CONVERGENCE OF AN UPWIND DIFFERENCE SCHEME FOR DEGENERATE PARABOLIC CONVECTION-DIFFUSION EQUATIONS WITH A DISCONTINUOUS COEFFICIENT

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ABSTRACT. We establish convergence of an upwind difference scheme (of Engquist-Osher type) for nonlinear degenerate parabolic convection-diffusion equations where the nonlinear convective flux function has a discontinuous coefficient $\gamma(x)$ and the diffusion function $A(u)$ is allowed to be strongly degenerate (the pure hyperbolic case is included in our setup). The main problem is obtaining a uniform bound on the total variation of the difference approximation u^{Δ}, which is a manifestation of resonance. To circumvent this analytical problem, we construct a singular mapping $\Psi(\gamma, \cdot)$ such that the total variation of the transformed variable $z^{\Delta} = \Psi(\gamma^{\Delta}, u^{\Delta})$ can be bounded uniformly in Δ. This establishes strong L^1 compactness of z^{Δ} and, since $\Psi(\gamma, \cdot)$ is invertible, also u^{Δ}. Our singular mapping is novel in that it incorporates a contribution from the diffusion function $A(u)$. We then show that the limit of a converging sequence of difference approximations is a weak solution as well as satisfying a Kružkov-type entropy inequality. We prove that the diffusion function $A(u)$ is Hölder continuous, implying that the constructed weak solution u is continuous in those regions where the diffusion is nondegenerate. Finally, some numerical experiments are presented and discussed.

1. INTRODUCTION AND STATEMENT OF RESULTS

We are interested in upwind finite difference approximations for nonlinear degenerate parabolic convection-diffusion initial value problems of the type

$$\begin{cases} u_t + f(\gamma(x), u)_x = A(u)_{xx}, & (x,t) \in \Pi_T = \mathbf{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where $T > 0$ is fixed, $u(x,t)$ is the scalar unknown function that is sought, and f, γ, A, u_0 are given functions to be detailed later. The special feature of the problem studied herein, which makes mathematical and numerical analysis more complicated, is the combination of a convection part that depends explicitly on the spatial location through a coefficient $\gamma(x)$ that may be *discontinuous* and a diffusion part that *strongly degenerates* in the sense that $A'(\cdot) \geq 0$. In fact, included in our setup are hyperbolic conservation laws with a discontinuous coefficient:

$$u_t + f(\gamma(x), u)_x = 0. \tag{1.2}$$

To facilitate the analysis, (1.2) is often written as a 2×2 nonstrictly hyperbolic system of equations:

$$\gamma_t = 0, \qquad u_t + f(\gamma, u)_x = 0. \tag{1.3}$$

Problems of the type (1.1) occur in several applications. Biased by our own interests, we mention here only flow in porous media (see, e.g., [8, 14]) and sedimentation-consolidation processes [4, 5]. The purely convective version of (1.1) ($A'(u) \equiv 0$) provides a simple model of traffic flow on a highway [47, 25], the spatially varying coefficient γ corresponding to varying road conditions. Scalar conservation laws with discontinuities in the flux also arise in radar shape-from-shading problems

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[39] and as building blocks in dimensional splitting methods for multi-dimensional Hamilton-Jacobi equations [29].

Before continuing, let us detail the assumptions that we need to impose on the "data" of the problem (1.1). For the coefficient γ, we assume that

$$\gamma(x) \in [\underline{\gamma}, \overline{\gamma}] \ \forall x \in \mathbf{R}; \quad \gamma \in BV(\mathbf{R}).$$

In particular, γ is allowed to be discontinuous. For the convective flux function f, we assume that

$$f(\gamma, 0) = f_0 \in \mathbf{R} \text{ for all } \gamma \text{ and } f(\gamma, 1) = f_1 \in \mathbf{R} \text{ for all } \gamma. \tag{1.4}$$

The purpose of this assumption is to guarantee that a solution initially in the interval $[0,1]$ remains in $[0,1]$ for all subsequent times. Furthermore, we assume that $f \in \mathrm{Lip}([\underline{\gamma}, \overline{\gamma}] \times [0,1])$. With this assumption the partial derivatives f_γ and f_u exist almost everywhere, and $\|f_\gamma\|_\infty$ and $\|f_u\|_\infty$ are Lipschitz constants of f with respect to γ and u. In what follows, we will use the notational shorthand $\|f_u\|_\infty = \|f_u\|$, $\|f_\gamma\|_\infty = \|f_\gamma\|$. Let

$$f_u^+(\gamma, u) = \max(0, f_u(\gamma, u)), \quad f_u^-(\gamma, u) = \min(0, f_u(\gamma, u)).$$

We will also require the technical assumption that f_u, f_u^+, and f_u^- are all Lipschitz continuous as a function of γ, with Lipschitz constant $L_{u\gamma}$. For example, if $f(\gamma, u) = \gamma \tilde{f}(u)$, where $\tilde{f} \in \mathrm{Lip}([0,1])$, this assumption will hold with $L_{u\gamma} = \|\tilde{f}'\|_\infty$. Finally, we assume that for each $\gamma \in [\underline{\gamma}, \overline{\gamma}]$, there is a unique maximum $u^*(\gamma) \in [0,1]$ such that $f(\gamma, \cdot)$ is strictly increasing for $u < u^*(\gamma)$ and strictly decreasing for $u > u^*(\gamma)$.

Regarding the diffusion function A, we assume that it belongs to $\mathrm{Lip}([0,1])$ with Lipschitz constant $\|A'\|$ and that the following *degenerate parabolicity* condition holds:

$$A(\cdot) \text{ is nondecreasing with } A(0) = 0. \tag{1.5}$$

Actually we shall be a bit more precise than (1.5). We assume that A degenerates (i.e., is constant) on a finite set of intervals, that is,

$$A'(w) = 0, \qquad \forall w \in \bigcup_{i=1}^{M} [\alpha_i, \beta_i],$$

where $\alpha_i < \beta_i$, $i = 1, \dots, M$, $M \geq 1$. On these intervals, (1.1) acts as a pure hyperbolic problem. We assume that A is non-degenerate (i.e., strictly increasing) off these intervals, that is,

$$A'(w) > 0, \qquad \forall w \notin \bigcup_{i=1}^{M} [\alpha_i, \beta_i],$$

so that (1.1) acts as a parabolic problem on $[0,1] \backslash \bigcup_i [\alpha_i, \beta_i]$. In view of (1.5), one often refers to (1.1) as a mixed hyperbolic-parabolic problem. In what follows, we assume (since the pure hyperbolic case has already been treated in [42, 43])

$$\max_{w \in [0,1]} A'(w) > 0.$$

Finally, we assume that the initial function u_0 satisfies

$$\begin{cases} u_0 \in L^1(\mathbf{R}) \bigcap BV(\mathbf{R}); \quad u_0(x) \in [0,1] \ \forall x \in \mathbf{R}; \\ TV(f(\gamma(x), u_0) - A(u_0)_x) < \infty. \end{cases} \tag{1.6}$$

Independently of the smoothness of γ, if (1.1) is allowed to degenerate at certain points, that is, $A'(s) = 0$ for some values of s, solutions are not necessarily smooth and weak solutions must be sought. A *weak solution* is here defined as follows:

Definition 1.1. *A measurable function $u(x,t)$ is a weak solution of the initial value problem* (1.1) *if it satisfies the following conditions:*

(**D**.1) $u \in L^1(\Pi_T) \bigcap L^\infty(\Pi_T) \bigcap C(0,T; L^1(\mathbf{R}))$ *and* $A(u) \in L^2(0,T; H^1(\mathbf{R}))$.

(**D**.2) *For all test functions* $\phi \in \mathcal{D}(\Pi_T)$ *such that* $\phi|_{t=T} = 0$,

$$\iint_{\Pi_T} \Big(u\phi_t + (f(\gamma(x), u) - A(u)_x)\phi_x \Big)\, dt\, dx + \int_{\mathbf{R}} u_0(x)\phi(x, 0) = 0. \tag{1.7}$$

Solutions behave even more dramatically if $A'(s)$ is zero on a whole interval $[\alpha, \beta]$. Then (weak) solutions may be discontinuous and they are not uniquely determined by their initial data. Consequently, an entropy condition must be imposed to single out the physically correct solution. If γ is "smooth", a weak solution u satisfies the *entropy condition* if for all convex C^2 functions $\eta : \mathbf{R} \to \mathbf{R}$,

$$\begin{aligned} &\eta(u)_t + (q(\gamma(x), u))_x + r(u)_{xx} \\ &\qquad + \gamma'(x)\big(\eta'(u) f_\gamma(\gamma(x), u) - q_\gamma(\gamma(x), u)\big) \le 0 \text{ in } \mathcal{D}'(\Pi_T), \end{aligned} \tag{1.8}$$

where $q, r : \mathbf{R} \to \mathbf{R}$ are defined by

$$q_u(\gamma(x), u) = \eta'(u) f_u(\gamma(x), u), \qquad r'(u) = \eta'(u) A'(u).$$

By a standard limiting argument, (1.8) implies that the Kružkov-type entropy condition

$$\begin{aligned} &|u - c|_t + [\operatorname{sign}(u - c)(f(\gamma(x), u) - f(\gamma(x), c))]_x + |A(u) - A(c)|_{xx} \\ &\qquad + \gamma'(x) \operatorname{sign}(u - c) f_\gamma(\gamma(x), c) \le 0 \end{aligned} \tag{1.9}$$

holds in $\mathcal{D}'(\Pi_T)$ for all $c \in \mathbf{R}$. The entropy condition (1.9) goes back to Kružkov [35], Vol'pert [44], and Vol'pert and Hudjaev [46]. Existence, uniqueness and stability results for entropy solutions of strongly degenerate parabolic equations with smooth coefficients can be found in [2, 6, 9, 27, 48, 46, 45]. For example, when the coefficients are sufficiently smooth and the initial function satisfies (1.6), there exists a unique entropy solution of (1.1) that belongs to $BV(\Pi_T)$ (i.e., u_x and u_t are finite measures on Π_T) and $A(u)$ belongs to the Hölder space $C^{1,\frac{1}{2}}(\Pi_T)$.

The entropy solution theory breaks down when γ is discontinuous. In Karlsen, Risebro, and Towers [30], we took a first step towards analyzing degenerate parabolic equations with a discontinuous coefficient. More precisely, we proved existence of a weak solution by passing to the limit in a problem where we had smoothed out the coefficient and added artificial viscosity. In contrast to the present paper, the convergence proof in [30] used the compensated compactness theory.

In this paper, we are interested in constructing a "simple" numerical scheme for (1.1) and proving its strong convergence towards a weak solution. When γ is constant or at least smooth, several numerical schemes have been proposed and analyzed already in the literature. Let us mention the operator splitting methods in [16, 24], the finite difference schemes in [18, 15, 17], the finite volume schemes in [1, 38, 20], and the kinetic BGK schemes in [3]. For a partial overview of mathematical and numerical theory for degenerate parabolic equations based on "hyperbolic" techniques, see [14].

We now present the numerical scheme that we propose for (1.1) when γ is possibly discontinuous. Let $\Delta x > 0$ and $\Delta t > 0$ denote the spatial and temporal discretization parameters respectively. We then let U_j^n denote the finite difference approximation of $u(j\Delta x, n\Delta t)$. The difference scheme, which uses the Engquist-Osher numerical flux [13] for the convection part and centered differencing for the parabolic part, takes the following (conservation) form

$$\begin{aligned} &\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - h(\gamma_{j-\frac{1}{2}}, U_j^n, U_{j-1}^n)}{\Delta x} \\ &\qquad = \frac{A(U_{j+1}^n) - 2A(U_j^n) + A(U_{j-1}^n)}{(\Delta x)^2}. \end{aligned} \tag{1.10}$$

Here the numerical flux h is the Engquist-Osher generalized upwind flux [13] (see Section 3 for precise statements). The scheme (1.10) is the one-dimensional version of the multidimensional algorithm presented in [28], where convergence was established for a "rough" but continuous coefficient γ. It is also closely related to the algorithm presented in [42] and [43], where a staggered mesh EO scheme was investigated for a purely hyperbolic problem with a discontinuous coefficient. In particular, the discretization of γ is staggered with respect to that of the conserved variable

u. As pointed out in [42, 43], this results in a significant reduction in complexity compared with the alternative of aligning the two discretizations. In the latter case, a more complicated 2×2 Riemann problem has to be solved (exactly or approximately) [36, 37, 21, 33, 32]. Staggering the discretizations also greatly simplifies the analysis, making it possible to apply, with some allowances for the parabolic terms, some of the analytical techniques developed for monotone difference schemes for purely hyperbolic problems. Another important feature of our scheme is its conservation form, i.e., it is a shock capturing algorithm in the purely hyperbolic regime where $A' = 0$. For the case of constant γ, Evje and Karlsen [18] provide numerical evidence that differencing the PDE (1.1) directly (i.e., not in conservation form) results in wrong solutions, specifically shocks may move with the wrong speed. Finally, our algorithm is a so-called upwind scheme, meaning that the differencing of the convective flux is biased in the direction of incoming waves, making it possible to resolve shocks without excessive smearing.

Let $u^{\Delta}(x,t)$ be the piecewise constant approximate solution generated by (1.10). Roughly speaking, our main results can be stated as follows

Theorem 1.1. *We have that u^{Δ} converges along a subsequence in $L^1_{\mathrm{loc}}(\Pi_T)$ to a weak solution u of the initial value problem*(1.1) *in the sense of Definition 1.1. Furthermore, if γ has finitely many discontinuities located at $\xi_1, \dots, \xi_{M'}$, the limit satisfies the following entropy condition in $\mathcal{D}'(\Pi_T)$ for all $c \in \mathbf{R}$:*

$$|u-c|_t + (\sigma(u-c)(f(\gamma,u) - f(\gamma,c)))_x + |A(u) - A(c)|_{xx} - |f(\gamma(x),c)_x| \leq 0 \ . \tag{1.11}$$

The convergence proof consists of establishing bounds on the solution and its L^1 space and time translates, measured with respect to a transformed variable z. Specifically, we prove that the scalar upwind difference scheme converges (along a subsequence) to a weak solution of (1.1) by constructing a singular mapping $\Psi : (\gamma, u) \mapsto (\gamma, z)$ such that strong compactness of $z^{\Delta} = \Psi(\gamma^{\Delta}, u^{\Delta})$ can be obtained. As in other problems concerning resonance phenomena, it is necessary to measure the space translates with respect to a nonlinear transformation Ψ, since there is generally no spatial variation bound for the conserved variable u itself. The singular mapping approach has been used for at least twenty years in the purely hyperbolic setting [33, 37, 41, 36, 42, 43]. However, the presence of the parabolic term in (1.1) requires a novel, and somewhat more complicated, singular mapping. Specifically, the singular mapping (3.1) includes a contribution from the diffusion term $A(u)$, which make the subsequent analysis a bit more intricate than in the purely hyperbolic case. We prove compactness for two separate parts of the singular mapping. One part, $\mathcal{F}(\gamma, u)$, is associated with the convective portion of the problem, and the other, $A(u)$, is associated with the diffusive portion of the problem. In the process of establishing compactness for the diffusive portion, we also prove that the limit u satisfies $A(u) \in L^2(0,T; H^1(\mathbf{R}))$. We then combine the two portions to recover the original singular mapping $\mathcal{F}(\gamma, u) + A(u)$, and conclude that since the mapping is strictly increasing as a function of the conserved variable u, convergence of the transformed variable implies convergence of u. We also establish regularity of the diffusion function $A(u)$, specifically that $A(u) \in C^{1,\frac{1}{2}}(\Pi_T)$, proving that the solution u itself is continuous in the regions where there is nonzero diffusion.

