

# Pattern Formation, Wave Propagation and Stability in Conservation Laws with Slow Diffusion and Fast Reaction

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## Abstract

The limiting behavior of the solution of scalar conservation laws with slow diffusion and fast bistable reaction is considered. In a short time the solution develops transition patterns connected by shock layers and rarefaction layers, when the initial data has finitely many monotone pieces. The existence and uniqueness of the front profiles for both shock layers and rarefaction layers are established. A variational characterization of the wave speed of these profiles is derived. These profiles are shown to be stable. Furthermore, it is proved that solutions with monotone intimal data approach to shock layer or rarefaction layer waves as time goes to infinity.

**Key Words:** Reaction-diffusion-convection equations, Pattern formation, transition fronts, asymptotic stability.

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# 1 Introduction

Front propagation and interface motion occur in many scientific areas such as chemical kinetics, combustion, biology, transport in porous media, geometry and image processing. In spite of these different applications, the basic phenomena can all be modeled by nonlinear parabolic PDEs, typically reaction-diffusion equations:

$$u_t = g(u) + \Delta_x u.$$

The field has gone through enormous growth and development since the pioneering work of Kolmogorov, Petrovsky, and Piskunov (KPP) [13] and Fisher [9] in 1937 on traveling fronts in reaction-diffusion equations.

In this article we consider a reaction-diffusion equation with nonlinear convection:

$$u_t + f(u)_x = \epsilon u_{xx} + \frac{1}{\gamma} g(u), \quad (1.1)$$

where  $f(u)$  is a smooth flux function and  $g(u) = -W'$ ,  $W$  being a double-wells potential,  $\gamma > 0$  and  $\epsilon > 0$  being small parameters.

The physical motivation for studying (1.1) is that general reacting flows or dynamics of phase transitions are combinations of fluid dynamic equations and reaction-diffusion equations [1]. Typically, the reaction time  $\gamma$  and viscosity  $\epsilon$  are proportional to mean free path. Thus, in this paper, we take

$$\gamma = \epsilon.$$

The fundamental question we address here is to understand how the nonlinear convection influence the characteristics of front propagation such as front speeds, front profiles, front locations and pattern formations.

Equation (1.1) can be written as a convective gradient flow

$$u_t + f(u)_x = -\frac{\delta E}{\delta u}$$

with the free energy  $E$  being given by

$$E(u) = \int_{\mathbb{R}} \left[ \frac{\epsilon^2}{2} u_x^2 - \int^u g(\xi) d\xi \right] dx.$$

A special case of (1.1) is the so called Allen-Cahn equation [1] which corresponds to the choice of

$$g(u) = u(1 - u^2), \quad f(u) = \text{Const.}$$

When the flux  $f(u) = 0$ , the equation (1.1) is a typical reaction-diffusion equation which is pretty well understood. In particular, the front propagation has been extensively studied. We refer to the review paper [19] and references cited therein for more information.

Our interest is in the limiting behavior of  $u = u^\epsilon$  as  $\epsilon \rightarrow 0$  when the initial data  $u_0(x) \in L^\infty(\mathbb{R})$  are assumed to have finitely many monotone pieces. We shall show that in a small time  $O(\epsilon)$  the transition layers are formed, outside the transition layers the solution  $u^\epsilon(x, t)$  behave as  $\pm 1$ . After this short time, the transition layer structure persists. These transition layer solution, i.e., traveling waves of (1.1), are asymptotically stable and

are approached by solutions starting with arbitrary monotone initial data. We further characterize the speed of the layers. The results on the reaction-convection equation, i.e, (1.1) without the term  $u_{xx}$ , [11], suggest that the speeds of different transition layers can be different and hence they can collide later.

The aim of this article is to study

- the development of the pattern by convergence analysis;
- the construction of traveling waves by matching argument;
- the characterization of the speed by min-max formulation;
- the time-asymptotic stability of the waves by the spectral method;

Our work is closely related to that of Fan, Jin and Teng [11], Fan and Jin [10], Fife and Hisao [8], Carr and Pego [5] and Fusco and Hale [7]. Fan, Jin and Teng uses the generalized characteristic method to study the zero reaction limit for the inviscid equation

$$u_t + f(u)_x = \frac{1}{\epsilon}g(u), \tag{1.2}$$

Fan and Jin [10] give the rigorous prediction of the slow motion of the shock layer and rarefaction layer generated by (1.2). The discussion of the traveling waves for (1.1) are also given in [10]. The equation without the nonlinear convection has been extensively studied in the literature. Fife and Hsiao [8] uses the maximum principle to predict the pattern formed after a short time and the velocity of the transition point for the initial data with single root. The papers of Carr and Pego [5], Fusco and Hale [7] give rigorous justification of the formal asymptotic analysis, in particular their results show that the transition points move with velocities of order  $e^{-Cl/\epsilon}$ , where  $C$  is a constant and  $l$  is the minimum distance between the transitions in the initial data. Bronsard and Kohn [2] first links the slow motion with the energy dissipation of the reaction-diffusion equation. Our convergence analysis has several advantages over those mentioned above: (1) it is natural to use BV norm for the limit is just a piecewise constant function; (2) it handles more general initial data.

The multidimensional analogue of (1.1), has also been the subject of the recent attention [6]. In particular the case without nonlinear convection has been extensively studied, consult [1, 16, 4, 3] and references therein. We would point out that the situation is rather different in  $\mathbb{R}^n$ ,  $n \geq 2$ , since the solution has transition surfaces rather than transition points. As  $\epsilon \rightarrow 0$  the transition surfaces has been proved to move with velocity  $\epsilon\kappa$ , where  $\kappa$  is the sum of the principal curvatures.

The main results and the plan of the paper is as follows: In §2, we prove that the solution  $u^\epsilon$  has a convergent subsequence  $u^{\epsilon_n}$  converging to  $\pm 1$  as  $\epsilon_n \rightarrow 0+$  almost everywhere for initial data having finitely many monotone pieces. This follows from the total variation bound of the solution and the bistable structure of the fast reaction term. The convergence result ensures the pattern formation in a short time  $O(\epsilon)$ . The uniqueness of limits and the structure of the limit are left for future investigation. §3 is devoted to the prediction of the speed of the front propagation, here the front is the internal layer formed near the transition points after a short time. We discuss the law of motion for the front and the link of the speed to the Lax-shock condition. Following [18, 12] we derive a variational characterization for the wave speed of the stable front profile.

In §4 we construct the monotone shock layer solution and rarefaction layer solution via a shooting method. These complement the previous study performed in [10]. The dynamic stability of the shock layer solution is investigated via the spectral method [17, 15]. Further approaching to a stable wave by solutions with arbitrary monotone initial data is proved along the line given in [18].

