

Some new domain decomposition and multigrid methods for variational inequalities

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1 Introduction

Domain decomposition (DD) and multigrid (MG) methods are powerful iterative methods for solving some partial differential equations. Some recent progress has shown that DD and MG methods are also efficient for some nonlinear elliptic problems and some convex minimization problems [15, 14, 18, 17]. Mesh independent convergence rate has been proved and it is shown that the convergence rate for some nonlinear problems is as good as the convergence rate of the methods when they are used for the Laplace equation. In many industrial applications, we need to solve nonlinear partial differential equations and at the same time the solutions of the equations need to satisfy some constraints. For such problems, the solutions always satisfy some variational inequalities. To apply DD and MG methods for variational inequalities is a difficult task, see [1, 3, 4, 5, 7, 8, 2, 9, 12] for some literature results. It is even more difficult to analyse the convergence rate. In this work, we shall propose some new algorithms using DD and MG methods for variational inequalities and at the same time it is shown that the proposed algorithms have a convergence rate that is as good as DD and MG are used for some linear elliptic equations. Another feature of our approach is that we interpret DD and MG methods as space decomposition techniques [19, 18] and our algorithms are proposed for general space decomposition techniques. Thus, the algorithms and the analysis cover both DD and MG in the same frame work. The algorithms proposed here are different from the algorithms of [16, 6, 7].

Algorithms and convergence rate analysis for DD method with a coarse mesh seem still missing in the literature. When no coarse mesh is used, DD method is essentially a block relaxation method and some results are available in the literature, see [13] for some reference. For MG method, the only uniform convergence rate estimate we know is [6, 7] which is valid in the asymptotic sense and need very special conditions.

2 Some subspace correction algorithms

Consider the nonlinear convex minimization problem

$$\min_{v \in K} F(v), \quad K \subset V, \quad (1)$$

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where F is a convex functional over a reflexive Banach space V and $K \subset V$ is a nonempty closed convex subset. The norm of V will be denoted by $\|\cdot\|$. In order to solve the minimization problem efficiently, we shall decompose V and K into a sum of subspaces and subsets of smaller sizes respectively as in [10] [17]. More precisely, we decompose

$$V = \sum_{i=1}^m V_i, \quad K = \sum_{i=1}^m K_i, \quad K_i \subset V_i \subset V, \quad (2)$$

where V_i are subspaces and K_i are convex subsets. We use two constants C_1 and C_2 to measure the quality of the decompositions. First, we assume that there exists a constant $C_1 > 0$ and some operators $R_i : K \mapsto K_i$, $i = 1, 2, \dots, m$, which are generally nonlinear operators, such that the following relations are correct for all $u, v \in K$

$$u = \sum_{i=1}^m R_i u, \quad v = \sum_{i=1}^m R_i v, \quad \text{and} \quad \left(\sum_{i=1}^m \|R_i u - R_i v\|^2 \right)^{\frac{1}{2}} \leq C_1 \|u - v\|. \quad (3)$$

We also need to assume that there is a $C_2 > 0$ such that for any $w_i \in V$, $\hat{v}_i \in V_i$, $\tilde{v}_j \in V_j$ it is true that

$$\sum_{i=1}^m \sum_{j=1}^m |\langle F'(w_{ij} + \hat{v}_i) - F'(w_{ij}), \tilde{v}_j \rangle| \leq C_2 \left(\sum_{i=1}^m \|\hat{v}_i\|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^m \|\tilde{v}_j\|^2 \right)^{\frac{1}{2}}. \quad (4)$$

In the above, F' is the Gâteaux differential of F and $\langle \cdot, \cdot \rangle$ is the duality pairing between V and its dual space V' , i.e. the value of a linear function at an element of V . We also assume that there exists a constant $\kappa > 0$ such that

$$\langle F'(v_1) - F'(v_2), v_1 - v_2 \rangle \geq \kappa \|v_1 - v_2\|^2, \quad \forall w, v \in V. \quad (5)$$

Under the assumption (5), problem (1) has a unique solution. For some nonlinear problems, the constant κ may depend on v_1 and v_2 and our algorithms and analysis are still valid for such cases [18]. For a given approximate solution $u \in K$, we shall find a better solution w using one of the following two algorithms.

