CRITICAL THRESHOLDS IN A CONVOLUTION MODEL FOR NONLINEAR CONSERVATION LAWS

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ABSTRACT. In this work we consider a convolution model for nonlinear conservation laws. Due to the delicate balance between the nonlinear convection and the nonlocal forcing this model allows for narrower shock layers than those in the viscous Burgers' equation and yet exhibits the conditional finite time break down as in the damped Burgers' equation. We show the critical threshold phenomena by presenting a lower threshold for the breakdown of the solutions and an upper threshold for the global existence of the smooth solution. The threshold condition depends only on the relative size of the gradient of the initial velocity and its magnitude. We show the exact blow up rate when the slope of the initial profile is below the lower threshold. We further prove the L^1 stability of the smooth shock profile provided the the slope of the initial profile is above the critical threshold.

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Key Words: Wave breakdown, critical threshold, shock profile, stability.

CONTENTS

1. INTRODUCTION

Consider the scalar equation of the form

$$u_t + uu_x = Q * u - u \tag{1.1}$$

where Q is a regular symmetric kernel, monotonically decreasing on $\mathbb{R}^+$, subject to initial data

$$u(0,x) = u_0(x), \quad u_0 \in C_b^1(\mathbb{R}), \tag{1.2}$$

We are concerned with the critical threshold phenomena supported by the balance between the nonlinear convection and the nonlocal source term in (1.1).

For the kernel Q we make the following assumption:

(H1) $Q \in C^1(\mathbb{R})$, $Q(-r) = Q(r) \geq 0$, $\int Q = 1$, $\int Q(y)|y|dy < \infty$ and $Q'(x) \leq 0$ for $x \geq 0$.

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To clarify the effect of the nonlocal term on the right of (1.1) we make a hyperbolic scaling

$$(t,x) \to (\frac{t}{\epsilon}, \frac{x}{\epsilon}), \quad \epsilon > 0,$$

which leads to

$$u_t + uu_x = \frac{1}{\epsilon}[Q_\epsilon * u - u], \tag{1.3}$$

where $Q_\epsilon := \frac{1}{\epsilon}Q(\frac{x}{\epsilon})$ converging to a delta function $\delta(x)$ as the scaled parameter ϵ tends to zero.

A typical example of the kernel Q is $\frac{1}{2}e^{-|x|}$, with this specific kernel the equation (1.3) can be written as

$$u_t + uu_x = \mathcal{F}^{-1}\left[\frac{-\epsilon\xi^2}{1+\epsilon^2\xi^2}\hat{u}(t,\xi)\right] = \epsilon\mathcal{F}^{-1}\left[\frac{1}{1+\epsilon^2\xi^2}\hat{u}(t,\xi)\right]_{xx}, \tag{1.4}$$

which is called R-C-E model for Rosenau's regularized version of the Chapman-Enskog-expansion for hydrodynamics [15]. The operator on the right side of (1.4)looks like the usual viscosity term ϵu_{xx} at low wave-number ξ, while for higher wave numbers it is intended to model a bounded approximation of a linearized collision operator, thereby avoiding the artificial instabilities that occur when the Chapman-Enskog expansion is truncated after a finite number of terms [15]. The rigorous analysis of this model, including the existence of the shock profiles, the smoothness as well as the upper-Lipschitz continuity, has been studied by Schochet and Tadmor [17]. We remark that as observed in [17] the solution sequence $\{u^\epsilon\}$ of (1.4) do not satisfy the Kružkov entropy inequality. The convergence of the solution u^ϵ of (1.4) to the entropy solution of the inviscid Burgers equation was proved in [17] via the L^1 contraction argument.

The equation (1.3) with $Q = \frac{1}{2}e^{-|x|}$ can also be written as a hyperbolic-elliptic system

$$u_t + uu_x = \phi_x, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.5}$$

$$\epsilon^2\phi_{xx} - \phi + \epsilon u_x = 0. \tag{1.6}$$

It is easy to see that the second equation enables one to express ϕ in terms of u formally as

$$\phi = (1-\epsilon^2\partial_x^2)^{-1}\epsilon u_x = \epsilon Q_\epsilon * u_x,$$

which in turn gives the right of (1.3)

$$\phi_x = \epsilon Q_\epsilon * u_{xx} = \frac{1}{\epsilon}[Q_\epsilon * u - u].$$

The system of equations (1.5)-(1.6) is derived as the third-order approximation of the full system describing the motion of radiating gas in therm-nonequilibrium, while the second-order approximation gives the viscous Burgers' equation $u_t + uu_x = \epsilon u_{xx}$, and the first-order approximation gives the inviscid Burgers equation $u_t + uu_x = 0$. Hamer [4] studied these equations in the physical respect, especially for the steady progressive shock-wave solutions. Noting that if ϵ in (1.6) is small one has $\phi \sim \epsilon u_x$, which leads to the usual viscous Burgers' equation. The viscous Burgers' equation admits smooth shock wave profile, but does not allow the finite breakdown. On the other hand, if the parameter ϵ is large, one finds from (1.6) that $\epsilon\phi_{xx} + u_x \sim 0$, which when combined with (1.5) gives the damped Burgers' equation $u_t + uu_x = -u/\epsilon$. This damped equation reflects the conditional breakdown in finite time, but does not support monotone travelling waves (shock profiles).