Reaction-diffusion-convection equations can serve as the prototypes for reactive flows, whose governing systems contains the effect of reaction, diffusion and convection terms. Although reaction-diffusion and convection-diffusion have been studied extensively, but their combinations are not. Further research in this area can be fruitful.

## 2 Convergence and pattern formation

We proceed to characterize the front propagation and the motion of the level set  $\Gamma(t)$ . This consists of two main steps:

- (1) Pattern formation (initialization of the front);
- (2) Front propagation.

This section is devoted to (1) by proving the convergence of  $u^\epsilon$  as  $\epsilon \rightarrow 0$ , which can be realized by the time dynamics with the time scales involved.

To simplify the presentation we make the following assumptions

- (a) Both  $f$  and  $g$  are smooth enough.
- (b)  $g(\pm 1) = g(0) = 0$ ,  $g'(\pm 1) < 0$  and  $g'(0) > 0$ .

(c) Initial data  $u_0(x) \in L^\infty(\mathbb{R})$  has finitely many monotone pieces such that its total variation is bounded, i.e.  $TV(u_0) \leq C$ .

Under the above assumptions we turn to prove the convergence of the solution  $u^\epsilon$  as  $\epsilon \rightarrow 0$ . This will follow from a series of lemmas.

**Lemma 1** *Number of monotone pieces of the solution  $u^\epsilon$  is non-increasing as time evolves.*

*Proof.* Set  $v = u_x$ , then  $v$  solves the following equation

$$v_t + (f'(u)v)_x = \epsilon v_{xx} + \frac{1}{\epsilon} g'(u)v.$$

Assume the contrary, i.e. that the number of zeros of  $v(x, t) : N(v, t)$  increase at time  $t = t_0 > 0$ , i.e.,

$$N(v, t_0-) \leq N(v, t_0+) - 1.$$

Then at the new zero point  $x_0$  of  $v(x, t)$  we have the following cases:

Case 1. In a neighborhood of  $(x_0, t_0)$  denoted by  $I_0 := I(x_0, t_0)$ ,  $v(x, t) > 0$  or ( $< 0$ ) for  $t < t_0$  and  $(x, t) \in I_0$ . In fact, at  $t = t_0$

$$v(x_0, t_0) = 0 = v_x(x_0, t_0), \quad v_{xx}(x_0, t_0) > 0 (< 0).$$

From the fact  $v(x_0, t) > 0$  for  $t < t_0$  and  $v(x_0, t_0) = 0$  it follows that

$$v_t(x_0, t_0) \leq 0.$$

However

$$v_t(x_0, t_0) = \epsilon v_{xx}(x_0, t_0) > 0$$

leads to a contradiction.

Case 2. In a neighborhood of  $(x_0, t_0)$ ,  $I(x_0, t_0) = I_0$ ,

$$v(x_0, t_0) = v_x(x_0, t_0) = v_{xx}(x_0, t_0) = 0, \quad v_{xxx}(x_0, t_0) > 0 (< 0)$$

and  $v_x(x, t) > 0 (< 0)$  for  $t < t_0$  in  $I_0$ .

Then  $w = v_x$  solves

$$w_t + (f'(u)v)_{xx} = \frac{1}{\epsilon}(g'(u)v)_x + \epsilon w_{xx}.$$

Thus at  $(x_0, t_0)$ ,

$$w_t(x_0, t_0) = (v_x)_t(x_0, t_0) > 0 (< 0). \quad (2.1)$$

However, from the fact that

$$v_x(x_0, t_0) = 0, \quad v_x(x_0, t) > 0 (< 0) \quad \text{for } t < t_0,$$

it follows that

$$(v_x)_t(x_0, t_0) \leq 0 (\geq 0),$$

which contradicts (2.1).

Case 3. Same as Case 1 except that

$$v(x_0, t_0) = v_x(x_0, t_0) = \cdots = \frac{\partial^{2k-1}}{\partial x^{2k-1}}v(x_0, t_0) = 0$$

with

$$\frac{\partial^{2k}}{\partial x^{2k}}v(x_0, t_0) > 0 (< 0).$$

In a neighborhood of  $(x_0, t_0)$

$$\frac{\partial^{2k-2}}{\partial x^{2k-2}}v(x, t) > 0 (< 0)$$

for  $t < t_0$  and  $(x, t) \in I_0$ . Then a similar derivation to the Case 1 leads to the contradiction.

Case 4. Similar to Case 2 except that

$$v(x_0, t_0) = v_x(x_0, t_0) = \cdots = \frac{\partial^{2k}}{\partial x^{2k}}v(x_0, t_0) = 0$$

with

$$\frac{\partial^{2k+1}}{\partial x^{2k+1}}v(x_0, t_0) > 0 (< 0).$$

In a neighborhood of  $(x_0, t_0)$

$$\frac{\partial^{2k-1}}{\partial x^{2k-1}}v(x, t) > 0 (< 0)$$

for  $t < t_0$  and  $(x, t) \in I_0$ . Then a similar derivation to Case 2 leads to the contradiction.

A combination of the above cases completes the proof. ■

**Lemma 2** *The solution  $u^\epsilon$  is uniformly bounded in  $\epsilon$  and*

$$\|u^\epsilon(x, t)\|_\infty \leq \max(1, \|u_0\|_\infty).$$

*Proof.* Note that the source term  $g(u)$  is bistable with  $u = 1$  and  $-1$  being stable zeros. Therefore if  $u_0(x) > 1$  for some  $x \in \mathbb{R}$  then the maximum principle gives

$$u(x, t) \leq \max u_0(x).$$

Hence

$$u(x, t) \leq \max(1, \|u_0(x)\|_{L^\infty})$$

Similarly

$$u(x, t) \geq \min(-1, -\|u_0(x)\|_{L^\infty})$$

■

**Lemma 3** *The total variation of the solution  $u^\epsilon$  is uniformly bounded in  $\epsilon$  and*

$$TV(u^\epsilon(\cdot, t)) \leq 2N(u_0) + TV(u_0).$$

*Proof.* This Lemma follows from a combination of Lemma 1 and Lemma 2. ■

**Lemma 4** *There exists a subsequence  $\{\epsilon_n\} \rightarrow 0$  such that*

$$u^{\epsilon_n} \rightarrow u(x, t), \quad \text{a.e. in } \mathbb{R} \times \mathbb{R}^+.$$

