Morphologically Invariant PDE Inpaintings

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Keywords

Inpainting, interpolation, PDE, transportation, diffusion, Bayesian and variational method, morphological invariance.

Abstract

This paper studies the PDE method for image inpaintings. Image inpainting is essentially an image interpolation problem, with wide applications in film and photo restoration, text removal, special effects in movies, disocclusion, digital zoom-in, and edge-based image compression and coding. Bertalmio, Sapiro, Caselles, and Ballester (2000) [3] first innovatively introduced the PDE method for the inpainting problem. Ever since, the authors of the present paper have worked along this line and developed the PDE method, mostly inspired by the Bayesian and variational method (especially by good image prior models). The current paper has two major goals. First, by surveying all the recent PDE inpainting techniques, we intend to develop a unified viewpoint based on two infinitesimal mechanisms: transportation and curvature driven diffusions (CDD). Furthermore, based this knowledge, we construct a new class of third order inpainting PDEs, which is derived from the set of axioms (or principles) refined from the existing works: morphological invariance, rotational invariance, stability principle, and linearity principle.

I. Introduction

The word "inpainting" is used among restoration artists which refers to the practice of manually filling or retouching the missing domains of an ancient or degraded painting in an undetectable manner. The missing domains are typically cracks or scratches caused by aging, unfavorable weather conditions, or accidental damages.

The term of "digital inpainting" was first introduced into image processing by Bertalmio, Sapiro, Caselles, and Ballester [3], who innovatively invented a 3rd order PDE model for digital inpainting. Equally important in [3] are the broad applications the authors demonstrated, which include film restoration, recovery of damaged photos, text removal, special effects in movies or pictures. Recently the same group of authors have also developed a new variational inpainting model based on a joint cost functional on the gradient vector field and grey values [2]. An earlier variational inpainting model was also proposed by Masnou and Morel [17] in the context of disocclusion in computer vision.

Image inpainting is essentially an image restoration problem. Researchers working on different applications have adopted different names: image interpolation [14], disocclusion [17], image replacement [11], and error concealment [12], [15], though each of them does carry its own individual characteristics. The universal approach to image restoration problems is the framework of Bayesian inference [10], or, in terms of the logarithm-likelihood functions or energy functions, the framework of variational method [20], [21], [25]. As in the classical PDE work of denoising and edge enhancement, variational models often lead to the right type of (Euler-Lagrange) partial differential equations for the task at hand. This viewpoint has motivated the recent works of Chan and Shen [9], [8], and Chan, Kang and Shen [6] on PDE inpainting models that are derived from or inspired by suitable Bayesian or variational formulations.

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One major advantage of the PDE approach for image inpainting is that the interpolation is done automatically by suitable numerical PDE schemes. It frees one from laboring on edge detection, T-junction detection and connection, or object segmentation. Moreover, it imposes no topological constraints on the shape of inpainting domains.

The current paper summarizes all the recent works on PDE based image inpaintings, and discuss their advantages and shortcomings, so that further work along this line can be carried out by the community. We shall explain that infinitesimally, all the existing works (Section II-V) on PDE based inpaintings are realized by two universal mechanisms: transportation and diffusion. Transportation allows gray scale information to propagate into the missing (inpainting) domain, while diffusion stabilizes the propagation and regularizes the geometry of the isophotes. In combination, they lead to efficient PDE inpainting models. This intuition initially motivated the inpainting algorithm of Bertalmio et al. [3] (also Section II). It gets more mathematically clear in the recent work of variational inpaintings based on Euler's elastica energy [6], whose Euler-Lagrange equation explicitly combines the two mechanisms (Section V).

In this paper, based on the knowledge from the existing works, we shall also derive a new class of third order inpainting PDEs that combines the two mechanisms (Section VI). Unlike all the previous inpainting PDEs which are either based on intuitions or Bayes and variational formulations, this new class is founded on a set of axioms, among which is the *morphological invariance*. (The axiomatic approach to image processing has been extensively studied by applied mathematicians recently [1], [5], who tried to put image processing on a firm mathematical foundation, as Euclidean did for classical geometry.) Morphological invariance means that the way you connect a broken isophote should not depend on its gray value. Or vividly speaking, given a damaged ancient painting with missing domains to be inpainted, the way a restoration artist retouches the paint should not depend on whether he stays inside a dim room or sits in the sun (since the pigments that the artist uses undergo the same luminance change). Morphological invariance is thus a very natural principle in image processing [1]. Its mathematical meaning can be put down rigorously. Let u_0 be the given observed data (for inpainting, u_0 is simply a portion of a complete image). An image processing tool (or *operator*) $T : u = T(u_0)$ is said to be morphologically invariant, if for any strictly increasing function (or *morphological transform*) $g : [0, 1] \to [0, 1]$, one always has $T(g(u_0)) = g(T(u_0))$.