The parameter ϵ in (1.3) does not play a role in our analysis, and so will be set to 1 for convenience. The equation (1.3) with $\epsilon = 1$, i.e., (1.1), is a physical model that allows for the shock wave profile and yet exhibits the finite time break down. For stability of shock profiles via energy method we refer to [9, 6]. The global weak solution to (1.1) was studied in [17].

As is known the typical well-posedness result asserts that either a solution of a time-dependent PDE exists for all time (global existence of the smooth solution) or else there is a finite time (called life span) such that some norm of the solution becomes unbounded as the life span is approached (called finite-time breakdown). The natural question is whether there is a critical threshold for the initial data such that the global existence of the smooth solution or the finite time breakdown depends only on crossing such critical threshold. This remarkable critical threshold phenomena was first observed and studied in [3] for a class of Euler-Poisson equations. In this paper we confirm such critical threshold phenomena for (1.1)-(1.2) by giving an upper threshold for the global existence of the smooth solution and a lower threshold for the finite time breakdown. We also show the exact blow up rate as the life span is approached.

In this paper we shall use the following notation for $g \in L^\infty(\mathbb{R})$ to denote the maximal variation

$$V(g) := \max_{x\in\mathbb{R}} g(x) - \min_{x\in\mathbb{R}} g(x).$$

The first result tells us the critical threshold phenomena in equation (1.1).

Theorem 1.1. *Consider the Cauchy problem (1.1)-(1.2) with initial data $u_0 \in C_b^1(\mathbb{R})$. Let the kernel Q satisfies (H_1), then*

- *if $V(u_0) < 1/2Q(0)$ and*

$$\inf_{x\in\mathbb{R}} \partial_x u_0(x) > -\frac{1}{2}\left[1+\sqrt{1-4Q(0)V(u_0)}\right],$$

then the smooth solution exists for all time;

- *if*

$$\inf_{x} \partial_x u_0(x) < -\frac{1}{2}\left[1+\sqrt{1+4Q(0)V(u_0)}\right],$$

then the solution u must break down at finite time T. Moreover,

$$\lim_{t\to T}(\min_{x\in\mathbb{R}}\{u_x(t,x)\}) = -\infty$$

and exact blow up rate is

$$\lim_{t\to T}((T-t)\min_{x\in\mathbb{R}}\{u_x(t,x)\}) = -1.$$

Concerning this theorem, several remarks are in order.

Remarks 1. The above results show that the solution behavior of (1.1)-(1.2) depends on the relative size of the initial velocity gradient and the magnitude of the initial velocity. If either the magnitude is too large or the initial velocity gradient is too negative, the solution would loss smoothness in finite time. This peculiar phenomena explains the result obtained in [17], in which additional constraints on the shock strength are imposed to ensure the smoothness of the shock profiles. Further relation between the smoothness of the shock profiles and the shock strength are given in [6]. The critical threshold phenomena was already partially observed in previous studies, see [17] and [7].

2. As an application we take $u_0^\theta(x) = \exp(-x^2/\theta)$ for $\theta > 0$. Note that

$$\inf_{x\in\mathbb{R}}[\partial_x u_0^\epsilon(x)] = -\sqrt{\frac{2}{e\theta}}, \quad V(u_0^\theta) = 1.$$

Therefore choosing θ so small that

$$\theta < \frac{4}{e(1+2Q(0)+\sqrt{1+4Q(0)})},$$

we see that $\partial_x u_0^\theta$ is below the lower threshold, and thereby the corresponding solution $u^\theta(t,x)$ breaks down in finite time.

2. Note that at the blow up time, the solution is still bounded, and the gradient of the solution becomes unbounded from below. Such breakdown is referred to as the wave breaking in the context of the shallow water waves. In [19] Whitham emphasized that wave breaking phenomena is one of the most intriguing long-standing problems of water theory. This issue was settled recently in [13] first for Whitham's equation. Another shallow water equation derived recently by Camassa and Holm [2] can be written as (1.5) coupled with the following equation

$$\phi_{xx} - \phi - u^2 - \frac{1}{2}u_x^2 = 0.$$

This equation as a completely integrable system has soliton solution and yet exhibits finite-time breakdown phenomena for a large class of initial data, which has been observed and justified by Holm [2], Contantin and Escher [1], and McKean [12]. The main tool used in above papers is to trace the solution gradient along a curve on which the minimum of the gradient is obtained. In this work we trace the dynamics of the solution gradient along the characteristics, which has been well known in the context of the hyperbolic equations, see, e.g. [10, 5, 11]. For the global weak solution to the above shallow water equation we refer to [18] and references therein.

