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MERTON'S PORTFOLIO OPTIMIZATION PROBLEM IN A BLACK & SCHOLES MARKET WITH NON-GAUSSIAN STOCHASTIC VOLATILITY OF ORNSTEIN-UHLENBECK TYPE

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ABSTRACT. We study Merton's classical portfolio optimization problem for an investor who can trade in a risk-free bond and a stock. The goal of the investor is to allocate money so that her expected utility from terminal wealth is maximized. The special feature of the problem studied in this paper is the inclusion of stochastic volatility in the dynamics of the risky asset. The model we use is driven by a superposition of non-Gaussian Ornstein-Uhlenbeck processes and it was recently proposed and intensively investigated for real market data by Barndorff-Nielsen and Shephard [3]. Using the dynamic programming method, explicit trading strategies and expressions for the value function via Feynman-Kac formulas are derived and verified for power utilities. Some numerical examples are also presented.

1. INTRODUCTION

Recently, Barndorff-Nielsen and Shephard [3] suggested to model the volatility in asset price dynamics as a weighted sum of non-Gaussian Ornstein-Uhlenbeck processes. This volatility model possesses many of the observed features of financial logreturn data, such as heavy tails and long-range dependency. Barndoff-Nielsen and Shephard [3] introduce subordinators (i.e., nondecreasing pure-jump Lévy process) as the noise driving the Ornstein-Uhlenbeck processes. That is, the volatility level is allowed to have sudden shifts in the upward direction, while decreasing exponentially between such shifts. Empirical investigations made by the authors [3] on exchange rates demonstrate that such volatility models show a remarkable fit to both the empirical autocorrelation and the leptokurtic behaviour of logreturn data. In the present paper we want to investigate Merton's classical portfolio optimization problem under this volatility model.

Merton [23, 24] explicitly solved the question of optimal portfolio allocation in a market with a risk-free bond and a stock as investment alternatives. The price of the risky asset (e.g., a stock) follows a geometric Brownian motion (or a geometric Brownian motion with Poisson jumps). The investor wants to maximize her terminal wealth under a power utility function. Using a stochastic volatility model as in Barndorff-Nielsen and Shephard [3], we prove that it is still possible to explicitly solve Merton's problem, however, now via a Feynman-Kac representation. Not surprisingly, the investor should follow Merton's original strategy of constant allocation (that is, an allocation independent of current level of wealth), but she should rebalance the portfolio according to changes in the volatility.

Our approach to solve the stochastic optimization problem goes via the dynamic programming method and the associated Hamilton-Jacobi-Bellman (HJB) integro-differential equation. Using a verification theorem, we are able to identify the expected value derived from utility as a solution of a second order integro-differential equation. The solution to this HJB equation can be written down in terms of the power utility function and a Feynman-Kac representation. In addition, it is possible to explicitly write down the optimal allocation strategy. All results are derived under exponential integrability assumptions on the Lévy measures coming from the volatility process. We suppose that there are borrowing and short-selling constraints in the market. In a companion paper [9] we study the portfolio problem with a general utility function and an extension of the

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stock price dynamics. In that paper we use viscosity solution methods to relate the value function to the HJB equation.

A major drawback with the dynamic programming approach is the fact that the allocation strategies must depend on the volatility (explicitly or through information generated by it). Of course, volatility is not directly observable in the market like the stock price, and it is therefore in practice impossible to follow portfolio rules where one must take the level of volatility explicitly into account. An alternative to our approach is stochastic control under partial observation. This would solve the problem of dependence of the volatility in the controls. Pham and Quenez [26] use the approach of partial observation to solve a portfolio problem similar to ours, however, with a stochastic volatility process driven by a Brownian motion correlated to the dynamics of the risky asset. Choosing their point of view in our context would lead to nonlinear filtering problems for jump processes. For more information on stochastic control under partial observation, see the monograph by Bensoussan [4].

Despite the (possible) practical shortcomings, *we still believe* our analysis contribute to further insight into portfolio optimization problem in markets with stochastic volatility. Investors *do* have a feeling for the current level of volatility, and it therefore may be desirable to include this (time-varying) information into their portfolio management strategies. Our results may also be useful for option pricing in incomplete markets. Following Hodges and Neuberger [19] and Davies, Panas and Zarpiphopoulou [12] one may use portfolio optimization techniques to determine the price of derivatives. Contrary to arbitrage techniques, this approach leads to a unique price. In the passing, we mention the work of Nicolato and Venardos [25] which deals with arbitrage pricing of European call options when the underlying asset follows the dynamics suggested by Barndorff-Nielsen and Shephard. In a forthcoming work [10], we will investigate the pricing problem from a utility maximization point of view.

Fouque, Papanicolaou and Sircar [16] study Merton's problem with stochastic volatility being the exponential of a mean-reverting process. The volatility process is driven by a Brownian motion correlated to the risky asset. Using the same line of argumentation as we do, they derive explicit solutions for both the optimal investment strategy and the investor's value function. The authors investigate their explicit solutions by means of asymptotic expansions around the (inverse of the) speed of mean-reversion, and thereby obtaining results of practical interest. Since our volatility process will be of a non-Gaussian nature, we face the problem of solving an *integro*-differential equation to find the value function. This equation can not be solved explicitly, but in general only by means of a Feynman-Kac formula. In addition, integrability conditions must be imposed in order to ensure the finiteness of the expectations that we encounter.

For implications of stochastic volatility models on derivatives pricing and hedging, the interested reader is advised to look into the monograph of Fouque, Papanicolaou and Sircar [16] and the references therein. See also the discussions and the reference list in Barndorff-Nielsen and Shephard [3]. Recently, many authors have studied portfolio optimization problems with asset dynamics going beyond the classical geometric Brownian motion (or the Samuelson model). We would like to mention Bank and Riedel [2], Benth, Karlsen and Reikvam [5, 6, 7, 8], Goll and Kallsen [14] and Kallsen [21] which model the risky asset as an exponential of a Lévy process, and Framstad, Øksendal and Sulem [17, 18] using a geometric stochastic differential equation driven by a Lévy process as stock price model. All these references are based on asset dynamics which do not take long-range dependency of logreturns into account, since the noise is driven by a Lévy process.

This paper is organized as follows: In Section 2 we give a rigorous formulation of the portfolio optimization problem together with some basic assumption. Section 3 shows some basic results on moments of the value process and the stochastic volatility model which are useful later. To prove the explicit solution derived in Section 4, we prove a verification theorem under some (natural) integrability conditions in Section 3. In Section 6, we discuss our results (and the conditions put forth) and relate them to the empirically based models of Barndorff-Nielsen and Shephard [3]. Finally, in Section 7 optimal investment strategies where varying volatility is taken into account are compared numerically with the classical solution of Merton [23, 24]. We demonstrate that stochastic volatility may lead to significantly different market behaviour.

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2. Formulation of optimization problem

For $0 \leq t < T < \infty$, let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $\{\mathcal{F}_s\}_{t \leq s \leq T}$ satisfying the usual conditions. Introduce m independent subordinators $Z_j(s)$ (i.e pure-jump Levy processes with no drift and positive increments) and denote the corresponding Lévy measures by $\ell_j(dz)$, $j = 1, \dots, m$. The Lévy measure of a subordinator satisfies the integrability condition

$$\int_{0+}^{\infty} \min(1, z)\, \ell(dz) < \infty.$$

We choose the (unique) cadlag (i.e., RCLL) version of $Z_j(s)$. Let $B(s)$ be a Wiener process independent of all the subordinators.

Let us introduce the stochastic volatility model proposed by Barndoff-Nielsen and Shephard [3]: Denote by $Y_j(s)$, $j = 1, \dots, m$, the Ornstein-Uhlenbeck stochastic processes with dynamics

$$dY_j(s) = -\lambda_j Y_j(s)\, ds + dZ_j(\lambda_j s),$$

where λ_j is a positive number. Observe that

$$Y_j(s) = y_j \mathrm{e}^{-\lambda_j(s-t)} + \int_t^s \mathrm{e}^{-\lambda_j(s-u)}\, dZ_j(\lambda_j u)\,,$$

and thus

$$Y(s) = (Y_1(s), \dots, Y_m(s)) \geq 0 \text{ a.s. for all } t \leq s \leq T.$$

Define the (stochastic) volatility of the stock to have the dynamics

$$\sigma^{t,y}(s) = \sum_{j=1}^{m} \omega_j Y_j(s), \qquad s \in [t, T],$$

where $\omega_j > 0$ are weights summing to one and $\sigma^{t,y}(t) = \sum_{j=1}^{m} \omega_j y_j =: \sigma$ is the initial volatility at time t. We shall frequently write $\sigma^y(s)$ for $\sigma^{0,y}(s)$. This will also be the case for other processes when they start at time zero. Inserting $Y_j(s)$ into the volatility process leads to

$$\sigma^{t,y}(s) = \sum_{j=1}^{m} \omega_j y_j \mathrm{e}^{-\lambda_j(s-t)} + \int_t^s \sum_{j=1}^{m} \omega_j \mathrm{e}^{\lambda_j(s-u)}\, dZ_j(\lambda_j u)\,.$$

The stock price dynamics is assumed to follow the stochastic differential equation (a geometric Brownian motion with stochastic volatility)

$$dS(s) = \left(\mu + \left(\tfrac{1}{2} + \beta\right) \sigma(s)\right) S(s)\, ds + \sqrt{\sigma(s)} S(s)\, dB(s),$$

where μ and β are constants. The risk-free asset has dynamics

$$dR(s) = rR(s)\, ds,$$

where $r > 0$. Let $\pi(s)$ be the fraction of wealth invested in the stock at time s, and assume that there are no transaction costs in the market. The wealth process $W(s)$ is the sum of the position in the stock and the risk-free asset:

$$W(s) = \frac{\pi(s)W(s)}{S(s)} S(s) + \frac{(1-\pi(s))W(s)}{R(s)} R(s).$$

The self-financing hypothesis yields the wealth dynamics:

$$dW(s) = \pi(s)\left(\left(\mu + (\tfrac{1}{2} + \beta)\sigma(s)\right) - r\right) W(s)\, ds + rW(s)\, ds + \pi(s)\sqrt{\sigma(s)} W(s)\, dB(s),$$

with initial wealth $W(t) = w$.

The set of admissible controls π is defined as follows:

Definition 2.1. An investment strategy (control) $\pi = \{\pi(s) : t \leq s \leq T\}$ is said to be *admissible*, and we write $\pi \in \mathcal{A}_t$, if π is progressively measurable and $\pi(s) \in [0,1]$ a.s. for all $t \leq s \leq T$.

Note that the control set $\mathcal{A}_t$ only depends on the current time, and not on the level of wealth nor volatility. We emphasize that the restriction $\pi \in [0,1]$ is just for mathematical convenience. Instead, we could have assumed $\pi \in [\underline{\pi}, \overline{\pi}]$ for some constants $-\infty < \underline{\pi} < \overline{\pi} < \infty$. All arguments below goes through with minor modifications in this case, however, the analysis is more transparent when $\underline{\pi} = 0$ and $\overline{\pi} = 1$. Since the processes $Y_j(s)$ are right-continuous it follows that $\sigma(s)$ is right-continuous. Thus $\int_0^T \sigma(s)\,ds < \infty$ a.s. , which together with $\pi \in [0,1]$ yields that $\int_0^t \pi(s)\sqrt{\sigma(s)}\,dB(s)$ is a well-defined local martingale. In this paper we need to impose exponential integrability conditions on the Lévy measures $\ell_j(dz)$. These conditions imply the martingale property of the Itô integral.