Algorithm 1 Choose a relaxation parameter $\alpha \in (0, 1/m]$. Find $\hat{w}_i \in K_i$ in parallel for $i = 1, 2, \dots, m$ such that

$$\hat{w}_i = \arg \min_{v_i \in K_i} G(v_i) \quad \text{with} \quad G(v_i) = F \left(\sum_{j=1, j \neq i}^m R_j u + v_i \right). \quad (6)$$

Set $w_i = (1 - \alpha)R_i u + \alpha \hat{w}_i$ and $w = (1 - \alpha)u + \alpha \sum_{i=1}^m \hat{w}_i$.

Algorithm 2 Choose a relaxation parameter $\alpha \in (0, 1]$. Find $\hat{w}_i \in K_i$ sequentially for $i = 1, 2, \dots, m$ such that

$$\hat{w}_i = \arg \min_{v_i \in K_i} G(v_i) \quad \text{with} \quad G(v_i) = F \left(\sum_{j < i} w_j + v_i + \sum_{j > i} R_j u \right) \quad (7)$$

where $w_j = (1 - \alpha)R_j u + \alpha \hat{w}_j$, $j = 1, 2, \dots, i - 1$. Set $w = (1 - \alpha)u + \alpha \sum_{i=1}^m \hat{w}_i$.

Denote u^* the unique solution of (1), the following convergence estimate is correct for Algorithms 1 and 2 (see Tai [13]):

Theorem 1 *Assuming that the space decomposition satisfies (3), (4) and that the functional F satisfies (5). Then for Algorithms 1 and 2, we have*

$$\frac{F(w) - F(u^*)}{F(u) - F(u^*)} \leq 1 - \frac{\alpha}{(\sqrt{1 + C^*} + \sqrt{C^*})^2}, \quad C^* = \left(C_2 + \frac{[C_1 C_2]^2}{2\kappa} \right) \frac{2}{\kappa}. \quad (8)$$

Algorithms 1 and 2 are written for general space decompositions. In implementation for a specific space decomposition technique, auxiliary functions may be introduced to make the implementation simpler and easier. For example, by defining $e_i = \hat{w}_i - R_i u$, Algorithms 1 and 2 can be written in the following equivalent form:

Algorithm 3 *Choose a relaxation parameter $\alpha \in (0, 1/m]$. Find $e_i \in V_i$ in parallel for $i = 1, 2, \dots, m$ such that*

$$e_i = \arg \min_{\substack{v_i + R_i u \in K_i \\ v_i \in V_i}} G(v_i) \quad \text{with} \quad G(v_i) = F\left(u + v_i\right). \quad (9)$$

Set $w = u + \alpha \sum_{i=1}^m e_i$.

Algorithm 4 *Choose a relaxation parameter $\alpha \in (0, 1]$. Find $e_i \in V_i$ sequentially for $i = 1, 2, \dots, m$ such that*

$$e_i = \arg \min_{\substack{v_i + R_i u \in K_i \\ v_i \in V_i}} G(v_i) \quad \text{with} \quad G(v_i) = F\left(u + \sum_{j < i} e_j + v_i\right) \quad (10)$$

Set $w = u + \alpha \sum_{i=1}^m e_i$.