3. From the above results we see that if the magnitude of the initial profile is small, both thresholds given in Theorem 1.1 are close to $\inf_{x\in\mathbb{R}} \partial_x u_0(x) = -1$, which is exactly the critical threshold for the damped Burgers' equation

$$u_t + uu_x = -u.$$

Indeed, along the particle path $x(\alpha, t)$ defined by

$$\frac{d}{dt}x(\alpha,t) = u(t, x(\alpha,t)), \quad x(\alpha,0) = \alpha, \quad \alpha \in \mathbb{R}.$$

the gradient of the solution to the above damped Burgers' equation can be written explicitly as

$$u_x(t,x) = [e^t(1 + (\partial_x u_0(\alpha))^{-1}) - 1]^{-1},$$

which is bounded from below for all time if and only if

$$\inf_{x\in\mathbb{R}} \partial_x u_0(x) \ge -1.$$

This remarkable critical threshold phenomena explains why the equation (1.1) admits a narrower shock layers than those in the viscous Burgers' equation. We now turn to discuss the asymptotic behavior os solutions as the initial data is above the critical threshold. We shall concentrate on the case $u_0(-\infty) = u_- > u_+ = u_0(+\infty)$. As shown in [17], the equation (1.1) with $Q = \frac{1}{2}e^{-|x|}$ admits a smooth shock profile $U(x - st)$ connecting u_+ to u_- if and only if the strength $|V(U)| = |u_+ - u_-| \le \sqrt{2}$. Considering the conservative form of the equation, the natural question is whether this shock profile is stable in $L^1(\mathbb{R})$.

Our stability result is summarized as

Theorem 1.2. *Let $U(x-st)$ be a continuous shock profile of (1.1) and $S(t)u_0$ be a solution to (1.1)-(1.2) with initial data $u_0 \in U + L^\infty(\mathbb{R})$ and $u_0 \in [\inf U, \sup U]$. If $\partial_x u_0 \geq -\frac{1}{2}[1 + \sqrt{1-4Q(0)V(u_0)}]$, then there exists a constant k such that*

$$\lim_{t\to\infty} \|S(t)u_0 - U(\cdot - st + k)\|_{L^1} = 0.$$

Remarks: 1. The $L^p(1 \leq p < \infty)$ stability is immediate from the above L^1 stability result and the L^∞ boundedness of $S(t)$. Consult [6] for the stability of travelling waves via the energy principle.

2. We assume that the initial data is above the upper critical threshold to ensure the regularity of the ω-limit set of the solution. This condition is expected to be relaxed since our upper threshold is not sharp.

We now conclude this section by outlining the rest of the paper. In Section 2, we recall several properties of (1.1) and give the estimate of the nonlocal term in (1.1), which paves the way for the next sections. The lower threshold for finite time breakdown will be given in Section 3, in which we also prove the exact blow up rate. The upper-threshold for global existence of the smooth solution is carried out in Section 4. The final section is devoted to the L^1 stability of the shock profiles.

2. PRELIMINARIES

This section is devoted to some estimates which will be used in the next two sections.

In order to formulate the problem, we denote the solution operator of (1.1) as $S(t)$ indexed with $t \in [0, \infty)$:

$$S(t) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}), \quad t \geq 0,$$

such that the solution $u(t, x)$ of (1.1) with initial data a can be expressed as

$$u(t) = S(t)a.$$

We recall from [17] that the solution operator $S(t)$ satisfies the following properties:

- (translate invariance) $S(t)a(x + k) = (S(t)a)(x + k)$ for any $k \in \mathbb{R}$;
- (conservative) If $a - b \in L^1(\mathbb{R})$, then for all $t > 0$, $S(t)a - S(t)b \in L^1(\mathbb{R})$ and $\int(S(t)a - S(t)b) = \int(a - b)$;
- (L^1 contraction) If $a - b \in L^1(\mathbb{R})$, then $S(t)a - S(t)b \in L^1(\mathbb{R})$ and $\|S(t)a - S(t)b\|_1$ is non-increasing of $t > 0$;
- (monotonicity) If $a(x) \geq b(x)$ for $x \in \mathbb{R}$, then $S(t)a \geq S(t)b$ for all $t > 0$.

The above monotonicity immediately gives us the following maximum principle.

Lemma 2.1. *Let $u_0 \in L^\infty(\mathbb{R})$. Then the solution $u(t, \cdot)$ is also bounded with*

$$\min_{x\in\mathbb{R}} u_0(x) \leq u(t, \cdot) \leq \max_{x\in\mathbb{R}} u_0(x).$$

This maximum principle leads to the following bounds which will be used in figuring out our threshold conditions.

Lemma 2.2. *Let u be the smooth solution in $[0, T]$. Then it holds*

$$\min_{x\in\mathbb{R}} u_0(x) \leq Q * u(t, \cdot) \leq \max_{x\in\mathbb{R}} u_0(x), \quad t \in [0, T], \tag{2.1}$$

$$-Q(0)V(u_0) \leq Q * u_x(t, \cdot) \leq Q(0)V(u_0). \tag{2.2}$$

Proof. The first inequality follows from the fact $Q*1 = 1$ and the L^∞ bound $\min_{x\in\mathbb{R}} u_0(x) \le u(t,\cdot) \le \max_{x\in\mathbb{R}} u_0(x)$. We shall prove the second inequality as follows:

$$
\begin{aligned}
Q * u_x &= \int_{\mathbb{R}} Q(x-y)u_y(t,y)dy \\
&= \int_{\mathbb{R}} Q_x(x-y)u(t,y)dy \\
&= \left[\int_{-\infty}^{x} Q_x(x-y)u(t,y)dy + \int_{x}^{+\infty} Q_x(x-y)u(t,y)dy\right] \\
&\le \min_{x\in\mathbb{R}} u_0(x) \int_{-\infty}^{x} Q_x(x-y)dy + \max_{x\in\mathbb{R}} u_0(x) \int_{x}^{+\infty} Q_x(x-y)dy \\
&\le Q(0)\left[-\min_{x\in\mathbb{R}} u_0(x) + \max_{x\in\mathbb{R}} u_0(x)\right] = Q(0)V(u_0).
\end{aligned}
$$

