kappa\tilde{w}_x^n(0,t)|^2 - |\tilde{z}_x^n(0,t)|^2) \\ \geq& |\kappa\tilde{w}^n(0,t)|^2 + |\tilde{z}^n(0,t)|^2 + \epsilon|\kappa\tilde{w}_t^n(0,t)|^2 + \epsilon|\tilde{z}_t^n(0,t)|^2 \\ &+ (\kappa^2 - 3b_w^2)|\tilde{w}^n(0,t)|^2 + 2(\kappa^2 - 2b_w^2)|\tilde{w}_x^n(0,t)|^2 \\ &- \epsilon(\kappa^2 + b_w^2)|\tilde{w}_t^n(0,t)|^2 - 4a^{-2}|\tilde{f}_2^n(0,t) - b_w\tilde{f}_1^n(0,t)|^2 \\ \geq& |\kappa\tilde{w}^n(0,t)|^2 + |\tilde{z}^n(0,t)|^2 + \epsilon|\kappa\tilde{w}_t^n(0,t)|^2 + \epsilon|\tilde{z}_t^n(0,t)|^2 \\ &+ (2\kappa^2 - 4b_w^2 - 2a^2\epsilon(\kappa^2 + b_w^2))|\tilde{w}_x^n(0,t)|^2 \\ &- 2\epsilon(\kappa^2 + b_w^2)|\tilde{f}_1^n(0,t)|^2 - 4a^{-2}|\tilde{f}_2^n(0,t) - b_w\tilde{f}_1^n(0,t)|^2 \\ \geq& |\kappa\tilde{w}^n(0,t)|^2 + |\tilde{z}^n(0,t)|^2 + \epsilon|\kappa\tilde{w}_t^n(0,t)|^2 + \epsilon|\tilde{z}_t^n(0,t)|^2 \\ &- 2\epsilon(\kappa^2 + b_w^2)|\tilde{f}_1^n(0,t)|^2 - 4a^{-2}|\tilde{f}_2^n(0,t) - b_w\tilde{f}_1^n(0,t)|^2. \end{aligned}$$

Thus, it follows from the inequality (5.12) that

$$\begin{aligned} &\frac{dE_\kappa^2(t)}{dt} + c_0(|\kappa\tilde{w}^n(0,t)|^2 + |\tilde{z}^n(0,t)|^2 + |\kappa\tilde{w}_t^n(0,t)|^2 + |\tilde{z}_t^n(0,t)|^2) \\ \leq& E_\kappa^2(t) + \|\big(\kappa\tilde{f}_1^n(t), \tilde{f}_2^n(t)\big)\|_1^2 + 2a^{-1}|\tilde{f}_2^n(0,t) - b_w\tilde{f}_1^n(0,t)|^2 + a^{-1}|\kappa\tilde{f}_1^n(0,t)|^2 \\ \leq& E_\kappa^2(t) + (1 + 5a^{-1})\|\big(\kappa\tilde{f}_1^n(t), \tilde{f}_2^n(t)\big)\|_1^2 \end{aligned}$$

with $c_0 = \min\{a/2, a^{-1}/4\}$. Here we have used the fact that $|b_w| \le \kappa$ and the familiar embedding inequality $|\cdot|_\infty \le \|\cdot\|_1$ (see (4.13)). Thus, we obtain

$$
\begin{aligned}
&E_\kappa^2(t) + c_0 \int_0^t e^{t-\tau}(|\kappa\tilde{w}^n(0,\tau)|^2 + |\tilde{z}^n(0,\tau)|^2 + |\kappa\tilde{w}_t^n(0,\tau)|^2 + |\tilde{z}_t^n(0,\tau)|^2)d\tau \\
&\le e^t E_\kappa^2(0) + (1+5a^{-1}) \int_0^t e^{t-\tau}\|\big(\kappa\tilde{f}_1^n(\tau), \tilde{f}_2^n(\tau)\big)\|_1^2 d\tau.
\end{aligned} \tag{5.13}
$$

Together with those in (5.6), the last inequality shows that (w^n, z^n) is a Cauchy sequence in the Banach space $C_{tr}(0,T;H^1)$ $(T < \infty)$ and therefore has a limit $(w,z) \in C_{tr}(0,T;H^1)$. It is easy to verify that this limit is the unique solution.

Since $(\hat{w}, \hat{z})$ satisfies the 0-th consistency condition and $(\hat{w}_x, \hat{z}_x) \in H^1$, the desired estimate can be obtained by repeating the above argument for $\tilde{w}^n \equiv w - \hat{w}$ and $\tilde{z}^n \equiv z - \hat{z}$. This completes the proof. □

Having Lemma 5.2 with the estimate, we can use the standard contraction mapping argument (see, e.g., [14]) for semilinear IVPs to get the following result, whose proof is omitted.