Condition A: For a constant $c_j > 0$,

$$\int_{0+}^{\infty} (\mathrm{e}^{c_j z} - 1)\, \ell_j(dz) < \infty.$$

Later we shall specify in detail the choice of the constant c_j in Condition A for each $j = 1, \ldots, m$. At this stage we only assume for every $j = 1, \ldots, m$ the existence of a c_j for which Condition A holds. Note that Condition A is a condition on the integrability of the tails of the Lévy measures, since $e^{c_j z} - 1 \sim z$ when z is in the neighborhood of zero. Under exponential integrability, we have

$$\mathbb{E}\left[\mathrm{e}^{aZ_j(\lambda_j t)}\right] = \exp\left(\lambda_j \int_{0+}^{\infty} (\mathrm{e}^{az} - 1)\, \ell_j(dz) t\right) \tag{2.1}$$

as long as Condition A holds with $c_j = a$.

Let $y := (y_1, \ldots, y_m) \in \mathbb{R}^m$. We will use the symbol $\mathbb{R}_+$ for the set $(0, \infty)$. The spatial domain of our stochastic control problem is

$$D := \Big\{(w, y) \in \mathbb{R} \times \mathbb{R}^m : w > 0, y_1, \ldots, y_m \geq 0\Big\}.$$

The functional to be optimized takes the form

$$J(t, w, y; \pi) = \mathbb{E}^{t,w,y}\left[U\left(W^\pi(T)\right)\right],$$

where the notation $\mathbb{E}^{t,w,y}$ means that we take the expectation conditioned on $W(t) = w$ and $Y_j(t) = y_j$, $j = 1, \ldots, m$. U is the investor's utility function, being concave, nondecreasing and bounded from below. In addition, we assume that U is of sublinear growth, i.e. there exist positive constants k and $\gamma \in (0,1)$ so that $U(w) \leq k(1 + w^\gamma)$ for all $w \geq 0$. Our optimal stochastic control problem consists of determining the value function

$$V(t, w, y) = \sup_{\pi \in \mathcal{A}_t} J(t, w, y; \pi), \qquad (t, x) \in [0, T] \times \overline{D}, \tag{2.2}$$

along with an optimal investment strategy $\pi^* \in \mathcal{A}_t$ such that

$$V(t, w, y) = J(t, w, y; \pi^*).$$

Observe that

$$V(T, w, y) = U(w) \quad \forall (w, y) \in \overline{D}, \qquad V(t, 0, y) \equiv U(0) \quad \forall (t, y) \in [0, T] \times \mathbb{R}^m_+. \tag{2.3}$$

The HJB equation associated to our stochastic control problem is

$$\begin{aligned} &v_t + \max_{\pi \in [0,1]}\left\{\pi\left(\mu + \left(\tfrac{1}{2} + \beta\right)\sigma - r\right) w v_w + \tfrac{1}{2}\pi^2 \sigma w^2 v_{ww}\right\} + r w v_w \\ &\quad - \sum_{j=1}^m \lambda_j y_j v_{y_j} + \sum_{j=1}^m \lambda_j \int_0^\infty \Big(v(t, w, y + z \cdot e_j) - v(t, w, y)\Big)\, \ell_j(dz) = 0, \end{aligned} \tag{2.4}$$

for $(t, w, y) \in [0, T) \times D$. In view of (2.3), we augment (2.4) with the terminal condition

$$v(T, w, y) = U(w) \quad \forall (w, y) \in \overline{D} \tag{2.5}$$

and the boundary condition

$$v(t,0,y)=U(0)\quad \forall (t,y)\in[0,T]\times\mathbb{R}_+^m. \tag{2.6}$$

We have used the following notational convention: $v_t=\partial v/\partial t$, $v_w=\partial v/\partial w$, $v_{ww}=\partial^2 v/\partial w^2$ and $v_{y_j}=\partial v/\partial y_j$, $j=1,\dots,m$.

3. Preliminary estimates

The following lemmas are useful in relating the existence of exponential moments of Y to exponential integrability conditions on the Lévy measures (see also [9]).

Lemma 3.1. *For $\xi_j>0$, assume Condition A holds with $c_j=\xi_j/\lambda_j$. Then*

$$\mathbb{E}\left[\exp\left(\xi_j\int_t^s Y_j(u)\,du\right)\right]\le \exp\left(\frac{\xi_j}{\lambda_j}y_j+\lambda_j\int_{0+}^{\infty}\{\exp(\xi_j z/\lambda_j)-1\}\,\ell_j(dz)(s-t)\right).$$

Proof. From the dynamics of Y_j we find

$$\begin{aligned}\lambda_j\int_t^s Y_j(u)\,du&=y_j+Z_j(\lambda_j s)-Z_j(\lambda_j t)-Y_j(s)\\&\le y_j+Z_j(\lambda_j s)-Z_j(\lambda_j t)\\&\overset{d}{=}y_j+Z_j(\lambda_j(s-t)),\end{aligned}$$

since $Y_j(s)\ge 0$. The last equality is in the sense of equality in distribution. Hence, from (2.1),

$$\begin{aligned}\mathbb{E}\left[\exp\left(\xi_j\int_t^s Y_j(u)\,du\right)\right]&\le\exp\left(\frac{\xi_j}{\lambda_j}y_j\right)\mathbb{E}\left[\exp\left(\frac{\xi_j}{\lambda_j}Z(\lambda_j(s-t))\right)\right]\\&=\exp\left(\frac{\xi_j}{\lambda_j}y_j+\lambda_j\int_{0+}^{\infty}\{\exp(\xi_j z/\lambda_j)-1\}\,\ell_j(dz)(s-t)\right),\end{aligned}$$

which proves the lemma. □

In a completely analogous way, we can show

Lemma 3.2. *Assume Condition A holds for some positive constant c_j. Then*

$$\mathbb{E}[\exp(c_jY_j(s))]\le\exp\left(c_jy_j+\lambda_j\int_{0+}^{\infty}\{\exp(c_jz)-1\}\,\ell_j(dz)(s-t)\right).$$

The following result provide us with a useful moment estimate on the wealth process.

Lemma 3.3. *For some $\theta>0$, assume Condition A holds with $c_j=2\theta\left(\left|\frac12+\beta\right|+\theta\right)\frac{\omega_j}{\lambda_j}$ for $j=1,\dots,m$. Then*

$$\sup_{\pi\in\mathcal{A}_t}\mathbb{E}^{t,w,y}\left[(W^\pi(s))^\theta\right]\le w^\theta\exp\left(K(\theta)\sum_{j=1}^m\frac{\omega_j}{\lambda_j}y_j+C(\theta)(s-t)\right),$$

where $K(\theta)=\theta\left(\left|\frac12+\beta\right|+\theta\right)$ and

$$C(\theta)=\theta(|\mu-r|+r)+\frac12\sum_{j=1}^m\lambda_j\int_{0+}^{\infty}\left(e^{\frac{2\omega_jK(\theta)}{\lambda_j}z}-1\right)\ell_j(dz).$$

Proof. First of all, we have

$$W^\pi(s)=w\exp\left(\int_t^s\alpha(u,\sigma(u))\,du+\int_t^s\pi(u)\sqrt{\sigma(u)}\,dB(u)\right),$$

where

$$\alpha(u,\sigma)=\pi(u)\left(\mu+\left(\tfrac12+\beta\right)\sigma-r\right)+r-\tfrac12(\pi(u))^2\sigma.$$

Define

$$X(s) = \exp\left(\int_t^s 2\theta\pi(u)\sqrt{\sigma(u)}\,dB(u) - \frac{1}{2}\int_t^s (2\theta)^2\left(\pi(u)\right)^2\sigma(u)\,du\right).$$

Then $\mathbb{E}[X(s)] = 1$ since $X(s)$ is a martingale by Novikov's condition. We have by Lemma 3.1 with $\xi_j = 2\theta^2\omega_j$ for $j = 1, \ldots, m$,

$$\mathbb{E}\left[e^{\frac{1}{2}\int_t^T (2\theta)^2(\pi(u))^2\sigma(u)\,du}\right] \le \mathbb{E}\left[e^{2\theta^2\int_t^T \sigma(u)\,du}\right] = \prod_{j=1}^m \mathbb{E}\left[e^{2\theta^2\omega_j\int_t^T Y_j(u)\,du}\right] < \infty.$$

Hence, by Hölder's inequality and using that $\pi \in [0,1]$,

$$\begin{aligned}
\mathbb{E}\left[(W^\pi(s))^\theta\right] &= w^\theta\mathbb{E}\left[\exp\left(\theta\int_t^s \alpha(u,\sigma(u))\,du + \theta\int_t^s \pi(u)\sqrt{\sigma(u)}\,dB(u)\right)\right] \\
&= w^\theta\mathbb{E}\left[\exp\left(\theta\int_t^s \alpha(u,\sigma(u))\,du + \theta^2\int_t^s \left(\pi(u)\right)^2\sigma(u)\,du\right)X(s)^{1/2}\right] \\
&\le w^\theta\mathbb{E}\left[\exp\left(\int_t^s 2\theta\alpha(u,\sigma(u)) + 2\theta^2\left(\pi(u)\right)^2\sigma(u)\,du\right)\right]^{1/2}\mathbb{E}[X(s)]^{1/2} \\
&\le w^\theta e^{(s-t)\theta(|\mu-r|+r)}\mathbb{E}\left[\exp\left(\left(2\theta\left|\tfrac{1}{2}+\beta\right| + 2\theta^2\right)\int_t^s \sigma(u)\,du\right)\right]^{1/2} \\
&= w^\theta e^{(s-t)\theta(|\mu-r|+r)}\prod_{j=1}^m \mathbb{E}\left[\exp\left(2\theta\left(\left|\tfrac{1}{2}+\beta\right| + \theta\right)\omega_j\int_t^s Y_j(u)\,du\right)\right]^{1/2}.
\end{aligned}$$

Choosing $\xi_j = 2\theta\left(\left|\frac{1}{2}+\beta\right| + \theta\right)\omega_j$, $j = 1, \ldots, m$, in Lemma 3.1 we obtain the desired estimate. □

The next proposition shows that the value function of our control problem is well-defined.

Proposition 3.4. *Assume Condition A holds with $c_j = 2\gamma\left(\left|\frac{1}{2}+\beta\right| + \gamma\right)\frac{\omega_j}{\lambda_j}$ for $j = 1, \ldots, m$. Then*

$$U(0) \le V(t,w,y) \le k\left(1 + w^\gamma e^{K(\gamma)\sum_{j=1}^m \frac{\omega_j}{\lambda_j}y_j + C(\gamma)(T-t)}\right),$$

where $K(\gamma)$ and $C(\gamma)$ are defined in Lemma 3.3 and k is a positive constant.

Proof. Since U is nondecreasing, we have $U(w) \ge U(0)$. Hence $\mathbb{E}[U(W^\pi(T))] \ge U(0)$ for any control π, which implies $V(t,w,y) \ge U(0)$. The upper bound follows from the sublinear growth of U and Lemma 3.3:

$$\begin{aligned}
V(t,w,y) &= \sup_{\pi\in\mathcal{A}}\mathbb{E}[U(W^\pi(T))] \le k\left(1 + \sup_{\pi\in\mathcal{A}}\mathbb{E}[(W^\pi(T))^\gamma]\right) \\
&\le k\left(1 + w^\gamma e^{K(\gamma)\sum_{j=1}^m \frac{\omega_j}{\lambda_j}y_j + C(\gamma)(T-t)}\right),
\end{aligned}$$

which proves the proposition. □

Remark. Under additional exponential integrability conditions on the Lévy measures $\ell_j(dz)$, local Hölder continuity of the value function in all variables is proven in [9]. To establish such a result, one needs of course that the utility function U is Hölder continuous.