The lower bound $-Q(0)V(u_0)$ is clear from the above estimate. □

The existence of T is ensured by the local existence theorem stated in the following

Lemma 2.3. *Consider the Cauchy problem (1.1)-(1.2) with initial data $u_0 \in C_b^1(\mathbb{R})$. Then there exists a positive constant T, depending only on $\|u_0\|_{C^1(\mathbb{R})}$ such that (1.1)-(1.2) has a unique smooth solution in $C_b^1(\mathbb{R}\times[0,T])$.*

The proof of this local existence is standard via iteration scheme, the details are omitted. This local existence provides a base for extending the solution or justify the finite time breakdown.

3. Blow up criterion—lower threshold

This section is devoted to a general discussion of wave breaking criterion.

Theorem 3.1. *Consider the Cauchy problem (1.1)-(1.2). The maximal existence time T is finite if and only if the gradient of the solution becomes unbounded from below in finite time.*

Proof. From the local existence in Lemma 2.3 it follows that if the gradient of the solution becomes unbounded from below in finite time, then $T < \infty$.

Let the life span $T < \infty$ and assume that for some constant $M > 0$ we have

$$u_x(t,x) \ge -M, \quad (t,x) \in [0,T) \times \mathbb{R}. \tag{3.1}$$

On the other hand by [17, Theorem 5.1] the solution $u(t,x)$ satisfies the one-sided Lipschitz condition, i.e.

$$u_x(t,x) \le \frac{1}{(\max_{x\in\mathbb{R}} u_{0x})^{-1} + t} \le \max_{x\in\mathbb{R}} u_{0x} < \infty.$$

Therefore the standard continuation argument enables us to extend solution to $[0, T+\delta)$ with $\delta > 0$, and thereby one must have $T = \infty$. □

The lower threshold is given in the following

Theorem 3.2. *Consider the Cauchy problem (1.1)-(1.2) with the initial profile $u_0 \in C_b^1(\mathbb{R})$. If u_0 is bounded and its gradient is negative with*

$$\inf_{x\in\mathbb{R}} \partial_x u_0(x) < -\frac{1}{2}\left[1 + \sqrt{1 + 4Q(0)V(u_0)}\right].$$

Then the life span T must be finite. Moreover

$$T \leq \left[-\frac{1}{2}\left(1+\sqrt{1+4Q(0)V(u_0)}\right) - \inf_{x\in\mathbb{R}} \partial_x u_0(x)\right]^{-1}$$

and

$$\lim_{t\to T}(\min_{x\in\mathbb{R}}\{u_x(t,x)\}) = -\infty.$$

Proof. Differentiation of (1.1) with respect to x leads to

$$d_t + ud_x + d^2 = Q * u_x - d, \quad t\in(0,T),$$

where $d := u_x(t,x)$. The smoothness of u ensures that there exists a smooth curve $x(\alpha,t)$ satisfying

$$\frac{d}{dt}x(\alpha,t) = u(t,x(\alpha,t)), \quad x(\alpha,0)=\alpha, \quad \alpha\in\mathbb{R}.$$

Evaluating the above $d-$ equation at $x(\alpha,t)$ and using $Q*u_x \leq A := Q(0)V(u_0)$ stated in Lemma 2.2 we have

$$d' + d^2 = Q*u_x(t,x(\alpha,t)) - d \leq A - d, \quad ' := \partial_t + u\partial_x$$

for $t\in(0,T)$. That is

$$d' \leq -(d-M_1)(d-M_2), \quad t\in(0,T) \tag{3.2}$$

with

$$M_1 := -\frac{1}{2}[1+\sqrt{1+4A}], \quad M_2 := -\frac{1}{2}[1-\sqrt{1+4A}].$$

For a fixed $\alpha\in\mathbb{R}$ if $d_0(\alpha) := u_0'(\alpha) < M_1$, then we claim that

$$d(t) < d_0(\alpha), \quad t\in(0,T). \tag{3.3}$$

If this would not be true, there is some $t_0\in(0,T)$ with $d(t)<d_0$ on $[0,t_0)$ and $d(t_0)=d_0$ by the continuity of $d=u_x$ in time. But in this case

$$d' \leq -(d_0-M_1)(d_0-M_2) < 0, \quad t\in(0,t_0).$$

An integration over $(0,t_0)$ yields

$$d(t_0) < d_0,$$

which contradicts with our assumption that $d(t_0)=d_0$ for $t_0<T$. This implies that (3.3) holds.