Finally, to be complete, we shall also point out that the PDE approach is aimed at inpainting *non-texture* images, since the explicit application of PDE's implicitly assumes that the underlying images satisfy certain regularity conditions. The problem of texture inpainting is closely connected to texture modeling and synthesis. Recent works can be found in [11], [26].

II. The First PDE inpainting model: smoothness transportation

The first PDE based inpainting model of Bertalmio, Sapiro, Caselles and Ballester [3] is based on the beautiful intuition of *smoothness transportation* along isophotes. Imagine to inpaint a broken smooth step edge, one naturally requests the intensity jump to propagate along the edge, so that a sharp edge can be restored (Fig. 1). Generally, Let $L(u)$ be a smoothness measure of an image u. For example, a second order smoothness measure can be expressed in the general form of

$$L(u) = f(\nabla u,\ \nabla \otimes \nabla u),$$

where ∇u is the gradient vector, and $\nabla \otimes \nabla u$ the Hessian matrix (2 by 2). The one experimented in [3] is the Laplacian:

$$L = \Delta u = \mathrm{trace}(\nabla \otimes \nabla u).$$

The Bertalmio-Sapiro-Caselles-Ballester inpainting model is then defined by a third order evolutionary equation:

$$\frac{\partial u}{\partial t} = \nabla^{\perp} u \cdot \nabla L(u), \tag{1}$$

where, $\nabla^{\perp} u = (-u_y, u_x) = |\nabla u| \, \vec{t}$ is the 90-degree-rotated normal vector (and thus points to the tangent $\vec{t}$). The model carries the transportation (or propagation) nature since as the evolution approaches its equilibrium state, we have (as long as $|\nabla u| \neq 0$)

$$\vec{t} \cdot \nabla L(u) = 0 \text{ or equivalently } \frac{\partial L(u)}{\partial \vec{t}} = 0, \tag{2}$$

which means, along an isophote, the smoothness measure is conserved. Thus in terms of the available boundary data, the inpainting process evolves like transporting the boundary smoothness information along the extended isophotes into the inpainting domain.

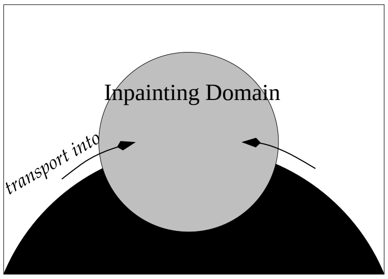

Fig. 1. The transportation model of Bertalmio, Sapiro, Caselles, and Ballester.

However, due to the lack of communications among the isophotes, the transportation may result in kinks or contradictions inside inpainting domains, just as shocks may develop in traffic models. Thus in [3], Eq. (1) is implemented with the help of intermediate steps of anisotropic diffusions. As we shall see below, such practice is well backed up by the elastica inpainting model.

The second issue with model (1) is the smoothness measure L. The choice of the Laplacian is convenient but less ideal in two aspects:

(a) The equilibrium equation $\partial(\Delta u)/\partial \vec{t} = 0$ is not morphologocially invariant. Let

$$g(\lambda) : [0, 1] \to [0, 1]$$

be a smooth morphological transform so that $g'(\lambda) > 0$, then

$$\Delta g(u) = g'(u)\Delta u + g''(u)(\nabla u)^2. \tag{3}$$

Thus

$$\frac{\partial(\Delta g(u))}{\partial \vec{t}} = g'(u)\frac{\partial(\Delta u)}{\partial \vec{t}} + g''(u)\frac{\partial(\nabla u)^2}{\partial \vec{t}}.$$

(Notice that u is a constant along $\vec{t}$.) Hence, generally, if u is the final equilibrium inpainting to a given image $u_0\big|_{\Omega\setminus D}$, then $g(u)$ is not the equilibrium inpainting to $g(u_0\big|_{\Omega\setminus D})$ due to the second term (unless that $g(\lambda)$ is a *linear* scaling: $g(\lambda) = a + b\lambda$).