For the purely hyperbolic problem, the the singular mapping approach can be traced back to Temple [41], who originated the technique in order to establish convergence of the Glimm scheme for a 2×2 resonant system of conservation laws modeling the displacement of oil in a reservoir by water and polymer. In addition to the Glimm scheme, convergence has been established for the 2×2 Godunov method by Lin, Temple, and Wang [36, 37]. Specifically, they applied the 2×2 Godunov method to the system (1.3) and used a version of the singular mapping to establish compactness (see also Hong [26] for an "improved" singular mapping). The front tracking method, which is based on the work of Dafermos [11] and Holden, Holden, and Høegh-Krohn [23], has been applied to a number of hyperbolic problems with discontinuous coefficients. Gimse and Risebro [21] used the front tracking method to study the two phase flow equation, proving compactness of the sequence of approximations via a bound on the spatial variation, measured with respect to the singular mapping. For the scalar conservation law with a concave flux, Klingenberg and Risebro [33] used the front tracking technique to establish existence, uniqueness, and asymptotic behavior for the Cauchy problem (1.3). The front tracking method has also been applied to the situation where the flux f is neither concave nor convex [32]. A version of the singular mapping was used in both [33] and [32]. The singular mapping has also been used to establish convergence of difference schemes for scalar conservation laws having a discontinuous coefficient [42, 43].

The following example, due to Lin, Temple, and Wang [36], helps to understand the impact of resonance in the purely hyperbolic setting. Assuming that the flux is smooth, they linearize the conservation law. Focusing on the case (for the sake of simplicity) where the original equation is $u_t + (\gamma(x)f(u))_x = 0$, the version that results from linearizing about u^* (where $f'(u^*) = 0$) is

$$u_t + f(u^*)\gamma_x = 0,$$

which has the solution $u(x,t) = -f(u^*)\gamma_x t + u_0(x)$. If γ belongs to C^1, this solution, along with its variation, grows linearly with time, i.e., neither the solution nor its variation is bounded. Furthermore, if γ is allowed to have jumps, then the solution only makes sense as a measure. If one instead linearizes about a point where $f'(u) \neq 0$ and the initial data u_0 is bounded, both the solution and its variation will remain bounded; this follows from [43] and Proposition 2.1 herein.

The present paper provides the groundwork for future work in several directions. The parabolic term forces a very small time step on our explicit scheme, and so we intend to investigate an implicit version, which will allow for a more efficient algorithm. We also plan to present a second order version of the scheme based on using flux limiters in a novel way that keeps the total variation bounded, as measured via the singular function. We will generalize to the situation where the diffusion term varies spatially, and incorporate more general invariant regions, making it possible to relax the condition (1.4), and allow for singular source terms. Additionally, we will prove uniqueness for piecewise smooth solutions of the initial value problem (1.1) satisfying the Kružkov-type entropy inequality (1.11). One more avenue of investigation is to relax the condition that the flux have a single maximum, allowing for any finite number of critical points.

The rest of this paper is organized as follows: Section 2 provides preliminary material concerning the definition of our algorithm and the resulting approximate solutions. In Section 3 we state and prove our main result, convergence of the scheme (2.4) to a weak solution of the initial value problem (1.1). Section 4 establishes that the diffusion function $A(u)$ is continuous of class $C^{1,\frac{1}{2}}$. In Section 5 we demonstrate that our scheme satisfies a cell entropy inequality, and that as a consequence, piecewise smooth limits of our algorithm satisfy a Kružkov-type entropy inequality. Section 6 provides the results of some numerical experiments.

2. Definition of approximate solutions

Let $\Delta x > 0$ and $\Delta t > 0$ be the spatial and temporal discretization parameters. The spatial domain $\mathbf{R}$ is discretized into cells

$$I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}),$$

where $x_k = k\Delta x$ for $k = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$. Similarly, the time interval $[0,T]$ is is discretized via $t_n = n\Delta t$ for $n = 0, \ldots, N$, where the integer N is chosen such that $N\Delta t = T$, resulting in the time strips

$$I^n = [t_n, t_{n+1}).$$

We let $\chi_j(x)$ and $\chi^n(t)$ be the characteristic functions for the intervals I_j and I^n, respectively. We let $\chi_j^n(x,t) = \chi_j(x)\chi^n(t)$ be the characteristic function for the rectangle

$$R_j^n = I_j \times I^n.$$

Also, we let $R^n_{j+\frac{1}{2}} = I_{j+\frac{1}{2}} \times [t_n, t_{n+1})$ with $I_{j+\frac{1}{2}} = [x_j, x_{j+1})$. To simplify the presentation, we use $\Delta_+$and Δ_- to designate the difference operators in the x direction, e.g.,

$$\Delta_+ f(\gamma_j, U_j^n) = f(\gamma_{j+1}, U_{j+1}^n) - f(\gamma_j, U_j^n) = \Delta_- f(\gamma_{j+1}, U_{j+1}^n).$$

Furthermore, Δ_+^u and Δ_-^u are spatial difference operators with respect to u only, keeping γ fixed, e.g.,

$$\Delta_+^u f(\gamma_j, U_j^n) = f(\gamma_j, U_{j+1}^n) - f(\gamma_j, U_j^n).$$

Before we can state the finite difference scheme, we need to introduce the Engquist-Osher (EO henceforth) numerical flux [13]

$$h(\gamma, v, u) = \frac{1}{2}(f(\gamma, u) + f(\gamma, v)) - \frac{1}{2}\int_u^v |f_u(\gamma, w)|\, dw.$$

The EO numerical flux is consistent with the actual flux in the sense that

$$h(\gamma, u, u) = f(\gamma, u).$$

In addition, for fixed γ, $h(\gamma, v, u)$ is a two-point monotone flux, meaning that it is nonincreasing with respect to v, and nondecreasing with respect to u. Due to the regularity assumptions about the flux f, the numerical flux h is Lipschitz continuous with respect to each of its arguments, and in fact satisfies

$$f_u^-(\gamma, v) = h_v(\gamma, v, u) \le 0 \le h_u(\gamma, v, u) = f_u^+(\gamma, u). \tag{2.1}$$

Thus, if the flux f is C^1 smooth, the numerical flux is also C^1 smooth as a function of the conserved variables u and v. From formula (2.1) it is clear that $\|f_u\|$ is a Lipschitz constant for the conserved variables u and v. It is not hard to check that $\|f_\gamma\| + \frac{1}{2}L_{u\gamma}$ is a Lipschitz constant for h with respect to the variable γ. We also recall the decomposition

$$\Delta_-^u h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) = \Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n) + \Delta_-^u h_+(\gamma_{j+\frac{1}{2}}, U_j^n), \tag{2.2}$$

where h_-, h_+ are defined in (2.3) via

$$h_-(\gamma_{j+\frac{1}{2}}, \xi) = \int_0^\xi f_u^-(\gamma_{j+\frac{1}{2}}, w)dw, \qquad h_+(\gamma_{j+\frac{1}{2}}, \xi) = \int_0^\xi f_u^+(\gamma_{j+\frac{1}{2}}, w)dw. \tag{2.3}$$

It is not hard to check that $h_-,\ h_+ \in \mathrm{Lip}([\underline{\gamma}, \overline{\gamma}] \times [0,1])$ with Lipschitz constants $\|f_u\|$ and $L_{u\gamma}$.

The difference scheme that is analyzed in this paper can then be stated as follows:

$$U_j^{n+1} = U_j^n - \lambda\Delta_- h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \mu\Delta_-\Delta_+ A(U_j^n), \tag{2.4}$$

for $j \in \mathbf{Z}$, $n = 0, \dots, N-1$, and λ, μ denoting the numbers

$$\lambda = \frac{\Delta t}{\Delta x}, \qquad \mu = \frac{\Delta t}{\Delta x^2} = \frac{\lambda}{\Delta x}.$$

The iteration (2.4) is started by setting

$$U_j^0 = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x)\, dx, \tag{2.5}$$

and the discretization of γ is *staggered* with respect to that of u:

$$\gamma_{j+\frac{1}{2}} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \gamma(x)\, dx. \tag{2.6}$$

It can be shown (see Lemma 3.2) that the difference scheme (2.4) is monotone and $U_j^n \in [0,1]$ with the CFL condition

$$2\lambda\|f_u\| + 2\mu\|A'\| \le 1.$$

In the case where $\|A'\| > 0$, we will have $\lambda = \mathcal{O}(\Delta x)$. In fact, then we can assume that there is a constant $0 < b \le 1$ with

$$\lambda = \frac{b\Delta x}{2\|f_u\|\Delta x + 2\|A'\|} \le \frac{b}{2\|A'\|}\Delta x.$$

In the totally degenerate case (pure convection), $\|A'\| = 0$, and then we can allow a less restrictive CFL condition of the form $\lambda = O(1)$. For the remainder of this paper we will assume that $\|A'\| > 0$.

In the purely convective setting, with constant γ, approximations generated by monotone schemes are well known to share many of the properties of the actual entropy solutions to the conservation law $u_t + f(u)_x = 0$, see [10, 22]. These properties include an ordering principle, L^1 contractiveness and L^1 time continuity, nonincreasing total variation, and satisfaction of a large (infinite) set of entropy inequalities. Some of these properties carry over to more general equations related to (1.1), as a number of works have demonstrated. Evje and Karlsen [18, 15] have extended the theory of monotone schemes to the more general setting of degenerate parabolic equations, with γ still constant. Karlsen and Risebro [28] have extended the monotone scheme approach to the multidimensional degenerate parabolic setting with coefficients that are spatially varying and continuous, but "rough". In the purely hyperbolic, one-dimensional setting, Towers [42, 43] extended the monotone scheme approach (for the EO scheme) to the case of a spatially varying discontinuous coefficient.

The difference solution $\{U_j^n\}$ constructed via the scheme (2.4) is extended to all of Π_T by defining

$$u^\Delta(x,t) = \sum_n \sum_j \chi_j^n(x,t) U_j^n, \qquad (x,t) \in \Pi_T, \tag{2.7}$$

where $\Delta = (\Delta x, \Delta t)$. Similarly, the discrete coefficient $\{\gamma_{j+\frac{1}{2}}\}$ is extended to all of $\mathbf{R}$ by defining

$$\gamma^\Delta(x) = \sum_j \chi_{j+\frac{1}{2}}(x) \gamma_{j+\frac{1}{2}}, \qquad x \in \mathbf{R},$$

where $\chi_{j+\frac{1}{2}}$ is the characteristic function for the interval $I_{j+\frac{1}{2}} = [x_j, x_{j+1})$.

Before proceeding to the proof of the main result of the paper, we provide the following as motivation for the singular mapping approach. Consider the conservation law (1.2) with $\underline{\gamma} > 0$. Assume that f is smooth, $f(u) \to 0$ as $u \to 0$, and f' is bounded away from zero, $f'(u) > f'_{min} > 0$. For example, the flux could be linear, $f(u) = u$. With these assumptions, the scheme (2.4) simplifies to

$$U_j^{n+1} = U_j^n - \lambda \left(\gamma_{j+\frac{1}{2}} f(U_j^n) - \gamma_{j-\frac{1}{2}} f(U_{j-1}^n) \right).$$

For bounded initial data u_0, it follows from results in [43] that (with an appropriate CFL condition) the scheme is monotone, produces solutions that are uniformly bounded, and a time continuity estimate of the type provided by Lemma 3.3 holds. In this case we can actually bound the spatial variation of U_j^n directly, as the following lemma shows. Note however that we are actually bounding the variation of $f(U_j^n)$, and because $f' > 0$, this leads to a variation bound on U_j^n.

Proposition 2.1. *The following spatial variation bounds hold uniformly in Δ:*

$$\sum_j \left| \gamma_{j+1/2} f(U_j^n) - \gamma_{j-1/2} f(U_{j-1}^n) \right| \le C_1, \quad \sum_j \left| U_{j+1}^n - U_j^n \right| \le C_2. \tag{2.8}$$

Proof. To prove the first estimate in (2.8),

$$\sum_j \left| \Delta_- \gamma_{j+1/2} f(U_j^n) \right| = \frac{1}{\lambda} \sum_j \left| U_j^{n+1} - U_j^n \right| \le C,$$

by Lemma 3.3. For the second estimate,

$$\begin{aligned} \sum_j \left| f(U_j^n) - f(U_{j-1}^n) \right| &= \frac{1}{\gamma_{j+1/2}} \sum_j \left| \gamma_{j+1/2} f(U_j^n) - \gamma_{j+1/2} f(U_{j-1}^n) \right| \\ &\le \frac{1}{\underline{\gamma}} \sum_j \left| \Delta_- \gamma_{j+1/2} f(U_j^n) \right| + \frac{1}{\underline{\gamma}} \|f\|_\infty \sum_j \left| \Delta_- \gamma_{j+1/2} \right|, \end{aligned}$$

and this is uniformly bounded, using the first estimate in (2.8) and the fact that $\gamma \in BV$. Now, by the mean value theorem,

$$\sum_j \left| f(U_j^n) - f(U_{j-1}^n) \right| = \sum_j \left| f'(\theta_j^n) \right| \left| U_j^n - U_{j-1}^n \right|,$$

for some θ_j^n between U_j^n and U_{j-1}^n, from which it follows that

$$C \ge \sum_j \left| f(U_j^n) - f(U_{j-1}^n) \right| \ge f'_{\min} \sum_j \left| U_j^n - U_{j-1}^n \right|,$$

completing the proof. ∎

For the conservation law (1.2), since $f' > 0$, the singular mapping Ψ defined by (3.1) in the next section reduces to

$$\Psi(\gamma, u) = \gamma f(u),$$

and the so first estimate in (2.8) gives a spatial variation bound for the transformed variable, $z_j^n = \Psi(\gamma_j, U_j^n)$. That bound is then used to derive a variation bound on the conserved quantity itself. Clearly, if f' is not bounded away from zero, the argument proving that $TV(u) < \infty$ breaks down, and it is possible to construct examples where $TV(u)$ actually blows up [43]. This variation blow-up can be viewed as resulting from resonance. On the other hand, the bound on $TV(z)$ remains valid even if f' vanishes. This leads to the idea of proving compactness for the transformed variable z, and then using the fact that the singular mapping $z = \Psi(\gamma, u)$ is strictly increasing as a function of u to recover the limiting value of the conserved quantity. Of course when f' is not bounded away from zero, the version of the singular mapping presented above is not monotone. To see how to circumvent this problem, suppose that the flux $f(u)$ has a single maximum at u^*, and notice that the *entropy* flux $F(u) = \operatorname{sign}(u - u^*)(f(u) - f(u^*))$ is strictly monotone, making it possible to use $\gamma F(u)$ as the singular mapping, at least in the purely hyperbolic case. When a degenerate diffusion term is present, a somewhat more complicated, but closely related mapping is required, as will be seen in the next section.

3. Convergence of approximate solutions

In this section, the goal is prove strong compactness of our approximate solution u^Δ and that any limit of a converging subsequence of u^Δ is a weak solution of (1.1).