From now on we suppose that Condition B – which ensures that the value function is well-defined – holds:

Condition B: For all $j = 1, \ldots, m$, Condition A holds with $c_j = \gamma\left(\left|\frac{1}{2}+\beta\right| + \gamma\right)\frac{\omega_j}{\lambda_j}$.

4. A VERIFICATION THEOREM

We state and prove the following verification theorem for our stochastic control problem:

Theorem 4.1. *Assume that* $v(t,w,y) \in C^{1,2,1}\left([0,T) \times (0,\infty) \times [0,\infty)^m\right) \bigcap C\left([0,T] \times \overline{D}\right)$ *is a solution of the HJB equation* (2.4) *with terminal and boundary conditions* (2.5) *and* (2.6). *For* $j = 1,\dots,m$, *assume*

$$\sup_{\pi\in\mathcal{A}_t} \int_0^T \int_{0+}^\infty \mathbb{E}[|v(s,W^\pi(s),Y(s)+z\cdot e_j) - v(s,W^\pi(s),Y(s))|]\, \ell_j(dz)\, ds < \infty$$

and

$$\sup_{\pi\in\mathcal{A}_t} \int_0^T \mathbb{E}\left[(\pi(s))^2 \sigma(s) \left(W^\pi(s)\right)^2 \left(v_w(s,W^\pi(s),Y(s))\right)^2\right] ds < \infty.$$

Then

$$v(t,w,y) \geq V(t,w,y), \qquad \forall (t,w,y) \in [0,T] \times \overline{D}.$$

If, in addition, there exists a measurable function $\pi^*(t,w,y) \in [0,1]$ *being a maximizer for the max-operator in* (2.4), *then* π^* *defines an optimal investment strategy in feedback form and*

$$V(t,w,y) = v(t,w,y) = \mathbb{E}^{t,w,y}\left[U\left(W^{\pi^*}(T)\right)\right], \qquad \forall (t,w,y) \in [0,T] \times \overline{D}.$$

Remark. The notation $C^{1,2,1}\left([0,T) \times (0,\infty) \times [0,\infty)^m\right)$ means twice continuously differentiable in w on $(0,\infty)$ and once continuously differentiable in t,y on $(0,T) \times (0,\infty)^m$ with continuous extensions of the derivatives to $t = 0$ and $y_j = 0$, $j = 1,\dots,m$.

Proof. Let $(t,w,y) \in [0,T) \times D$ and $\pi \in \mathcal{A}_t$, and introduce the operator

$$\mathcal{M}^\pi v = \pi\left(\mu + \left(\tfrac{1}{2} + \beta\right)\sigma - r\right) w v_w - \tfrac{1}{2}\sigma w^2 \pi^2 v_{ww} + r w v_w - \sum_{j=1}^m \lambda_j y_j v_{y_j}.$$

Itô's Formula (see, e.g., Ikeda and Watanabe, [20]) yields (with $t \leq s \leq T$)

$$\begin{aligned}
v(s,W^\pi(s),Y(s)) &= v(t,w,y) + \int_t^s \left\{v_t(u,W^\pi(u),Y(u)) + \mathcal{M}^\pi v(u,W^\pi(u),Y(u))\right\} du \\
&+ \int_t^s \pi(u)\sqrt{\sigma(u)}W^\pi(u) v_w(u,W^\pi(u),Y(u))\, dB(u) \\
&+ \sum_{j=1}^m \int_t^s \int_{0+}^\infty \Big(v(u,W^\pi(u),Y(u-)+z\cdot e_j) - v(u,W^\pi(u),Y(u-))\Big) N_j(\lambda_j du, dz),
\end{aligned}$$

where N_j is the (homogeneous) Poisson random measure coming from the Lévy-Kinchine representation of the subordinator Z_j. From the assumptions we know that the Itô integral is a martingale and that the integrals with respect to N_j are semimartingales (not only *local* semimartingales). Hence, taking expectations on both sides implies

$$\begin{aligned}
\mathbb{E}[v(s,W^\pi(s),Y(s))] &= v(t,w,y) + \mathbb{E}\left[\int_t^s (v_t + \mathcal{L}^\pi v)(u,W^\pi(u),Y(u))\, du\right] \\
&\leq v(t,w,y) + \mathbb{E}\left[\int_t^s \left(v_t + \max_{\pi\in[0,1]} \mathcal{L}^\pi v\right)(u,W^\pi(u),Y(u))\, du\right] \\
&= v(t,w,y),
\end{aligned}$$

where

$$\mathcal{L}^\pi v = \mathcal{M}^\pi v + \sum_{j=1}^m \lambda_j \int_{0+}^\infty \Big(v(t,w,y+z\cdot e_j) - v(t,w,y)\Big) \ell_j(dz).$$

Putting $s = T$ and invoking the terminal condition for v, we find

$$v(t,w,y) \geq \mathbb{E}[U(W^\pi(T))],$$

for all $\pi \in \mathcal{A}_t$. Therefore the first conclusion in the theorem holds for $(t,w,y) \in [0,T) \times D$.

To prove the second part, observe that since $\pi^*(t,w,y)$ is assumed to be a measurable function, we have that $\pi^*(s,W(s),Y(s))$ is $\mathcal{F}_s$-measurable for $t \le s \le T$. This together with $\pi^* \in [0,1]$ implies that $\pi^*(s,W(s),Y(s))$ is an admissible (feedback) control, i.e., an element of $\mathcal{A}_t$. Moreover, since π^* is a maximizer, $\max_{\pi\in[0,1]} \mathcal{L}^\pi v = \mathcal{L}^{\pi^*} v$. The above calculations using Itô's Formula above go through with equality by letting $\pi = \pi^*$. Hence,

$$v(t,w,y) = \mathbb{E}\left[U\left(W^{\pi^*}(T)\right)\right] \le V(t,w,y).$$

This together with the first part of the theorem yields

$$v(t,w,y) = V(t,w,y) = \mathbb{E}\left[U\left(W^{\pi^*}(T)\right)\right],$$

for $(t,w,y) \in [0,T) \times D$.

Observe that from the terminal and boundary conditions (2.5) and (2.6), the two conclusions of the theorem obviously hold when $t = T$ and $w = 0$. Hence the theorem is proven. □

Remark. In Section 5 we construct an explicit solution of the HJB-equation 2.4 when U is a power utility. Theorem 4.1 is used to prove that this solution coincides with the value function.

5. Explicit solution

In this section we shall construct and verify an explicit solution to the control problem (2.2) together with an explicit optimal control π^* when the utility function is of power type, i.e.,

$$U(w) = \gamma^{-1} w^\gamma,$$

where $1-\gamma$ is known as the relative risk aversion of the investor and $\gamma \in (0,1)$. These power functions are also known as HARA-utility functions.

5.1. **Reduction of the HJB-equation.** Define

$$v(t,w,y) = \gamma^{-1} w^\gamma h(t,y), \qquad (t,w,y) \in [0,T] \times \overline{D},$$

for some function $h(t,y)$. Observe that $v(t,0,y) = U(0)$. Inserted into the HJB-equation (2.4) we get a first-order integro-differential equation for h:

$$h_t(t,y) + \gamma \Pi(\sigma) h(t,y) - \sum_{j=1}^m \lambda_j y_j h_{y_j}(t,y) + \sum_{j=1}^m \lambda_j \int_{0+}^\infty \Big(h(t,y+z\cdot e_j) - h(t,y)\Big)\,\ell_j(dz) = 0, \tag{5.1}$$

where $(t,y) \in [0,T) \times [0,\infty)^m$. The terminal condition is $h(T,y) = 1$ for all $y \in [0,\infty)^m$, since $v(T,w,y) = U(w) = \gamma^{-1} w^\gamma$. Recall here that $\sigma = \sum_{j=1}^m \omega_j y_j$. The function $\Pi : [0,\infty) \to \mathbb{R}$ is defined as

$$\Pi(\sigma) = \max_{\pi\in[0,1]} \left\{\pi\left(\mu + \left(\tfrac{1}{2}+\beta\right)\sigma - r\right) - \tfrac{1}{2}\pi^2 \sigma(1-\gamma)\right\} + r. \tag{5.2}$$

We calculate an explicit representation of Π. A first order condition for an interior optimum is

$$\left(\mu + \left(\tfrac{1}{2}+\beta\right)\sigma - r\right) - \pi\sigma(1-\gamma) = 0.$$

If we denote the interior optimum by $\bar\pi = \bar\pi(\sigma)$, then this gives

$$\bar\pi(\sigma) = \frac{1}{1-\gamma}\left(\frac{\mu-r}{\sigma} + \frac{1}{2} + \beta\right).$$

Note that $\bar\pi(\sigma)$ is a function in σ only, and not in its different components y_j explicitly. We can thus treat this interior optimum as a function on $(0,\infty)$. Note from the constraints that $\bar\pi(\sigma)$ is an optimum if $\bar\pi(\sigma) \in [0,1]$. If this is not the case, the optimum is reached either in 0 or in 1, depending on the parameters of the problem. We now investigate this more closely.

In the rest of this section we assume $\mu > r$ (the analysis for $\mu < r$ is analogous) and aim at finding π^*, i.e., the value of π for which the maximum is reached in the expression of Π. Observe that $\bar{\pi}(\sigma)$ is *non-increasing*, $\lim_{\sigma\downarrow 0}\bar{\pi}(\sigma) = +\infty$ and

$$\lim_{\sigma\to\infty}\bar{\pi}(\sigma) = \frac{\frac{1}{2}+\beta}{1-\gamma}.$$

We separate the further discussion into 3 cases:

Case I: $\frac{\frac{1}{2}+\beta}{1-\gamma} \geq 1$. Under this assumption we see that $\bar{\pi} \geq 1$ for all σ, and hence

$$\pi^*(\sigma) = 1, \qquad \sigma \in [0,\infty). \tag{5.3}$$

Inserting this into the expression for Π we get

$$\Pi(\sigma) = \mu + \left(\frac{\gamma}{2}+\beta\right)\sigma, \qquad \sigma \in [0,\infty).$$

Define the constant $b_1 := \beta + \frac{1}{2}\gamma$, and observe that $b_1 > 0$.