(b) For the equilibrium inpainting u, according to (2), the smoothness measure $L(u)$ must be a constant along the isophotes. Therefore, if p and q are two pixels along the inpainting boundary and belong to the same isophote, but with different L values computed from the available data $u_0\big|_{\Omega\setminus D}$, then theoretically there shall be no equilibrium inpainting. Such situation often occurs in large-scale inpainting problems, which is not caused by noise, but instead, by the natural variations of L itself along an isophote. Thus asking L to be a constant along the isophotes, like the gray value u itself, is perhaps demanding too much.

The new model we derive in Section VI shall remedy these drawbacks.

III. The TV inpainting model: anisotropic diffusion

In [9], Chan and Shen studied a Bayesian or variational inpainting model which is based on the Total Variation (TV) prior image model.

In the literature of image restoration, the Bayesian or variational framework is the most general framework or principle, explicitly expressed or implicitly hidden in any successful classical model, such as the Geman-Geman's model [10], the Mumford-Shah segmentation model [21], and Rudin-Osher-Fatemi's TV denoising and deblurring model [25]. the viewpoint inspired Chan and Shen's TV inpainting model.

Let D be the inpainting domain, where the image information is missing. Typically the image available outside $u_0\big|_{\Omega\setminus D}$ is noisy. The TV inpainting model proposed in [9] is to minimize

$$J[u] = \int_\Omega |\nabla u| \, dx + \frac{\lambda}{2} \int_{\Omega\setminus D} (u - u_0)^2 dx, \tag{4}$$

where dx is the 2-D area element, and λ a fitting constant or Lagrange multiplier, which is inversely proportional to the variation of the noise [7]. Also the formulation approximates the noise by Gaussian. The mathematical issue of uniqueness and existence is recently answered in Chan, Kang and Shen [6].

The Euler-Lagrange equation of the TV inpainting energy is given by

$$\frac{\partial u}{\partial t} = \nabla \cdot \left[\frac{\nabla u}{|\nabla u|}\right] + \lambda_e (u - u^0), \tag{5}$$

valid on the entire image domain Ω. The extended Lagrange multiplier $\lambda_e = \lambda(1 - \chi_D)$, where χ_D is the characteristic function (or mask) of the inpainting domain D. Therefore, inside the inpainting domain, the model employs a simple anisotropic diffusion process:

$$\frac{\partial u}{\partial t} = \nabla \cdot \left[\frac{\nabla u}{|\nabla u|}\right], \tag{6}$$

which has been studied extensively ([25], [24], [18]). The application of anisotropic diffusions in image denoising and enhancement now has become a classical topic since Perona and Malik [23] (also see [27], [18]).

From Eq. (5), in the absence of noise (i.e., $\lambda_e = \infty$ outside the inpainting domain), the equilibrium inpainting of the TV model is indeed morphologically invariant since the right hand side of Eq. (6) is exactly the curvature of the isophotes and is independent of the relative gray values. On the other hand, if one requires the entire time evolution (6) to be morphologically invariant, then the factor $|\nabla u|$ should be added to balance the time derivative:

$$\frac{\partial u}{\partial t} = |\nabla u| \, \nabla \cdot \left[\frac{\nabla u}{|\nabla u|}\right],$$

which is exactly the *mean curvature motion* [18], and is also very useful for the speeding-up of numerical convergence, as studied by Marquina and Osher [16] recently.

The TV inpainting model also offers alternative explanations to some aspects of the human disocclusion process in vision psychology [9], including the *entanglement illusions* gathered and analyzed by Kanizsa ([9], [13], [22]).

Fig. 2 shows one example of TV inpainting for text removal. The main advantage of inpainting algorithms based on numerical PDE's, as Bertalmio et al. [3] pointed out, is the permission of a wide range of domain topology. It becomes unnecessary to first couple the pixels along the boundary,

Fig. 2. An example of TV inpainting for text removal.

and then connect the broken isophotes individually, and also make sure that the interpolants indeed stay inside the domain.

Due to its Bayesian foundation, the TV inpainting model, after being properly digitized, also finds its successful applications in digital zoom-in and edge-based image coding schemes. More details can be found in Chan and Shen [9]. Fig. 3 shows how the TV model can reconstruct (lossily) the whole image based only on the intensity values on a narrow tube (1 or 2-pixel wide) surrounding the detected edges.

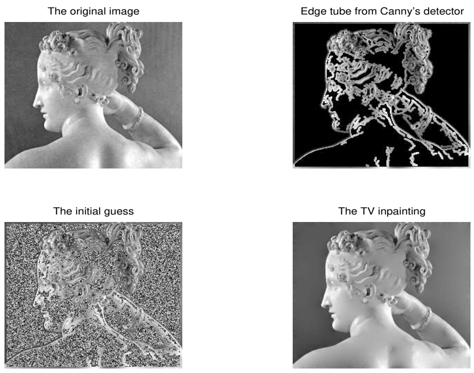

Fig. 3. An example of TV inpainting for edge decoding. (Image source: test image of the Computational Vision Lab at California Institute of Technology.)