In what follows we will be studying approximate solutions as the mesh size $\Delta = (\Delta x, \Delta t)$ decreases. We will always assume that the mesh refinement parameter Δ is decreasing with λ constant if $\|A'\|_\infty = 0$ (the purely hyperbolic case), or μ constant if $\|A'\|_\infty \neq 0$, the constant in each case determined by an appropriate CFL condition.

As previously mentioned, our approach is to prove a uniform variation bound with respect to a transformed quantity $z = \Psi(\gamma, u)$. The singular mapping $\Psi(\gamma, u)$ is designed to be Lipschitz continuous and strictly increasing as a function of u. Due to the presence of the diffusion term $A(u)$, it is necessary to modify the singular mapping somewhat from the purely hyperbolic setting, where the singular mapping would simply be

$$\Psi_{\text{hyp}}(\gamma, u) = \int_0^u |f_u(\gamma, w)|\, dw.$$

We add the diffusion term, and at the same time, zero out the contribution of the convective flux wherever $A(u)$ is nondegenerate. This allows us to analyze the convective portion $\mathcal{F}(\gamma, u)$ and the diffusive portion $A(u)$ separately. Let $\mathcal{S}$ be the characteristic function for $\bigcup_i [\alpha_i, \beta_i]$. The singular mapping is then

$$\Psi(\gamma, u) = \int_0^u \mathcal{S}(w)|f_u(\gamma, w)|\, dw + A(u) =: \mathcal{F}(\gamma, u) + A(u). \tag{3.1}$$

Lemma 3.1. *The mapping $\Psi(\gamma, u)$ is strictly increasing as a function of u. Furthermore, both Ψ and $\mathcal{F}$ belong to* $\mathrm{Lip}([\underline{\gamma}, \overline{\gamma}] \times [0, 1])$.

Proof. By definition,

$$\partial_u \Psi(\gamma, u) = \mathcal{S}(u)|f_u(\gamma, u)| + (1 - \mathcal{S}(u))A'(u) > 0 \quad \text{a.e.},$$

which proves strict monotonicity. For Lipschitz continuity of Ψ in u,

$$\begin{aligned} |\Psi(\gamma, u) - \Psi(\gamma, v)| &\leq \left| \int_v^u |f_u(\gamma, w)| dw \right| + |A(u) - A(v)| \\ &\leq (\|f_u\| + \|A'\|), \end{aligned}$$

and for Lipschitz continuity in γ,

$$|\Psi(\gamma_1, u) - \Psi(\gamma_2, u)| = \left| \int_0^u \mathcal{S}(w)|f_u(\gamma_1, w)| dw - \int_0^u \mathcal{S}(w)|f_u(\gamma_2, w)| dw \right.$$

$$\leq \left| \int_0^u |f_u(\gamma_1, w) - f_u(\gamma_2, w)| dw \right|$$
$$\leq |u - 0| L_{u\gamma} |\gamma_1 - \gamma_2| \leq L_{u\gamma} |\gamma_1 - \gamma_2|.$$

It is clear that essentially the same estimates prove that also $\mathcal{F} \in \mathrm{Lip}([\underline{\gamma}, \overline{\gamma}] \times [0, 1])$. □

We will use the notation $\|\Psi_u\|$, $\|\Psi_\gamma\|$, $\|\mathcal{F}_u\|$, and $\|\mathcal{F}_\gamma\|$ for the Lipschitz constants provided by the previous lemma. In what follows, C will denote a generic positive constant that that can depend on the data of the problem but not on Δ.

The approach will be to show compactness for the sequence of transformed functions

$$z^\Delta(x, t) = \Psi(\gamma^\Delta(x), u^\Delta(x, t)),$$

where $u^\Delta(x, t)$ denotes the numerical approximation generated by the scheme.

A finite difference scheme such as the scheme (2.4) is monotone [10, 22] if

$$U_j^n \leq V_j^n \quad \forall j \quad \Longrightarrow \quad U_j^{n+1} \leq V_j^{n+1} \quad \forall j. \tag{3.2}$$

Lemma 3.2. *For initial data $u_0(\cdot) \in [0, 1]$, if the parameters λ and μ are chosen so that the following CFL condition is satisfied*

$$2\lambda \|f_u\| + 2\mu \|A'\| \leq 1, \tag{3.3}$$

then the computed solutions remain in the interval $[0, 1]$, the CFL condition (3.3) holds for each succeeding time step and the scheme (2.4) is monotone.

Proof. The formula (2.4) defines U_j^{n+1} as a function

$$U_j^{n+1} = G_j(U_{j+1}^n, U_j^n, U_{j-1}^n, \gamma_{j+\frac{1}{2}}, \gamma_{j-\frac{1}{2}}).$$

The partial derivatives with respect to the conserved variables are

$$\partial U_j^{n+1} / \partial U_{j+1}^n = -\lambda f_u^-(\gamma_{j+\frac{1}{2}}, U_{j+1}^n) + \mu A'(U_{j+1}^n) \geq 0,$$
$$\partial U_j^{n+1} / \partial U_{j-1}^n = \lambda f_u^+(\gamma_{j-\frac{1}{2}}, U_{j-1}^n) + \mu A'(U_{j-1}^n) \geq 0,$$
$$\partial U_j^{n+1} / \partial U_j^n = 1 + \lambda f_u^-(\gamma_{j+\frac{1}{2}}, U_j^n) - \lambda f_u^+(\gamma_{j-\frac{1}{2}}, U_j^n) - 2\mu A'(U_j^n).$$

Thus U_j^{n+1} is a nondecreasing function of the conserved variables at the lower time level if

$$1 + \lambda f_u^-(\gamma_{j+\frac{1}{2}}, U_j^n) - \lambda f_u^+(\gamma_{j-\frac{1}{2}}, U_j^n) - 2\mu A'(U_{j+1}^n) \geq 0.$$

This will hold if the CFL condition (3.3) is satisfied for the solution at level n. If the CFL condition holds for the initial data, then since each of the functions $G_j(\cdot, \cdot, \cdot, \gamma_{j+\frac{1}{2}}, \gamma_{j-\frac{1}{2}})$ is nondecreasing as a function of its first three arguments, and $U_j^0 \in [0, 1]$

$$\begin{aligned} 0 = G_j(0, 0, 0, \gamma_{j+\frac{1}{2}}, \gamma_{j-\frac{1}{2}}) &\leq G_j(U_{j+1}^0, U_j^0, U_{j-1}^0, \gamma_{j+\frac{1}{2}}, \gamma_{j-\frac{1}{2}}) \\ &= U_j^{n+1} \leq G_j(1, 1, 1, \gamma_{j+\frac{1}{2}}, \gamma_{j-\frac{1}{2}}) = 1. \end{aligned} \tag{3.4}$$

The first and last equalities in this relationship result from the fact that for all γ, $f(\gamma, 0) = f_0$ and $f(\gamma, 1) = f_1$. Proceeding inductively, it is clear that the solution $U_j^n \in [0, 1]$ for each $n \geq 0$, and thus the CFL condition remains satisfied at each succeeding time level. That the scheme is monotone is clear from the fact that U_j^{n+1} is a nondecreasing function of U_{j+1}^n, U_j^n, and U_{j-1}^n. □

The next lemma is of fundamental importance for the subsequent analysis. In addition to providing for L^1 time continuity of the numerical approximations, it also plays a key role in our bound on the space translates of the transformed variable. With respect to the spatial variation, the situation is somewhat different here than in the case where γ is constant. Specifically, the variation (measured via the transformed variable) may actually increase from one time step to the next, and so the now classical total variation decreasing (TVD) argument is not available. Instead, we bound the spatial variation in terms of the L^1 time translates.

Lemma 3.3. *Assume that the CFL condition* (3.3) *is satisfied and that* $TV(u_0)$ *as well as* $TV(f(\gamma, u_0) - \partial_x A(u_0))$ *are finite. Then there exists a constant* C, *independent of* Δ, *such that*

$$\Delta x \sum_j \left|U_j^{n+1} - U_j^n\right| \le \Delta x \sum_j \left|U_j^1 - U_j^0\right| \le C\Delta t.$$

Proof. Starting from the marching formula (2.4), the time differences can be expressed as follows:

$$\begin{aligned} U_j^{n+1} - U_j^n &= U_j^n - U_j^{n-1} - \lambda\Delta_-\Big(h(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n) - h(\gamma_{j+\frac12}, U_{j+1}^{n-1}, U_j^{n-1})\Big) \\ &\quad + \mu\Delta_-\Delta_+(A(U_j^n) - A(U_j^{n-1})) \\ &= \Big(1 - \lambda C_{j+\frac12}^{n-\frac12} + \lambda B_{j-\frac12}^{n-\frac12} - 2\mu D_j^{n-\frac12}\Big)(U_j^n - U_j^{n-1}) \\ &\quad + \Big(-\lambda B_{j+\frac12}^{n-\frac12} + \mu D_{j+1}^{n-\frac12}\Big)(U_{j+1}^n - U_{j+1}^{n-1}) \\ &\quad + \Big(\lambda C_{j-\frac12}^{n-\frac12} + \mu D_{j-1}^{n-\frac12}\Big)(U_{j-1}^n - U_{j-1}^{n-1}), \end{aligned}$$

where

$$\begin{aligned} B_{j+\frac12}^{n-\frac12} &= \int_0^1 f_u^-(\gamma_{j+\frac12}, \theta U_{j+1}^n + (1-\theta)U_{j+1}^{n-1})d\theta \le 0, \\ C_{j+\frac12}^{n-\frac12} &= \int_0^1 f_u^+(\gamma_{j+\frac12}, \theta U_j^n + (1-\theta)U_j^{n-1})d\theta \ge 0, \\ D_j^{n-\frac12} &= \frac{A(U_j^n) - A(U_j^{n-1})}{U_j^n - U_j^{n-1}} \ge 0. \end{aligned}$$

Due to the CFL condition (3.3),

$$1 - \lambda C_{j+\frac12}^{n-\frac12} + \lambda B_{j-\frac12}^{n-\frac12} - 2\mu D_j^{n-\frac12} \ge 0, \tag{3.5}$$

and so

$$\begin{aligned} |U_j^{n+1} - U_j^n| \le &\Big(1 - \lambda C_{j+\frac12}^{n-\frac12} + \lambda B_{j-\frac12}^{n-\frac12} - 2\mu D_j^{n-\frac12}\Big)|U_j^n - U_j^{n-1}| \\ &+ \Big(-\lambda B_{j+\frac12}^{n-\frac12} + \mu D_{j+1}^{n-\frac12}\Big)|U_{j+1}^n - U_{j+1}^{n-1}| \\ &+ \Big(\lambda C_{j-\frac12}^{n-\frac12} + \mu D_{j-1}^{n-\frac12}\Big)|U_{j-1}^n - U_{j-1}^{n-1}|. \end{aligned} \tag{3.6}$$

Summing this inequality over j and multiplying by Δx gives

$$\Delta x \sum_j |U_j^{n+1} - U_j^n| \le \Delta x \sum_j |U_j^n - U_j^{n-1}|.$$

Continuing this way by induction yields

$$\Delta x \sum_j |U_j^{n+1} - U_j^n| \le \Delta x \sum_j |U_j^1 - U_j^0|.$$

Then, using Lipschitz continuity of h and the fact that $\lambda \Delta x = \Delta t$,

$$\begin{aligned} \Delta x \sum_j |U_j^1 - U_j^0| &= \Delta t \sum_j |\Delta_- h(\gamma_{j+\frac12}, U_{j+1}^0, U_j^0) - \Delta_- \frac{1}{\Delta x}\Delta_+ A(U_j^0)| \\ &\le \Delta t \sum_j |\Delta_- f(\gamma_j, U_j^0) - \Delta_- \frac{1}{\Delta x}\Delta_+ A(U_j^0)| \\ &\quad + \Delta t \sum_j |\Delta_- h(\gamma_{j+\frac12}, U_{j+1}^0, U_j^0) - \Delta_- h(\gamma_{j+\frac12}, U_j^0, U_j^0)| \\ &\quad + \Delta t \sum_j |\Delta_- f(\gamma_{j+\frac12}, U_j^0) - \Delta_- f(\gamma_j, U_j^0)| \end{aligned}$$

$$\le \Delta t \sum_j |\Delta_- f(\gamma_j, U_j^0) - \Delta_- \frac{1}{\Delta x} \Delta_+ A(U_j^0)|$$
$$+ 2\Delta t \|f_u\| \sum_j |U_{j+1}^0 - U_j^0| + 2\Delta t \|f_\gamma\| \sum_j |\gamma_{j-\frac{1}{2}} - \gamma_j|$$
$$\le \Delta t \Big(C_1 TV(f(\gamma, u_0) - \partial_x A(u_0)) + C_2 TV(u_0) + C_3 TV(\gamma) \Big).$$

□

Lemma 3.4. *Under the assumptions in Lemma 3.3, the computed solutions $u^\Delta(\cdot, t^n)$ satisfy a uniform $L^1(\mathbf{R})$ bound for $t^n \in [0,T]$:*

$$\|u^\Delta(\cdot, t^n)\|_{L^1(\mathbf{R})} \le CT + \|u_0\|_{L^1(\mathbf{R})}, \tag{3.7}$$

and if v^Δ is another solution generated using the same discretization of Π_T, the following discrete L^1 contraction property holds:

$$\|u^\Delta(\cdot, t^n) - v^\Delta(\cdot, t^n)\|_{L^1(\mathbf{R})} \le \|u^\Delta(\cdot, 0) - v^\Delta(\cdot, 0)\|_{L^1(\mathbf{R})}. \tag{3.8}$$

Proof. Using the triangle inequality and the result of Lemma 3.3 yields

$$\|u^\Delta(\cdot, t^n)\|_{L^1(\mathbf{R})} = \Delta x \sum_j |U_j^n| \le \Delta x \sum_j |U_j^n - U_j^{n-1}| + \Delta x \sum_j |U_j^{n-1}|$$
$$\le \Delta x \sum_j |U_j^1 - U_j^0| + \Delta x \sum_j |U_j^{n-1}|.$$

Proceeding by induction,

$$\|u^\Delta(\cdot, t^n)\|_{L^1(\mathbf{R})} \le n\Delta x \sum_j |U_j^1 - U_j^0| + \Delta x \sum_j |U_j^0|.$$

By Lemma 3.3, $n\Delta x \sum_j |U_j^1 - U_j^0| \le nC\Delta t \le CT$. Using that $\Delta x \sum_j |U_j^0| = \int_{\mathbf{R}} |u_0(x)|\,dx$, the proof of (3.7) is complete. The discrete L^1 contraction property (3.8) follows from the Crandall-Tartar lemma [10], using the fact that the operator which advances the initial approximation to time level n is monotone, conservative, and takes L^1 mesh functions into L^1 mesh functions. □

The next three lemmas provide a proof of compactness for the sequence of functions $\mathcal{F}^\Delta$ defined by

$$\mathcal{F}^\Delta(x,t) = \mathcal{F}(\gamma^\Delta(x), u^\Delta(x,t)).$$

In what follows, we will use the Kružkov entropy-entropy flux pair indexed by c:

$$V(u) = |u - c|, \qquad F(\gamma, u) = \sigma(u - c)(f(\gamma, u) - f(\gamma, c)),$$

where $\sigma(w) = w/|w|$ if $w \ne 0$ and $\sigma(0) = 0$. We use the notation $\mathcal{O}(\Delta\gamma_j)$ to mean terms which sum (over j) to $\mathcal{O}(TV(\gamma))$.