*Proof.* The convergence is ensured by the Helley Theorem. ■

**Theorem 2.1** *Assume (a), (b) and (c) hold. Let  $u^\epsilon$  be the solution of (1.1) associated with the initial datum  $u^\epsilon = u_0$  on  $\mathbb{R} \times \{0\}$ , where  $u_0 \in C_B^1(\mathbb{R})$  is such that  $\Gamma_0 = \{x, u_0(x) = 0\}$  contains finitely many single points in  $\mathbb{R}$ . Then, as  $\epsilon \rightarrow 0$ , the solution converges to a piecewise constant function taking only the values  $\pm 1$ , with finitely many discontinuities.*

*Proof.* Let  $\phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$  be a smooth test function, then the weak formulation of the solution to

$$u_t^\epsilon + f(u^\epsilon)_x = \frac{1}{\epsilon}g(u^\epsilon) + \epsilon u_{xx}^\epsilon$$

can be written as

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}} \left[ u_t^\epsilon + f(u^\epsilon)_x - \frac{1}{\epsilon}g(u^\epsilon) - \epsilon u_{xx}^\epsilon \right] \phi(x, t) dx dt \\ &= \left( \int_{\mathbb{R}} u^\epsilon \phi dx \right) \Big|_0^T + \int_0^T \int_{\mathbb{R}} \left[ -\phi_t u^\epsilon - f(u^\epsilon) \phi_x - \epsilon u^\epsilon \phi_{xx} - \frac{1}{\epsilon}g(u^\epsilon) \phi \right] dx dt. \end{aligned}$$

Since  $\|u^\epsilon(\cdot, t)\|_\infty \leq C$  independent of  $\epsilon$  and  $t$ , we have

$$\frac{1}{\epsilon} \int_0^T \int_{\mathbb{R}} g(u^\epsilon) \phi dx dt \leq C_1,$$

where  $C_1$  is independent of  $\epsilon$ .

Assume the contrary, i.e.  $u(x, t) \neq \pm 1$  or 0 in a subset of  $\mathbb{R} \times \mathbb{R}^+$  with measure  $\mu > 0$ . Then we can get a open ball  $B$  in  $\mathbb{R} \times \mathbb{R}^+$  over which  $u(x, t) \neq -1, +1$  or 0. Since the number of the zero points of  $u^\epsilon(x, t)$  denoted by  $N(u^\epsilon(x, t))$  decrease as  $t$  increase. Then the number of components of  $\{x, u^\epsilon > 0\}$  and  $\{x, u^\epsilon < 0\}$  are decreasing as  $t$  increase. Thus we can choose a ball such that  $u^\epsilon(x, t)$  does not change sign over  $t \in [0, T]$ .

Let  $\phi$  be a smooth approximation of  $\text{sign}(g(u^\epsilon))$  with  $\phi \in C_0^\infty(B)$ , then

$$C_1 \geq \frac{1}{\epsilon} \int_0^T dt \int_{\mathbb{R}} g(u^\epsilon) \phi dx dt \rightarrow \infty$$

as  $\epsilon \rightarrow 0$ , leading to a contradiction. The proof is complete. ■

### 3 Propagation speed of fronts

#### 3.1 Law of motion of fronts

Armed with the convergence theorem we turn to the study of the front propagation. Formally in a short time the solution  $u^\epsilon$  evolves according to the ordinary differential equation

$$u_t = \frac{1}{\epsilon} g(u).$$

For  $x \in D_0^+ := \{x, u_0(x) > 0\}$ ,  $u_0$  is in the domain of attraction of the equilibrium point  $u = 1$ ; Similarly it is attracted to  $u = -1$  for  $x \in D_0^- := \{x, u_0(x) < 0\}$ . To gain further insight we take  $g = u(1 - u^2)$ , the solution of the above ODE can be solved explicitly

$$u^2 = \frac{u_0^2}{\exp(-\frac{2t}{\epsilon}) + u_0^2(1 - \exp(-\frac{2t}{\epsilon}))}.$$

A simple calculation shows that for  $u_0 \neq 0$  the solution  $u^\epsilon$  will approach  $\pm 1$  for a quite short time in the sense that

$$|u^2 - 1| \leq \epsilon$$

if

$$t \geq \frac{1}{2} \log \left| \left(1 - \frac{1}{u_0^2}\right) \frac{\epsilon}{1 - \epsilon} \right| \sim \epsilon |\ln \epsilon|.$$

Therefore as  $t$  increase, the values of  $u^\epsilon$  of equation (1.1) are pulled apart near  $\Gamma_0$  and layers (a zone near  $\Gamma(t)$  where  $u^\epsilon$  has a large  $x$ -gradient) are formed at  $\Gamma_0$ , called shock layer or rarefaction layer depending on the relative position of  $\Gamma_0$  and  $D_0^\pm$ .

After this initial formation, the fronts will propagate. Let  $x = \xi(t)$  be the position of the fronts (to be determined) as a function of  $t$ . Set  $z = (x - \xi(t))/\epsilon$  and look for a formally approximation of the form

$$u^\epsilon(x, t) = U(z, t).$$

Upon substitution into (1.1) one has

$$-U_z \xi'(t) + \epsilon U_t + f'(U) U_z = U_{zz} + g(U),$$

whose leading order approximation is obtained by setting  $\epsilon = 0$

$$f'(U)U_z - U_z\xi'(t) = U_{zz} + g(U).$$

On the other hand let  $u(x, t) = U(z)$  with  $z = (x - ct)$  be the traveling wave solution of

$$u_t + f(u)_x = u_{xx} + g(u).$$

Then

$$(f'(U) - c)U_z = U_{zz} + g(U), \quad U(\pm\infty) = \pm 1 \quad (\mp 1).$$

It follows that the profile with desired limiting behavior as  $z \rightarrow \pm\infty$  can exist if

$$\xi'(t) = c$$

with initial data  $\xi(0) = z_0(0)$  for  $i = 1, \dots, N$ . This is the law of motion of the fronts.

### 3.2 Lax Shock condition

In order to figure out the propagation speed of the transition layers, we proceed to seek solution of the form  $u = U(z)$ ,  $z = (x - ct)/\epsilon$  for equation (1.1) to obtain

$$-sU' + f(U)' = U'' + g(U) \tag{3.1}$$

with boundary conditions

$$U(\pm\infty) = \mp 1, \quad U'(\pm\infty) = 0, \tag{3.2}$$

$$\text{or} \quad U(\pm\infty) = \pm 1, \quad U'(\pm\infty) = 0. \tag{3.3}$$

As pointed in [10], the first boundary condition yields the shock layer and the second is for the rarefaction layer connecting equilibrium points  $-1$  and  $+1$ .