Case II: $\frac{\frac{1}{2}+\beta}{1-\gamma} \in (0,1)$. Under this assumption we see that there exists a $\hat{\sigma}_1$ such that $\bar{\pi}(\hat{\sigma}_1) = 1$ and $\bar{\pi}(\sigma) \in (0,1)$ for all $\sigma > \hat{\sigma}_1$. A straightforward calculation gives

$$\hat{\sigma}_1 = \frac{\mu - r}{(1-\gamma) - \left(\frac{1}{2}+\beta\right)}.$$

Hence, the optimal π is given as

$$\pi^*(\sigma) = \begin{cases} 1, & \sigma \in [0,\hat{\sigma}_1), \\ \bar{\pi}(\sigma), & \sigma \in [\hat{\sigma}_1,\infty). \end{cases} \tag{5.4}$$

The expression for Π now becomes

$$\Pi(\sigma) = \begin{cases} \mu + \left(\frac{\gamma}{2}+\beta\right)\sigma, & \sigma \in [0,\hat{\sigma}_1), \\ \frac{(\mu-r)^2}{2(1-\gamma)\sigma} + \frac{(\mu-r)\left(\frac{1}{2}+\beta\right)}{(1-\gamma)} + \frac{\left(\frac{1}{2}+\beta\right)^2\sigma}{2(1-\gamma)} + r, & \sigma \in [\hat{\sigma}_1,\infty). \end{cases}$$

Moreover, it is easily seen that

$$|\Pi(\sigma)| \leq a + b_2\sigma \text{ for some constant } a.$$

Case III: $\frac{\frac{1}{2}+\beta}{1-\gamma} < 0$. Observe that this assumption is equivalent to $\frac{1}{2}+\beta < 0$ since $\gamma \in (0,1)$. In this situation there will exist a $\hat{\sigma}_1$ such that $\bar{\pi}(\hat{\sigma}_1) = 1$ and a $\hat{\sigma}_0$ such that $\bar{\pi}(\hat{\sigma}_0) = 0$. The former we calculated above, while the latter is easily found to be

$$\hat{\sigma}_0 = -\frac{\mu - r}{\frac{1}{2}+\beta}.$$

The optimal π is

$$\pi^*(\sigma) = \begin{cases} 1, & \sigma \in [0,\hat{\sigma}_1), \\ \bar{\pi}(\sigma), & \sigma \in [\hat{\sigma}_1,\hat{\sigma}_0], \\ 0, & \sigma \in (\hat{\sigma}_0,\infty). \end{cases} \tag{5.5}$$

Hence the expression for Π becomes

$$\Pi(\sigma) = \begin{cases} \mu + \left(\frac{\gamma}{2}+\beta\right)\sigma, & \sigma \in [0,\hat{\sigma}_1), \\ \frac{(\mu-r)^2}{2(1-\gamma)\sigma} + \frac{(\mu-r)\left(\frac{1}{2}+\beta\right)}{(1-\gamma)} + \frac{\left(\frac{1}{2}+\beta\right)^2\sigma}{2(1-\gamma)} + r, & \sigma \in [\hat{\sigma}_1,\hat{\sigma}_0], \\ r, & \sigma \in (\hat{\sigma}_0,\infty). \end{cases}$$

We now prove that $\Pi(\sigma)$ is, in fact, continuously differentiable on $[0,\infty)$ in all three cases.

Lemma 5.1. *Assume $\mu > r$. Then the function $\Pi(\sigma)$ defined in (5.2) is continuously differentiable on $[0,\infty)$.*

Proof. It is obvious that Π is continuous since π^* defined in (5.3), (5.4) and (5.5) are continuous, and

$$\Pi(\sigma) = \pi^*(\sigma)\left(\mu + \left(\tfrac{1}{2}+\beta\right)\sigma - r\right) - \tfrac{1}{2}\pi^*(\sigma)^2\sigma(1-\gamma) + r.$$

In Case I, Π is trivially differentiable. Furthermore, to prove differentiability of Π in the two subsequent cases, we must show that Π is differentiable at $\sigma = \hat{\sigma}_1$ in Case II and at $\sigma = \hat{\sigma}_1$ and $\sigma = \hat{\sigma}_0$ in Case III. But it is then sufficient to only consider case III:

$$\Pi'(\sigma) = \begin{cases} \frac{\gamma}{2}+\beta, & \sigma \in [0,\hat{\sigma}_1) \\ -\frac{(\mu-r)^2}{2(1-\gamma)\sigma^2} + \frac{\left(\frac{1}{2}+\beta\right)^2}{2(1-\gamma)}, & \sigma \in (\hat{\sigma}_1,\hat{\sigma}_0) \\ 0, & \sigma \in (\hat{\sigma}_0,\infty). \end{cases}$$

Straightforward calculations show

$$\lim_{\sigma\uparrow\hat{\sigma}_1} \Pi'(\sigma) = \frac{\gamma}{2}+\beta = \lim_{\sigma\downarrow\hat{\sigma}_1} \Pi'(\sigma)$$

and

$$\lim_{\sigma\uparrow\hat{\sigma}_0} \Pi'(\sigma) = 0 = \lim_{\sigma\downarrow\hat{\sigma}_0} \Pi'(\sigma),$$

implying the differentiability on $[0,\infty)$ of Π. This proves the lemma. □

5.2. **A Feynman-Kac formula for** $h(t,y)$**.** Define the function $g(t,y)$ by

$$g(t,y) = \mathbb{E}^y\left[\mathrm{e}^{\int_0^t \gamma\Pi(\sigma(s))\,ds}\right], \qquad (t,y)\in[0,T]\times[0,\infty)^m, \tag{5.6}$$

and recall that $\sigma^y(0) = \sum_{j=1}^m \omega_j y_j =: \sigma$. Note that $g(0,y) = 1$. We first show that g is of exponential growth in σ and thus well-defined under a growth hypothesis.

Lemma 5.2. *Assume Condition A holds with $c_j = \gamma b\frac{\omega_j}{\lambda_j}$ for $j = 1,\dots,m$, where b is equal to b_1 in Case I, b_2 in Case II and $b_3 = 0$ in Case III. Then*

$$g(t,y) \le \exp\left(kt + \gamma b \sum_{j=1}^m \frac{\omega_j}{\lambda_j} y_j\right),$$

for some positive constant k.

Proof. From the discussion in the previous subsection, we know that there exist positive constants a and b as given in the assumptions such that $|\Pi(\sigma)| \le a + b\sigma$. Thus

$$g(t,y) = \mathbb{E}^y\left[\mathrm{e}^{\int_0^t \gamma\Pi(\sigma(s))\,ds}\right] \le \mathbb{E}^y\left[\mathrm{e}^{\int_0^t (\gamma a+\gamma b\sigma(s))\,ds}\right] = \mathrm{e}^{\gamma a t}\mathbb{E}\left[\prod_{j=1}^m \mathrm{e}^{\gamma b\omega_j \int_0^t Y_j^{y_j}(s)\,ds}\right].$$

By independence of the Y_j's we get

$$g(t,y) \le \mathrm{e}^{\gamma a t}\prod_{j=1}^m \mathbb{E}\left[\mathrm{e}^{\gamma b\omega_j\int_0^t Y_j^{y_j}(s)\,ds}\right] \le \mathrm{e}^{\gamma a t}\prod_{j=1}^m \mathrm{e}^{\frac{\gamma b\omega_j}{\lambda_j}y_j + t\int_{0+}^\infty\left(\mathrm{e}^{\frac{\gamma b\omega_j}{\lambda_j}z}-1\right)\ell_j(dz)}.$$

To derive the last inequality, we used Lemma 3.1 with $\xi_j = \gamma b\omega_j$. Hence, there exists a positive constant k such that

$$g(t,y) \le \mathrm{e}^{kt+\gamma b\sum_{j=1}^m \frac{\omega_j}{\lambda_j}y_j},$$

and the lemma is proven. □

We show next that g is continuously differentiable in y.

Lemma 5.3. *Assume Condition A holds with $c_j = \gamma b\frac{\omega_j}{\lambda_j}$ for $j = 1,\dots,m$, where b is equal to b_1 in Case I, b_2 in Case II and $b_3 = 0$ in Case III. Then $g \in C^{0,1}([0,T]\times[0,\infty)^m)$, i.e., $g(\cdot,y)$ is continuous for all $y\in[0,\infty)^m$ and $g(t,\cdot)$ is once continuously differentiable for all $t\in[0,T]$.*

Proof. To prove differentiability, we will use the dominated convergence theorem to show that we may interchange expectation and differentiation. The condition for this is contained in Theorem 2.27 of Folland [15], which essentially says that we need to bound the derivative by an integrable function independent of y.

Let $(t,y) \in [0,T] \times \mathbb{R}_+^m$ and introduce the function

$$F(t,y) = \mathrm{e}^{\int_0^t \gamma \Pi(\sigma^y(s))\,ds}.$$

For each $j = 1,\dots,m$, we have

$$\frac{\partial F(t,y)}{\partial y_j} = \left(\frac{\partial}{\partial y_j}\int_0^t \gamma\Pi(\sigma^y(s))\,ds\right)\mathrm{e}^{\int_0^t \gamma \Pi(\sigma^y(s))\,ds}.$$

By Lemma 5.1, Π is continuously differentiable and Π' is bounded. Hence

$$\gamma\Pi'(\sigma^y(s))\frac{\partial\sigma^y(s)}{\partial y_j} = \gamma\Pi'(\sigma^y(s))\omega_j\mathrm{e}^{-\lambda_j s} \le c\,\mathrm{e}^{-\lambda_j s},$$

for some strictly positive constant c. Theorem 2.27 b) in Folland [15] says that differentiation and integration now commutes,

$$\frac{\partial F(t,y)}{\partial y_j} = \left(\gamma\omega_j\int_0^t \Pi'(\sigma^y(s))\mathrm{e}^{-\lambda_j s}\,ds\right)\mathrm{e}^{\int_0^t \gamma \Pi(\sigma^y(s))\,ds}.$$

From the discussion in the previous subsection we know there exists constants a and b such that $|\Pi(\sigma)| \le a + b\sigma$ in Cases I and II, and $\Pi(\sigma) \le a$ in Case III, where $b = \frac{\gamma}{2} + \beta$ and $b = \frac{\left(\frac{1}{2}+\beta\right)^2}{2(1-\gamma)}$ in Cases I and II, respectively. Hence

$$\begin{aligned}\left|\frac{\partial F(t,y)}{\partial y_j}\right| &\le \left(c\int_0^t \mathrm{e}^{-\lambda_j s}\,ds\right)\mathrm{e}^{\int_0^t \gamma \Pi(\sigma^y(s))\,ds}\\ &\le \frac{c}{\lambda_j}\mathrm{e}^{\gamma a T}\begin{cases}\mathrm{e}^{\gamma b\int_0^t \sigma^y(s)\,ds}, & \text{Cases I and II},\\ 1, & \text{Case III}.\end{cases}\end{aligned}$$

As in Lemma 3.1,

$$\omega_j\gamma b\int_0^t Y_j^{y_j}(s)\,ds \le \omega_j\gamma b\left(\frac{y_j}{\lambda_j} + \frac{1}{\lambda_j}Z_j(\lambda_j t)\right) \le \frac{\omega_j\gamma b}{\lambda_j}\left(y_j + Z_j(\lambda_j t)\right).$$

Whence, there exists a positive constant k such that

$$\left|\frac{\partial F(t,y)}{\partial y_j}\right| \le k\begin{cases}\mathrm{e}^{\sum_{j=1}^m\left(\frac{\omega_j\gamma b}{\lambda_j}y_j + \frac{\omega_j\gamma b}{\lambda_j}Z_j(\lambda_j t)\right)}, & \text{Cases I and II},\\ 1, & \text{Case III}.\end{cases}$$

By (2.1), we have

$$\mathbb{E}\left[\mathrm{e}^{\frac{\omega_j\gamma b}{\lambda_j}Z_j(\lambda_j t)}\right] = \int_{0+}^\infty\left(\mathrm{e}^{\frac{\omega_j\gamma b}{\lambda_j}z} - 1\right)\ell_j(dz),$$

which is assumed finite.

Cases I and II: Choose a compact set where y is in the interior. On this compact we have that $|\partial F(t,y)/\partial y_j|$ is uniformly bounded (in y) by the random variable

$$\exp\left(\frac{\omega_j\gamma b}{\lambda_j}Z_j(\lambda_j t)\right),$$

which is integrable since by assumption $\int_0^\infty\left(\mathrm{e}^{\omega_j\gamma b z/\lambda_j} - 1\right)\ell_j(dz) < \infty$, for $j = 1,\dots,m$. Theorem 2.27 b) in [15] implies that $g(t,y) = \mathbb{E}[F(t,y)]$ is differentiable in y. Differentiability is a local notion, hence the result is independent of the choice of the compact set. We conclude that

$$\frac{\partial g(t,y)}{\partial y_j} = \mathbb{E}\left[\frac{\partial F(t,y)}{\partial y_j}\right], \qquad \forall y \in \mathbb{R}_+^m, \qquad j = 1,\dots,m.$$

Moreover, we have that $y \mapsto \partial F(t,y)/\partial y_j$ is continuous since $y \mapsto \sigma^y(s)$, $\sigma \mapsto \Pi(\sigma)$ and $\sigma \mapsto \Pi'(\sigma)$ all are continuous mappings. Using Theorem 2.27 a) in Folland [15] we conclude that the mapping $(t,y) \mapsto \partial g(t,y)/\partial y_j$ is continuous.