Lemma 3.5. *For each $c \in \mathbf{R}$,*

$$V(U_j^{n+1}) \le V(U_j^n) - \lambda \Delta_-^u H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \mu \Delta_- \Delta_+ |A(U_j^n) - A(c)| + \lambda \mathcal{O}(\Delta\gamma_j), \tag{3.9}$$

where the EO numerical entropy flux is given by

$$H(\gamma, v, u) = \frac{1}{2}(F(\gamma, u) + F(\gamma, v)) - \frac{1}{2} \int_u^v \sigma(w - c)|f_u(\gamma, w)|\,dw. \tag{3.10}$$

Proof. Let $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$. With

$$\rho_j^{n+1} = U_j^n - \lambda \Delta_-^u h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \mu \Delta_- \Delta_+ A(U_j^n),$$

the following discrete entropy inequality follows from Lemma 3.7 of [18]:

$$V(\rho_j^{n+1}) \le V(U_j^n) - \lambda \Delta_-^u H(\gamma_{j+\frac{1}{2}}, U_{j+1}, U_j^n) + \mu \Delta_- \Delta_+ |A(U_j^n) - A(c)|,$$

where

$$H(\gamma, v, u) = h(\gamma, v \vee c, u \vee c) - h(\gamma, v \wedge c, u \wedge c). \tag{3.11}$$

For a derivation of the explicit formula (3.10) for the EO numerical entropy flux from (3.11), see [43]. Then

$$
\begin{aligned}
V(U_j^{n+1}) \le V(U_j^n) - \lambda\Delta_-^u H(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n) + \mu\Delta_-\Delta_+|A(U_j^n) - A(c)| \\
- V(\rho_j^{n+1}) + V(U_j^{n+1}).
\end{aligned}
$$

It remains to show that $V(\rho_j^{n+1}) - V(U_j^{n+1}) = \lambda\mathcal{O}(\Delta\gamma_j)$:

$$
\begin{aligned}
|V(\rho_j^{n+1}) - V(U_j^{n+1})| &\le |\rho_j^{n+1} - U_j^{n+1}| \\
&= \lambda|\Delta_- h(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n) - \Delta_-^u h(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n)| \\
&\le \lambda(\|f_\gamma\| + \frac{1}{2}L_{u\gamma})|\gamma_{j+\frac12} - \gamma_{j-\frac12}| \\
&= \lambda\mathcal{O}(\Delta\gamma_j). \qquad (3.12)
\end{aligned}
$$

□

In what follows, $\chi_r(w;c)$ is the characteristic function for the interval $[c,+\infty)$, $\chi_l(w;c)$ is the characteristic function for $(-\infty,c]$, and $\chi_{(\alpha,\beta]}(w)$ is the characteristic function of the interval $(\alpha,\beta]$. The following identities (see [43] for a derivation) will be required in the proof of Lemma 3.6:

$$
\begin{aligned}
&\frac{1}{2}\Big(\Delta_-^u H(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n) + \Delta_-^u h(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n)\Big) \\
&\quad = \int_{U_j^n}^{U_{j+1}^n} \chi_r(w;c) f_u^-(\gamma_{j+\frac12}, w)\,dw + \int_{U_{j-1}^n}^{U_j^n} \chi_r(w;c) f_u^+(\gamma_{j+\frac12}, w)\,dw, \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\Big(\Delta_-^u H(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n) - \Delta_-^u h(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n)\Big) \\
&\quad = -\int_{U_j^n}^{U_{j+1}^n} \chi_l(w;c) f_u^-(\gamma_{j+\frac12}, w)\,dw - \int_{U_{j-1}^n}^{U_j^n} \chi_l(w;c) f_u^+(\gamma_{j+\frac12}, w)\,dw. \qquad (3.14)
\end{aligned}
$$

Lemma 3.6. *The following inequality holds for $c \ge u^*(\gamma_{j+\frac12})$:*

$$
\begin{aligned}
-\Delta_-^u\Big(\int_0^{U_{j+1}^n} \chi_r(w;c)|f_u(\gamma_{j+\frac12}, w)|\,dw + \frac{1}{\Delta x}\Delta_+(A(U_j^n) - A(c))_+\Big) \\
\le \frac{1}{\lambda}|U_j^{n+1} - U_j^n| + \mathcal{O}(\Delta\gamma_j), \qquad (3.15)
\end{aligned}
$$

and the following inequality holds for $c \le u^*(\gamma_{j-\frac12})$:

$$
\begin{aligned}
-\Delta_-^u\Big(\int_0^{U_j^n} \chi_l(w;c)|f_u(\gamma_{j-\frac12}, w)|\,dw - \frac{1}{\Delta x}\Delta_+(A(U_j^n) - A(c))_-\Big) \\
\le \frac{1}{\lambda}|U_j^{n+1} - U_j^n| + \mathcal{O}(\Delta\gamma_j). \qquad (3.16)
\end{aligned}
$$

Proof. Using "$\Delta_-^u - \Delta_- = \mathcal{O}(\Delta\gamma_j)$", we write the scheme (2.4) as

$$
\begin{aligned}
\Delta_-^u h(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n) - \frac{1}{\Delta x}\Delta_-\Delta_+(A(U_j^n) - A(c)) \\
= \frac{1}{\lambda}(U_j^n - U_j^{n+1}) + \mathcal{O}(\Delta\gamma_j). \qquad (3.17)
\end{aligned}
$$

Using (3.12), we write the discrete entropy inequality (3.9) as

$$
\begin{aligned}
\Delta_-^u H(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n) - \frac{1}{\Delta x}\Delta_-\Delta_+|A(U_j^n) - A(c)| \\
\le \frac{1}{\lambda}(V(U_j^n) - V(U_j^{n+1})) + \mathcal{O}(\Delta\gamma_j) \\
\le \frac{1}{\lambda}\left|U_j^n - U_j^{n+1}\right| + \mathcal{O}(\Delta\gamma_j). \qquad (3.18)
\end{aligned}
$$

Inequality (3.15) is derived by adding (3.17) and (3.18), and then dividing by two. The right hand side of this combination is

$$\begin{aligned}\frac{1}{2}\Big(\frac{1}{\lambda}(U_j^n - U_j^{n+1}) + \mathcal{O}(\Delta\gamma_j) + \frac{1}{\lambda}\left|U_j^n - U_j^{n+1}\right| + \mathcal{O}(\Delta\gamma_j)\Big)\\ = \frac{1}{\lambda}\left(U_j^n - U_j^{n+1}\right)_+ + \mathcal{O}(\Delta\gamma_j)\\ \le \frac{1}{\lambda}\left|U_j^n - U_j^{n+1}\right| + \mathcal{O}(\Delta\gamma_j).\end{aligned}$$

Applying (3.13), the left side is

$$\begin{aligned}&\frac{1}{2}\Big(\Delta_-^u h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \Delta_-^u H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n)\Big)\\ &\qquad - \frac{1}{2}\Big(\frac{1}{\Delta x}\Delta_-\Delta_+(A(U_j^n) - A(c)) + \frac{1}{\Delta x}\Delta_-\Delta_+\left|A(U_j^n) - A(c)\right|\Big)\\ &= \int_{U_j^n}^{U_{j+1}^n} \chi_r(w;c) f_u^-(\gamma_{j+\frac{1}{2}}, w)\,dw + \int_{U_{j-1}^n}^{U_j^n} \chi_r(w;c) f_u^+(\gamma_{j+\frac{1}{2}}, w)\,dw\\ &\qquad - \frac{1}{\Delta x}\Delta_-\Delta_+\left(A(U_j^n) - A(c)\right)_+.\end{aligned}$$

With $c \ge u^*(\gamma_{j+\frac{1}{2}})$, the integral involving $\chi_r(w;c)f_u^+(\gamma_{j+\frac{1}{2}}, w)$ is zero, and

$$\chi_r(w;c)f_u^-(\gamma_{j+\frac{1}{2}}, w) = -\chi_r(w;c)|f_u(\gamma_{j+\frac{1}{2}}, w)|,$$

completing the proof of (3.15). Inequality (3.16) is proven in a similar way by subtracting (3.17) from (3.18) and dividing by two, and then using (3.14). □

Let

$$c^R(\gamma) = u^*(\gamma) \vee c, \quad c^L(\gamma) = u^*(\gamma) \wedge c$$

and

$$\psi^R(\gamma, c, u) = \int_0^u \chi_r(w; c^R(\gamma))|f_u(\gamma, w)|\,dw + \frac{1}{\Delta x}\Delta_+(A(u) - A(c^R(\gamma)))_+, \tag{3.19}$$

$$\psi^L(\gamma, c, u) = \int_0^u \chi_l(w; c^L(\gamma))|f_u(\gamma, w)|\,dw - \frac{1}{\Delta x}\Delta_+(A(u) - A(c^L(\gamma)))_-. \tag{3.20}$$

Lemma 3.7. *For some constant C, independent of the mesh size Δ,*

$$\sum_j \left|\Delta_-^u \mathcal{F}(\gamma_{j-\frac{1}{2}}, U_j^n)\right| \le C.$$

Proof. Recalling Lemma 3.6:

$$-\Delta_-^u \psi^L(\gamma_{j-\frac{1}{2}}, c, U_j^n) \le \frac{1}{\lambda}|U_j^{n+1} - U_j^n| + O(\Delta\gamma_j), \tag{3.21}$$

$$-\Delta_-^u \psi^R(\gamma_{j+\frac{1}{2}}, c, U_{j+1}^n) \le \frac{1}{\lambda}|U_j^{n+1} - U_j^n| + O(\Delta\gamma_j). \tag{3.22}$$

Since the right hand sides of (3.21) and (3.22) are positive, both $\psi^L(\gamma_{j-\frac{1}{2}}, c, U_j^n)$ and $\psi^R(\gamma_{j+\frac{1}{2}}, c, U_{j+1}^n)$ have uniformly bounded negative variation. Moreover, both functions are bounded uniformly in Δ, and so the total variations are also uniformly bounded:

$$\sum_j \left|\Delta_-^u \psi^L(\gamma_{j-\frac{1}{2}}, c, U_j^n)\right| \le C, \quad \sum_j \left|\Delta_-^u \psi^R(\gamma_{j-\frac{1}{2}}, c, U_j^n)\right| \le C,$$

where the constant C is independent of the mesh size Δ. Fix i, and take $c = \alpha_i$, $c = \beta_i$. An application of the triangle inequality yields

$$\begin{aligned}\sum_j \Big| \Delta_-^u \Big(\psi^L(\gamma_{j-\frac{1}{2}}, \beta_i, U_j^n) - \psi^L(\gamma_{j-\frac{1}{2}}, \alpha_i, U_j^n) \\ + \psi^R(\gamma_{j-\frac{1}{2}}, \beta_i, U_j^n) - \psi^R(\gamma_{j-\frac{1}{2}}, \alpha_i, U_j^n) \Big) \Big| \le C. \end{aligned} \tag{3.23}$$

Using the fact that A is constant on the interval $[\alpha_i, \beta_i]$,

$$\psi^L(\gamma, \beta_i, u) - \psi^L(\gamma, \alpha_i, u) = \int_0^u \chi_{(\alpha_i^L(\gamma), \beta_i^L(\gamma)]}(w) |f_u(\gamma, w)| \, dw,$$

$$\psi^R(\gamma, \beta_i, u) - \psi^R(\gamma, \alpha_i, u) = \int_0^u \chi_{(\alpha_i^R(\gamma), \beta_i^R(\gamma)]}(w) |f_u(\gamma, w)| \, dw,$$

and so

$$\begin{aligned}\Big(\psi^L(\gamma, \beta_i, u) - \psi^L(\gamma, \alpha_i, u) + \psi^R(\gamma, \beta_i, u) - \psi^R(\gamma, \alpha_i, u) \Big) \\ = \int_0^u \Big(\chi_{(\alpha_i^L(\gamma), \beta_i^L(\gamma)]}(w) + \chi_{(\alpha_i^R(\gamma), \beta_i^R(\gamma)]}(w) \Big) |f_u(\gamma, w)| \, dw.\end{aligned}$$

Finally, applying the identity

$$\chi_{(\alpha_i^L(\gamma), \beta_i^L(\gamma)]} + \chi_{(\alpha_i^R(\gamma), \beta_i^R(\gamma)]} = \chi_{(\alpha_i, \beta_i]},$$

inequality (3.23) becomes

$$\sum_j \left| \Delta_-^u \int_0^{U_j^n} \chi_{(\alpha_i, \beta_i]}(w) |f_u(\gamma_{j-\frac{1}{2}}, w)| \, dw \right| \le C.$$

The proof is completed by observing that

$$\mathcal{F}(\gamma, u) = \sum_{i=1}^M \int_0^u \chi_{(\alpha_i, \beta_i]}(w) |f_u(\gamma, w)| \, dw,$$

and so the desired variation bound follows from the triangle inequality. □

Lemma 3.8. *There exists a subsequence of* $\{\mathcal{F}^\Delta\}$, *also denoted by* $\{\mathcal{F}^\Delta\}$, *and a function*

$$\overline{\mathcal{F}} \in L^1(\Pi_T) \bigcap L^\infty(\Pi_T)$$

such that $\mathcal{F}^\Delta \to \overline{\mathcal{F}}$ *in* $L^1_{\mathrm{loc}}(\Pi_T)$ *and a.e. in* Π_T. *Furthermore,* $\overline{\mathcal{F}}(\cdot, t) \in L^1(\mathbf{R})$ *for all* $t \in [0, T]$.

Proof. The first step of the proof is to establish a uniform variation bound for $\mathcal{F}^\Delta(\cdot, t^n)$. The step function $\mathcal{F}^\Delta(\cdot, t^n)$ has jumps at cell centers, due to jumps in γ^Δ, plus jumps at cell boundaries, due to jumps in u^Δ. The contribution of the jumps in γ^Δ is $\mathcal{O}(TV(\gamma))$, and so

$$TV(\mathcal{F}^\Delta(\cdot, t^n)) = \sum_j \left| \Delta_-^u \mathcal{F}(\gamma_{j-\frac{1}{2}}, U_j^n) \right| + \mathcal{O}(TV(\gamma)),$$

yielding the uniform variation bound as a result of Lemma 3.7.