Another advantage the TV inpainting equation (5), compared to all the other PDE models, is that it is only of second order, and the numerical scheme is simple and converges much faster. Therefore, in applications, TV inpainting can cheaply provides a valuable initial guess, if the output is less ideal to human observers.

There are also two major drawbacks. The first one is that the TV model is only a linear interpolant, i.e., the broken isophotes are interpolated by straight lines. Thus it can generate corners along the inpainting boundary. The second one is that TV often fails to make the connection of *widely* separated parts of a whole object, due to the high cost of long-distance communication [9], [6] (See Fig. 4).

IV. The CDD inpainting model: curvature driven diffusion

The failure of the TV inpainting on the *Connectivity Principle* [9] due to the high cost on long-distance connection inspired the CDD (curvature driven diffusion) inpainting model of Chan and

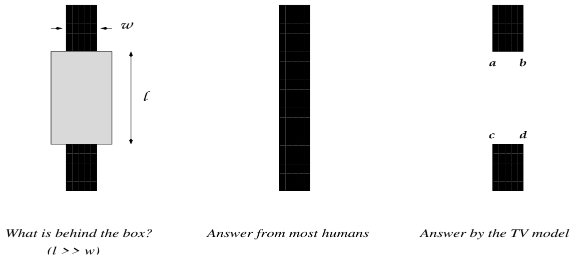

Fig. 4. TV fails to realize the *Connectivity Principle* in inpaintings with large scales (or aspect ratios).

Shen [8].

The CDD inpainting model is a further refinement of the TV anisotropic diffusion (5). To encourage long-distance connections, the CDD employs the curvature information for the diffusion. It is based on the simple observation (such as from Fig. 4) that when the TV gets lazy in connection, the edge isophotes typically contain corners (a, b, c, d in Fig. 4) which has large curvatures. Thus, from the optimistic point of view, large curvatures can be incorporated into the diffusion process to "push" out the false edges (ab and cd in Fig. 4) formed in the TV diffusion:

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(\frac{g(\kappa)}{|\nabla u|} \nabla u \right), \qquad \kappa = \nabla \cdot \left[\frac{\nabla u}{|\nabla u|} \right], \tag{7}$$

where $g : R \to [0, +\infty)$ is a continuous function satisfying $g(0) = 0$ and $g(\pm\infty) = +\infty$. The introduction of $g(\kappa)$ is to penalize large curvatures and encourage small ones (or flatter and smoother isophotes), since $D = g(\kappa)/|\nabla u|$ denotes the diffusion strength. A simple example would be $g(s) = |s|^p$ for some positive power p. Fig. 5 shows one example of CDD inpainting, where even very weak edges are connected successfully (like the shadow of the nose).

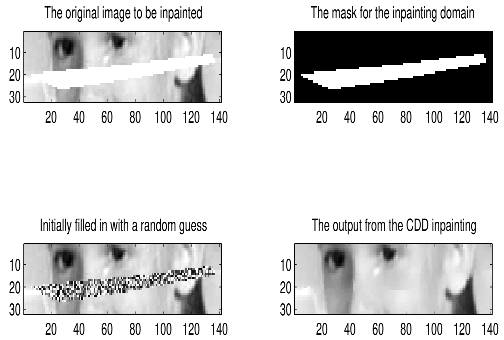

Fig. 5. An example of CDD inpainting for scratch removal.

CDD inpainting is a third order PDE model, and is indeed morphologically invariant since both the curvature κ and the normal vector $\vec{n}$ are. The model encourages long-distance connections. But one drawback of the TV inpainting model still stays. That is, the isophotes are still approximated by straight lines.

It is this drawback that has eventually driven Chan, Kang, and Shen [6] to the re-investigation of the earlier proposal of Masnou and Morel [17] on inpaintings based on Euler's elastica energy.