As is well known for the reaction-diffusion equation the traveling wave speed is in general unknown. We first discuss the determination of the wave speed for a given traveling wave.

Let  $U$  be a traveling wave solution satisfying (3.1), (3.2). Multiplication of (3.1) by  $V = U'$  and integration over  $\mathbb{R}$  gives

$$c \int_{\mathbb{R}} V^2(\xi) d\xi = \int_{\mathbb{R}} [f'(U)V^2(\xi) - g(U)U'] d\xi.$$

Thus

$$c = \frac{\int_{-1}^1 g(u) du + \int_{\mathbb{R}} f'(U)V^2(\xi) d\xi}{\int_{\mathbb{R}} V^2 d\xi}.$$

Noting that  $g(u) = u(1 - u^2)$  yields  $\int_{-1}^1 g(u) du = 0$  and the speed  $c$  satisfies

$$c = \frac{\int_{\mathbb{R}} f'(U)V^2(\xi) d\xi}{\int_{\mathbb{R}} V^2 d\xi}.$$

Clearly for convex flux  $f$ , it follows that

$$f'(-1) < c < f'(1),$$



which is in agreement with Lax' shock condition in the context of scalar conservation laws.

As is well known for the viscous conservation laws ( $g = 0$ ) the shock speed is uniquely determined by the Rankine-Hugoniot condition  $c[u] = [f]$ . To clarify the contribution of the source term we integrate the equation directly,

$$c = \frac{[f(u)] + \int_{\mathbb{R}} g(U(\xi)) d\xi}{[u]} = \frac{f(1) - f(-1)}{2} + \frac{1}{2} \int_{\mathbb{R}} g(U(\xi)) d\xi.$$

To sum up, we state the following

**Lemma 5** *Let  $U$  be a monotone shock layer solution satisfying  $U(\pm\infty) = \mp 1$ . Then the speed  $c$  satisfies*

$$c = \frac{\int_{-1}^1 g(u) du + \int_{\mathbb{R}} f'(U) V^2(\xi) d\xi}{\int_{\mathbb{R}} V^2 d\xi}$$

or

$$c = \frac{f(1) - f(-1)}{2} + \frac{1}{2} \int_{\mathbb{R}} g(U(\xi)) d\xi.$$

For the case  $\int_{-1}^1 g(u) du = 0$ , the Lax shock condition still holds

$$f'(-1) < c < f'(1).$$

For the monotone rarefaction layer, the speed can be similarly obtained.

**Lemma 6** *Let  $U$  be a monotone rarefaction layer solution satisfying  $U(\pm\infty) = \pm 1$ . Then the speed  $c$  satisfies*

$$c = \frac{-\int_{-1}^1 g(u) du + \int_{\mathbb{R}} f'(U) V^2(\xi) d\xi}{\int_{\mathbb{R}} V^2 d\xi}$$

or

$$c = \frac{f(1) - f(-1)}{2} - \frac{1}{2} \int_{\mathbb{R}} g(U(\xi)) d\xi.$$

The above formulae are implicit because either the solution  $U$  or  $V = U'$  depends on  $c$ .

### 3.3 Variational formula

We proceed to obtain an explicit formula for the speed of the unique stable traveling waves with  $U(\pm\infty) = \mp 1$ .

To this end we assume that

- (a) there exists unique traveling wave  $(U, c)$  of (1.1);
- (b) the wave is stable with respect to some subset of the initial data

$$\limsup_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} |u(x, t) - U(x - ct + \theta)| = 0$$

holds for some shift  $\theta \in \mathbb{R}$ .

For the verification of these two assumptions we refer to Section 4.

Define the admissible set

$$K := \{v \in C^2(\mathbb{R}), \quad v_x < 0, \quad -1 < v(x) < 1, \quad v(\pm\infty) = \mp 1, \quad v \in I_s\},$$

where  $I_s$  is the attractor domain of the stable traveling wave  $U$ .

We then have

**Theorem 3.1** *Assume (a) and (b). Then the speed  $c$  is uniquely given by*

$$\sup_{v \in K} \inf_{x \in \mathbb{R}} \phi(v(x)) = c = \inf_{v \in K} \sup_{x \in \mathbb{R}} \phi(v(x)),$$

where

$$\phi(v(x)) = f'(v(x)) + \frac{v'' + g(v)}{-v'(x)}.$$

*Proof.* By assumption the wave  $(U, c)$  satisfies  $\phi(U) = c$ . Then, we have

$$\sup_{v \in K} \inf_{x \in \mathbb{R}} \phi(v(x)) \geq \inf_{v \in K} \sup_{x \in \mathbb{R}} \phi(v(x)).$$

Thus, it suffices to show that for all  $v \in K$

$$\inf_{x \in \mathbb{R}} \phi(v(x)) \leq c \leq \sup_{x \in \mathbb{R}} \phi(v(x))$$

holds. Assume the right inequality fails, then there exists  $c_1 < c$  and a  $v \in K$  such that

$$\phi(v(x)) \leq c_1, \quad x \in \mathbb{R},$$

which gives

$$c_1 v' + v'' + g(v) - f'(v)v' \leq 0.$$

Let  $\bar{v} = v(x - c_1 t)$ , then

$$\bar{v}_t - \bar{v}_{xx} - g(\bar{v}) + f(\bar{v})_x \geq 0.$$

Let  $w$  be the solution of the equation

$$w_t - w_{xx} - g(w) + f(w)_x = 0$$

with initial data  $\bar{v}(x)$ , the comparison principle ensures that

$$w(x, t) \leq \bar{v}(x, t) = v(x - c_1 t),$$

and in particular

$$w(x + c_1 t, t) \leq v(x) < 1, \quad \forall t > 0. \tag{3.4}$$

By the stability assumption we have

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} |w(x, t) - U(x - ct + \theta)| = 0.$$

Hence

$$|w(x + c_1 t, t) - U(x + (c_1 - c)t + \theta)| \rightarrow 0 \quad t \rightarrow \infty.$$

Since  $c_1 < c$  and  $U(-\infty) = 1$  one arrives that

$$w(x + c_1 t, t) \rightarrow 1 \quad \text{as } t \rightarrow \infty$$

which contradicts (3.4) and finishes the proof. ■

## 4 Existence and stability of layer solutions

This section is devoted to the study of existence and stability of layer solutions for the following scaled equation

$$u_t + f(u)_x = Au_{xx} + g(u). \quad (4.1)$$

Its traveling wave  $U(z)$ ,  $z = x - ct$  solves a second order ODE,

$$AU'' + (c - f'(U))U' + g(U) = 0 \quad (4.2)$$

with boundary condition

$$U(\pm\infty) = u_{\pm}$$

where  $u_{\pm} = \pm 1$  or  $\mp 1$ , corresponding to the rarefaction layer or shock layer solution, respectively.