Combining (3.3) with (3.2) we obtain

$$d' \leq -(d-M_1)^2, \quad t\in(0,T),$$

and integration yields

$$d(t) \leq M_1 + \left[t - \frac{1}{M_1-d_0}\right]^{-1}.$$

From this we find that $d(t)\to-\infty$ before t reaches $\frac{1}{M_1-d_0}$. This proves that the solution breaks down in finite time once $\partial_x u_0 \geq M_1$ fails. □

The blow up rate at the breaking time is summarized in the next

Theorem 3.3. *Let T be the maximal existence time of (1.1)-(1.2). If the life span T is finite, then*

$$\lim_{t\to T}((T-t)(\min_{x\in\mathbb{R}}\{u_x(t,x)\})) = -1.$$

Proof. By Theorem 3.1 one has

$$\lim_{t\to T}(\min_{x\in\mathbb{R}}\{u_x(t,x)\}) = -\infty.$$

For $t \in [0,T)$ the solution u is smooth, then the curve $x(\alpha,t)$ is well defined by

$$\frac{d}{dt}x(\alpha,t) = u(t,x(\alpha,t)), \quad x(\alpha,0)=\alpha, \quad \alpha\in\mathbb{R}.$$

This implies

$$\frac{\partial}{\partial\alpha}x(\alpha,t) = \exp\left(\int_0^t u_x(\tau,x(\alpha,\tau))d\tau\right) > 0, \quad t\in(0,T),$$

and hence $x(\alpha,t)$ is a 1-1 mapping from $\mathbb{R}$ to $\mathbb{R}$. From these facts follow that there exists an $\alpha\in\mathbb{R}$ such that

$$\min_{x\in\mathbb{R}}\{u_x(t,x)\} = u_x(t,x(\alpha,t)).$$

As done previously we consider dynamics of $d=u_x$ along the curve $x(\alpha,t)$, using $-A\le Q*u_x\le A=Q(0)V(u_0)$ to obtain

$$-A-d\le d'+d^2\le A-d, \quad t\in(0,T).$$

Let $\epsilon\in(0,1)$ be suitably small. Since $\lim_{t\to T}d(t)=-\infty$, there exists $t_0\in(0,T)$ such that

$$d(t) < B^-(\epsilon), \quad t\in[t_0,T) \tag{3.4}$$

with

$$B^-(\epsilon) = \frac{-2A}{\sqrt{1+4A\epsilon(2-\epsilon)}-1}$$

being the smaller root of $(\epsilon^2-2\epsilon)d^2-d+A=0$. Otherwise there exists $\delta>0$ such that

$$d(t)<B^-(\epsilon), \quad t\in(t_0,t_0+\delta)$$

and for $\delta<T-t_0$

$$d(t_0+\delta)=B^-(\epsilon).$$

Hence for $d(t)<B^-(\epsilon)$ on $(t_0,t_0+\delta)$

$$\frac{d}{dt}d(t)\le A-d-d^2\le -(1-\epsilon)^2d^2<0, \quad t\in(t_0,t_0+\delta).$$

Integration gives

$$d(t_0+\delta)<d(t_0)<B^-(\epsilon).$$

This contradiction shows that

$$d\le B^-(\epsilon), \quad t\in[t_0,T),$$

therefore

$$d'\le -(1-\epsilon)^2d^2, \quad t\in[t_0,T). \tag{3.5}$$

On the other hand let

$$B^+(\epsilon) = \frac{-2A}{\sqrt{1+4A\epsilon(2+\epsilon)}+1},$$

which is the bigger root of $(\epsilon^2+2\epsilon)d^2-d-A=0$. We find that $B^-(\epsilon)<B^+(\epsilon)$ and

$$d(t)<B^+(\epsilon), \quad t\in(t_0,T).$$

This gives $(\epsilon^2+2\epsilon)d^2-d-A>0$, yielding

$$d'\ge -(d^2+d+A)\ge -(1+\epsilon)^2d^2, \quad t\in(t_0,T). \tag{3.6}$$

A combination of (3.5) with (3.6) gives

$$-(1+\epsilon)^2 d^2 \le d' \le -(1-\epsilon)^2 d^2, \quad t \in (t_0, T).$$

Note that d is locally Lipschitz on (t_0, T) and so is $1/d$ on (t_0, T). The above inequality leads to

$$(1-\epsilon)^2 \le \left(\frac{1}{d}\right)' \le (1+\epsilon)^2, \quad t \in (t_0, T).$$

For $t \in (t_0, T)$, integrate the above over (t, T) to obtain

$$-(1-\epsilon)^2 (T-t) \le \frac{1}{d(t)} \le -(1+\epsilon)^2 (T-t), \quad t \in (t_0, T).$$

Optimizing the above in terms of ϵ one then has

$$\lim_{t \to T} (T-t) d(t) = -1.$$

This completes the proof. □

4. Global smoothness—upper threshold

With the breakdown criterion in Section 2, we are ready to discuss the upper threshold for the global existence of the smooth solution to (1.1)-(1.2).