3 Some Applications

We apply the algorithms for the following obstacle problem:

$$\text{Find } u \in K, \quad \text{such that} \quad a(u, v - u) \geq f(v - u), \quad \forall v \in K, \quad (11)$$

with

$$a(v, w) = \int_{\Omega} \nabla v \cdot \nabla w \, dx, \quad K = \{v \in H_0^1(\Omega) \mid v(x) \geq \psi(x) \text{ a.e. in } \Omega\}.$$

It is well known that the above problem is equivalent to the minimization problem (1) assuming that $f(v)$ is a linear functional on $H_0^1(\Omega)$. For the obstacle problem (11), the minimization space $V = H_0^1(\Omega)$. Correspondingly, we have $\kappa = 1$ for assumption (5). Later, $|\cdot|_1$ and $\|\cdot\|_1$ are used to denote the semi-norm and norm of $H_0^1(\Omega)$. The finite element method shall be used to solve (11). It shall be shown that domain decomposition and multigrid methods satisfy the conditions (3) and (4). For simplicity of the presentation, it will be assumed that

$$\psi = 0.$$

3.1 Overlapping domain decomposition methods

For the domain Ω , let \mathcal{T}_H be a shape regular quasi-uniform finite element division, or a coarse mesh, of Ω , with a mesh size H . Further more, assume that $\{\Omega_i\}_{i=1}^M$ is a non-overlapping decomposition of Ω where each Ω_i has a diameter of order H and is the union of several coarse mesh elements. We further refine \mathcal{T}_H to get a fine mesh

partition \mathcal{T}_h with mesh size h . We assume that \mathcal{T}_h forms a shape regular quasi-uniform finite element subdivision of Ω . We call this the fine mesh or the h -level subdivision of Ω . We denote by $S_H \subset W_0^{1,\infty}(\Omega)$ and $S_h \subset W_0^{1,\infty}(\Omega)$ the continuous, piecewise linear finite element spaces over the H -level and h -level subdivisions of Ω respectively. For each Ω_i , we consider an enlarged subdomain Ω_i^δ consisting of elements $\tau \in \mathcal{T}_h$ with $\text{distance}(\tau, \Omega_i) \leq \delta$. The union of Ω_i^δ covers $\bar{\Omega}$ with overlaps of size δ . For the overlapping subdomains, assume that there exist m colors such that each subdomain Ω_i^δ can be marked with one color, and the subdomains with the same color will not intersect with each other. Let Ω_i^c be the union of the subdomains with the i^{th} color, and $V_i = \{v \in S_h \mid v(x) = 0, x \notin \Omega_i^c\}$, $i = 1, 2, \dots, m$. By denoting the subspaces $V_0 = S_H$, $V = S_h$, we find that

$$a). \quad V = \sum_{i=1}^m V_i \quad \text{and} \quad b). \quad V = V_0 + \sum_{i=1}^m V_i. \quad (12)$$

Note that the summation index is now from 0 to m instead of from 1 to m when the coarse mesh is added. For the constraint set K , we define

$$K_0 = \{v \in V_0 \mid v \geq 0\}, \quad \text{and} \quad K_i = \{v \in V_i \mid v \geq 0\}, \quad i = 1, 2, \dots, m. \quad (13)$$

Under the condition that $\psi = 0$, it is easy to see that (2) is correct both with or without the coarse mesh. When the coarse mesh is added, the summation index is from 0 to m . Let $\{\theta_i\}_{i=1}^m$ be a partition of unity with respect to $\{\Omega_i^c\}_{i=1}^m$, i.e. $\theta_i \in V_i$, $\theta_i \geq 0$ and $\sum_{i=1}^m \theta_i = 1$. It can be chosen so that

$$|\nabla \theta_i| \leq C/\delta, \quad \theta_i(x) = \begin{cases} 1 & \text{if } x \in \tau, \text{ distance}(\tau, \partial\Omega_i^c) \geq \delta \text{ and } \tau \subset \Omega_i^c, \\ 0 & \text{on } \bar{\Omega} \setminus \Omega_i^c. \end{cases} \quad (14)$$