### 4.1 Connecting orbit

Let us write the second order equation (4.1) of traveling wave into a closed dynamical system

$$\begin{aligned} U' &= V, \\ AV' &= (f'(U) - c)V - g(U). \end{aligned}$$

To construct the traveling waves connecting  $\pm 1$  it suffices to construct a connecting orbit between the equilibrium points  $(1, 0)$  and  $(-1, 0)$ .

The nature of the connecting orbit depends strongly on the type of three equilibrium points  $(u^*, 0) = (-1, 0), (0, 0), (1, 0)$ . Linearize the system about  $(u^*, 0)$  to obtain a system

$$\begin{pmatrix} U \\ V \end{pmatrix}' = B(u^*) \begin{pmatrix} U \\ V \end{pmatrix} + O|(U, V)|^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where the coefficient matrix is given by

$$B(u^*) = \begin{pmatrix} 0 & 1 \\ -\frac{g'(u^*)}{A} & \frac{f'(u^*) - c}{A} \end{pmatrix}.$$

A simple calculation gives the eigenvalues of  $B(u^*)$

$$\lambda_{\pm}(u^*) = \frac{1}{2A} \left[ (f'(u^*) - c) \pm \sqrt{(f'(u^*) - c)^2 - 4Ag'(u^*)} \right].$$

Since  $g'(\pm 1) < 0$ , both  $(\pm 1, 0)$  are both saddle points and the type of  $(0, 0)$  depends on the relative size of  $c$  and  $A$  as well as  $f'(0)$ .

We shall use the above facts to construct the connecting orbit via the phase plane analysis.

The connection orbit for the case  $U(\pm\infty) = \pm 1$  has been established by Fan and Jin [10], which we state below for the readers convenience.

**Theorem 4.1** *For any  $A > 0$ , the boundary value problem (4.1), (3.3) has a monotone rarefaction layer solution when  $c = f'(0)$ .*

We now focus on the construction of the traveling waves for the case  $U(\pm\infty) = \mp 1$ . The argument in [10] does not work for this case.

In the phase plane the system reduces to

$$AV \frac{dV}{du} = (f'(u) - c)V - g(u), \quad V(\pm 1) = 0. \quad (4.3)$$

**Theorem 4.2** *There exists unique traveling wave  $(U, c)$  of (4.1), (3.2) with the speed determined by the min-max formula given in Theorem 3.1.*

*Proof.* Let  $\gamma_u(+1, c)$  be the portion of the unstable manifold of (4.3) in  $V < 0$  issued from  $(1, 0)$  and entering  $\{v < 0\}$ ,  $\gamma_s(-1, c)$  be the portion of the stable manifold of (4.3) in  $V < 0$  coming into  $(-1, 0)$  from  $\{v < 0\}$ . In order to prove Theorem 4.2 it suffices to show there exists a unique  $c$  such that

$$\gamma_u(+1, c) \cap \gamma_s(-1, c)$$

is not empty.

Proving this consists of the following four steps:

**Step 1.** The unstable manifold  $\gamma_u(+1, c)$  denoted by  $v = V(u, +1, c)$  moves upward as  $c$  increase.

This follows from the following two observations:

(1) The slope of the unstable manifold  $\gamma_u(+1, c)$  at  $(+1, 0)$  is  $\frac{dV}{du}|_{u=1} = \lambda_+(+1, c)$  and the eigenvalue  $\lambda_+(+1, c)$  is decreasing in  $c$ ; Indeed

$$\frac{d\lambda(+1, c)}{dc} = -\frac{f'(1) - c + \sqrt{(f'(1) - c)^2 + 8A}}{2\sqrt{(f'(1) - c)^2 + 8A}} < 0.$$

(2)  $\gamma_u(+1, c_1)$  does not intersect with  $\gamma_u(+1, c_2)$  for  $c_1 \neq c_2$ ;

Assume the contrary, i.e., that the two trajectories intersect at  $(u^*, v^*)$  with  $v^* < 0$  and  $u^* < 1$ . More precisely

$$AV_1 \frac{dV_1}{du} = (f'(u) - c_1)V_1 - g(u), \quad V_1(u^*) = v^*$$

and

$$AV_2 \frac{dV_2}{du} = (f'(u) - c_2)V_2 - g(u), \quad V_2(u^*) = v^*.$$

Set  $w = V_1^2 - V_2^2$ , the above two equations yield

$$\frac{dw}{du} = \frac{2(f'(u) - c_1)}{V_1 + V_2} w + \frac{2V_2}{A}(c_2 - c_1), \quad w(u^*) = 0.$$

Integration over  $[u^*, u]$  with  $u^* \leq u < 1$  gives

$$w(u) = \frac{2(c_2 - c_1)}{A} \int_{u^*}^u V_2(s) \exp\left(\int_s^u \frac{2(f'(\xi) - c_1)}{A(V_1 + V_2)} d\xi\right) ds.$$

Letting  $u$  tend to 1, one has  $\lim_{u \rightarrow 1} w(u) \neq 0$ , which contradicts the fact that  $V_1(1, c_1) = 0 = V_2(1, c_2)$ .

**Step 2.** The trajectory  $\gamma_u(+1, c)$  fills the line  $l = \{(0, v), v \in (-\infty, 0)\}$  as  $c$  runs in  $\mathbb{R}$ . Observe that the unstable manifold  $\gamma_u(+1, c)$  can not enter the upper-plane from  $\{(u, 0), 0 < u < 1\}$ , where  $V' = g(u) < 0$ . Thus all bounded orbits will cross the line  $\{(0, v), -\infty < v < 0\}$ . Assume the contrary to the claim, i.e., there exists a  $c_0$  such that

$$V(0, +1, c_{0+}) > V(0, +1, c_{0-}).$$

Pick  $v^* \in (V(0, +1, c_{0-}), V(0, +1, c_{0+}))$  and consider the trajectory through this point:

$$AV \frac{dV}{du} = (f'(u) - c_0)V - g(u), \quad V(0) = v^*.$$

The uniqueness of the ODE shows that the above trajectory must enter  $(1, 0)$  as  $z \rightarrow -\infty$ . Since the equilibrium point  $(1, 0)$  is a saddle, these three trajectories must coincide as a separatrix.