Using that $u^\Delta \in [0, 1]$ and the definition (3.1) of $\mathcal{F}$ provides an L^∞ bound: $\|\mathcal{F}\|_\infty \le \|f_u\|$. To show that $\mathcal{F}^\Delta(\cdot, t)$ is in $L^1(\mathbf{R})$ (uniformly) on each of the time slices $0 \le t \le T$, observe that $\mathcal{F}(\gamma, 0) = 0$, and so, with n chosen so that $t \in [t^n, t^{n+1})$,

$$\begin{aligned}\int_{\mathbf{R}} \left| \mathcal{F}^\Delta(x, t) \right| \, dx &= \int_{\mathbf{R}} \left| \mathcal{F}^\Delta(x, t^n) \right| \, dx \\ &= \int_{\mathbf{R}} \left| \mathcal{F}(\gamma^\Delta(x), u^\Delta(x, t^n)) \right| \, dx \\ &= \int_{\mathbf{R}} \left| \mathcal{F}(\gamma^\Delta(x), u^\Delta(x, t^n)) - \mathcal{F}(\gamma^\Delta(x), 0) \right| \, dx \\ &\le \|f_u\| \int_{\mathbf{R}} \left| u^\Delta(x, t^n) - 0 \right| \, dx\end{aligned}$$

$$\leq \|f_u\|\Big(CT + \|u_0\|_{L^1(\mathbf{R})}\Big), \tag{3.24}$$

by Lemma 3.4. For time continuity, Lipschitz continuity of $\mathcal{F}$ with respect to its second argument gives

$$\begin{aligned}
&\int_{\mathbf{R}} |\mathcal{F}^\Delta(x, t^n + \Delta t) - \mathcal{F}^\Delta(x, t^n)|\, dx \\
&= \frac{\Delta x}{2} \sum_j \Big(|\mathcal{F}(\gamma_{j+\frac{1}{2}}, U_j^{n+1}) - \mathcal{F}(\gamma_{j+\frac{1}{2}}, U_j^n)| + |\mathcal{F}(\gamma_{j-\frac{1}{2}}, U_j^{n+1}) - \mathcal{F}(\gamma_{j-\frac{1}{2}}, U_j^n)| \Big) \\
&\leq C_1 \Delta x \sum_j |U_j^{n+1} - U_j^n| \leq C \Delta t,
\end{aligned}$$

by Lemma 3.3. Using this estimate it is easy to check that

$$\|\mathcal{F}^\Delta(\cdot, t+\tau) - \mathcal{F}^\Delta(\cdot, t)\|_{L^1(\mathbf{R})} \leq C(|\tau| + \Delta t). \tag{3.25}$$

Fix $X > 0$. From standard compactness results [40], there is a subsequence (which we do not bother to relabel) of $\{\mathcal{F}^\Delta\}$ such that for any fixed $X \geq 0$,

$$\int_{-X}^{X} \left|\mathcal{F}^\Delta(x,t) - \overline{\mathcal{F}}(x,t)\right| dx \to 0, \quad t \in [0,T], \quad \text{and} \tag{3.26}$$

$$\int_0^T \int_{-X}^{X} |\mathcal{F}^\Delta(x,t) - \overline{\mathcal{F}}(x,t)|\, dx\, dt \to 0, \tag{3.27}$$

for some measurable function $\overline{\mathcal{F}}$. By passing to a further subsequence if necessary, we may assume that $\mathcal{F}^\Delta$ converges a.e. in $[-X,X] \times [0,T]$ and a.e. in $[-X,X]$ for each $t \in [0,T]$. Let X_k be a countable set with $X_k \to \infty$. By letting $k \to \infty$, and taking further subsequences we can arrange that $\mathcal{F}^\Delta$ converges a.e. in $\mathbf{R} \times [0,T]$ and a.e. in $\mathbf{R}$ for each $t \in [0,T]$. To show that each $\overline{\mathcal{F}}(\cdot,t) \in L^1(\mathbf{R})$,

$$\begin{aligned}
\int_{-X}^{X} |\overline{\mathcal{F}}(x,t)|\, dx &\leq \int_{-X}^{X} |\overline{\mathcal{F}}(x,t) - \mathcal{F}^\Delta(x,t)|\, dx + \int_{-X}^{X} |\mathcal{F}^\Delta(x,t)|\, dx \\
&\leq \int_{-X}^{X} |\overline{\mathcal{F}}(x,t) - \mathcal{F}^\Delta(x,t)|\, dx + \|f_u\|\Big(CT + \|u_0\|_{L^1(\mathbf{R})}\Big).
\end{aligned}$$

Letting first $\Delta \to 0$ and then $X \to \infty$, we get

$$\|\overline{\mathcal{F}}(\cdot,t)\|_{L^1(\mathbf{R})} \leq \|f_u\|\Big(CT + \|u_0\|_{L^1(\mathbf{R})}\Big),$$

proving that $\overline{\mathcal{F}}(\cdot,t) \in L^1(\mathbf{R})$. It is clear from this estimate that

$$\|\overline{\mathcal{F}}\|_{L^1(\Pi_T)} \leq T\|f_u\|\Big(CT + \|u_0\|_{L^1(\mathbf{R})}\Big),$$

and so $\overline{\mathcal{F}} \in L^1(\Pi_T)$. That $\overline{\mathcal{F}} \in L^\infty(\Pi_T)$ is clear from the bound on $\mathcal{F}^\Delta$. □

To show strong compactness of $A(u^\Delta)$, we shall in the following three lemmas obtain uniform $L^2(\Pi_T)$ estimates on the space and time translates of $A(u^\Delta)$.

Lemma 3.9. *There exists a constant C, independent of Δ, such that*

$$\left\{\Delta t\, \Delta x \sum_{j,n} (\Delta_+ A(U_j^n))^2\right\}^{\frac{1}{2}} \leq C\Delta x. \tag{3.28}$$

Proof. The proof is a discrete energy argument similar to the one used in [28] (but the proof here is a bit simpler since we know that the difference approximations are L^1 Lipschitz continuous in time). Multiplying (2.4) by $\Delta t\, \Delta x U_j^n$, summing over n, j, and doing summation by parts in j, we find that

$$\Delta x \sum_{j,n} U_j^n (U_j^{n+1} - U_j^n) + \Delta t \sum_{j,n} U_j^n \Delta_- h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) + \frac{\Delta t}{\Delta x} \sum_{j,n} \Delta_+ U_j^m \Delta_+ A(U_j^n) = 0. \tag{3.29}$$

Observe that we can write

$$U_j^n(U_j^{n+1}-U_j^n)=\frac{1}{2}\Big((U_j^{n+1})^2-(U_j^n)^2-(U_j^{n+1}-U_j^n)^2\Big).$$

Since $A'(\cdot)\geq 0$, we also have

$$\frac{1}{\|A'\|}(\Delta_+A(U_j^n))^2\leq \left(\Delta_+U_j^n\right)\left(\Delta_+A(U_j^n)\right).$$

Using these observations in (3.29) as well as (2.2), (3.12) and Lemma 3.3, we get

$$\begin{aligned}
\Delta t\sum_{j,n}&U_j^n\Delta_+^u h_-(\gamma_{j+\frac12},U_j^n)+\Delta t\sum_{j,n}U_j^n\Delta_-^u h_+(\gamma_{j+\frac12},U_j^n)\\
&+\frac{1}{\|A'\|}\frac{\Delta t}{\Delta x}\sum_{j,n}(\Delta_+A(U_j^n))^2\\
&\leq\frac{-\Delta x}{2}\sum_{j,n}\Big((U_j^{n+1})^2-(U_j^n)^2-(U_j^{n+1}-U_j^n)^2\Big)+\mathcal{O}\left(TV(\gamma)\right)\\
&\leq\frac{\Delta x}{2}\sum_{j,n}(U_j^{n+1}-U_j^n)^2+\frac{\Delta x}{2}\sum_j(U_j^0)^2+\mathcal{O}\left(TV(\gamma)\right)\\
&\leq\max_{n,j}\left|U_j^n\right|\Delta x\sum_{j,n}\left|U_j^{n+1}-U_j^n\right|+\frac{\Delta x}{2}\sum_j(U_j^0)^2+\mathcal{O}\left(TV(\gamma)\right)\\
&\leq C_1,
\end{aligned}\tag{3.30}$$

for some finite constant C_1 that is independent of Δ.

To continue our analysis, we need to introduce the functions

$$\mathcal{H}^{\pm}(\gamma,\xi)=\int_0^\xi w\,\partial_w h_\pm(\gamma,w)\,dw=\xi h_\pm(\gamma,\xi)-\int_0^\xi h_\pm(\gamma,\xi)\,d\xi.$$

Then the following equalities hold

$$\begin{aligned}
U_j^n\Delta_+^u h_-(\gamma_{j+\frac12},U_j^n)&=\Delta_+^u\mathcal{H}_-(\gamma_{j+\frac12},U_j^n)-\int_{U_{j+1}^n}^{U_j^n}\Big(h_-(\gamma_{j+\frac12},w)-h_-(\gamma_{j+\frac12},U_{j+1}^n)\Big)\,dw,\\
U_j^n\Delta_-^u h_+(\gamma_{j+\frac12},U_j^n)&=\Delta_-^u\mathcal{H}_+(\gamma_{j+\frac12},U_j^n)+\int_{U_{j-1}^n}^{U_j^n}\Big(h_+(\gamma_{j+\frac12},w)-h_+(\gamma_{j+\frac12},U_{j-1}^n)\Big)\,dw.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{j,n}U_j^n\Delta_+^u h_-(\gamma_{j+\frac12},U_j^n)&=-\sum_{j,n}\int_{U_{j+1}^n}^{U_j^n}\Big(h_-(\gamma_{j+\frac12},w)-h_-(\gamma_{j+\frac12},U_{j+1}^n)\Big)\,dw+\mathcal{O}\left(TV(\gamma)\right),\\
\sum_{j,n}U_j^n\Delta_-^u h_+(\gamma_{j+\frac12},U_j^n)&=\sum_{j,n}\int_{U_{j-1}^n}^{U_j^n}\Big(h_+(\gamma_{j+\frac12},w)-h_+(\gamma_{j+\frac12},U_{j-1}^n)\Big)\,dw+\mathcal{O}\left(TV(\gamma)\right).
\end{aligned}\tag{3.31}$$

To bound the terms involving integrals, we need the following technical result (an easy proof can be found in [19]): Let $g:\mathbf{R}\to\mathbf{R}$ be a monotone Lipschitz continuous function with Lipschitz constant L_g. Then we have

$$\left|\int_{\xi_1}^{\xi_2}\big(g(w)-g(\xi_1)\big)\,dw\right|\geq\frac{1}{2L_g}\big(g(\xi_2)-g(\xi_1)\big)^2,\qquad\forall\xi_1,\xi_2\in\mathbf{R}.$$

Applying this to h_-, h_+ we find that

$$
\begin{aligned}
&-\int_{U_{j+1}^n}^{U_j^n} \Big(h_-(\gamma_{j+\frac{1}{2}}, w) - h_-(\gamma_{j+\frac{1}{2}}, U_{j+1}^n)\Big)\, dw \geq \frac{1}{2\|f_u\|}(\Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n))^2, \\
&\int_{U_{j-1}^n}^{U_j^n} \Big(h_+(\gamma_{j+\frac{1}{2}}, w) - h_+(\gamma_{j+\frac{1}{2}}, U_{j-1}^n)\Big)\, dw \geq \frac{1}{2\|f_u\|}(\Delta_-^u h_+(\gamma_{j+\frac{1}{2}}, U_j^n))^2.
\end{aligned} \tag{3.32}
$$

Inserting (3.32) into (3.30) yields via (3.31) the following inequality:

$$
\begin{aligned}
&\frac{\Delta t}{2\|f_u\|}\sum_{j,n}(\Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n))^2 + \frac{\Delta t}{2\|f_u\|}\sum_{j,n}(\Delta_-^u h_+(\gamma_{j+\frac{1}{2}}, U_j^n))^2 \\
&\qquad + \frac{1}{\|A'\|}\frac{\Delta t}{\Delta x}\sum_{j,n}(\Delta_+ A(U_j^n))^2 \leq C_1 + \mathcal{O}\left(TV(\gamma)\right).
\end{aligned} \tag{3.33}
$$

From (3.33), we conclude that (3.28) holds. □

Remark 3.1. Although we did not need it in the proof of Lemma 3.9, (3.33) also provides us with the estimates

$$
\left\{\Delta t\,\Delta x \sum_{j,n}(\Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n))^2\right\}^{\frac{1}{2}}, \left\{\Delta t\,\Delta x \sum_{j,n}(\Delta_-^u h_+(\gamma_{j+\frac{1}{2}}, U_j^n))^2\right\}^{\frac{1}{2}} \leq C\sqrt{\Delta x}, \tag{3.34}
$$

which imply, for any finite index set $\mathcal{J} \subset \mathbf{Z}$, an estimate of the type

$$
\Delta t\,\Delta x \sum_n \sum_{j\in\mathcal{J}} \Big(|\Delta_+^u h_-(\gamma_{j+\frac{1}{2}}, U_j^n))| + |\Delta_-^u h_+(\gamma_{j+\frac{1}{2}}, U_j^n))|\Big) \leq C(\mathcal{J})\sqrt{\Delta x}, \tag{3.35}
$$

where $C(\mathcal{J})$ is a constant depending on $\mathcal{J}$ but not Δ. This is a sort of variation bound and for that reason the estimates in (3.34) are sometimes called weak BV estimates in the literature [7, 19, 34]. We mention that under a stronger CFL condition it is possible to prove Lemma 3.10 without using L^1 Lipschitz continuity in time of the approximate solutions as stated in Lemma 3.3, see [28].

Lemma 3.10. *There exists a constant C, independent of Δ, such that*

$$
\|A(u^\Delta(\cdot + y, \cdot)) - A(u^\Delta(\cdot, \cdot))\|_{L^2(\Pi_T)} \leq C(|y| + 2\Delta x), \qquad \forall y \in \mathbf{R}.
$$

Proof. Let $\mathcal{I}(x) \in \mathbf{Z}$ be the integer such that $x \in [x_{\mathcal{I}(x)}, x_{\mathcal{I}(x)+1})$. Then we have that $\mathcal{I}(x_I + y) - I =: J$ for some integer $J \in \mathbf{Z}$ and $|J\Delta x| \leq |y| + \Delta x$. Equipped with this and Lemma 3.9, we calculate as follows:

$$
\begin{aligned}
\iint_{\Pi_T} \Big(A(u^\Delta(x+y,t)) - A(u^\Delta(x,t))\Big)^2\, dt\, dx &= \Delta t \sum_n \int_{\mathbf{R}} \Big(A(u^\Delta(x+y,t_n)) - A(U_j^n)\Big)^2 dx \\
&= \Delta t \sum_n \int_{\mathbf{R}} \left(\sum_{\ell=0}^{\mathcal{I}(x+y)-\mathcal{I}(x)-1} \Delta_+ A\left(U_{\mathcal{I}(x)+\ell}^n\right)\right)^2 dx \\
&\leq \Delta t \sum_n \int_{\mathbf{R}} \left(\sum_{\ell=0}^{J} \Delta_+ A\left(U_{\mathcal{I}(x)+\ell}^n\right)\right)^2 dx \\
&\leq (|J|+1)\,\Delta t \sum_n \int_{\mathbf{R}} \sum_{\ell=0}^{J} \Big(\Delta_+ A\left(U_{\mathcal{I}(x)+\ell}^n\right)\Big)^2 dx \\
&= (|J|+1)\,\Delta t\,\Delta x \sum_{n,I} \sum_{\ell=0}^{J} \Big(\Delta_+ A\left(U_{j+\ell}^n\right)\Big)^2, \\
&\leq C^2\,(|J|+1)\,|^2 (\Delta x)^2 \\
&\leq C^2 (|y| + 2\Delta x)^2,
\end{aligned} \tag{3.36}
$$

where C is the constant in Lemma 3.9. This concludes the proof of the lemma. □

Lemma 3.11. *There exists a constant C, independent of Δ, such that*

$$\|A(u^{\Delta}(\cdot,\cdot+\tau)) - A(u^{\Delta}(\cdot,\cdot))\|_{L^2(\Pi_{T-\tau})} \leq C\sqrt{\tau+\Delta t}, \qquad \forall \tau \in (0,T).$$

Proof. From Lemma 3.3 it follows that

$$\int_{\mathbf{R}} |u^{\Delta}(x,t+\tau) - u^{\Delta}(x,t)|\, dx = \mathcal{O}\left(\tau + \Delta t\right),$$

for all t,τ such that $t, t+\tau \in (0,T)$. Now "interpolating between L^1 and L^∞", we obtain the desired result:

$$\begin{aligned}\int_0^{T-\tau}\int_{\mathbf{R}} &(A(u^{\Delta}(x,t+\tau)) - A(u^{\Delta}(x,t)))^2\, dx\, dt \\ &\leq C_1 \int_0^{T-\tau}\int_{\mathbf{R}} |u^{\Delta}(x,t+\tau) - u^{\Delta}(x,t)|\, dx \leq C_2(\tau+\Delta t),\end{aligned}$$

where the constants C_1 and C_2 do not depend on Δ. □

Remark 3.2. Again under a stronger CFL condition, it is possible to prove Lemma 3.11 without using L^1 Lipschitz continuity in time of the approximate solutions (Lemma 3.3), see [28].