Theorem 4.1. *Consider the Cauchy problem (1.1)-(1.2) with the initial profile* $u_0 \in C_b^1(\mathbb{R})$. *If* u_0 *is bounded with amplitude* $V(u_0) \le \frac{1}{4Q(0)}$ *and its gradient is above an upper threshold, i.e.*

$$\inf_{x \in \mathbb{R}} \partial_x u_0(x) \ge -\frac{1}{2}\left[1 + \sqrt{1 - 4Q(0)V(u_0)}\right].$$

Then the smooth solution exists for all time and satisfies

$$u(t,x) \ge -\frac{1}{2}\left[1 + \sqrt{1 - 4Q(0)V(u_0)}\right].$$

Proof. To show the global existence of the smooth solution it suffices to establish a priori lower bound for the gradient of solution u_x. As argued earlier we evaluate $d := u_x$ along the particle path $x(\alpha, t)$ to obtain

$$d' + d^2 = Q * u_x(t, x(\alpha, t)) - d(t).$$

Noting that the lower bound of Ku_x is $-A = -V(u_0)Q(0)$, we find that

$$d' \ge -A - d - d^2 = -(d - A_1)(d - A_2),$$

where

$$A_1 = -\frac{1}{2}[1 + \sqrt{1 - 4A}], \quad A_2 = -\frac{1}{2}[1 - \sqrt{1 - 4A}].$$

Now let q solve the following problem

$$\frac{d}{dt} q(t) = -(q - A_1)(q - A_2), \quad q(0) = d_0.$$

Then the comparison of the above differential relations yields

$$d - q \ge (d_0 - q(0)) \exp\left(-\int_0^t (d + q + 1) d\tau\right) = 0, \quad t > 0.$$

However, q can be solved explicitly as

$$q(t) = \left[A_1 - A_2 \frac{d_1 - A_1}{d_0 - A_2} \exp\left(A_2 - A_1\right)t\right] \left[1 - \frac{d_1 - A_1}{d_0 - A_2} \exp\left(A_2 - A_1\right)t\right]^{-1}.$$

Therefor for $A_2 > d_0 \geq A_1$ one has $d(t) \geq q(t) \geq A_1$; for $d_0 \geq A_2$ one has $d(t) \geq q(t) \geq A_2$. The possible breakdown occurs only when $d_0 < A_1$ because

$$q(t^*) = -\infty, \qquad t^* = \frac{1}{A_2 - A_1} \log \frac{d_1 - A_2}{d_0 - A_1} > 0.$$

The lower bound of d can not be ensured for $d_0 < A_1$. However, $d_0 \geq A_1$ is sufficient to ensure the global existence of the smooth solution. □

5. L^1 STABILITY OF SHOCK PROFILES

Let us rewrite the equation (1.1) as

$$u_t + f(u)_x = Q * u - u, \quad f = u^2/2. \tag{5.1}$$

A shock wave with speed $s \in \mathbb{R}$ is a solution of (5.1) of the form $U(x - st)$, with U approaching two different shock states $u_\pm$ at far field. The function U formally satisfies the equation

$$-sU' + f(U)' = Q * U - U, \quad U(\pm\infty) = u_\pm.$$

The critical threshold phenomena revealed in the previous sections suggests that the smooth shock profile is possible subject to some constraints on the shock strength.

Indeed the existence of the shock profiles for (5.1) with convex flux function f has been proved in [17, Theorem 3.1], which we state, for $Q = \frac{1}{2}e^{-|x|}$, below for reader's convenience.

Theorem 5.1. *Assume $f'' > 0$. Then the Lax shock condition*

$$f'(u_+) < s < f'(u_-) \tag{5.2}$$

and the Rankine-Hugoniot shock condition

$$H(u_+) = 0, \quad H(u) \equiv -s(u - u_-) + f(u) - f(u_-), \tag{5.3}$$

are necessary conditions for the existence of a travelling wave solution

$$U(z \equiv x - st), \qquad \lim_{z \to \pm\infty} U(z) = u_\pm,$$

for (5.1). Conversely, if (5.2), (5.3) hold, then a sufficient condition for the existence of such a travelling wave is

$$4 \sup_{u_+ < u < u_-} \{-f''(u)H(u)\} \leq 1,$$

and a necessary condition is

$$4\{-f''(u^*)H(u^*)\} \leq 1.$$

Here u^* is defined by

$$f'(u^*) = s.$$

Note that for the Burgers' flux $f = u^2/2$, the shock speed by Rankine-Hugoniot relation (5.3) becomes $s = \frac{u_+ + u_-}{2}$. If the shock condition (5.2) i.e.,

$$u_+ < u_-$$

holds, then there exists such travelling wave if and only if

$$|u_+ - u_-| \leq \sqrt{2}. \tag{5.4}$$

This shows that the travelling wave solutions of the R-C-E equation give narrower shock layers than those of the viscous Burgers' equation.

Recall that the solution operator

$$S(t) : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}), \quad t \geq 0$$

satisfies the nice properties listed in Section 2, which ensure that $S(t)$ can be well extended to $L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ and preserves all those properties.