V. Euler's elastica inpainting: transportation and CDD combined

Euler's elasticas were first studied by Euler in 1744 for modeling the steady shape of a thin and torsion free rod. It was Mumford who first introduced this class of smooth curves (or *nonlinear* *splines* [4]) as a prior curve model in computer vision [19]. Later, Masnou and Morel applied the elastica curves to interpolate broken isophotes. A dynamic programming algorithm was created to first couple the boundary pixels and then connect each pair of pixels with an elastica curve. In search of the most suitable image prior model for inpaintings, Chan, Kang and Shen [6] recently re-investigated this model from the functional and PDE point of view. The most inspiring result is that the associated Euler-Lagrange equation perfectly unifies all the previous works on PDE inpaintings.

A general curvature based variational inpainting energy, which includes Euler's elastica energy as a special case, is given by

$$J[u] = \int_D \phi(\kappa)|\nabla u|\, dx, \qquad \kappa = \nabla \cdot \left[\frac{\nabla u}{|\nabla u|}\right],$$

where $\phi(s)$ can be any non-decreasing smooth function of $|s|$, and $\phi(\pm\infty) = +\infty$. For example, for Euler's elastica,

$$\phi(s) = a + bs^2, \qquad \text{for two positive weights} \quad a, b.$$

The Euler-Lagrange equation is shown to be [6]:

$$\frac{\partial u}{\partial t} = \nabla \cdot \vec{V}, \quad \vec{V} = \phi(\kappa)\vec{n} - \frac{\vec{t}}{|\nabla u|}\frac{\partial(\phi'(\kappa)|\nabla u|)}{\partial \vec{t}},$$

where $\vec{n} = \nabla u / |\nabla u|$ is the gradient, and $\vec{t}$ the tangent. The vector field $\vec{V}$ is called the *flux field.* We can easily show that $\vec{V}$ is morphologically invariant: first, all the three quantities $\kappa, \vec{n}$, and $\vec{t}$ are morphologically invariant; and then, the second term involving $|\nabla u|$ is also morphologically invariant since under any morphological transform $g : [0,1] \to [0,1]$ with $g' > 0$, we have

$$\frac{1}{|\nabla g(u)|}\frac{\partial(\phi'(\kappa)|\nabla g(u)|)}{\partial \vec{t}} = \frac{1}{g'(u)|\nabla u|}\frac{\partial(g'(u)\phi'(\kappa)|\nabla u|)}{\partial \vec{t}} = \frac{g'(u)}{g'(u)|\nabla u|}\frac{\partial(\phi'(\kappa)|\nabla u|)}{\partial \vec{t}}.$$

(Notice that along the tangent direction u is a constant, and so is $g'(u)$.)

We now explain that this Euler-Lagrange equation offers a unified view on the transportation inpainting of Bertalmio, Sapiro, Caselles, and Ballester [3] and the CDD inpainting of Chan and Shen [8] discussed in earlier sections. The flux field $\vec{V}$ has two components: the normal component

$$\vec{V}_n = \phi(\kappa)\, \vec{n},$$

and the tangential part

$$\vec{V}_t = -\frac{1}{|\nabla u|}\frac{\partial(\phi'(\kappa)|\nabla u|)}{\partial \vec{t}}\, \vec{t}.$$

Therefore, the normal flux $\vec{V}_n$ exactly corresponds to Chan and Shen's CDD program (7) with

$$g(\kappa) = \phi(\kappa).$$

On the other hand, the tangential component can be written as

$$\vec{V}_t = -\left(\frac{1}{|\nabla u|^2}\frac{\partial(\phi'(\kappa)|\nabla u|)}{\partial \vec{t}}\right)\, \nabla^{\perp} u,$$

and thus its divergence is

$$\nabla \cdot \vec{V}_t = \nabla^{\perp} u \cdot \nabla \left(\frac{-1}{|\nabla u|^2}\frac{\partial(\phi'(\kappa)|\nabla u|)}{\partial \vec{t}}\right)$$

since $\nabla^{\perp} u$ is divergence free, which corresponds exactly to the transportation scheme (1) with the smoothness measure given by

$$L_\phi = \frac{-1}{|\nabla u|^2}\frac{\partial(\phi'(\kappa)|\nabla u|)}{\partial \vec{t}}.$$

Unlike the Laplacian mentioned in Section II, L_ϕ is generally a smoothness measure based on up to third order differentials.

In summary, the elastica inpainting scheme combines both the transportation mechanism of Bertalmio et al.'s model and the CDD mechanism of Chan and Shen's model. It thus provides a theoretical foundation for these two earlier works on PDE based image inpaintings. In return, the earlier works also shed lights on the meaning of the flux field $\vec{V}$ and the interpretation of the elastica or curvature based variational inpaintings.