Lemma 3.12. *There exists a subsequence of $\{A^{\Delta}\}$, also denoted by $\{A^{\Delta}\}$, and a function*

$$\overline{A} \in L^2(0,T;H^1(\mathbf{R}))$$

such that $A^{\Delta} \to \overline{A}$ in $L^2_{\mathrm{loc}}(\Pi_T)$ and a.e. in Π_T. Furthermore, $\overline{A} = A(u)$ a.e. in Π_T, where u denotes the L^∞ weak-$*$ limit of u^{Δ}.

Proof. The first part of the lemma is a straightforward consequence of Lemmas 3.10 and 3.11 and Kolmogorov's L^p compactness criterion (see Theorem IV.8.21 in [12]). The proof of the second part can be found in [30]. □

We are now in a position to prove our main convergence theorem.

Theorem 3.1. *Let u^{Δ} be defined by* (2.7) *and the scheme* (2.4), (2.5), (2.6). *Assume that*

$$TV(u_0) \quad \text{and} \quad TV\left(f(\gamma,u_0) - \partial_x A\left(u_0\right)\right)$$

are finite, and that the CFL condition (3.3) *holds. Then there exists a subsequence of $\{u^{\Delta}\}$, also denoted by $\{u^{\Delta}\}$, and a function u such that $u^{\Delta} \to u$ in $L^1_{\mathrm{loc}}(\Pi_T)$ and a.e. in Π_T. The limit function u is a weak solution of the initial value problem* (1.1) *in the sense of Definition 1.1.*

Proof. Let $z^{\Delta} = \Psi(\gamma^{\Delta},u^{\Delta}) = \mathcal{F}^{\Delta} + A^{\Delta}$. Both of the sequences $\{\mathcal{F}^{\Delta}\}$ and $\{A^{\Delta}\}$ have subsequences converging boundedly a.e. in Π_T, by Lemma 3.8 and Lemma 3.12, and therefore in $L^1_{\mathrm{loc}}(\Pi_T)$. By passing to a further subsequence on which both $\{\mathcal{F}^{\Delta}\}$ and $\{A^{\Delta}\}$ converge, there is a subsequence, also denoted by $\{z^{\Delta}\}$, such that for some $z \in L^1_{\mathrm{loc}}(\Pi_T) \bigcap L^\infty(\Pi_T)$, $z^{\Delta} \to z$ in $L^1_{\mathrm{loc}}(\Pi_T)$ and a.e. Let $u(x,t) = \Psi^{-1}(\gamma(x), z(x,t))$, which is well-defined a.e. in Π_T, thanks to the fact that $\Psi(\gamma,w)$ is strictly increasing as a function of w. The immediate goal is to show that we have pointwise convergence of u^{Δ} a.e. in Π_T. Suppressing the dependence on the point (x,t),

$$\begin{aligned}\left|\Psi(\gamma,u^{\Delta}) - \Psi(\gamma,u)\right| &\leq \left|\Psi(\gamma,u^{\Delta}) - \Psi(\gamma^{\Delta},u^{\Delta})\right| + \left|\Psi(\gamma^{\Delta},u^{\Delta}) - \Psi(\gamma,u)\right| \\ &\leq \|\Psi_\gamma\| \left|\gamma - \gamma^{\Delta}\right| + \left|z^{\Delta} - z\right|.\end{aligned}$$

Thus, since $\gamma^{\Delta} \to \gamma$ a.e. and $z^{\Delta} \to z$ a.e., $\Psi(\gamma,u^{\Delta}) \to \Psi(\gamma,u)$ a.e. in Π_T. Since $\Psi(\gamma,\cdot)$ is strictly increasing, it follows that $u^{\Delta} \to u$ boundedly a.e., from which convergence in $L^1_{\mathrm{loc}}(\Pi_T)$ follows.

Since each $u^{\Delta} \in [0,1]$, it is clear that $u \in L^\infty(\Pi_T)$. Also, it is immediate from Lemma 3.12 that $A(u) \in L^2(0,T;H^1(\mathbf{R}))$. To prove that $u \in L^1(\Pi_T)$, fix $X > 0$, and set $\Pi_T^X = [-X,X] \times [0,T]$. Then

$$\iint_{\Pi_T^X} |u(x,t)|\, dt\, dx \leq \iint_{\Pi_T^X} \left|u(x,t) - u^{\Delta}(x,t)\right|\, dt\, dx + \iint_{\Pi_T^X} \left|u^{\Delta}(x,t)\right|\, dt\, dx$$

$$\leq \iint_{\Pi_T^X} \left| u(x,t) - u^\Delta(x,t) \right| \, dt\, dx + \iint_{\Pi_T} \left| u^\Delta(x,t) \right| \, dt\, dx.$$

After invoking Lemma 3.4, then letting $\Delta \to 0$ and subsequently $X \to \infty$, we get

$$\iint_{\Pi_T} |u(x,t)| \, dt\, dx \leq T\Big(CT + \|u(\cdot,0)\|_{L^1(\mathbf{R})}\Big),$$

proving that $u \in L^1(\Pi_T)$. As a result of the time continuity estimate of Lemma 3.3, and by passing to a further subsequence if necessary, $u^\Delta(\cdot,t) \to u(\cdot,t)$ in $L^1(\mathbf{R})$ for each $t \in [0,T]$ (see, e.g., the proof of Lemma 16.8 of [40]). To show that $u \in C(0,T;L^1(\mathbf{R}))$, let $\tau > 0$, and apply the triangle inequality:

$$\begin{aligned} \|u(\cdot,t+\tau) - u(\cdot,t)\|_{L^1(\mathbf{R})} &\leq \|u(\cdot,t+\tau) - u^\Delta(\cdot,t+\tau)\|_{L^1(\mathbf{R})} \\ &\quad + \|u^\Delta(\cdot,t+\tau) - u^\Delta(\cdot,t)\|_{L^1(\mathbf{R})} + \|u^\Delta(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R})}. \end{aligned}$$

It is a simple consequence of Lemma 3.3 that

$$\|u^\Delta(\cdot,t+\tau) - u^\Delta(\cdot,t)\|_{L^1(\mathbf{R})} \leq C(\tau + \Delta t).$$

Using this fact, and letting $\Delta \to 0$ in the preceding inequality gives the desired L^1 time continuity estimate for the limit function u, proving that $u \in C(0,T;L^1(\mathbf{R}))$.

It remains to show that the limit solution u is a weak solution to the initial value problem (1.1), for which a version of the Lax-Wendroff theorem is required. Let $\phi \in C_0^\infty$ with $\phi(x,T) = 0$. Fix $X > 0$ such that ϕ vanishes for $|x| \geq X$. Let $\phi_j^n = \phi(x_j,t^n)$. Multiplying the difference scheme (2.4) by $\phi_j^n \Delta x$, then summing by parts results in

$$\begin{aligned} &- \Delta x\, \Delta t \sum_{j,n} U_j^n \frac{\phi_j^n - \phi_j^{n-1}}{\Delta t} - \Delta x \sum_j U_j^0 \phi_j^0 \\ &\qquad - \Delta x\, \Delta t \sum_{j,n} h(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n) \frac{1}{\Delta x} \Delta_+ \phi_j^n + \Delta x\, \Delta t \sum_{j,n} A(U_j^n) \frac{1}{\Delta x^2} \Delta_- \Delta_+ \phi_j^n = 0, \end{aligned}$$

where $J\Delta x = X$ and $N\Delta t = T$, and $j \in \{-J,\dots,J\}$, $n \in \{0,\dots,N\}$. For $(x,t) \in R_j^n$ we have that

$$\left.\begin{aligned} \frac{\phi_j^n - \phi_j^{n-1}}{\Delta t} &= \phi_t(x,t) \\ \frac{1}{\Delta x}\Delta_+ \phi_j^n &= \phi_x(x,t) \\ \frac{1}{\Delta x^2}\Delta_- \Delta_+ \phi_j^n &= \phi_{xx}(x,t) \end{aligned}\right\} + \mathcal{O}\left(\Delta t + \Delta x\right),$$

and

$$\phi_j^0 = \phi(x,0) + \mathcal{O}\left(\Delta x\right).$$

Therefore we have that

$$\begin{aligned} &\Delta x \Delta t \sum_{j,n} U_j^n \frac{\phi_j^n - \phi_j^{n-1}}{\Delta t} + \Delta x \Delta t \sum_{j,n} A(U_j^n) \frac{1}{\Delta x^2} \Delta_- \Delta_+ \phi_j^n + \Delta x \sum_j U_j^0 \phi_j^0 \\ &\qquad = \iint_{\Pi_T} \Big(u^\Delta \phi_t + A(u^\Delta)\phi_{xx}\Big)\, dt\, dx + \int_{\mathbf{R}} u^\Delta(x,0)\phi(x,0)\, dx + \mathcal{O}\left(\Delta x + \Delta t\right). \end{aligned} \tag{3.37}$$

Now defining

$$\gamma_j = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \gamma(x)\, dx,$$

we compute

$$\begin{aligned} h(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n) - f(\gamma_j, U_j^n) &= h(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n) - h(\gamma_j, U_{j+1}^n, U_j^n) \\ &\quad + h(\gamma_j, U_{j+1}^n, U_j^n) - h(\gamma_j, U_j^n, U_j^n) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\gamma_j}^{\gamma_{j+\frac12}} h_\gamma(w, U_{j+1}^n, U_j^n)\,dw + \int_{U_j^n}^{U_{j+1}^n} f_u^-(\gamma_j, w)\,dw \\
&= \int_{\gamma_j}^{\gamma_{j+\frac12}} h_\gamma(w, U_{j+1}^n, U_j^n)\,dw + \Delta_+^u h_-(\gamma_j, U_j^n).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\Delta x \Delta t \sum_{j,n} h(\gamma_{j-\frac12}, U_j^n, U_{j-1}^n) \frac{1}{\Delta x} \Delta_- \phi_j^n \\
&= \iint_{\Pi_T} f(\gamma^\Delta, u^\Delta) \phi_x(x,t)\,dt\,dx + \Delta x \mathcal{O}\left(TV(\gamma)\right) + \Delta x \Delta t \sum_{j,n} \Delta_+^u h_-(\gamma_{j+\frac12}, U_j^n) \frac{1}{\Delta x} \Delta_- \phi_j^n.
\end{aligned}
$$

Now, using (3.35), we find that the last term above can be bounded by

$$
C_\phi \sqrt{\Delta x}, \tag{3.38}
$$

where the constant C_ϕ depends on ϕ but not on Δ. Collecting these bounds we find that

$$
\iint_{\Pi_T} \left(u^\Delta \phi_t + f(\gamma^\Delta, u^\Delta)\phi_x + A(u^\Delta)\phi_{xx}\right) dt\,dx + \int_{\mathbf{R}} u^\Delta(x,0)\phi(x,0)\,dx = \mathcal{O}\left(\sqrt{\Delta}\right), \tag{3.39}
$$

where the $\mathcal{O}\left(\right)$ term on the right depends only on ϕ. Since $A(u) \in L^2(0,T; H^1(\mathbf{R}))$ it is possible to integrate by parts in x, and hence

$$
\iint_{\Pi_T} A(u^\Delta)\phi_{xx}\,dt\,dx \to \iint_{\Pi_T} A(u)\phi_{xx}\,dt\,dx = -\iint_{\Pi_T} A(u)_x \phi_x\,dt\,dx.
$$

Letting $\Delta \downarrow 0$, we thus find that u is a weak solution, and that we have a "weak convergence rate" of $1/2$. □

Remark 3.3. Note that $TV_{[-X,X]} z^\Delta(\cdot, t) \le C$, for some constant C that is independent of Δ but dependent on X. Here the R dependence comes from only having an L^2 space translation estimate on $A(u^\Delta)$. This bound could have been used directly to get strong compactness of z^Δ.

4. Additional regularity

In this section we show that $A(u)$, where u is the limit constructed in Theorem 3.1, can be identified a.e. with a $C^{1,\frac12}(\Pi_T)$ function. To this end, we will in the next two lemmas obtain uniform L^∞ estimates of the space and time translates of $\{A(U_j^n)\}$.

Lemma 4.1. *There exists a constant C, independent of Δ, such that*

$$
\left|A(U_j^n) - A(U_i^n)\right| \le C|j-i|\Delta x, \qquad \forall i,j \in \mathbf{Z}.
$$

Proof. Assuming that M is a positive integer such that $U_j^n = 0$ for $j \le -M+1$, from the definition of the scheme (2.4),

$$
\begin{aligned}
\frac{1}{\Delta x}\left|\Delta_+ A(U_j^n)\right| - \left|h(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n)\right| &\le \left|h(\gamma_{j+\frac12}, U_{j+1}^n, U_j^n) - \frac{1}{\Delta x}\Delta_+ A(U_j^n)\right| \\
&= \left|\sum_{\ell=-M}^{j} \Delta_- \left(h(\gamma_{\ell+\frac12}, U_{\ell+1}^n, U_\ell^n) - \frac{1}{\Delta x}\Delta_+ A(U_\ell^n)\right)\right| \\
&= \frac{1}{\lambda}\left|\sum_{\ell=-M}^{j} \left(U_\ell^{n+1} - U_\ell^n\right)\right| \\
&\le \frac{\Delta x}{\Delta t}\sum_j \left|U_j^{n+1} - U_j^n\right| \le C.
\end{aligned}
$$

Since $h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n)$ is bounded,

$$\left|\Delta_+ A(U_j^n)\right| \le C\Delta x,$$

and the lemma follows. □

Although the above $L^\infty(\Pi_T)$ space translation estimate was an easy consequence of Lemma 3.3, the $L^\infty(\Pi_T)$ time translation estimate is a bit trickier to obtain, as shown by the proof of the next lemma.