We observe that the unstable manifold  $\gamma_u(+1, c)$  can not enter the upper-plane from  $\{(u, 0), 0 < u < 1\}$ , where  $V' = g(u) < 0$ . Thus we see that

$$V(0, 1, c) \leq 0 \quad \text{for all } c \in \mathbb{R}.$$

We further claim that

$$\lim_{c \rightarrow -\infty} V(0; 1, c) = -\infty.$$

To this end, we calculate from (4.3) to get that

$$A \frac{dV}{du} = f'(u) - c - \frac{g(u)}{V} \geq f'(u) - c - \left| \frac{u-1}{V} \right| \max_{|u| \leq 1} |g'(u)|.$$

As long as

$$c \leq \min_{|u| \leq 1} f'(u) - 2 \max_{|u| \leq 1} |g'(u)| \tag{4.4}$$

and

$$V \leq u - 1 \quad \text{for } |u| \leq 1,$$

we have

$$A \frac{dV}{du} \geq f'(u) - c - \max_{|u| \leq 1} |g'(u)| \geq \frac{1}{2} (\min_{|u| \leq 1} f'(u) - c).$$

This is equivalent to

$$-V(u^*) = \int_{u^*}^1 \frac{dV}{du} du \geq \frac{1}{2A} (\min_{|u| \leq 1} f'(u) - c) (1 - u^*).$$

When

$$\min_{|u| \leq 1} f'(u) - 4A \geq c, \tag{4.5}$$

then above inequality implies that

$$V(u^*) \leq \frac{1}{2A} (\min_{|u| \leq 1} f'(u) - c) (u^* - 1) \leq 2(u^* - 1) \tag{4.6}$$

if

$$V(u^*) \leq (u^* - 1) \tag{4.7}$$

for  $|u^*| \leq 1$ . Since

$$\frac{dV}{du} = \lambda_+(1) \rightarrow \infty$$

as  $c \rightarrow -\infty$ , we see that (4.7) is satisfied for  $u^* < 1$  and close to 1. Then, above reasoning shows that when  $-c$  is large enough so that (4.4) and (4.5) hold, then (4.6) will be valid for all  $|u^*| \leq 1$ . The claim now follows from (4.6) immediately.

**Step 3.** Similarly, we can prove that the stable manifold  $\gamma_s(-1, c)$  denoted by  $v = V(u; -1, c)$  moves downward as  $c$  increase and that

$$\lim_{c \rightarrow \infty} V(0; -1, c) = -\infty.$$

Furthermore, the set  $\{V(0; -1, c) \mid c \in \mathbb{R}\}$  is an interval.

**Step 4.** Combining the above three steps we conclude that there exists a unique  $c$  such that

$$V(0; -1, c) = V(0; 1, c),$$

and hence there is a unique connecting orbit for the problem (4.1). ■

**Remark 4.1** *In fact, the proof of Theorem 4.2 also applies to the case where  $u(\pm\infty) = \pm 1$  to yield the existence of the unique traveling wave.*

In next section, we shall prove that the solution with monotone initial data will converge to a traveling wave of (1.1). This and the uniqueness of the traveling wave proved above provide the conditions needed for the min-max formula in Theorem 3.1 to hold.

## 4.2 Dynamic Stability and Approaching to a Wave

We now turn to the study of the dynamic asymptotic stability of the propagating fronts, which is important because the fronts should be stable if they are to be experimentally observed.

**Definition 1** *Let  $u(x, t)$  be a solution to (4.1) with initial data  $u_0(x) = U(x) + \tilde{u}(x)$  with  $\tilde{u}(x)$  being a smooth spatial decaying perturbation. If for some  $\theta \in \mathbb{R}$*

$$\lim_{t \rightarrow \infty} \|u(x, t) - U(x - ct + \theta)\|_\infty = 0,$$

*the front profile  $U(x - ct)$  is called asymptotically stable.*

**Remark 4.2** *The shift  $\theta$  in the definition is clear since if we take the initial perturbation  $\phi = U(x + x_0) - U(x)$  to the profile  $U(x)$ , then the solution is  $U(x + x_0 - ct)$  instead of  $U(x - ct)$ .*

A theorem concerning stability may be stated as follows.

**Theorem 4.3** *Let  $U(x - ct)$  be the traveling wave connecting  $u_+$  and  $u_-$  constructed in §4.1. Then there exists a positive number  $\delta$ , such that for any function  $u_0(x)$ , satisfying the condition  $\|u_0 - U\|_\infty \leq \delta$ , the solution  $u(x, t)$  of equation (4.1), with initial data  $u(x, 0) = u_0(x)$  exists for all  $t > 0$  and asymptotically approach a shifted traveling wave, i.e.*

$$\lim_{t \rightarrow \infty} \|u(x, t) - U(x - ct + \theta)\|_\infty = 0,$$

where  $\theta$  is some number.

*Proof.* Following Sattinger [17] we linearize the problem (4.1) (set  $A = 1$  here and below) by writing

$$u(x, t) = U(z) + w(z, t), \quad z = x - ct.$$

Thus the perturbation  $w$  satisfies

$$w_t = w_{zz} + (c - f'(U))w_z + (g'(U) - f''(U)U')w + N(w) = Lw + N(w),$$

where

$$N(w) = [g(U+w) - g(U) - g'(U)w] - [f'(U+w) - f'(U)]w_z - [f''(U+w) - f''(U)]wU'$$

contains quadratic or higher order nonlinear terms. The location of the spectrum of  $L$ , denoted by  $\sigma(L)$ , carries information on the decay of  $w$ .

Moreover a straightforward computation shows that

$$U' \in \ker L.$$

In the sequel we assume that

$$\ker L = U'\mathbb{R}.$$

The essential spectrum is included in the union of the essential spectra of the limiting operators

$$L_\pm = \lim_{z \rightarrow \pm\infty} L = \partial_z^2 + (c - f'(\pm 1))\partial_z + g'(\pm 1).$$

Since  $g'(\pm 1) < 0$ , the essential spectrum of  $L_\pm$  is strictly in the left-half plane, with a gap from the imaginary axis (by Fourier transform).

Take  $e^L$ , a positive bounded linear operator from  $L^2$  to itself. The essential spectrum of  $L$  is contained in  $B(0, r)$ , a disc about 0 of radius  $r < 1$ . The remaining spectrum of  $e^L$  consists of isolated eigenvalues of finite multiplicities. Noting that

$$LU' = 0, \quad \text{and } U' \in L^2 \text{ does not change sign.}$$

Thus  $-U'$  or  $U'$  is a positive eigenvalue function of  $e^L$  with eigenvalue 1. By Perron-Frobenius Theorem, 1 is a simple eigenvalue, and the rest of the point spectrum of  $e^L$  lies strictly inside the unit disc.