Fig. 6 and 7 show two examples of curvature inpaintings based on Euler's elastica: $\phi(s) = a + bs^2$. The comparison on different ratios of b/a clearly shows two effects of elastica inpaintings. First, as the ratio b/a increases, the connection becomes smoother (Fig. 6). Notice that the extreme case of $b/a = 0$ corresponds to the TV inpainting. Secondly, as the ratio b/a increases, long distance connection gets cheaper according to the energy function (Fig. 7). Thus the *Connectivity Principle* is realized.

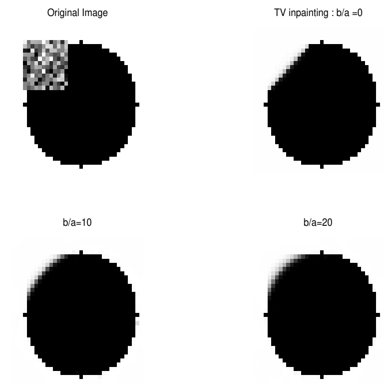

Fig. 6. Effect (I) of elastica inpaintings: a larger weight b against the curvature term produces smoother isophotes and edges, and better visual effect.

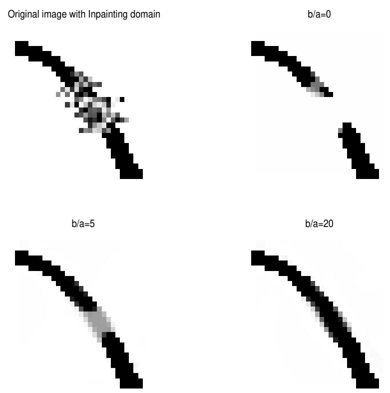

Fig. 7. Effect (II) of elastica inpaintings: a larger weight b against the curvature term favors the *Connectivity Principle*: the model encourages the long-distance connection. (The inpainting domain is the middle random shape in the upper left panel.)

The elastica inpainting is a fourth order PDE model. Thus the numerical computation is much more challenging than the previous three models and the convergence is often slow if the inpainting domain is large. These are the aspects that can be further worked out. The detailed numerical PDE scheme can be found in [6].

VI. The axiomatic approach to PDE inpaintings

Based on the experience gained from all the above works, in this section, we derive a new third order inpainting PDE from a set of principles or axioms. From the mathematical point of view, the axiomatic approach is an important step in the big blueprint for putting image processing on a firm mathematical foundation. Previous works can be found in [1], [5].

A. Geometric representation of differentials: the curvature κ and deviation rate σ

Given an image u, its Cartesian differentials up to the second order are given by

$$\nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix}, \qquad \nabla \otimes \nabla u = \begin{bmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{bmatrix}. \tag{8}$$

They are easy to compute if the image u is given in the x and y coordinates, yet less ideal from the invariant (or geometric) point of view. For example, if the observer rotates by some angle, then both ∇u and $\nabla \otimes \nabla u$ change. Another simpler reason for the less idealness is that for a given image u, the x and y directions have no specific significance as far as visual information is concerned.

However, near a regular pixel of a given image u, we do have two orthogonal directions that come naturally with the image itself: the normal $\vec{n}$ and the tangent $\vec{t}$. Let $\boldsymbol{p} = \nabla u = p\ \vec{n}$ ($p \geq 0$) and $H = \nabla \otimes \nabla u$ denote the Cartesian differentials. We make the following transform from $R^2 \backslash \{(0,0)\} \times R^{2\times 2}$ to $R^+ \times S^1 \times R^{2\times 2}$:

$$(\boldsymbol{p}, H) \to \left(p,\ \vec{n},\ \frac{1}{p}[\vec{t}, \vec{n}\,]^T H [\vec{t}, \vec{n}\,]\right) = (p,\ \vec{n},\ G). \tag{9}$$

Apparently the transform is invertible and smooth (where $p \neq 0$). The transformed differentials have nicer geometric or morphological properties:

(a) $p = |\boldsymbol{p}| = |\nabla u|$ is rotationally invariant while $\vec{n}$ is morphologically invariant. and even more importantly,