Lemma 4.2. *There exists a constant C, independent of Δ, such that*

$$\left|A(U_j^n) - A(U_j^m)\right| \le C\sqrt{|n-m|\Delta t}.$$

Proof. The proof is an adaptation of a technique used in [17]. To prove the lemma, we shall need to work with an interpolant of of the discrete values $\{U_j^n\}$ that is continuous everywhere and differentiable almost everywhere. For this purpose, define $\tilde{u}^n(x)$ as

$$\tilde{u}^n(x) = \frac{1}{\Delta x}\Big((x - x_{j-1})\, U_j^n + (x_j - x)\, U_{j-1}^n\Big), \qquad x \in [x_{j-1}, x_j].$$

Then define

$$\tilde{u}^\Delta(x,t) = \frac{1}{\Delta t}\Big((t - t_n)\, \tilde{u}^{n+1}(x) + (t_{n+1} - t)\, \tilde{u}^n(x)\Big), \qquad t \in [t_n, t_{n+1}].$$

As before, let $\mathcal{I}(x) \in \mathbf{Z}$ be the integer satisfying $x \in [x_{\mathcal{I}(x)}, x_{\mathcal{I}(x)+1})$, so that $\mathcal{I}(x_j + \alpha) - j =: J$ for some $J \in \mathbf{Z}$. Since $\tilde{u}^\Delta$ is differentiable in time almost everywhere on Π_T, we can proceed as follows:

$$\begin{aligned}
&\int_{x_j}^{x_j+\alpha} \left(\tilde{u}^\Delta(x,t_n) - \tilde{u}^\Delta(x,t_m)\right)\, dx \\
&\quad = \int_{x_j}^{x_j+\alpha}\int_{t_m}^{t_n} \partial_t \tilde{u}^\Delta(x,\tau)\, d\tau\, dx \\
&\quad = \sum_{k=j}^{\mathcal{I}(x_j+\alpha)-1} \sum_{\ell=m}^{n-1} \iint_{R_{k-1/2}^\ell} \partial_t \tilde{u}^\Delta(x,\tau)\, d\tau\, dx \\
&\quad = \sum_{k=j}^{\mathcal{I}(x_j+\alpha)-1} \sum_{\ell=m}^{n-1} \int_{x_k}^{x_{k+1}} \left(\tilde{u}^{\ell+1}(x) - \tilde{u}^\ell(x)\right)\, dx \\
&\quad = \sum_{k=j}^{\mathcal{I}(x_j+\alpha)-1} \sum_{\ell=m}^{n-1} \int_{x_k}^{x_{k+1}} \frac{1}{\Delta x}\Big((x - x_k)\left(U_{k+1}^{\ell+1} - U_{k+1}^\ell\right) + (x_{k+1} - x)\left(U_k^{\ell+1} - U_k^\ell\right)\Big)\, dx \\
&\quad = \frac{\Delta x}{2} \sum_{\ell=m}^{n-1} \sum_{k=j}^{\mathcal{I}(x_j+\alpha)-1} \Big(\left(U_{k+1}^{\ell+1} - U_{k+1}^\ell\right) + \left(U_k^{\ell+1} - U_k^\ell\right)\Big).
\end{aligned}$$

Hence, using Lemma 3.3, we find that

$$\left|\int_{x_j}^{x_j+\alpha} \left(\tilde{u}^\Delta(x,t_n) - \tilde{u}^\Delta(x,t_m)\right)\, dx\right| \le C\,|n-m|\,\Delta t. \tag{4.1}$$

Now set $\alpha = \sqrt{|n-m|\Delta t}$. By the mean value theorem, there exists a number x^* in $[x_j, x_j+\alpha]$ such that

$$\left|\tilde{u}^\Delta(x^*,t_n) - \tilde{u}^\Delta(x^*,t_m)\right| = \frac{1}{\alpha}\left|\int_{x_j}^{x_j+\alpha} \left(\tilde{u}^\Delta(\xi,t_n) - \tilde{u}^\Delta(\xi,t_m)\right)\, d\xi\right| = \mathcal{O}\left(\sqrt{|n-m|\,\Delta t}\right), \tag{4.2}$$

where we have used (4.1). From this we derive the following estimate:

$$
\begin{aligned}
E_2 = \left|A(\tilde{u}^\Delta(x^*,t_n)) - A(\tilde{u}^\Delta(x^*,t_m))\right| &\le \|A'\| \left|\tilde{u}^\Delta(x^*,t_n) - \tilde{u}^\Delta(x^*,t_m)\right| \\
&= \mathcal{O}\left(\sqrt{|n-m|\Delta t}\right).
\end{aligned} \tag{4.3}
$$

By the triangle inequality,

$$\left|A(U_j^n) - A(U_j^m)\right| = \left|A(\tilde{u}^\Delta(x_j,t_n)) - A(\tilde{u}^\Delta(x_j,t_m))\right| \le E_1 + E_2 + E_3,$$

where

$$
\begin{aligned}
E_1 &= |A(\tilde{u}^\Delta(x_j,t_n)) - A(\tilde{u}^\Delta(x^*,t_n))|, \\
E_2 &= |A(\tilde{u}^\Delta(x^*,t_n)) - A(\tilde{u}^\Delta(x^*,t_m))|, \\
E_3 &= |A(\tilde{u}^\Delta(x^*,t_m)) - A(\tilde{u}^\Delta(x_j,t_m))|.
\end{aligned}
$$

By Lemma 4.1,

$$E_1 + E_3 = \mathcal{O}\left(|x^* - x_j|\right) = \mathcal{O}\left(\alpha\right) = \mathcal{O}\left(\sqrt{|n-m|\,\Delta t}\right),$$

which finishes the proof of the lemma. □

Introduce the piecewise bilinear interpolant $A^\Delta(x,t)$ interpolating the values $A(U_j^n)$ in the same way that $\tilde{u}^\Delta$ interpolates the values U_j^n. Then the main theorem of this section can be stated as follows:

Theorem 4.1. *There exists a subsequence of A^Δ, also denoted by A^Δ, and a function*

$$\bar{A} \in C^{1,\frac{1}{2}}(\Pi_T)$$

such that $A^\Delta \to \bar{A}$ in $L^\infty_{\mathrm{loc}}(\Pi_T)$. Furthermore, $\bar{A} = A(u)$ a.e. in Π_T, where u is the weak solution constructed in Theorem 3.1.

Proof. Let (j,n) and (i,m) be integers such that $(x,t) \in R^n_{j+\frac{1}{2}}$ and $(x+y,t+\tau) \in R^m_{i+\frac{1}{2}}$. Then

$$\left|A^\Delta(x+y,t+\tau) - A^\Delta(x,t)\right| \le E_1 + E_2 + E_3,$$

where

$$
\begin{aligned}
E_1 &= \left|A^\Delta(x+y,t+\tau) - A^\Delta(x_i,t_m)\right|, \\
E_2 &= \left|A^\Delta(x_i,t_m) - A^\Delta(x_j,t_n)\right|, \\
E_3 &= \left|A^\Delta(x_j,t_n) - A^\Delta(x,t)\right|.
\end{aligned}
$$

Lemmas 4.1 and 4.2 imply that $E_2 = \mathcal{O}\left(|i-j|\Delta x + \sqrt{|n-m|\Delta t}\right)$. Obviously, for $(x,t) \in R^n_{j+\frac{1}{2}}$ we have

$$
\begin{aligned}
&\min\left\{A(U_j^n), A(U_{j+1}^n), A(U_j^{n+1}), A(U_{j+1}^{n+1})\right\} \\
&\qquad \le A^\Delta(x,t) \le \max\left\{A(U_j^n), A(U_{j+1}^n), A(U_j^{n+1}), A(U_{j+1}^{n+1})\right\}.
\end{aligned} \tag{4.4}
$$

From this and again Lemmas 4.1 and 4.2, we have $E_1 + E_3 = \mathcal{O}\left(\Delta x + \sqrt{\Delta t}\right)$. Thus there exists a constant C, independent of Δ, such that

$$\left|A^\Delta(x+y,t+\tau) - A^\Delta(x,t)\right| \le C\left(y + \tau + \Delta x + \sqrt{\Delta t}\right).$$

Equipped with this estimate, we repeat the proof of the Ascoli-Arzela theorem to conclude that there is a subsequence of $\{A^\Delta\}$, still denoted by $\{A^\Delta\}$, and a limit function $\bar{A} \in C^{1,\frac{1}{2}}(\Pi_T)$ such that $A^\Delta \to \bar{A}$ uniformly on compact sets and pointwise on Π_T.

Next we show that $\bar{A} = A(u)$ almost everywhere. Without loss of generality, assume for some sequence $\Delta \to 0$ that $u^\Delta \to u$ a.e. and $A^\Delta \to \bar{A}$ pointwise everywhere. Pick an arbitrary but fixed point (x,t) such that $u^\Delta(x,t) \to u(x,t)$. We have

$$\left|A(u(x,t)) - \bar{A}(x,t)\right| \le E_1 + E_2 + E_3,$$

where

$$E_1 = \left|A(u(x,t)) - A(u^\Delta(x,t))\right|,$$
$$E_2 = \left|A(u^\Delta(x,t)) - A^\Delta(x,t)\right|,$$
$$E_3 = \left|A^\Delta(x,t) - \bar{A}(x,t)\right|.$$

Obviously, E_1 and E_3 vanish as $\Delta \to 0$. Let j and n be integers such that $(x,t) \in R^n_{j+\frac{1}{2}}$. Then, in view of (4.4) and Lemmas 4.1 and 4.2, $E_2 = \left|A(U_j^n) - A^\Delta(x,t)\right| = \mathcal{O}\left(\Delta x + \sqrt{\Delta t}\right)$, which also tends to zero as $\Delta \to 0$. This concludes the proof of the lemma. □

5. Entropy satisfaction

Because the diffusion term is strongly degenerate, solutions to the initial value problem (1.1) can develop discontinuities, and so solutions are not a priori unique. In the case where the coefficient γ is continuous, an entropy condition is used to single out the physically relevant solution. In the strictly hyperbolic, constant γ setting, discrete entropy inequalities for monotone schemes were first established by Harten, Hyman, and Lax [22], and Crandall and Majda [10]. For one-dimensional degenerate parabolic equations, with γ constant, Evje and Karlsen [15, 18] established cell entropy inequalities for both explicit and implicit monotone finite difference schemes. Karlsen and Risebro [28] proved a cell entropy inequality for the multidimensional version of the scheme (2.4), with spatially varying coefficients that are continuous but "rough". In each of these cases, there is a continuous analog of the discrete entropy inequality which can be used to prove uniqueness of the computed limit solutions, generally via some variant of the classical Kružkov [35] doubling of variables argument.

The presence of discontinuities in the coefficient γ presents some analytical difficulties. In [33], Klingenberg and Risebro observed that the Kružkov entropy inequality does not make sense in this situation, and they used a so-called wave entropy inequality to prove uniqueness for an initial value problem very close to the purely hyperbolic version of (1.1). Also in the hyperbolic setting, Towers [42] derived a cell entropy inequality similar to the one to be established below, and used it to prove uniqueness within the class of piecewise smooth solutions. Another approach is to prove uniqueness within the class of solutions that are the (strong) limits of an equation with a smoothed coefficient γ^ε, as the smoothing parameter ε tends to zero. Klausen and Risebro [31] and Klingenberg and Risebro [32] used this approach for the case of zero diffusion, and Karlsen, Risebro, and Towers [30] used it for degenerate parabolic equations of the type (1.1).

In the remainder of this section we establish a cell entropy inequality for solutions of our finite difference algorithm, and then show that piecewise smooth limit solutions satisfy a Kružkov-type entropy inequality.

Lemma 5.1. *The following cell entropy inequality is satisfied by approximate solutions U_j^n generated by the scheme* (2.4)*:*

(5.1)
$$V(U_j^{n+1}) \le V(U_j^n) - \Delta_-\left(\lambda H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - \mu\Delta_+\left|A(U_j^n) - A(c)\right|\right) + \lambda\left|\Delta_+ f(\gamma_{j-\frac{1}{2}}, c)\right|,$$

where the numerical entropy flux $H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n)$ is defined by

$$H(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) = f(\gamma_{j+\frac{1}{2}}, U_{j+1}^n \vee c, U_j^n \vee c) - f(\gamma_{j+\frac{1}{2}}, U_{j+1}^n \wedge c, U_j^n \wedge c). \tag{5.2}$$

Proof. The proof is an adaptation of a portion of the proof of Lemma 4.2 of [10]. Let

$$G_j(U_{j+1}^n, U_j^n, U_{j-1}^n) = U_j^{n+1} = U_j^n - \lambda\Delta_-\left(h(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n) - \frac{1}{\Delta x}\Delta_+ A(U_j^n)\right),$$

and observe that

$$G_j(c,c,c) = c - \lambda\Delta_- f(\gamma_{j+\frac{1}{2}}, c).$$

The following inequalities are a consequence of monotonicity of the numerical scheme:

$$G_j(c,c,c) \vee G_j(U_{j+1}^n, U_j^n, U_{j-1}^n) \le G_j(c \vee U_{j+1}^n, c \vee U_j^n, c \vee U_{j-1}^n), \tag{5.3}$$

$$-G_j(c,c,c) \wedge G_j(U^n_{j+1},U^n_j,U^n_{j-1}) \le -G_j(c\wedge U^n_{j+1}, c\wedge U^n_j, c\wedge U^n_{j-1}). \tag{5.4}$$

Following [10], (5.3) and (5.4) are added, and the identity $a\vee b - a\wedge b = |a-b|$ is applied, giving

$$\begin{aligned}&\left|G_j(U^n_{j+1},U^n_j,U^n_{j-1}) - G_j(c,c,c)\right| \\ &\quad\le G_j(c\vee U^n_{j+1}, c\vee U^n_j, c\vee U^n_{j-1}) - G_j(c\wedge U^n_{j+1}, c\wedge U^n_j, c\wedge U^n_{j-1}).\end{aligned} \tag{5.5}$$

Take the left side of (5.5):

$$\begin{aligned}\left|G_j(U^n_{j+1},U^n_j,U^n_{j-1}) - G_j(c,c,c)\right| &= \left|G_j(U^n_{j+1},U^n_j,U^n_{j-1}) - c + \lambda\Delta_- f(\gamma_{j+\frac12},c)\right| \\ &\ge \left|U^{n+1}_j - c\right| - \lambda\left|\Delta_- f(\gamma_{j+\frac12},c)\right|.\end{aligned} \tag{5.6}$$

Now take the right side of (5.5):

$$\begin{aligned}&G_j(c\vee U^n_{j+1}, c\vee U^n_j, c\vee U^n_{j-1}) - G_j(c\wedge U^n_{j+1}, c\wedge U^n_j, c\wedge U^n_{j-1}) \\ &\quad= c\vee U^n_j - c\wedge U^n_j \\ &\qquad - \lambda\Delta_-\Big(h(\gamma_{j+\frac12}, c\vee U^n_{j+1}, c\vee U^n_j) - h(\gamma_{j+\frac12}, c\wedge U^n_{j+1}, c\wedge U^n_j)\Big) \\ &\qquad + \frac{1}{\Delta x}\lambda\Delta_-\Delta_+(A(U^n_j\vee c) - A(U^n_j\wedge c)) \\ &\quad= \left|U^n_j - c\right| - \lambda\Delta_-\Big(H(\gamma_{j+\frac12}, U^n_{j+1}, U^n_j) - \frac{1}{\Delta x}\Delta_+\left|A(U^n_j) - A(c)\right|\Big).\end{aligned} \tag{5.7}$$

The last step in (5.7) used the fact that $A(U^n_j\vee c) - A(U^n_j\wedge c) = |A(U^n_j) - A(c)|$, which results from the fact that A is nondecreasing. The proof is completed by comparing (5.6) with (5.7). ☐

Let γ be piecewise C^1, with finitely many jumps (in γ and γ'), located at $\xi_1 < \xi_2 < \ldots < \xi_{M'}$. Assume that γ' has a bounded derivative, $|\gamma'(x)| \le \beta$, away from the points of discontinuity, with bounded (also by β) one sided limits at the jumps. The solution u is assumed to be piecewise C^1 with any possible jumps occurring along a finite number of piecewise C^1 curves which intersect in at most a finite number of locations.