Case III: In this case $|\partial F_t(y,\omega)/\partial y_j| \leq k$ for some positive constant k. Hence, Theorem 2.27 a)-b) in [15] immediately applies to conclude that $(t,y) \mapsto \partial g(t,y)/\partial y_j$ is continuous and

$$\frac{\partial g(t,y)}{\partial y_j} = \mathbb{E}\left[\frac{\partial F_t(y,\cdot)}{\partial y_j}\right], \qquad j = 1,\ldots,m.$$

Since the limit of $\partial F(t,y)/\partial y_j$ exists when $y_i \downarrow 0$ for any $i = 1,\ldots,m$, we can argue as above to show that $\partial g(t,y)/\partial y_j$ has a limit when $y_i \downarrow 0$. This concludes the proof of the Lemma. $\square$

Remark. Note that for Case III in Lemmas 5.2 and 5.3 we do not impose any integrability condition on the Lévy measures $\ell_j(dz)$.

Lemma 5.4. *Assume Condition A holds with $c_j = 2\gamma b\frac{\omega_j}{\lambda_j}$ for $j = 1,\ldots,m$, where b is equal to b_1 in Case I and b_2 in Case II. In Case III, assume $\int_{0+}^{\infty} z\,\ell_j(dz) < \infty$, $j = 1,\ldots,m$. Then*

$$\sum_{j=1}^{m} \mathbb{E}\left[\int_0^T \int_{0+}^{\infty} |g(u,Y(u)+z\cdot e_j) - g(u,Y(u))|\;\ell_j(dz)\,du\right] < \infty. \tag{5.7}$$

Proof. By the mean value theorem and differentiability of g we have,

$$\begin{aligned}
|g(u,y+z\cdot e_j) - g(u,y)| &\leq \sup_{x\in[0,z]} \left|\frac{\partial g(u,y+x\cdot e_j)}{\partial y_j}\right| z \\
&\leq kz \begin{cases} \mathrm{e}^{\sum_{j=1}^m \frac{b\gamma\omega_j}{\lambda_j}(y_j + Z_j(\lambda_j u))}, & \text{Cases I and II} \\ 1, & \text{Case III,} \end{cases}
\end{aligned}$$

where k is a positive constant only dependent on T and the parameters of the problem. Since

$$\frac{b\gamma\omega_j}{\lambda_j}\left(Z_j(\lambda_j u) + Y_j^{y_j}(u)\right) \leq \frac{b\gamma\omega_j}{\lambda_j}\left(Z_j(\lambda_j u) + y_j + Z_j(\lambda_j u)\right) = \frac{b\gamma\omega_j}{\lambda_j} y_j + \frac{2b\gamma}{\lambda_j} Z_j(\lambda_j u),$$

we have

$$|g(u,Y(u)+z\cdot e_j) - g(u,Y(u))| \leq kz \begin{cases} \mathrm{e}^{\sum_{j=1}^m \left(\frac{b\gamma\omega_j}{\lambda_j} y_j + \frac{2b\gamma\omega_j}{\lambda_j} Z_j(\lambda_j u)\right)}, & \text{Cases I and II,} \\ 1, & \text{Case III.} \end{cases}$$

From the integrability assumptions on $\ell_j(dz)$ in Cases I and II, we have from (2.1)

$$\begin{aligned}
&\int_0^T \mathbb{E}\left[\int_{0+}^{\infty} kz\,\ell_j(dz) \mathrm{e}^{\sum_{j=1}^m \frac{2b\gamma\omega_j}{\lambda_j} Z_j(\lambda_j u)}\right] du \\
&\qquad = k\int_{0+}^{\infty} z\,\ell_j(dz)\,\mathrm{e}^{\sum_{j=1}^m \lambda_j \int_{0+}^{\infty}\left(\mathrm{e}^{\frac{2b\gamma\omega_j}{\lambda_j}z} - 1\right)\ell_j(dz)} < \infty.
\end{aligned}$$

The exponential integrability conditions in Cases I and II imply $\int_{0+}^{\infty} z\,\ell_j(dz) < \infty$. In Case III we have $b = b_3 = 0$ and hence

$$\int_0^T \mathbb{E}\left[\int_{0+}^{\infty} kz\ell_j(dz)\right] du \leq kT\int_{0+}^{\infty} z\,\ell_j(dz),$$

which is finite by assumption. This proves the lemma. $\square$

We now prove that $g(t,y)$ is a (classical) solution to the related forward problem of (5.1).

Proposition 5.5. *Assume there exists $\varepsilon > 0$ such that Condition A is satisfied with $c_j = 2\gamma b\frac{\omega_j}{\lambda_j}+\varepsilon$ for $j = 1,\ldots,m$, where $b = b_1$ in Case I and $b = b_2$ in Case II. Then $g(t,\cdot)$ belongs to the domain of the infinitesimal generator of Y and*

$$\frac{\partial g}{\partial t}(t,y) = \gamma\Pi(\sigma)g(t,y) - \sum_{j=1}^{m}\lambda_j y_j\frac{\partial g}{\partial y_j}(t,y) + \sum_{j=1}^{m}\lambda_j\int_{0+}^{\infty}\Big(g(t,y+z\cdot e_j)-g(t,y)\Big)\,\ell_j(dz)\,, \tag{5.8}$$

for $(t,y)\in(0,T]\times[0,\infty)^m$. Moreover, $\partial g(t,y)/\partial t$ is continuous, so that $g\in C^{1,1}\left((0,T]\times[0,\infty)^m\right)$.

Proof. First of all, observe that the conditions in Lemmas 5.3 and 5.7 are fulfilled. The two first terms on the right-hand side of (5.8) are continuous since Π is continuous and $g(t,\cdot)\in C^1$ by Lemma 5.3 for all $t\in[0,T]$. The integral operator also defines a continuous function in time and space. This follows from the integrability conditions on the Lévy measures $\ell_j(dz)$ and Theorem 2.27 in [15] together with arguments along the lines of the proofs of Lemmas 5.3 and 5.7. Hence if g solves (5.8), then $\partial g(t,y)/\partial t$ is continuous for $(t,y)\in(0,T)\times[0,\infty)^m$, and may be continuously extended to $t=T$. Hence, $g\in C^{1,1}\left((0,T]\times[0,\infty)^m\right)$.

¿From Lemma 5.3 we have that $y\mapsto g(t,y)$ is a continuously differentiable map. Hence, we know from Itô's lemma (see, e.g., Ikeda and Watanabe [20]) that the mapping $s\mapsto g(t,Y(s))$ is a (local) semimartingale with dynamics

$$\begin{aligned}g(t,Y(s)) = g(t,y) &- \sum_{j=1}^{m}\lambda_j\int_0^s Y_j(u)\frac{\partial g}{\partial y_j}(t,Y(u))\,du\\ &+\sum_{j=1}^{m}\int_0^s\int_{0+}^{\infty}\Big(g(t,Y(u-)+z\cdot e_j)-g(t,Y(u-))\Big)\,N_j(\lambda_j du,dz),\end{aligned}$$

where $N_j(\lambda_j du,dz)$ is the Poisson random measure in the Lévy-Kintchine representation of $Z_j(\lambda_j u)$. From Lemma 5.7 we know that

$$\mathbb{E}\left[\int_0^T\int_{0+}^{\infty}|g(t,Y(u)+z\cdot e_j)-g(t,Y(u))|\;\ell_j(dz)\,du\right]<\infty,$$

and thus $g(t,Y(u)+z\cdot e_j)-g(t,Y(u))\in\mathbf{F}^1$ (see Ikeda and Watanabe [20], page 61-62, for this notation). This implies that $g(t,Y(s))$ is a semimartingale (and not only *local* semimartingale). Taking expectations on both sides and rearranging terms give

$$\begin{aligned}\frac{\mathbb{E}[g(t,Y(s))]-g(t,y)}{s} &= -\sum_{j=1}^{m}\lambda_j\frac{1}{s}\int_0^s\mathbb{E}\left[Y_j(u)\frac{\partial g}{\partial y_j}(t,Y(u))\right]\,du\\ &+\sum_{j=1}^{m}\frac{\lambda_j}{s}\int_0^s\int_{0+}^{\infty}\mathbb{E}\left[g(t,Y(u)+z\cdot e_j)-g(t,Y(u))\right]\,\ell_j(dz)\,du.\end{aligned}$$

Hence, by letting $s\downarrow 0$ we get that $g(t,\cdot)$ is in the domain of the infinitesimal generator of Y, which is denoted by $\mathcal{G}$, and

$$\mathcal{G}g(t,y) = -\sum_{j=1}^{m}\lambda_j y_j\frac{\partial g}{\partial y_j}(t,y)+\sum_{j=1}^{m}\lambda_j\int_{0+}^{\infty}\Big(g(t,y+z\cdot e_j)-g(t,y)\Big)\,\ell_j(dz).$$

Since $g(t,Y(s))\in L^1(\Omega,P)$ for all $s>0$ in a neighborhood of zero, we can calculate

$$\begin{aligned}\mathbb{E}[g(t,Y(s))] &= \mathbb{E}\left[\mathbb{E}\left[e^{\int_0^t\gamma\Pi(\sigma^{Y^y(s)}(u))\,du}\right]\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[e^{\int_0^t\gamma\Pi(\sigma^y(u+s))\,du}\Big|\mathcal{F}_s\right]\right]\\ &= \mathbb{E}\left[e^{\int_s^{t+s}\gamma\Pi(\sigma^y(u))\,du}\right]\\ &= \mathbb{E}\left[e^{\int_0^{t+s}\gamma\Pi(\sigma^y(u))\,du}\cdot e^{-\int_0^s\gamma\Pi(\sigma^y(u))\,du}\right],\end{aligned}$$

where we have used the Markov property of Y together with the law of double expectation. Hence

$$\begin{aligned}\frac{\mathbb{E}[g(t,Y(s))]-g(t,y)}{s} &= \frac{1}{s}\mathbb{E}\left[\mathrm{e}^{\int_0^{t+s}\gamma\Pi(\sigma^y(u))\,du}\mathrm{e}^{-\int_0^{s}\gamma\Pi(\sigma^y(u))\,du}-\mathrm{e}^{\int_0^{t}\gamma\Pi(\sigma^y(u))\,du}\right]\\ &= \frac{1}{s}\mathbb{E}\left[\mathrm{e}^{\int_0^{t+s}\gamma\Pi(\sigma^y(u))\,du}\mathrm{e}^{-\int_0^{s}\gamma\Pi(\sigma^y(u))\,du}-\mathrm{e}^{\int_0^{t+s}\gamma\Pi(\sigma^y(u))\,du}\right]\\ &\quad+\frac{1}{s}\left\{\mathbb{E}\left[\mathrm{e}^{\int_0^{t+s}\gamma\Pi(\sigma^y(u))\,du}\right]-\mathbb{E}\left[\mathrm{e}^{\int_0^{t}\gamma\Pi(\sigma^y(u))\,du}\right]\right\}\\ &= \mathbb{E}\left[\mathrm{e}^{\int_0^{t+s}\gamma\Pi(\sigma^y(u))\,du}\frac{1}{s}\left\{e^{-\int_0^{s}\gamma\Pi(\sigma^y(u))\,du}-1\right\}\right]+\frac{g(t+s,y)-g(t,y)}{s}.\end{aligned}$$