Later in this paper, we use I_h as the linear Lagrangian interpolation operator which uses the function values at the h -level nodes. In addition, we also need a nonlinear interpolation operator $I_H^\ominus : S_h \mapsto S_H$. Assume that $\{x_0^i\}_{i=1}^{n_0}$ are all the interior nodes for \mathcal{T}_H and let ω_i be the support for the nodal basis function of the coarse mesh at x_0^i . The nodal values for $I_H^\ominus v$ for any $v \in S_h$ is defined as $(I_H^\ominus v)(x_0^i) = \min_{x \in \omega_i} v(x)$, c.f [13]. This operator satisfies

$$I_H^\ominus v \leq v, \quad \forall v \in S_h, \quad \text{and} \quad I_H^\ominus v \geq 0, \quad \forall v \geq 0, v \in S_h. \quad (15)$$

Moreover, it has the following monotonicity property

$$I_{h_1}^\ominus v \leq I_{h_2}^\ominus v, \quad \forall h_1 \geq h_2 \geq h, \quad \forall v \in S_h. \quad (16)$$

As $I_H^\ominus v$ equals v at least at one point in ω_i , it is thus true that for any $u, v \in S_h$

$$\|I_H^\ominus u - I_H^\ominus v - (u - v)\|_0 \leq c_d H |u - v|_1, \quad |I_H^\ominus v|_1 \leq c_d |v|_1, \quad (17)$$

where d indicates the dimension of the physical domain Ω , i.e. $\Omega \subset \mathbb{R}^d$, and

$$c_d = \begin{cases} C & \text{if } d = 1; \\ C \left(1 + \left|\log \frac{H}{h}\right|^{\frac{1}{2}}\right) & \text{if } d = 2, \\ C \left(\frac{H}{h}\right)^{\frac{1}{2}} & \text{if } d = 3, \end{cases}$$

With C being a generic constant independent of the mesh parameters. See Tai [13] for a detailed proof.

3.2 Decompositions with or without the coarse mesh

We first give the definition for the operators $R_i : K \mapsto K_i$ for the decomposition (12.a), i.e. we consider the domain decomposition method without using the coarse mesh. For any given $v \in S_h$, we decompose v as

$$v = \sum_{i=1}^m v_i, \quad v_i = I_h(\theta_i v), \quad (18)$$

and we define the mapping from v to v_i as R_i , i.e. $R_i v = v_i, \forall v \in S_h$. In case that $v \geq 0$, it is true that $v_i \geq 0$, i.e. R_i is a mapping from K to K_i . In addition,

$$\sum_{i=1}^m \|R_i u - R_i v\|_1^2 \leq C \left(1 + \frac{1}{\delta^2}\right) \|u - v\|_1^2,$$

which shows that $C_1 \leq C(1 + \delta^{-1})$. It is known that $C_2 \leq \sqrt{m}$ with m being the number of colors. From Theorem 1, the following rate is obtained for the one level domain decomposition method (c.f. (12.a)):

$$\frac{F(w) - F(u^*)}{F(u) - F(u^*)} \leq 1 - \frac{\alpha}{1 + C(1 + \delta^{-2})}.$$

For Algorithm 2, we can take $\alpha = 1$.

Numerical experiments and the convergence analysis for the two-level domain decomposition method, i.e. an overlapping domain decomposition with a coarse mesh, seem still missing in the literature. To apply our algorithms for the two-level domain decomposition method, i.e. for the decomposition (12.b), the operators R_i are defined as

$$R_0 v = I_H^\ominus v, \quad R_i v = I_h(\theta_i(v - I_H^\ominus v)), \quad i = 1, 2, \dots, m \quad \forall v \in S_h. \quad (19)$$

For a given $v \geq 0$, it is true using (15) that $0 \leq R_0 v \leq v$ and so $R_i v \geq 0$, $i = 1, 2, \dots, m$, which indicates that $R_0 v \in K_0$ and $R_i v \in K_i$, $i = 1, 2, \dots, m$ for any $v \in K$. It follows from (17) that for any $u, v \in K$ $\|R_0 u - R_0 v\|_1 \leq c_d \|u - v\|_1$. Note that $R_i u - R_i v = I_h(\theta_i(u - v - I_H^\ominus u + I_H^\ominus v))$. Using estimate (17) and a proof similar to those for the unconstrained cases, c.f. [17], [18], it can be proven that $\|R_i u - R_i v\|_1^2 \leq c_d \left(1 + \frac{H}{\delta}\right) \|u - v\|_1^2$. Thus

$$\left(\|R_0 u - R_0 v\|_1^2 + \sum_{i=1}^m \|R_i u - R_i v\|_1^2 \right)^{\frac{1}{2}} \leq C(m) c_d \left(1 + \left(\frac{H}{\delta} \right)^{\frac{1}{2}} \right) \|u - v\|_1.$$