(b) The new second order differential matrix G carries much more explicit geometric information about the image. The first diagonal of G:

$$\frac{1}{p}\,\vec{t}^{\,T}\ H\ \vec{t} = \frac{1}{|\nabla u|}(\nabla \otimes \nabla u)(\vec{t}\,, \vec{t}\,) := \kappa$$

is exactly the scalar curvature of the oriented (by the gradient) isophotes. It is a geometric quantity characterizing each individual isophote, and thus is both rotationally and morphologically invariant. The off-diagonal of G:

$$\frac{1}{p}\,\vec{t}^{\,T}\ H\ \vec{n} = \frac{1}{|\nabla u|}(\nabla \otimes \nabla u)(\vec{t}\,, \vec{n}) := \sigma$$

is also a rotationally and morphologically invariant scalar, which has played almost no role in the classical scale-space or filtering theory due to the ellipticity constraint [1], [5]. However, as shown below, for inpainting, it can play an important role for the transportation mechanism. In this paper, we shall call this scalar the *deviation rate* of the image or the associated isophotes. It can easily be shown that

$$\sigma = \frac{1}{|\nabla u|}\frac{\partial |\nabla u|}{\partial \vec{t}} = \frac{\partial (\ln |\nabla u|)}{\partial \vec{t}},$$

from which the rotational and morphological invariances are immediate. (To the best knowledge of the authors, the deviation rate σ was first mentioned in the classical paper of Rudin and

Osher [24] on total variation based image denoising and deblurring. The name *deviation rate* is given here for the first time, however.) The last diagonal of G:

$$\frac{1}{p}\,\vec{n}^T\; H\;\vec{n} = \frac{1}{|\nabla u|}(\nabla\otimes\nabla u)(\vec{n}\;,\vec{n}) = \frac{1}{|\nabla u|}\Delta u - \kappa$$

is only rotationally invariant and generally not morphologically invariant due to the formula (3).

By considering images u of the general quadratic form:

$$u = \frac{1}{2}\,\boldsymbol{x}^T\;H\;\boldsymbol{x} + \boldsymbol{p}^T\;\boldsymbol{x} + c,$$

we can easily establish the following theorem, with the help of formula (3) and the one-to-one transform (or change of variables) (9).

Theorem 1 *Let $f = f(\nabla u, \nabla\otimes\nabla u)$ be a function of up to the second order differentials. Then f is morphologically invariant if and only if it can be written in the form of*

$$f = f(\vec{n},\;\kappa,\;\sigma).$$

If furthermore, f is also rotationally invariant, then

$$f = f(\kappa,\sigma).$$

In other words, f is both rotationally and morphologically invariant if and only if it is a function of the curvature κ and the deviation rate σ.

B. The axiomatic approach to a class of third order inpainting PDEs

As inspired by all the previous four inpainting models, we look for a third order inpainting PDE of the divergence form:

$$\frac{\partial u}{\partial t} = \nabla\cdot\vec{V}.$$

Therefore, the flux field $\vec{V}$ shall be of second order only:

$$\vec{V} = \vec{V}(\nabla u, \nabla\otimes\nabla u).$$

It can be naturally decomposed in the normal and tangent directions:

$$\vec{V} = f\;\vec{n} + g\;\vec{t}\;,$$

and

$$f = f(\nabla u, \nabla\otimes\nabla u), \qquad g = g(\nabla u, \nabla\otimes\nabla u).$$

Axiom 1: Morphological invariance.

This first axiom requires that the equilibrium equation $0 = \nabla\cdot\vec{V}$ is morphologically invariant. Since both $\vec{n}$ and $\vec{t}$ are already morphologically invariant, it amounts to saying that

both f and g are morphologically invariant.

Then by Theorem 1, we must have

$$f = f(\vec{n},\kappa,\sigma) \qquad \text{and} \qquad g = g(\vec{n},\kappa,\sigma). \tag{10}$$

Axiom 2: Rotational invariance.

The second axiom requires that the equilibrium equation $0 = \nabla \cdot \vec{V}$ is rotationally invariant. Since all the following scalars and operators are rotationally invariant:

$$\nabla \cdot \vec{n}, \quad \nabla \cdot \vec{t}, \quad \vec{n} \cdot \nabla, \quad \text{and} \quad \vec{t} \cdot \nabla,$$

it requires that both f and g are rotationally invariant. Therefore, by Theorem 1, we must have

$$f = f(\kappa, \sigma) \qquad \text{and} \qquad g = g(\kappa, \sigma). \tag{11}$$

The following two axioms or principles are imposed on the normal flux $f\vec{n}$ and tangential flux $g\vec{t}$ individually. (It is more inspired by all the practical approaches discussed above, rather than by the *superimposition principle*, which we do not have due to nonlinearity.)

Axiom 3: Stability principle for the pure diffusion.

As well known in the PDE theory, backward diffusion is unstable. Thus this principle asks for the stability of the pure diffusion term

$$\frac{\partial u}{\partial t} = \nabla \cdot (f\vec{n}) = \nabla \cdot \left(\frac{f}{|\nabla u|} \nabla u \right).$$

Stability requires that $f \geq 0$, or the strong stability $f \geq a > 0$.