By the fundamental theorem of calculus we have that

$$\mathrm{e}^{\int_0^{t+s}\gamma\Pi(\sigma^y(u))\,du}\frac{1}{s}\left(\mathrm{e}^{-\int_0^{s}\gamma\Pi(\sigma^y(u))\,du}-1\right)\to-\gamma\Pi(\sigma)\mathrm{e}^{\int_0^{t}\gamma\Pi(\sigma^y(u))\,du}\text{ as } s\downarrow 0.$$

In order to show that limit and integration commute, define the function

$$f(s)=\mathrm{e}^{-\int_0^{s}\gamma\Pi(\sigma^y(u))\,du}.$$

The mean value theorem gives

$$\begin{aligned}\frac{1}{s}\left|(f(s)-f(0))\right| &\le \frac{1}{s}\sup_{s\in[0,T]}|f'(s)|\,s=\sup_{s\in[0,T]}\left|\gamma\Pi(\sigma^y(s))\mathrm{e}^{-\int_0^{s}\gamma\Pi(\sigma^y(u))\,du}\right|\\ &\le \gamma\mathrm{e}^{\gamma\int_0^T(a+b\sigma^y(u))\,du}\left(a+b\sup_{s\in[0,T]}\sigma^y(s)\right).\end{aligned}$$

In the last estimation we have used the linear growth of Π. The constant b is b_1 in Case I, b_2 in Case II and $b_3=0$ in Case III. Since each Z_j is a nondecreasing process,

$$\sup_{s\in[0,T]}\sigma^y(s)\le\sigma+\sum_{j=1}^m\omega_jZ_j(\lambda_jT).$$

This implies

$$\mathrm{e}^{\int_0^{t+s}\gamma\Pi(\sigma^y(u))\,du}\frac{1}{s}\left(\mathrm{e}^{-\int_0^{s}\gamma\Pi(\sigma^y(u))\,du}-1\right)\le k\sum_{j=1}^m\omega_j\mathrm{e}^{2\gamma\int_0^T(a+b\sigma^y(u))\,du}Z_j(\lambda_jT),$$

for some positive constant k. But from (2.1)

$$\begin{aligned}\sum_{j=1}^m\omega_j\mathbb{E}\left[\mathrm{e}^{2\gamma\int_0^T(a+b\sigma^y(u))\,du}Z_j(\lambda_jT)\right] &\le k\sum_{j=1}^m\omega_j\mathbb{E}\left[\mathrm{e}^{\left(\frac{2\gamma\omega_jb}{\lambda_j}+\varepsilon\right)Z_j(\lambda_jT)}\right]\\ &= k\sum_{j=1}^m\omega_j\mathrm{e}^{\lambda_j\int_{0+}^\infty\left(\mathrm{e}^{\frac{2\gamma b\omega_j}{\lambda_j}z}-1\right)\ell_j(dz)},\end{aligned}$$

where k is some positive constant (different than above). In our estimation, we have used that there exists a positive constant k_ε such that $z\le k_\varepsilon\mathrm{e}^{\varepsilon z}$ for all $z\ge 0$. The last sum is finite by our integrability assumption. Hence by dominated convergence (see Theorem 2.27 a) in [15]) $\partial g/\partial t$ exists and

$$\mathcal{G}g(t,y)=-\gamma\Pi(\sigma)g(t,y)+\frac{\partial g(t,y)}{\partial t}.$$

This concludes the proof of the proposition. □

Let $h(t,y)=g(T-t,y)$, i.e.,

$$h(t,y)=\mathbb{E}^y\left[\mathrm{e}^{\int_0^{T-t}\gamma\Pi(\sigma(s))\,ds}\right].$$

We can represent this equivalently as

$$h(t,y)=\mathbb{E}^{t,y}\left[\mathrm{e}^{\int_t^{T}\gamma\Pi(\sigma(s))\,ds}\right],\tag{5.9}$$

by using the time-homogeneity of Y. Hence our explicit solution candidate is

$$v(t,w,y) = \gamma^{-1} w^{\gamma} h(t,y). \tag{5.10}$$

The optimal feedback control $\pi^*(\sigma)$ is given in (5.3), (5.4) or (5.5), depending on the size of $\left(\frac{1}{2}+\beta\right)/(1-\gamma)$. In the next section we prove that (5.10) coincides with the value function (2.2).

5.3. **Explicit solution of the control problem.** We apply the verification theorem to connect our explicit solution to the value function of the control problem. But first we need the integrability results stated in the following two lemmas.

Lemma 5.6. *Assume Condition A holds with* $c_j = 8\gamma\left(\left|\frac{1}{2}+\beta\right| + 4\gamma\right)\frac{\omega_j}{\lambda_j}$ *for* $j = 1,\dots,m$. *Then*

$$\int_0^T \mathbb{E}\left[(\pi(u))^2 (W^\pi(u))^2 \sigma(u) (W^\pi(u))^{2(\gamma-1)} h(u,Y(u))\right] du < \infty, \qquad \forall \pi \in \mathcal{A}_0.$$

Proof. Observe that the function h has the same growth as g. Hence, by Lemma 5.2 and $\pi \in [0,1]$,

$$\begin{aligned}
&\int_0^T \mathbb{E}\left[(\pi(u))^2 (W^\pi(u))^2 \sigma(u) (W^\pi(u))^{2(\gamma-1)} h(u,Y(u))\right] du \\
&\leq \int_0^T \mathbb{E}\left[(W^\pi(u))^{2\gamma} \sigma(u) \mathrm{e}^{ku+\gamma b \sum_{j=1}^m \frac{\omega_j}{\lambda_j} Y_j(u)}\right] du \\
&\leq k_\varepsilon \mathrm{e}^{kT} \int_0^T \mathbb{E}\left[(W^\pi(u))^{2\gamma} \mathrm{e}^{(\gamma b+\varepsilon)\sum_{j=1}^m \frac{\omega_j}{\lambda_j} Y_j(u)}\right] du,
\end{aligned}$$

where k_ε is a positive constant such that $\sigma \leq k_\varepsilon \mathrm{e}^{\varepsilon \sum_{j=1}^m \frac{\omega_j}{\lambda_j} y_j}$. Hölder's inequality gives

$$\begin{aligned}
&\int_0^T \mathbb{E}\left[(\pi(u))^2 (W^\pi(u))^2 \sigma(u) (W^\pi(u))^{2(\gamma-1)} h(u,Y(u))\right] du \\
&\leq k_\varepsilon \int_0^T \prod_{j=1}^m \mathbb{E}\left[\mathrm{e}^{2(\gamma b+\varepsilon)\frac{\omega_j}{\lambda_j} Y_j(u)}\right]^{1/2} \mathbb{E}\left[(W^\pi(u))^{4\gamma}\right]^{1/2} du \\
&\leq k_\varepsilon \int_0^T \prod_{j=1}^m \mathbb{E}\left[\mathrm{e}^{\left(2\gamma b\frac{\omega_j}{\lambda_j}+\varepsilon\right) Y_j(u)}\right]^{1/2} \mathbb{E}\left[(W^\pi(u))^{4\gamma}\right]^{1/2} du,
\end{aligned}$$

where b is b_1 in Case I, b_2 in Case II and $b_3 = 0$ in Case III. In the last estimation we have redefined ε to get a more tractable integrability condition.

We now argue that the two expectations are finite. Note that we are free to choose ε as long as it is positive. Let $\xi_j = 2\gamma b\omega_j/\lambda_j + \varepsilon$. In Case I, we have $\xi_j = 2\gamma\left(\frac{\gamma}{2}+\beta\right) + \varepsilon < c_j$ for a suitably chosen ε, where c_j is defined in the lemma. In Case II, $\xi_j = 2\gamma\frac{\left(\frac{1}{2}+\beta\right)^2}{2(1-\gamma)}\frac{\omega_j}{\lambda_j} + \varepsilon$, and since we have $\frac{\frac{1}{2}+\beta}{1-\gamma} < 1$, $\xi_j < \gamma\left(\frac{1}{2}+\beta\right)\frac{\omega_j}{\lambda_j} + \varepsilon < c_j$ if we choose ε small enough. In Case III, we obviously have $\xi_j = \varepsilon < c_j$ when choosing ε smaller than c_j. Thus, the integrability condition in Lemma 3.2 holds and the terms involving the expectation of $Y_j(u)$ above are finite. Finally, invoking Lemma 3.3 yields the desired result. □

Lemma 5.7. *Assume Condition A holds with* $c_j = 8\gamma\left(\left|\frac{1}{2}+\beta\right| + 4\gamma\right)\frac{\omega_j}{\lambda_j}$ *for* $j = 1,\dots,m$. *Then*

$$\int_0^T \int_{0+}^\infty \mathbb{E}\left[(W^\pi(s))^\gamma \left|h(u,Y(u)+z\cdot e_j) - h(u,Y(u))\right|\right] \ell_j(dz)\, du < \infty, \qquad j = 1,\dots,m.$$

Proof. We follow the arguments in the proof of Lemma 5.7:

$$\begin{aligned}
&\int_0^T \int_{0+}^\infty \mathbb{E}[(W^\pi(u))^\gamma |h(u,Y(u)+z) - h(u,Y(u))|]\, \ell_j(dz)\, du \\
&\leq \int_0^T \mathbb{E}\left[(W^\pi(u))^\gamma \int_{0+}^\infty Kz\mathrm{e}^{\gamma \sum_{j=1}^m \frac{\omega_j}{\lambda_j} b(y_j+2Z_j(\lambda_j u))}\, \ell_j(dz)\right] du
\end{aligned}$$

$$\leq k_1 e^{k_2\sigma} \int_{0+}^{\infty} z\,\ell_j(dz) \int_0^T \mathbb{E}\left[(W^\pi(u))^\gamma \, e^{2\gamma b \sum_{j=1}^m \frac{\omega_j}{\lambda_j} Z_j(\lambda_j u)}\right] du,$$

where k_1, k_2 are positive constants. Using Hölder's inequality with $p = 4$ and $q = 4/3$ gives

$$\begin{aligned}
&\int_0^T \int_{0+}^{\infty} \mathbb{E}\left[(W^\pi(u))^\gamma \left|h(u, Y(u) + z) - h(u, Y(u))\right|\right] \ell_j(dz)\, du \\
&\quad \leq k_1 e^{k_2\sigma} \int_{0+}^{\infty} z\,\ell_j(dz) \int_0^T \mathbb{E}\left[(W^\pi(u))^{4\gamma}\right]^{1/4} \mathbb{E}\left[e^{\frac{8}{3}\gamma b \sum_{j=1}^m \frac{\omega_j}{\lambda_j} Z_j(\lambda_j u)}\right]^{3/4} du \\
&\quad \leq k_1 e^{k_2\sigma} \int_{0+}^{\infty} z\,\ell_j(dz) \prod_{j=1}^m \mathbb{E}\left[e^{\frac{8}{3}\gamma b \frac{\omega_j}{\lambda_j} Z_j(\lambda_j T)}\right]^{3/4} \int_0^T \mathbb{E}\left[(W^\pi(u))^{4\gamma}\right]^{1/4} du \\
&\quad = k_1 e^{k_2\sigma} \int_{0+}^{\infty} z\,\ell_j(dz) e^{\frac{3}{4}\sum_{j=1}^m \lambda_j \int_{0+}^{\infty}\left(e^{\frac{8}{3}\gamma b \frac{\omega_j}{\lambda_j} z} - 1\right)\ell_j(dz)} \int_0^T \mathbb{E}\left[(W^\pi(u))^{4\gamma}\right]^{1/4} du.
\end{aligned}$$