The estimate for C_2 is known, c.f. [17], [18]. Thus, for the two-level domain decomposition method, we have $C_1 = C(m) c_d \left(1 + \frac{\sqrt{H}}{\sqrt{\delta}}\right)$, $C_2 = C(m)$, where $C(m)$ is a constant only depending on m , but not on the mesh parameters and the number of subdomains. An application of Theorem 1 will show that the following convergence rate estimate is correct for the two-level domain decomposition method (12.b):

$$\frac{F(w) - F(u^*)}{F(u) - F(u^*)} \leq 1 - \frac{\alpha}{1 + c_d^2(1 + H\delta^{-1})}.$$

3.3 Multigrid decomposition

Multigrid methods can be regarded as a repeated use of the two-level method. We assume that the finite element partition \mathcal{T}_h is constructed by a successive refinement process. More precisely, $\mathcal{T}_h = \mathcal{T}_J$ for some $J > 1$, and \mathcal{T}_j for $j \leq J$ is a nested

sequence of quasi-uniform finite element partitions, see [13], [17], [18]. We further assume that there is a constant $\gamma < 1$, independent of j , such that h_j is proportional to γ^{2j} . Corresponding to each finite element partition \mathcal{T}_j , a finite element space \mathcal{M}_j can be defined by

$$\mathcal{M}_j = \{v \in W_0^{1,\infty}(\Omega) : v|_\tau \in \mathcal{P}_1(\tau), \forall \tau \in \mathcal{T}_j\}.$$

let $\{x_j^k\}_{k=1}^{n_j}$ be the set of all the interior nodes. Denoted by $\{\phi_j^i\}_{i=1}^{n_j}$ the nodal basis functions satisfying $\phi_j^i(x_j^k) = \delta_{ik}$. We then define a one dimensional subspace $V_j^i = \text{span}(\phi_j^i)$. Letting $V = \mathcal{M}_J$ and $K_j^i = \{v \in V_j^i | v \geq 0\}$, we have the following trivial space decomposition:

$$V = \sum_{j=1}^J \sum_{i=1}^{n_j} V_j^i, \quad K = \sum_{j=1}^J \sum_{i=1}^{n_j} K_j^i. \quad (20)$$

Each subspace V_j^i is a one dimensional subspace. Let $I_{h_j}^\ominus$ to be the nonlinear interpolation operator from \mathcal{M}_j to \mathcal{M}_J . For any $v \geq 0$ and $j \leq J-1$, define $v_j = I_{h_j}^\ominus v - I_{h_{j-1}}^\ominus v \in \mathcal{M}_j$. Let $v_J = v - I_{h_{J-1}}^\ominus v \in \mathcal{M}_J$. A further decomposition of v_j is given by $v_j = \sum_{i=1}^{n_j} v_j^i$ with $v_j^i = v_j(x_j^i)\phi_j^i$. It is easy to see that

$$v = \sum_{j=1}^J v_j = \sum_{j=1}^J \sum_{i=1}^{n_j} v_j^i. \quad (21)$$

We define the mapping from v to v_j^i to be R_j^i , i.e. $R_j^i v = v_j^i$. It follows from (15) and (16) that $v_j^i \geq 0$ for all $v \geq 0$, i.e. R_j^i is a mapping from K to K_j^i under the condition that $\psi = 0$. Define

$$\tilde{c}_d = \begin{cases} C, & \text{if } d = 1; \\ C(1 + |\log h|^{\frac{1}{2}}), & \text{if } d = 2; \\ Ch^{-\frac{1}{2}}, & \text{if } d = 3. \end{cases}$$