Theorem 5.1. *Let u be a limit point of the sequence $\{u^\Delta\}$ generated by the scheme* (2.4) *(see Theorem 3.1). The following entropy inequality holds for all nonnegative test functions $\phi \in \mathcal{D}'(\Pi_T)$ such that $\phi|_{t=0} = \phi|_{t=T} = 0$:*

$$\begin{aligned}&\iint_{\Pi_T} \Big(|u-c|\phi_t + \sigma(u-c)(f(\gamma,u) - f(\gamma,c))\phi_x + |A(u) - A(c)|\phi_{xx}\Big)\,dt\,dx \\ &\qquad\qquad + \iint_{\Pi_T} |f(\gamma(x),c)_x|\phi\,dtdx \ge 0.\end{aligned} \tag{5.8}$$

Remark 5.1. Note that in the last integral in (5.8), since γ is not continuous, but only of bounded variation, the term $|f(\gamma(x),c)_x|$ must be interpreted as a measure. If we label this measure ν, then for any set $E \subseteq \mathbf{R}$,

$$\nu(E) = T.V.\left.\right|_E (f(\gamma(\cdot),c)) = \int_E |f(\gamma(x),c)_x|\,dx.$$

Proof. Let $V(u) = |u-c|$ and $F(\gamma,u) = \sigma(u-c)(f(\gamma,u) - f(\gamma,c))$. Following the proof of the Lax-Wendroff theorem, the discrete entropy inequality (5.1) is multiplied by $\phi^n_j\Delta x$, then summed over j and n. Here $\phi^n_j = \phi(x_j,t^n)$ and ϕ is a test function of type described in the statement of the theorem. This yields

(5.9)

$$\Delta t\,\Delta x\sum_{j,n}\phi_j^n\left(\frac{V(U_j^{n+1})-V(U_j^n)}{\Delta t}+\frac{1}{\Delta x}\Delta_-H(\gamma_{j+\frac12},U_{j+1}^n,U_j^n)-\frac{1}{\Delta x^2}\Delta_-\Delta_+\left|A(U_j^n)-A(c)\right|\right)$$
$$-\Delta t\Delta x\sum_{j,n}\phi_j^n\left|\frac{1}{\Delta x}\Delta_+f(\gamma_{j-\frac12},c)\right|\le 0.$$

Summing by parts and letting $\Delta\to 0$ gives, by the bounded convergence theorem,

(5.10)

$$\Delta t\,\Delta x\sum_{j,n}\phi_j^n\left(\frac{V(U_j^{n+1})-V(U_j^n)}{\Delta t}+\frac{1}{\Delta x}\Delta_-H(\gamma_{j+\frac12},U_{j+1}^n,U_j^n)\right)\to-\iint_{\Pi_T}\Big(V(u)\phi_t+F(\gamma,u)\phi_x\Big)\,dt\,dx,$$

as in the proof of the Lax-Wendroff theorem. For the term containing $|A(U_j^n)-A(c)|$, summing by parts and letting $\Delta\to 0$ gives

$$\Delta t\,\Delta x\sum_{j,n}\phi_j^n\frac{1}{\Delta x^2}\Delta_-\Delta_+\left|A(U_j^n)-A(c)\right|\to\iint_{\Pi_T}|A(u)-A(c)|\phi_{xx}\,dtdx. \tag{5.11}$$

For the remaining term,

(5.12)

$$\Delta t\,\Delta x\sum_{j,n}\phi_j^n\left|\frac{1}{\Delta x}\Delta_+f(\gamma_{j-\frac12},c)\right|$$
$$\to\iint_{\Pi_T\setminus\{\xi_m\}}|f(\gamma(x),c)_x|\,\phi(x,t)\,dtdx+\int_0^T\sum_{m=1}^{M'}\left|f(\gamma(\xi_m^+),c)-f(\gamma(\xi_m^-),c)\right|\phi\left(\xi_m,t\right)\,dt,$$

which follows by breaking the spatial portion of the sum into sums over intervals where γ is differentiable and isolating the finite number of cells where the jumps in γ are located. The proof is complete once (5.10), (5.11), and (5.12) are combined. □

6. Numerical examples

This section discusses some numerical examples for the equation

$$u_t+(\gamma(x)f(u))_x=A(u)_{xx},$$

each using the convective flux $f(u)=u(1-u)$. We will focus on Riemann problems, with initial data denoted by (u_L,u_R), meaning that $u_0(x)=u_L$ for $x<0$, $u_0(x)=u_R$ for $x>0$. Similarly the coefficient has a single jump at the origin, which we denote by (γ_L,γ_R).

Example 1. Figure 1 shows the result of two runs of the scheme (2.4) with the Riemann problem having constant initial data $(u_L,u_R)=(0.6,0.6)$, and the coefficient given by $(\gamma_L,\gamma_R)=(0.05,0.1)$. The scheme was run for 500 time steps, with $\Delta x=.02$ and $\Delta t=.04$. In (a), the diffusion term was $A(u)=0$, i.e., the purely hyperbolic problem. In (b), the diffusion term was

$$A(u)=.0025\Big(u\chi_{[0,.45)}(u)+.45\chi_{[.45,.55)}(u)+(u-0.1)\chi_{[.55,1]}(u)\Big), \tag{6.1}$$

which is degenerate ($A'(u)=0$) in the interval $(.45,.55)$, and linear with $A'(u)=1$ elsewhere.

In (a), the constant state on the left $u_L=0.6$ is is connected to a steady jump at $x=0$ by a rarefaction, which is connected to a constant state of approximately $u=0.15$. This constant state is connected to $u_R=0.6$ by a shock moving to the right. All of these waves are created by the jump in the coefficient γ. In (b), the shock moving to the right is smaller, due to the diffusion, and the steady jump at $x=0$ has been replaced by a shock moving to the left. Both shocks have end states at approximately $u=0.45$ and $u=0.55$, providing numerical evidence that discontinuities can only occur in regions of state space where the diffusion $A(u)$ degenerates to a constant.

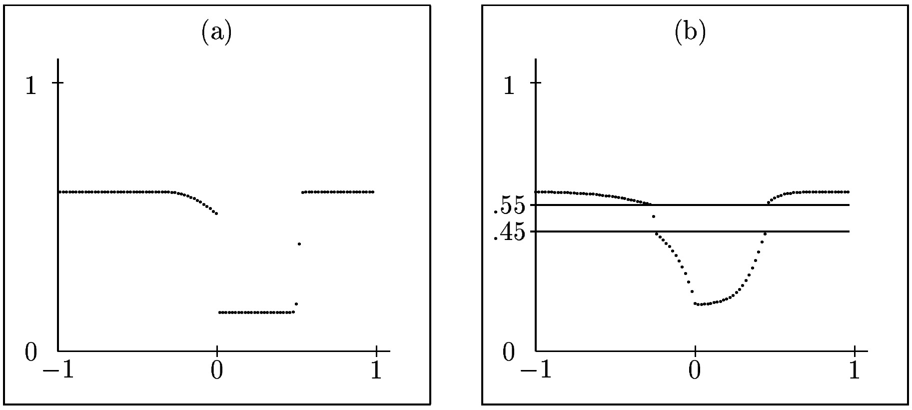

FIGURE 1. Example 1. (a) Purely hyperbolic problem; $A(u) = 0$. (b) With degenerate diffusion term (6.1).

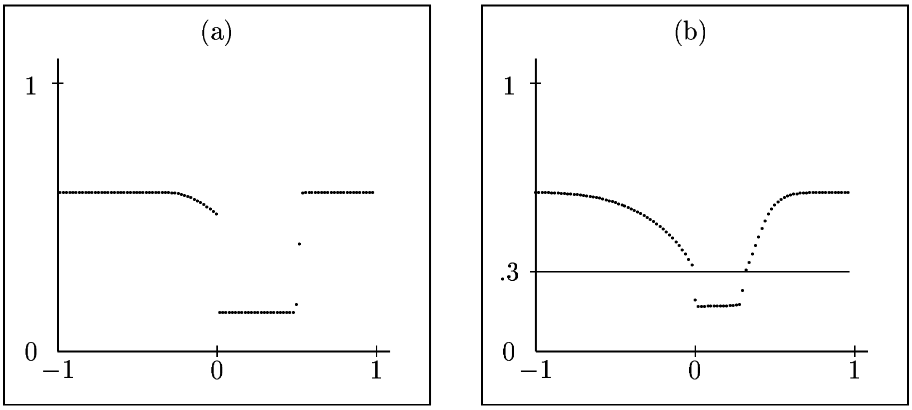

FIGURE 2. Example 2. (a) Purely hyperbolic problem; $A(u) = 0$. (b) With the degenerate diffusion term (6.2).

Example 2. The initial data u_0 and the coefficient γ are the same as in the previous example. The diffusion term is now given by

$$A(u) = .0025(u - 0.3)\chi_{[.3,1]}(u), \tag{6.2}$$

which is degenerate for $u < .3$, and allows for a stationary jump in the solution u at $x = 0$. Figure 2 (a) shows the solution for the purely hyperbolic problem, and (b) shows the effect of adding the diffusion term. The scheme was run for 500 time steps, with $\Delta x = .02$ and $\Delta t = .04$, as in the previous example. The diffusion term has the effect of changing the upper end states for both the steady jump and the shock moving to the right, lowering them to approximately $u = 0.3$, providing more numerical evidence that the solution is continuous wherever $A(u)$ is nondegenerate.

Example 3. Figure 3 shows the result of two runs of the scheme (2.4) with the Riemann problem having initial data $(u_L, u_R) = (0.8, 0.2)$, and the coefficient given by $(\gamma_L, \gamma_R) = (0.05, 0.1)$. Both plots in Figure 3 show the purely hyperbolic case. In (a), $\Delta x = 0.02$, $\Delta t = 0.04$, with 100 time steps. In (b), $\Delta x = 0.01$, $\Delta t = 0.01$, with 400 time steps. Both plots show a spurious bump that starts out as a kink, and then moves to the right at the edge of the rarefaction. Refining the mesh causes the bump to decrease in amplitude and width, as can be seen by comparing Figure 3 (a) with (b). We have found that these spurious bumps turn up in certain (but not all) Riemann problems. As predicted by our convergence theory, they shrink as the mesh size diminishes. For a Riemann problem the discretizations (2.5) and (2.6) result in an intermediate state $u_M = (u_L + u_R)/2$ and a sharp jump from γ_L to γ_R. We have found from experience that

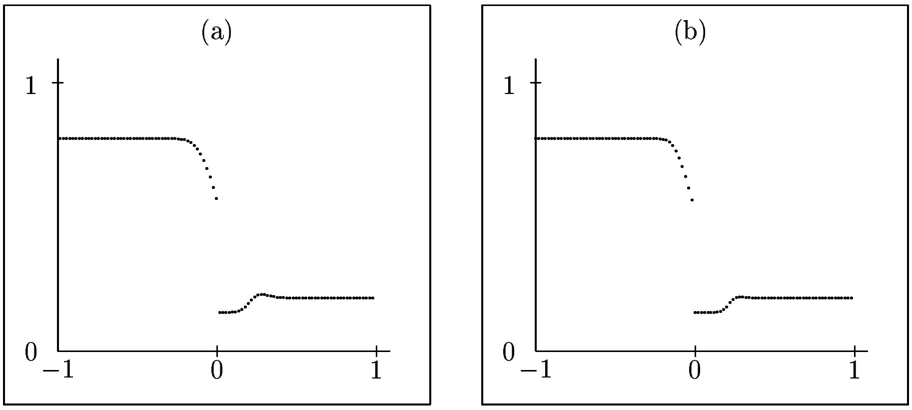

FIGURE 3. Example 3. (a) Purely hyperbolic problem, showing spurious bump; $A(u) = 0$. (b) Purely hyperbolic problem. Mesh size reduced to show that the spurious bump reduces in width and amplitude.

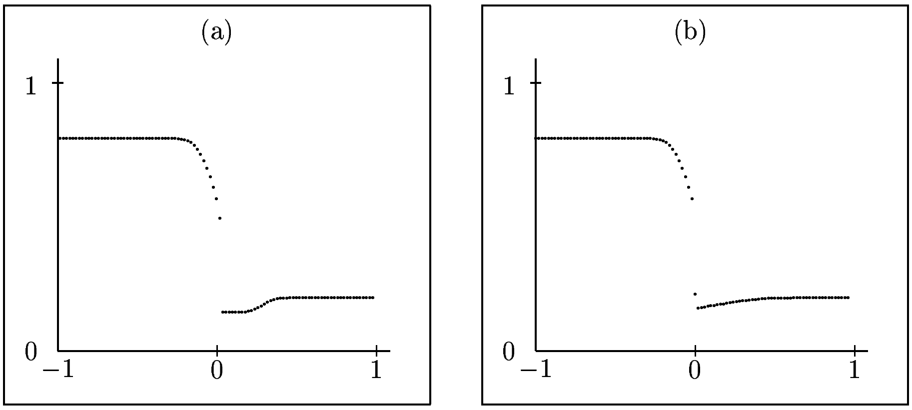

FIGURE 4. Numerical example 3. (a) Purely hyperbolic problem, with jump in γ moved by one mesh width to the right to get rid of spurious bump. (b) With degenerate diffusion term (6.3).

the bump can be removed by moving the jump in γ one mesh width in the direction of the bump. All of our convergence theory remains valid with a change of this type, since we still have $\gamma^{\Delta} \to \gamma$ in L^1_{loc} and boundedly a.e. Figure 4 (a) is the same as Figure 3 (a) with the exception that the jump in u_0 has been moved one mesh width to the right, with the result that the spurious bump does not appear. Figure 4 (b) is the same as Figure 4 (a), except that the degenerate diffusion term

$$A(u) = .0025\Big(u\chi_{[0,.2]}(u) + .2\chi_{(.2,1]}(u)\Big) \tag{6.3}$$

is incorporated.

The effect of the diffusion is to smear out the corners where the rarefaction meets the constant states. Although not shown in this plot, when the mesh size is reduced sufficiently, the small jump between the minimum point on the graph and $u = .2$ fills in, so that the solution is continuous in the region where $A'(u) > 0$.

Example 4. In this example we study the convergence rate for the problem in the previous example, with the diffusion term included. We used the discretizations (2.5) and (2.6), so that the spurious bump was present, but diminished as the mesh shrank. Table 1. shows the results of the test. The last row in the table was used as the "true" solution; L^1 differences with this solution

Δx	Δt	Number of steps	L^1 error estimate
1/16	1.6	4	0.1973
1/32	0.4	16	0.0137
1/64	0.1	64	0.0105
1/128	0.025	256	0.0042
1/256	0.00625	1024	0.0018
1/512	0.0015625	4096	
Example 4.			

were computed, and appear in the last column of the table. Although the test is not conclusive, it appears that for this particular example there is linear convergence as $\Delta x \to 0$.

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