Let $\xi_j = \frac{8}{3}\gamma b \frac{\omega_j}{\lambda_j}$. In Case I, $\xi_j = \frac{8}{3}\gamma\left(\frac{\gamma}{2} + \beta\right)\frac{\omega_j}{\lambda_j}$, which obviously is less than the c_j given in the lemma. In Case II, we have $\xi_j < \frac{8}{6}\gamma\left(\frac{1}{2} + \beta\right)\frac{\omega_j}{\lambda_j} < c_j$. Finally, in Case III, $\xi_j = 0 < c_j$. Hence, the integrability condition in the Lemma implies that $\int_{0+}^{\infty}\left(e^{\frac{8}{3}\gamma b \frac{\omega_j}{\lambda_j} z} - 1\right)\ell_j(dz) < \infty$ for $j = 1, \ldots, m$. From Lemma 3.3 the desired result follows. □

We sum up our results in this section in the following theorem:

Theorem 5.8. *Assume Condition A holds with* $c_j = 8\gamma\left(\left|\frac{1}{2} + \beta\right| + 4\gamma\right)\frac{\omega_j}{\lambda_j}$ *for* $j = 1, \ldots, m$. *Then the value function of the control problem is*

$$V(t, w, y) = \gamma^{-1} w^\gamma h(t, y)$$

where h *is defined in* (5.9). *Furthermore, the optimal investment strategy is* $\pi^*(\sigma) = 1$ *in Case I,*

$$\pi^*(\sigma) = \begin{cases} 1, & \sigma \in [0, \hat{\sigma}_1), \\ \bar{\pi}(\sigma), & \sigma \in [\hat{\sigma}_1, \infty) \end{cases}$$

in Case II and

$$\pi^*(\sigma) = \begin{cases} 1, & \sigma \in [0, \hat{\sigma}_1), \\ \bar{\pi}(\sigma), & \sigma \in [\hat{\sigma}_1, \hat{\sigma}_0], \\ 0, & \sigma \in (\hat{\sigma}_0, \infty). \end{cases}$$

in Case III. The function $\bar{\pi}(\sigma)$ *is defined as*

$$\bar{\pi}(\sigma) = \frac{1}{1-\gamma}\left(\frac{\mu - r}{\sigma} + \frac{1}{2} + \beta\right)$$

and

$$\hat{\sigma}_1 = \frac{\mu - r}{(1-\gamma) - \left(\frac{1}{2} + \beta\right)}, \qquad \hat{\sigma}_0 = -\frac{\mu - r}{\frac{1}{2} + \beta}.$$

Proof. Observe that Condition B holds under our assumption. Moreover, the integrability condition implies by Lemma 5.2 that h is well-defined.

Let $v(t, w, y) = \gamma^{-1} w^\gamma h(t, y)$, and observe that the condition in Lemmas 5.6 and 5.7 hold. Moreover, we claim that Proposition 5.5 holds true. Let $\xi_j = 2\gamma b \frac{\omega_j}{\lambda_j} + \varepsilon$, where ε is any positive number which we are free to choose. Going through all the three cases for the constant b, we see that $\xi_j < c_j$, where c_j is given in the theorem for ε appropriately chosen. Hence, the integrability condition assumed in the theorem is stronger than the required integrability in Proposition 5.5. We also see that the integrability conditions in Lemma 5.3 holds, which imply $v \in C\left([0, T] \times \overline{D}\right)$, since v obviously is continuous in w on $[0, \infty)$. Therefore v is a classical solution of the HJB equation (2.4) and we can apply the verification theorem (Theorem 4.1) to conclude the proof. □

Remark. The reader should note that the above arguments are valid also for $\mu < r$. However, the Cases I, II and III will be slightly different and the optimal solution must be changed accordingly. Also observe that by letting $\lambda = 0$, $\beta = 0$ and $m = 1$ we get back the classical Merton solution with $\sigma \equiv \sigma(t)$ as the constant volatility (t is the starting time).

Remark. Note that π^* is exactly like in traditional Merton, except that we know react on changes in σ, i.e., $\pi^* = \pi^*(\sigma_t)$, where σ_t is the underlying volatility. It is still optimal to choose a fraction inversely proportional to volatility, however, now it varies with the changing volatility rather than being fixed, as is the assumption in the classical Merton case. In some sense this is how investors following the Merton optimal strategy behave. At every instant of rebalancing of the portfolio, they will calculate the current volatility and invest according to the Merton optimal strategy. But then they have in effect invested according to a strategy with changing volatility, and not according to a strategy where the level of volatility is fixed from the start.

In Case I, it is possible to calculate the value function explicitly. This is the content of the following proposition.

Proposition 5.9. *Assume Condition A holds with* $c_j = 8\gamma\left(|\frac{1}{2}+\beta| + 4\gamma\right)\frac{\omega_j}{\lambda_j}$ *for* $j = 1, \dots, m$. *Then the solution* $h \in C^{1,1}\left([0,T) \times [0,\infty)^m\right)$ *of the first-order integro-differential equation*

$$h_t(t,y) + \gamma h(t,y) - \sum_{j=1}^m \lambda_j y_j h_{y_j}(t,y) + \sum_{j=1}^m \lambda_j \int_0^\infty \Big(h(t, y + z\cdot e_j) - h(t,y)\Big)\, \ell_j(dz) = 0,$$

with terminal condition $h(T,y) = 1$ $\forall y \in [0,\infty)^m$, *is*

$$h(t,y) = \exp\Bigg(\gamma\mu(T-t) + \gamma\left(\tfrac{\gamma}{2}+\beta\right)\sum_{j=1}^m \frac{\omega_j}{\lambda_j} y_j\left(1 - e^{-\lambda_j(T-t)}\right) + \lambda_j \int_t^T \int_{0+}^\infty \left(e^{\gamma\left(\frac{\gamma}{2}+\beta\right)\frac{\omega_j}{\lambda_j}\left(1-e^{-\lambda_j s}\right)z} - 1\right)\ell_j(dz)\, ds\Bigg). \tag{5.11}$$

Furthermore, the value function (2.2) *is explicitly given by*

$$V(t,w,y) = \gamma^{-1} w^\gamma h(t,y), \tag{5.12}$$

where h *is given in* (5.11).

Proof. Let us calculate the function $g(t,y)$ defined in (5.6):

$$\begin{aligned} g(t,y) &= \mathbb{E}\left[e^{\int_0^t \gamma\Pi(\sigma(s))\,ds}\right] = e^{\gamma\mu t}\mathbb{E}\left[e^{\gamma\left(\frac{\gamma}{2}+\beta\right)\int_0^t \sigma(s)\,ds}\right] \\ &= e^{\gamma\mu t + \gamma\left(\frac{\gamma}{2}+\beta\right)\sum_{j=1}^m \omega_j y_j \int_0^t e^{-\lambda_j s}\,ds}\mathbb{E}\left[e^{\gamma\left(\frac{\gamma}{2}+\beta\right)\sum_{j=1}^m \int_0^t e^{-\lambda_j s}\int_0^s e^{\lambda_j u}\,dZ_j(\lambda_j u)\,ds}\right] \\ &= e^{\gamma\mu t + \gamma\left(\frac{\gamma}{2}+\beta\right)\sum_{j=1}^m \frac{\omega_j}{\lambda_j} y_j\left(1-e^{-\lambda_j t}\right)} \prod_{j=1}^m \mathbb{E}\left[e^{\gamma\left(\frac{1}{2}+\beta\right)\omega_j \int_0^t\int_0^s e^{-\lambda_j(s-u)}\,dZ_j(\lambda_j u)\,ds}\right]. \end{aligned}$$

The Fubini theorem yields

$$\int_0^t \int_0^s e^{-\lambda_j(s-u)}\,dZ_j(\lambda_j u)\,ds = \frac{1}{\lambda_j}\int_0^t \left(1 - e^{-\lambda_j(t-s)}\right) dZ_j(\lambda_j s),$$

so that

$$\begin{aligned} g(t,y) &= e^{\gamma\mu t + \gamma\left(\frac{\gamma}{2}+\beta\right)\sum_{j=1}^m \frac{\omega_j}{\lambda_j} y_j\left(1-e^{-\lambda_j t}\right)} \prod_{j=1}^m \mathbb{E}\left[e^{\gamma\left(\frac{\gamma}{2}+\beta\right)\frac{\omega_j}{\lambda_j}\int_0^t\left(1-e^{-\lambda_j(t-s)}\right)dZ_j(\lambda_j s)}\right] \\ &= \exp\Bigg(\gamma\mu t + \gamma\left(\tfrac{\gamma}{2}+\beta\right)\sum_{j=1}^m \frac{\omega_j}{\lambda_j} y_j\left(1 - e^{-\lambda_j t}\right) \\ &\qquad + \lambda_j \int_0^t \int_{0+}^\infty \left(e^{\gamma\left(\frac{\gamma}{2}+\beta\right)\frac{\omega_j}{\lambda_j}\left(1-e^{-\lambda_j(t-s)}\right)z} - 1\right)\ell_j(dz)\,ds\Bigg) \end{aligned}$$

$$= \exp\Bigg(\gamma\mu t + \gamma\left(\tfrac{\gamma}{2}+\beta\right)\sum_{j=1}^{m}\frac{\omega_j}{\lambda_j}y_j\left(1-\mathrm{e}^{-\lambda_j t}\right) \\ + \lambda_j\int_0^t\int_{0+}^{\infty}\left(\mathrm{e}^{\gamma\left(\frac{\gamma}{2}+\beta\right)\frac{\omega_j}{\lambda_j}\left(1-\mathrm{e}^{-\lambda_j s}\right)z}-1\right)\ell_j(dz)\,ds\Bigg).$$

Hence, by recalling that $h(t,y) = g(T-t,y)$ and after a change of variables in the ds-integration, we recover (5.11). □

Remark. In Case I, we could have first calculated the function (5.12), and then afterwards verified that this coincides with the value function of our control problem. This approach would have given us weaker conditions on exponential integrability of the Lévy measures. In Cases II and III, explicit results seem to be impossible to obtain due to the much more complicated structure of $\Pi(\sigma)$. Indeed, the dependency on the level and inverse of $\sigma(s)$ makes it a difficult task to calculate the expectation.

6. Discussion

Almost all of our results are based on sufficient exponential integrability of the Lévy measures. The question arises, how are these conditions related to the models that we would like to use for stochastic volatility dynamics? To be able to discuss this question, we need to give a brief description of the modelling approaches suggested by Barndorff-Nielsen and Shephard [3]. Two ways of finding a stochastic volatility model is suggested. In their first (and main) approach, they start out with the stationary distribution (on the positive axis) of the volatility and from this derive the Lévy processes Z_j, where Z_j is coined the *background driving Lévy process.* To achieve this, the stationary distribution must be chosen among the so-called *self-decomposable* distributions. Some examples of such distributions are given in [3]. In the class of generalized inverse Gaussian distributions, the authors calculate the upper tail integral of the Lévy densities (i.e., the density of the Lévy measure) for the background driving Lévy process in several cases, thereby in effect giving the tail behaviour of the Lévy measure. Formulas for the inverse Gaussian, positive hyperbolic, reciprocal gamma and gamma distributions are given, all showing an exponential damping in the tails. However, the rate of damping is given by (some combination) of the parameters in the respective distributions which lead to restrictions on the choices of risk aversion $1-\gamma$ and skewness β when applied in our control problem. The choice of stationary distribution for the volatility is based on empirical decisions. For example, Barndorff-Nielsen and Shephard [3] note that if we choose $\sigma_j(t)$ to have an inverse Gaussian distribution, then the logreturns will be approximately normal inverse Gaussian distributed, a distribution which models semi-heavy tails observed in market data very well (see Barndorff-Nielsen [2] for more on this class of distributions and, e.g., Eberlein and Keller [13], Rydberg [28] and Bølviken and Benth [11] for applications to empirical finance). Their statistical studies of Deutsche Mark-Dollar exchange rate (data on 5 minutes periods over 10 years) suggest to use inverse Gaussian marginals for the volatility. In addition, one has to use a superposition of Ornstein-Uhlenbeck processes to correctly model the dependency in the logreturns. The analysis shows a good fit for the autocorrelation structure when $m = 4$, that is, a superposition of four non-Gaussian Ornstein-Uhlenbeck processes.