Axiom 4: Linearity principle for the pure transportation.

For the pure transportation term

$$\frac{\partial u}{\partial t} = \nabla \cdot (g\, \vec{t}\,),$$

as learned from the drawback of Bertalmio et al.'s model, we now impose the *linear interpolation* constraint, or simply, the linearity principle. First notice that

$$\begin{aligned} \nabla \cdot (g\, \vec{t}\,) &= \nabla \cdot ((g|\nabla u|)\nabla^{\perp} u) = \nabla^{\perp} u \cdot \nabla (g|\nabla u|) \\ &= |\nabla u|\, \vec{t} \cdot \nabla (g|\nabla u|) = |\nabla u| \, \frac{\partial}{\partial\, \vec{t}} (g|\nabla u|). \end{aligned}$$

The linearity principle means that there must exist some smoothness measure L so that

$$g|\nabla u| = \frac{\partial L}{\partial\, \vec{t}}, \tag{12}$$

and thus the equilibrium solution u satisfies

$$\frac{\partial^2 L}{\partial \vec{t}^{\,2}} = 0.$$

Therefore, along any inpainted isophote, L must be linear: $L = a + bs$, where s denotes the arclength parameter of the isophote, and a and b are two constants that are determined by the L values at the two boundary pixels. As discussed in Section II, Bertalmio et al.'s pure transportation model demands a constant value for the smoothness measure along any inpainted isophote. But for a generic image function u, level lines of u (i.e. isophotes) are generally different from those of L.

Since g is a second order feature, by (12) L must only involve the first order differential ∇u, or $L = L(\nabla u)$. Furthermore, since g, $|\nabla u|$, and $\partial/\partial\vec{t}$ are all rotationally invariant, so must be L by (12), which means $L = L(|\nabla u|)$. Therefore,

$$g = \frac{1}{|\nabla u|}\frac{\partial L(|\nabla u|)}{\partial\,\vec{t}} = L'(|\nabla u|)\frac{1}{|\nabla u|}\frac{\partial |\nabla u|}{\partial\,\vec{t}} = L'(|\nabla u|)\,\sigma.$$

Together with Eq. (11), it implies that

$$L'(|\nabla u|) = a, \qquad \text{a constant.}$$

(Notice that g/σ is a function of κ and σ only, and from the transform (9), generically, $p = |\nabla u|$ is independent of κ and σ).) Therefore,

$$L(|\nabla u|) = a|\nabla u| + b, \qquad \text{and} \quad g = a\,\sigma = a\,\frac{\partial(\ln|\nabla u|)}{\partial\vec{t}}.$$

In summary, we have established the following theorem.

Theorem 2 *Under the previous four principles, a third order inpainting model in the divergence form must be given by*

$$\frac{\partial u}{\partial t} = \nabla\cdot(f(\kappa,\sigma)\,\vec{n} + a\sigma\,\vec{t}\,), \tag{13}$$

where a is a non-zero constant, $f(\kappa,\sigma)$ a positive function, and

$$\kappa = \nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right) \quad \text{and} \quad \sigma = \frac{\partial(\ln|\nabla u|)}{\partial\vec{t}},$$

are the scalar curvature and the deviation rate.

For instance, as inspired by Euler's elastica inpainting model in Section V, one can choose $f = a + b\kappa^2$ for two positive constants a and b, or even $f = \exp(c\kappa^2)$ for some positive constant c. As discussed in Section IV for the CDD (curvature driven diffusion) model, such choices penalize large curvatures and thus realize the connectivity principle.

VII. Conclusion

In this paper, we have surveyed four classes of PDE based inpainting models: the first transportation model of Bertalmio, Sapiro, Caselles, and Ballester [3], the TV (total variation) inpainting model and CDD (curvature driven diffusion) model of Chan and Shen [9], [8], and the elastica inpainting model of Masnou and Morel [17], and Chan, Kang and Shen [6]. The PDE model coming from the elastica inpainting combines both the transportation and CDD mechanisms. By extracting a set of four principles (or axioms) from the existing works, we are then able to establish a new class of third order inpainting PDEs, which are both morphologically and rotationally invariant.

All these PDE models only apply directly to the inpainting of non-texture (or low-texture) images. Texture inpainting will be closely connected to texture modeling and synthesis. Recent works can be found in Igehy and Pereira [11] and Wei and Levoy [26].

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