Theorem 5.3. *Suppose $f_i(w,z) \in C^1(G)$ satisfies $f_i(0,0) = 0$, $(w_0, z_0) \in H^1$ takes values in a bounded convex subset $G_0 \subset\subset G$, $b_w(w,t)$ is constant, and the 0-th consistency condition $z_0(0) = b(w_0(0), 0)$ holds.*

Then there is a positive T_*, depending only on G_0 and $\|(w_0, z_0)\|_1$, such that the problem (5.1)-(5.2) has a unique solution $(w,z) \in C_{tr}(0,T_*;H^1)$.

Note that Remark 3.1 in Section 3 applies here.

Having Theorem 5.3, Lemma 3.1 can be proven as follows. Consider the IBVP (3.1)-(3.3). With $w := \bar{\psi} - a\bar{\phi}$ and $z := \bar{\psi} + a\bar{\phi}$, (3.1) can be rewritten as the diagonal form (5.1) with

$$f_1(w,z) = f_2(w,z) = f\big(U + (z-w)/(2a)\big) - f(U) - (z+w)/2.$$

Moreover, the boundary conditions have an equivalent form like that in (5.2) with

$$b(w,t) = \frac{(B_u - aB_v)w}{B_u + aB_v}.$$

Note that $B_u + aB_v \ne 0$ and b_w is constant. Recall (3.4) that the 0-th consistency condition hold. According to Theorem 5.3, there is a positive constant T_*, depending only on the boundary-layer (G_0) and any upper bound of $\|\big(\bar{\phi}(0), \bar{\psi}(0)\big)\|_1$, such that the above problem has a unique solution $(\bar{\phi}, \bar{\psi}) \in C_{tr}(0,T_*;H^1)$. Furthermore, T_* can be chosen so that the estimate holds (see the estimate in Lemma 5.2 with $(\hat{w}, \hat{z}) = (0,0)$).

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References

[1] H. Freistühler and D. Serre, *The L1-stability of boundary layers for scalar viscous conservation laws*, Preprint.

[2] S. Jin and Z. Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure Appl. Math. **48** (1995), 235-276.

[3] H.-O. Kreiss and J. Lorenz, *Initial-boundary value problems and the Navier-Stokes equations*, Academic Press, Boston, 1989.

[4] W. Kreß, *Asymptotisches Verhalten von hyperbolischen Randwertproblemen mit Relaxationsterm*, Diplom Thesis, Universität Heidelberg, 1999.

[5] T.-T, Li and W.-C. Yu, *Boundary value problems for quasilinear hyperbolic systems*, Duke University Math. Ser. V, Duke University Press, Durham, 1985.

[6] H.L. Liu, *Asymptotic stability of shock profiles for non-convex convection diffusion equation*, Appl. Math. Lett. **10** (1997), 129-134.

[7] H.L. Liu, J. Wang, *Asymptotic stability of traveling wave solution for a hyperbolic system with relaxation terms*, Beijing Math. **2** (1996), 119-130.

[8] H.L. Liu, J. Wang, and T. Yang, *Stability of a relaxation model with a nonconvex flux*, SIAM J. Math. Anal. **29** (1998), 18-29.

[9] T.-P. Liu, *Hyperbolic conservation laws with relaxation*, Comm. Math. Phys. **108** (1987), 153-175.

[10] T.-P. Liu and K. Nishihara, *Asymptotic behavior for scalar viscous conservation laws with boundary effect*, J. Differ. Equations **133** (1997), 296-320.

[11] A. Matsumura and K. Nishihara, *Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity*, Comm. Math. Phys. **165** (1994), 83-96.

[12] R. Natalini and A. Terracina, *Convergence of a Relaxation Approximation to a Boundary Value Problem for Conservation Laws*, Quaderno IAC n.22/1999, Roma, 1999.

[13] S. Nishibata and S.-H. Yu, *The asymptotic behavior of the hyperbolic conservation laws with relaxation on the quarter-plane*, SIAM J. Math. Anal. **28** (1997), 304-321.

[14] W.-A. Yong, *Singular perturbations of first-order hyperbolic systems*, Ph.D. Thesis, Universität Heidelberg, 1992.

[15] ———, *Boundary conditions for hyperbolic systems with stiff source terms*, Indiana Univ. Math. J. **48** (1999), 115-137.

[16] ———, *Existence and asympotic stability of traveling wave solutions of a model system for reacting flow*, Nonlinear Anal.: Theory Methods Appl. **26** (1996), 1791-1809.

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