A second way of introducing stochastic volatility dynamics through non-Gaussian Ornstein-Uhlenbeck models is to directly model the Lévy process, as also suggested by the authors. Even more, one may model the Lévy process by directly specifying the Lévy measure. This approach may give more room for the parameters in the control problem. However, in both the suggested approaches, we see there will be a competition between the parameters of the control problem and the parameters in the model of the risky asset. But these constraints need not be too binding, as can be seen from the examples below. We would finally like to remark that our conditions on the exponential integrability of the Lévy measures may be weakened by going through the estimates more carefully. This task will not be pursued in any further detail here.

7. Numerical examples

In this section, we present some numerical examples. We focus on the variability of the volatility and how an investor (with complete knowledge of the present level of volatility) should optimally diversify her portfolio. Our purpose is to demonstrate some effects incurred by a varying volatility. In order to simplify matters, we choose to model the stochastic volatility by *one* non-Gaussian Ornstein-Uhlenbeck process, i.e., we set $m = 1$, and thus have

$$\sigma(s) = \mathrm{e}^{-\lambda s}\sigma(0) + \mathrm{e}^{-\lambda s}\int_0^s \mathrm{e}^{\lambda u}\, dZ(\lambda u),$$

when we start the process at $t = 0$ (which we indeed shall do in the examples below). Furthermore, since we want to simulate numerically the volatility it is convenient to choose the stationary probability distribution for $\sigma(t)$ to be in the class of Gamma distribution, i.e., $\sigma(t) \sim \Gamma(\nu, \alpha)$. The Gamma distribution is a member of the family of generalized inverse Gaussian distributions and has a density

$$\frac{\alpha^\nu}{\Gamma(\nu)} x^{\nu-1}\exp(-\alpha x), \quad x > 0.$$

As is calculated in [3], the background driving Lévy process Z will then have a Lévy density

$$\ell(dz) = \nu\alpha\exp(-\alpha z)\, dz.$$

As will be explained below, this density makes it particularly easy to simulate $\sigma(t)$. In the examples below we choose $\alpha = 10$, $\lambda = 0.01$ and $\nu = 3$.

We describe the procedure which we use to simulate paths of $\sigma(s)$. The suggested algorithm is introduced by Marcus [22] and Rosinski [27], and explained in our context of stochastic volatility in Barndorff-Nielsen and Shephard [3]. We adopt here the notation in [3]. Assume we discretize the time line $[0, T]$ by homogeneous time intervals of length $\Delta > 0$. Then a straightforward calculation shows

$$\sigma(s + \Delta) = \mathrm{e}^{-\lambda\Delta}\sigma(s) + \mathrm{e}^{-\lambda\Delta} z_s,$$

where

$$z_s = \mathrm{e}^{-\lambda s}\int_{\lambda s}^{\lambda s + \lambda\Delta} \mathrm{e}^u\, dZ(u)$$

and s is a time point in our discretization of $[0, T]$. Note that z_s is independent of z_t when $t \neq s$. By a change of variables, we find that

$$z_s = \int_0^{\lambda\Delta} e^u\, dZ(u),$$

where equality is in distribution. The integral z_s can be represented as an infinite series which are suitable for simulation: Let $\{r_i\}_i$ be independent samples from a uniform probability distribution on $[0, 1]$ and $a_1 < a_2 < \cdots < a_i < \ldots$ be the arrival times of a Poisson process with intensity 1. Then (in distribution)

$$z_s = \sum_{i=1}^\infty W^{-1}(a_i)e^{\lambda r_i \Delta}.$$

In the above expansion, the function $W^{-1}(x)$ appears, which is the inverse of $W^+(x)$[1], where $W^+(x)$ is the upper tail integral of the Lévy density of Z. For the Gamma distribution this is explicitly invertible:

$$W^{-1}(x) = \max\left(0, -\frac{1}{\alpha}\ln\left(\frac{x}{\nu}\right)\right).$$

Introducing this function in the series expansion, Barndorff-Nielsen and Shephard [3] suggest to simulate z_s from the representation (in law)

$$z_s = \frac{1}{\alpha}\sum_{i=1}^{N(\nu)} \ln\left(\frac{\nu}{a_i}\right)\mathrm{e}^{\lambda\Delta r_i},$$

[1]Note that we follow the notation of Barndorff-Nielsen and Shephard [3] here. The function W^{-1} is not to be confused with the (inverse of the) wealth dynamics in the present paper.

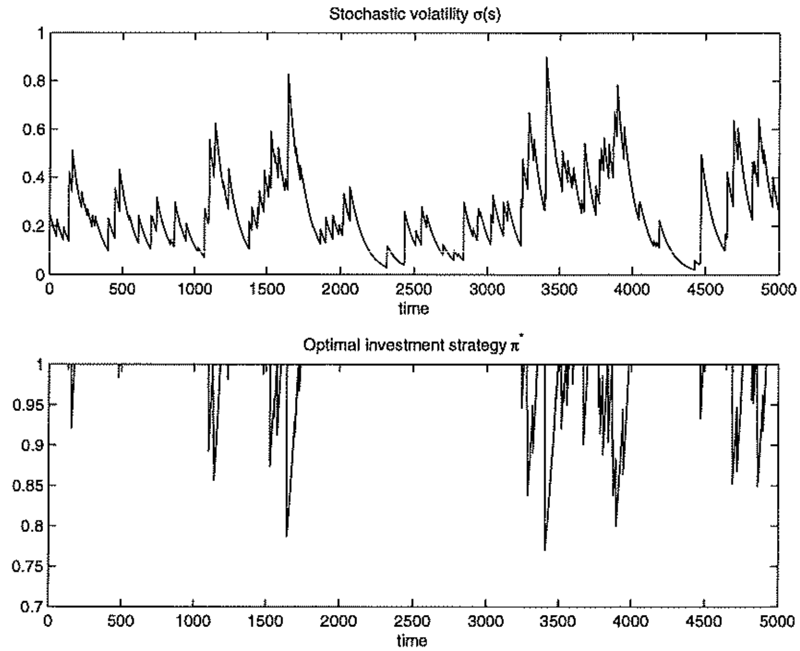

FIGURE 1. The stochastic volatility and optimal investment strategy in Example 1.

where $N(\nu)$ is the number of arrivals a_i before time ν. The inversion of the upper tail integral $W^+(x)$ is in general not analytically possible, thus leading to more complicated simulation algorithms. The Gamma-distribution is the only case where we can invert $W^+(x)$ among the examples of distributions suggested by Barndorff-Nielsen and Shephard [3]. This is the reason why we use this distribution for our numerical investigations. In all the simulations below, $\Delta = 1$.

We present two examples: one where the logreturns are symmetric and one with skew logreturns. The purpose with the numerical examples is to show how optimal allocation taking stochastic volatility into account may dramatically deviate from the classical Merton investment strategy. We choose as $\sigma(0)$ the historical volatility, i.e., the volatility a Merton investor would choose to pin down her strategy at time 0 and follow until time T. We choose $\sigma(0) = 0.25$. The investment horizon is assumed to be $T = 5000$. The optimal strategy in the stochastic volatility case is found by simulating one path of the volatility using the method described above, and then calculating the optimal strategy using the rules in Theorem 5.8. Note that the parameters we choose in the examples below satisfy the integrability condition in Theorem 5.8. The numerical algorithms were implemented in MATLAB, and simulation of one path of 5000 points took about 4-5 sec.

Example 1. Let $\beta = 0$, the risk aversion $1 - \gamma = \frac{7}{8}$ (i.e., the investor is very risk averse) and $\mu - r = \frac{5}{32}$. An investor following the classical Merton strategy, would choose $\pi_M = 1$, i.e., invest her whole wealth in the risky asset. Since $\frac{1}{2} + \beta/1 - \gamma = \frac{4}{7} \in (0, 1)$, we are in Case II where π^* is not necessarily equal to one. Figure 1 shows one possible scenario, where we observe that in fact the optimal strategy may be to invest below 100% in the risky asset. Indeed, we see that in periods of times we should go down to about 80%, significantly more conservative than putting all the money in the risky asset. However, this happens in very short periods, compared to the long periods where the investor is advised to place 100% of her total wealth in the risky asset.

Example 2. Again we choose parameters such that the classical Merton investor puts all her money in the risky asset. Let $\beta = -\frac{2}{3}$ and $1 - \gamma = \frac{1}{2}$, i.e., the logreturns are skew to the left and the investor is moderately risk averse. Furthermore, we assume $\mu - r = \frac{1}{10}$. The Merton investor choose $\pi_M = 1$. Since $\frac{1}{2} + \beta = -\frac{1}{6} < 0$, we are in Case III, where the optimal strategy may be to choose $\pi^* = 0$ in periods of very high volatility. And indeed this may happen, as is seen in Figure

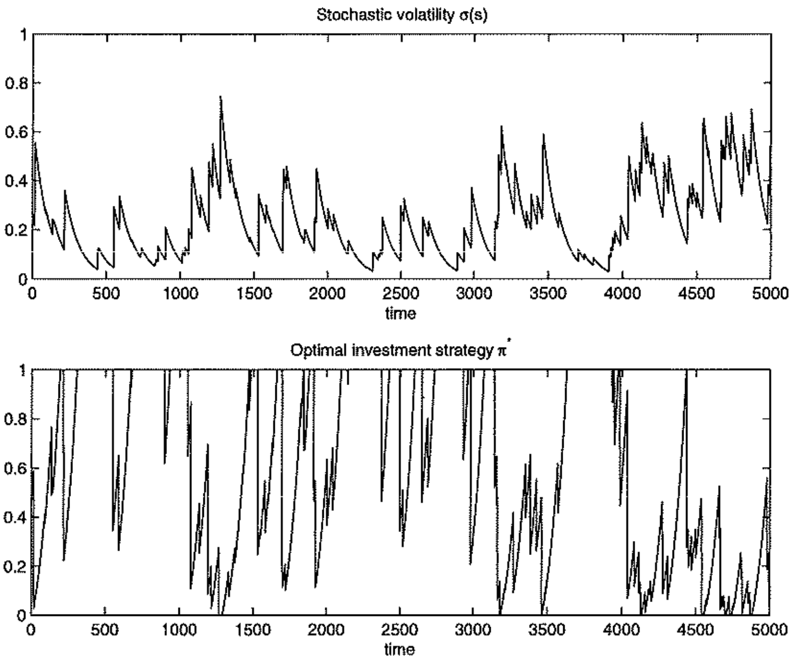

FIGURE 2. The stochastic volatility and optimal investment strategy in Example 2.

2. Note also that periods where the investor should allocate less than 100% in the risky assets are dominating. We conclude form these two examples that varying volatility both with and without skew logreturns may lead to significantly different optimal investment behaviour. Remark that we have used different seeds in the simulation of the stochastic volatility in the two examples, thus the paths are different. The average of $\sigma(s)$ over the paths are about 0.25 in both examples.

Remark. Note that the Gamma-model that we have chosen in the above two numerical examples is not necessarily market relevant. The parameters are not estimated from empirically observed prices, but admittedly chosen to highlight some deviations from the classical Merton case. We do believe, however, that similar observations can be made when operating with stochastic volatility models which are statistically fitted to observed price data. Since the procedure of estimating the parameters in the stochastic volatility model is rather involved, we leave such considerations to future research.

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