Thus the perturbation can be decomposed as

$$w(x, t) = u_1 + u_2,$$

where  $\langle u_1, U' \rangle = 0$  and  $u_2 = U'$ . The above spectral analysis shows that  $\lim_{t \rightarrow \infty} u_1 = 0$  and  $u_2$  leads to the translation of  $U$  by  $x_0$ . Moreover, due to the exponential decay

rate in time (spectral gap), the nonlinear term  $N$  is governed by the linear part if the perturbation is suitably small, i.e. choosing  $\delta$  suitably small. ■

Armed with the above stability result we will be able to show that we have approach to a wave from arbitrary monotone initial conditions with the same behavior at infinity as the wave. The key in extending the known stability result is to utilize a global tool–comparison principle.

We now establish a more general theorem concerning such extension in possibly weighted space. Let  $w(x) > 0$  be a given weight function and weighted norm  $\|\cdot\|_w$  be denoted by

$$\|f\|_w = \|f \cdot w\|_\infty.$$

We say  $U(x - ct)$  is a  $w$ -stable traveling wave of (4.1) if there exists  $\delta > 0$  and  $\theta \in \mathbb{R}$  such that

$$\lim_{t \rightarrow \infty} \|u(x, t) - U(x + \theta - ct)\|_w = 0,$$

provided  $\|u(x, 0) - U(x)\|_w \leq \delta$ .

The traveling wave is assumed to approach equilibrium states  $u_\pm$  in  $\|\cdot\|_w$  norm, i.e.,

$$\lim_{x \rightarrow \pm\infty} \|U(x) - u_\pm\|_w = 0,$$

for the initial data  $u_0(x)$  we also require existence of the limits

$$\lim_{x \rightarrow \pm\infty} \|u_0(x) - u_\pm\|_w = 0. \tag{4.8}$$

We prove the following theorem.

**Theorem 4.4** *Let  $U$  be a  $w$ -stable traveling wave described above,  $u_0(x)$  be a monotone function satisfying (4.8). Then the solution  $u(x, t)$  of the Cauchy problem (4.1) with initial data  $u_0(x)$  asymptotically approach a shifted wave in  $\|\cdot\|_w$  norm:*

$$\lim_{t \rightarrow \infty} \|u(x, t) - U(x + \theta - ct)\|_w = 0,$$

where  $\theta \in \mathbb{R}$  is some number.

**Remark 4.3** (1) *There are typically three choices of  $w$ :  $w = 1$ ,  $w(x) = 1 + e^{\alpha x}$  (exponential decay) or  $w(x) = (1 + x^2)^{\beta/2}$  (algebraic decay).*

(2) *The above approaching to a wave in weighted norm and its proof below is valid for any system that comparison principle holds.*

The following lemma will be used for the proof of above theorem.

**Lemma 7** [18, Lemma 6.1, page 245] *Let  $g_i(x), i = 1, 2$  be two monotonically decreasing continuous function, given on the interval  $[a, b]$ , where  $g_1(a) > g_2(b)$ . Then there exists a continuous monotonically decreasing function  $\phi(x)$ , such that  $\phi(a) = g_1(a)$ ,  $\phi(b) = g_2(b)$  and*

$$|\phi(x) - g_i(x)| \leq |g_2(x) - g_1(x)|, \quad i = 1, 2.$$



**Proof of the Theorem 4.4:** Without loss of generality, we assume that  $c = 0$ ,  $u_+ < u_-$  and  $u_0(x)$  is a continuous monotonically decreasing function. Let  $\delta$  be the amplitude of the admissible perturbation in the statement of w-stability above. We now introduce a monotonically decreasing function  $u_0^*$  such that

$$\|U(x) - u_0^*\|_w < \delta \quad (4.9)$$

and  $u_0^*(x) = u_0(x)$  for large  $|x|$ . Indeed, from the assumption it follows that

$$\lim_{x \rightarrow \pm\infty} \|u_0(x) - U(x)\|_w = 0.$$

Therefore, there exists a number  $M_1 > 0$  such that

$$\|u_0(x) - U(x)\|_w < \delta \quad \text{for } |x| > M_1.$$

On the other hand the monotonicity of  $U$  and  $u_0(x)$  and their approaching the same limits at infinity ensure that there exists  $M_2 > M_1$  such that

$$U(M_1) > u_0(M_2), \quad U(-M_1) < u_0(-M_2).$$

In the interval  $\pm x \in [M_1, M_2]$  we apply Lemma 7 to  $U$  and  $u_0(x)$ . We obtain two continuous decreasing functions  $g_{\pm}(x)$  such that  $g_{\pm}(M_1) = U(\pm M_1)$ ,  $g_{\pm}(M_2) = u_0(\pm M_2)$  and

$$\|g_{\pm}(x) - U(x)\|_w \leq \|u_0(x) - U(x)\|_w < \delta, \quad \pm x \in [M_1, M_2].$$

Thus we can take

$$u_0^*(x) = \begin{cases} U(x), & |x| \leq M_1, \\ g_{\pm}(x), & M_1 \leq \pm x \leq M_2, \\ u_0(x), & |x| \geq M_2. \end{cases}$$

which satisfied the indicated property (4.9) and  $u_0(x) = u_0^*(x)$  for  $|x| \geq M_2$ .

Next we introduce the function

$$\phi_{\tau}(x) = \min(u_0(x), u_0^*(x - \tau)).$$

It is continuous and monotonically decreasing. A simple check shows that

$$\phi_{-2M_2}(x) = u_0^*(x + 2M_2), \quad \phi_{2M_2}(x) = u_0(x).$$

Moreover, we also obtain the inequality

$$u_0^*(x + 2M_2) \leq \phi_{\tau}(x) \leq u_0^*(x - 2M_2), \quad -2M_2 \leq \tau \leq 2M_2. \quad (4.10)$$

Let  $u_{\tau}(x, t)$  be the solution of the Cauchy problem (4.1) with initial condition

$$u_{\tau}(x, 0) = \phi_{\tau}(x).$$

From  $\|u_0^*(x + 2M_2) - U(x + 2M_2)\|_w < \delta$  and the stability result it follows that for  $\tau = -2M_2$  the solution  $u_{\tau}(x, t)$  of this problem approaches to a shifted wave. In order to prove the theorem, i.e. approach to a wave for  $\tau = 2M_2$ , we advance with respect to parameter  $\tau$  with suitable step  $h$  until the value  $\tau = 2M_2$  is obtained.