To reformulate the stability problem we introduce the following set

$$A := U + L^1(\mathbb{R}),$$

which is a complete metric space with the metric

$$\rho(a_1, a_2) = \|a_1 - a_2\|_1.$$

We also set two subspaces of A,

$$A_1 := \{U(\cdot + k), \quad k \in \mathbb{R}\}.$$

and

$$A_2 = \{a \in A : \lim_{t\to\infty} S(t)a \quad \text{exists and} \lim_{t\to\infty} S(t)a \in A_1\}.$$

Equipped with the above notations we see that proving the stability result in Theorem 1.2 boils down to proving the following relation:

$$A \cap [u+, u_-] \subset A_2, \tag{5.5}$$

provided $S(t)a$ is smooth.

We introduce the $\omega-$limit set of a as

$$\omega(a) = \cap_{s\geq 0}\overline{\cup_{t\geq s}\{S(t)a\}}.$$

This ω limit set is invariant for $S(t)$. In fact, the definition implies that $b \in \omega(a)$ if and only if there is a sequence $\{t_k\} \to \infty$ such that

$$\rho(S(t_k)a, b) \to 0.$$

The following lemma plays a critical role in proving (5.5).

Lemma 5.2. *If $a, b \in A \cap [u_+, u_-]$ and $a - b$ does not keep same sign on $\mathbb{R}$. Then*

$$\|S(t)a - S(t)b\|_1 < \|a - b\|_1, \quad t > 0.$$

Proof. By Kružkov's argument ([8]) we have

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \{|u - v|\phi_t + sgn(u - v)[f(u) - f(v)]\phi_x\}dxdt \\ &\geq \int_0^T \int_{\mathbb{R}} \{|u - v| - sgn(u - v)G * (u - v)\}\phi dxdt,\end{aligned}$$

where ϕ is an arbitrary nonnegative test function. Thus, by taking $\phi(x,t) = \chi(t)\psi(x,t)$ and letting $\psi = 1 - g_\epsilon(|x - x_0| - M(T - t))$ with $M = sup|f'|$ tend to the function that is identically one; and letting $\chi(t)$ approximate the indicator function of the interval $[0, t]$ to conclude

(5.6)

$$\|a - b\|_1 - \|S(t)a - S(t)b\|_1 \geq \int_{\mathbb{R}} |S(t)a - S(t)b| - sign(a - b)G * (S(t)a - S(t)b)dx.$$

Using the monotonicity of $S(t)$ we see that if $a - b$ changes sign on $\mathbb{R}$ then so does $S(t)a - S(t)b$. Note that $\|Q\|_1 = 1$ we find that

$$\int_{\mathbb{R}} |u| - sign(u)Q * udx = 0$$

if and only if u does not change sign or $u \equiv 0$. This shows that the right side of (5.6) is positive if $a - b$ changes sign on $\mathbb{R}$. ☐

Armed with the above lemma we proceed to complete the stability proof via the following steps, which have become standard since the work by Osher and Ralston [14] and Serre [16].

First we restrict to the initial data in

$$N(U,k_1,k_2) := \{a \in A, \quad U(x+k_1) \le a(x) \le U(x+k_2), \quad \text{for some} \quad k_1,k_2 \in \mathbb{R}\},$$

and we can later extend to a larger class using the following dense lemmas.

Step 1.(Dense argument)

We first show both A_1 and A_2 are complete subspaces of A.

Lemma 5.3. *Let U be the monotone shock profile, then $A_i, \quad i=1,2$ are close in A.*

Proof. We first show the closeness of A_1. It is easy to see that for any $k \in \mathbb{R}$, $U(x+k) \in A$ since

$$\|U(\cdot+k) - U(\cdot)\|_1 = |k(u_+ - u_-)| < \infty.$$

We assume $U(x+k_n)$ converges in A. Then it is a Cauchy sequence. Note that

$$\|U(\cdot+k_n) - U(\cdot+k_m)\|_{L^1} = |(k_n-k_m)(u_+ - u_-)|$$

implies k_n is also a Cauchy sequence in $\mathbb{R}$. Let its limit be k, then letting $m \to \infty$ in the above one finds that the limit of $U(x+k_n)$ is $U(x+k) \in A_1$.

We now turn to show the closeness of A_2. Let $a_k \in A_2$ be a Cauchy sequence with limit being a in A. We need to show $a \in A_2$. Note that for each $a_k \in A_2$ we have that $\lim_{t\to\infty} S(t)a_k = \tilde{a}_k \in A_1$ exists. Hence $\tilde{a}_k$ is a Cauchy sequence in the complete metric space A_1, for

$$\|\tilde{a}_k - \tilde{a}_l\|_1 = \lim_{t\to\infty} \|S(t)a_k - S(t)a_l\|_1 \le \|a_k - a_l\|_1$$

We denote the limit of $\tilde{a}_k$ by $\tilde{a}$ as $k \to \infty$, which when combined with the closeness of A_1 implies that $\tilde{a} \in A_1$. Therefore $a \in A_2$ since

$$\|S(t)a - \tilde{a}\|_1 \le \|S(t)a - S(t)a_k\|_1 + \|S(t)a_k - \tilde{a}_k\|_1 + \|\tilde{a}_k - \tilde{a}\|_1 \to 0$$

as $k \to \infty$ and $t \to \infty$. □

Lemma 5.4. *For any given $k_1, k_2 \in \mathbb{R}$, the set $N(U,k_1,k_2)$ is dense in $A \cap [u_+, u_-]$.*

The proof can be done as in [14], the details are omitted.

Step2 (Compact criteria)

Lemma 5.5. *For any $k_1,k_2 \in \mathbb{R}$, the ω-limit set $\omega(N(U,k_1,k_2))$ is not empty.*

Proof. It suffices to show that $\cup_{t\ge 0}\{S(t)a\}$ is pre-compact for any $a \in N(U,k_1,k_2)$. Indeed, due to $a - U \in L^1$ and the L^1 contraction of $S(t)$ we have

$$\|S(t)a - U\|_1 = \|S(t)a - S(t)U\|_1 \le \|a-U\|_1 < \infty, \quad t \ge 0.$$

The L^1 equi-continuity follows from the fact that

$$\|S(t)a(x+h) - S(t)a(x)\|_1 \le \|a(x+h) - a(x)\|_1 \to 0$$

uniformly in time as h goes to zero. Using semigroup property of $S(t)$ we have

$$U(x+k_1) \le S(t)a \le U(x+k_2), \quad t \ge 0.$$

Hence

$$||S(t)a - U(x)||_{L^1(|x|>M)} \le \max\{\|U(\cdot+k_1) - U\|_{L^1(|x|>M)}, \|U(\cdot+k_2) - U\|_{L^1(|x|>M)}\} \to 0$$

uniformly in t as M goes to ∞.

The above facts when recalling the Frechet-Kolmogorov-Rieze compactness theorem yield that $\cup_{t\ge 0}\{S(t)a\}$ is pre-compact. □

Step 3.(Time-Invariance)

Lemma 5.6. *Let $b \in \omega(N(U, k_1, k_2))$. Then for any given $k \in \mathbb{R}$*

$$\|b - U(\cdot + k)\|_1 = \|S(t)b - U(\cdot + k)\|_1.$$

Proof. Since $b \in \omega(N(U, k_1, k_2))$, we see that there exists $a \in N(U, k_1, k_2)$ and a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \|S(t_n)a - b\|_1 = 0.$$

Given any $k \in \mathbb{R}$, by contraction of $S(t)$ we know that

$$\|S(t)a - U(x + k)\|_1 = \|S(t)a - S(t)U(x + k)\|_1$$

is decreasing in time and thus admits a limit $c_k \geq 0$ as $t \to \infty$, i.e.

$$\lim_{t\to\infty} \|S(t)a - U(x + k)\|_1 = c_k \geq 0.$$

Letting $t = t_n$ in the above and passing to the limit we have

$$\|b - U(\cdot + k)\|_1 = c_k.$$

Note $b \in \omega(a)$, then $S(t)b \in \omega(a)$ (ω is invariant under the flow) thereby

$$\|S(t)b - U(\cdot + k)\|_1 = c_k.$$

Therefore

$$\|S(t)b - U(\cdot + k)\|_1 = \|b - U(\cdot + k)\|_1, \quad \forall t > 0, \quad k \in \mathbb{R}.$$

□

We are now ready to prove (5.5). We first prove

$$N(U, k_1, k_2) \subset A_2.$$

By Lemma 5.5 we know that $\omega(N(U, k_1, k_2))$ is not empty. For $a \in N(U, k_1, k_2)$ and $b \in \omega(a)$. We need to show that there exists a $k \in \mathbb{R}$ such that

$$b = U(x + k).$$

Lemma 5.6 shows that

$$\|b - U(\cdot + k)\|_1 = \|S(t)b - U(\cdot + k)\|_1 = c_k.$$

Noting that $U(x + k)$ is the fixed point of $S(t)$, Lemma 5.2 shows that $b - U(x + k)$ must stay with one sign.

Therefore choosing

$$k = \int_{\mathbb{R}} (a - U)dx/(u_+ - u_-)$$

gives

$$c_k = \int_{\mathbb{R}} b - U(\cdot + k) = \int_{\mathbb{R}} a - U(x + k) = 0$$

On the other hand since the initial data a is assumed to be above the critical threshold, $\partial_x(S(t)a)$ is uniformly bounded with respect to t, and hence b is Lipschitz continuous. This regularity combined with the above fact yields

$$b = U(x + k).$$

We now conclude the proof of (5.5). Let $a \in A \cap [u_+, u_-]$. We need to show $a \in A_2$.

Using Lemma 5.4 there exists $a_n \in N(U, k_1, k_2) \in A$ such that $\|a_n - a\|_1 \to 0$ as $n \to \infty$. By the above proved fact we see that there exists k_n such that

$$\lim_{t\to\infty} \|S(t)a_n - U(\cdot + k_n)\|_1 = 0.$$

This tells that $a_n \in A_2$. Due to the closeness of A_2, the limit a also belong to A_2. Therefore there exists a k such that

$$\lim_{t\to\infty} \|S(t)a - U(\cdot + k)\|_1 = 0,$$

as argued above, the constant k as the limit of $\int_{\mathbb{R}}(a_n - U)dx/(u_+ - u_-)$ is

$$\int_{\mathbb{R}}(a - U)dx/(u_+ - u_-)$$

since $|\int(a_n - a)dx| \le \|a_n - a\| \to 0$ as $n \to \infty$. This completes the proof of (5.5), and thereby of Theorem 1.2.

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