Thus, along with function  $u_0(x)$  we consider the function  $\phi_{\tau+h}(x)$  ( $h > 0$ ). From the definition it follows that

$$\phi_\tau(x) \leq \phi_{\tau+h}(x) \leq \phi_\tau(x-h).$$

By the comparison principle we obtain

$$u_\tau(x, t) \leq u_{\tau+h}(x, t) \leq u_\tau(x-h, t), \quad t > 0.$$

This gives

$$\|u_{\tau+h}(x, t) - u_\tau(x, t)\|_\infty \leq \|u_\tau(x-h, t) - u_\tau(x, t)\|_\infty \leq Kh,$$

where  $K = \sup |\partial_x u_\tau(x, t)| < \infty$ , which does not depend on  $\tau, h, t$  for  $t > 1$  since  $u_+ \leq u_\tau(x, t) \leq u_-$  and the boundedness of  $u_x$ .

Applying the comparison principle to (4.10) we have

$$u^*(x+2M_2, t) \leq u_\tau(x, t) \leq u^*(x-2M_2, t). \quad (4.11)$$

This yields

$$\|u_{\tau+h}(x, t) - u_\tau(x, t)\|_\infty \leq u^*(x-2M_2, t) - u^*(x+2M_2, t),$$

for  $-2M_2 \leq \tau \leq 2M_2 - h$ , where  $u^*(x, t)$  denotes the solution of (4.1) with initial data  $u_0^*(x)$ . W-stability result tells that  $u^*(x, t)$  approaches a shifted wave in  $\|\cdot\|_w$  norm, that is there exists  $T_1 > 0$  and a shift  $\theta_1 \in \mathbb{R}$  such that

$$\|u^*(\cdot, t) - U(\cdot + \theta_1)\|_w < \delta/6 \quad \text{for } t > T_1.$$

On the other hand from  $\lim_{\pm x \rightarrow \infty} \|U(x) - u_\pm\|_w = 0$  it follows that for large  $|x|$

$$\|U(\cdot + \theta_1) - U(\cdot + x_0 + \theta_1)\|_w < \delta/6.$$

The above estimates lead to

$$\|u_{\tau+h}(x, t) - u_\tau(x, t)\|_w < \delta/2$$

for all  $-2M_2 \leq \tau \leq 2M_2 - h$ ,  $|x| > M$  and  $t > T_1$ .

Note that for  $|x| \leq M$  one has

$$\|u_{\tau+h}(x, t) - u_\tau(x, t)\|_w \leq Kh \sup_{|x| \leq M} w(x).$$

Taking  $h$  so that

$$Kh \sup_{|x| \leq M} w(x) = \delta/2,$$

we thus obtain

$$\|u_{\tau+h}(x, t) - u_\tau(x, t)\|_w \leq \delta/2$$

for all  $-2M_2 \leq \tau \leq 2M_2 - h$  and  $t > T_1$ .

We now conclude the proof by induction argument. Let  $u_\tau(x, t)$  approach to a wave. Then for some number  $T_2 > 0$  and a shift number  $\theta_2 \in \mathbb{R}$  we have

$$\|u_\tau(\cdot, t) - U(\cdot + \theta_2)\|_w < \delta/2$$

for  $\tau > T_2$ . Then it follows that

$$\|u_{\tau+h}(\cdot, t) - U(\cdot + \theta_2)\|_w < \delta$$

for  $t > \max(T_1, T_2)$ , which implies that  $u_{\tau+h}(x, t)$  approaches to a wave in  $\|\cdot\|_w$  norm. This completes the proof of the theorem.

□

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## References

- [1] S. ALLEN AND J. CAHN, *A microscopic theory for anti phase motion and its application to anti phase domain coarsening*, Acta Metal. **27** (1979), 1084-1095.
- [2] L. BRONSARD AND R.V. KOHN, *On the slowness of phase boundary motion in one space dimension*, Comm. Pure Appl. Math. **43** (1990), 983-997.
- [3] G. BARLES AND P.E. SOUGANIDIS, *A new approach to front propagation problems: theory and applications*, Arch. Rat. Mech. Anal. **141** (1998), 237-296.
- [4] X. CHEN, *Generation and propagation of interfaces in reaction-diffusion systems*, J. Diff. Equ. **96** (1992), 116-142.
- [5] J. CARR AND P.L. PEGO, *Metastable patterns in solutions of  $u_t = \epsilon u_{xx} - f(u)$* , Comm. Pure Appl. Math. **42** (1989), 523-576.
- [6] H.T. FAN AND S. JIN, *Front motion in multi-dimensional conservation laws with stiff source terms driven by mean curvature and variation of front thickness*, preprint (2001).
- [7] G. FUSCO AND J.K. HALE, *Slow motion manifolds, dormant instability, and singular perturbations*, J. Dyn. Diff. Eqn. **1** (1989), 75-94.
- [8] P.C. FIFE AND L. HSIAO, *The generation and propagation of internal layers*, Non-linear Analysis, **12** (1988), 19-41.
- [9] R. A. FISHER, *The wave of advance of advantageous*, Ann. Eugenics, **7** (1937), 355-369.
- [10] H.T. FAN AND S. JIN, *Wave patterns and slow motions in inviscid and viscous hyperbolic equations with stiff reaction terms*, (2000),
- [11] H. FAN, S. JIN AND Z-H TENG, *Zero reaction limit for hyperbolic conservation laws with source terms*, J. Diff. eqs. **168** (2000), 270-294.
- [12] S. HEINZE, G. PAPANICOLAOU AND A. STEVENS, *Variational principles for propagation speeds in inhomogeneous media*, Preprint (1999).

- [13] A. N. KOLMOGOROV, I. G. PETROVSKY AND N.S. PISKUNOV, Moskow Univ. Math. Bull. **1** (1937), 1-25.
- [14] S. OSHER AND J. SETHIAN, *Fronts moving with curvature dependent speed: Algorithm based on Hamilton-Jacobi equations*, J. Comp. Phys. **79** (1988), 12-49.
- [15] J. PAN AND H.L. LIU, *On stability of traveling waves of Burgers-Fisher equation*, Ann. Differential Equations, **14**, (1998), 37-47.
- [16] J. RUBINSTEIN, P. STERNBERG AND J.B. KELLER , *Fast reaction, slow diffusion, and curve shortening*, SIAM J. Appl. Math. **49** (1989), 111-131.
- [17] D.H. SATTINGER, *On the stability of waves of nonlinear parabolic systems*, Adv. math. **22** (1976), 312-355.
- [18] A. VOLPERT, V. VOLPERT AND V. VOLPERT , *Traveling wave solutions of parabolic systems*, Transl. Math. Monographs **140** , AMS, Providence, RI, (1994).
- [19] J. XIN, *Front propagation in heterogeneous media*, SIAM Review, **42** (2000), 161-230.