For any given $u \in S_h$, we define u_j and u_j^i in a similar way as we did for v . The following estimate can be obtained using approximation properties (17) (see [13]):

$$\sum_{j=1}^J \sum_{i=1}^{n_j} \|R_j^i u - R_j^i v\|_1^2 \leq C \sum_{j=1}^J h_j^{-2} \|u_j - v_j\|_0^2 \leq \tilde{c}_d^2 \sum_{j=1}^J h_j^{-2} h_{j-1}^2 |u - v|_1^2 \leq \tilde{c}_d^2 \gamma^{-2} J |u - v|_1^2,$$

which proves that

$$C_1 \cong \tilde{c}_d \gamma^{-1} J^{\frac{1}{2}} \cong \tilde{c}_d \gamma^{-1} |\log h|^{\frac{1}{2}}.$$

The estimate for C_2 is known, i.e. $C_2 = C(1 - \gamma^d)^{-1}$, see Tai and Xu [18]. Thus for the multigrid method, the error reduction factor for the algorithms is

$$\frac{F(w) - F(u^*)}{F(u) - F(u^*)} \leq 1 - \frac{\alpha}{1 + \tilde{c}_d^2 \gamma^{-2} J}.$$

3.4 Numerical experiments

For the implementation of the proposed algorithms, we need some subroutines to carry out the actions of the decomposition operators and some other subroutines to solve the sub-minimization problems in the algorithms. If we implement the algorithms in the form as given in Algorithms 3 and 4, it can be seen the decomposition operators R_i are only needed to determine the obstacles for the subproblems.

Let us first sketch the implementation for the DD methods with or without the coarse mesh. Without the coarse mesh, we need to have some subroutines to calculate the θ_i functions for the decomposition operators R_i given in (18). The construction of the θ_i functions are not unique and we just choose one of them. The sub-minimization problems over the subdomains are solved by the augmented Lagrangian method as stated in [11] (but without the dimensional splitting). Once the coarse mesh is added, we need a subroutine to calculate $I_H^\ominus v$ for any given $v \in S_h$ and this will give the decomposition operator R_0 as given in (19). Once this is done, the other operators R_i can be done using the functions θ_i . The sub-minimization problem over the coarse mesh is also solved by the augmented Lagrangian method, see [16, 11]. The subproblems need more computing time in the first iteration. From the second iteration, very good initial guess is available and the iterations are terminated after a few iterations.

For the multigrid decomposition (20), we need to calculate $I_{h_j}^\ominus$ to get the actions of operators R_j^i . The cost for calculating this is very cheap. For any $v \in S_h$ and $v \geq 0$, we use a vector z_j to store the values $\min_{\tau_j^i} v$ for all the elements $\tau_j^i \subset \mathcal{T}_j$. As the meshes are nested, the vectors z_j can be computed recursively starting from the finest mesh and ending with the coarsest mesh. From the vectors z_j , it is easy to compute $I_{h_j}^\ominus v$ on each level. The value of $I_{h_j}^\ominus v$ at a given node is just the smallest value of z_j in the neighboring elements. The sub-minimization problems are just some minimization problems over a one-dimensional subspace and explicit formula can be given for these sub-minimization problems and they can be implemented similarly as for unconstrained problems, see [16]. The operation cost per iteration for the algorithms is $O(n_J)$.

We shall test our algorithms for the obstacle problem (11) with $\Omega = [-2, 2] \times [-2, 2]$, $f = 0$ and $\psi(x, y) = \sqrt{x^2 + y^2}$ when $x^2 + y^2 \leq 1$ and $\psi(x, y) = -1$ elsewhere. This problem has an analytical solution [13]. Note that the continuous obstacle function ψ is not even in $H^1(\Omega)$. Even for such a difficult problem, uniform linear convergence has been observed in our experiments. In the discrete case, the non-zero obstacle can be shifted to the right hand side.

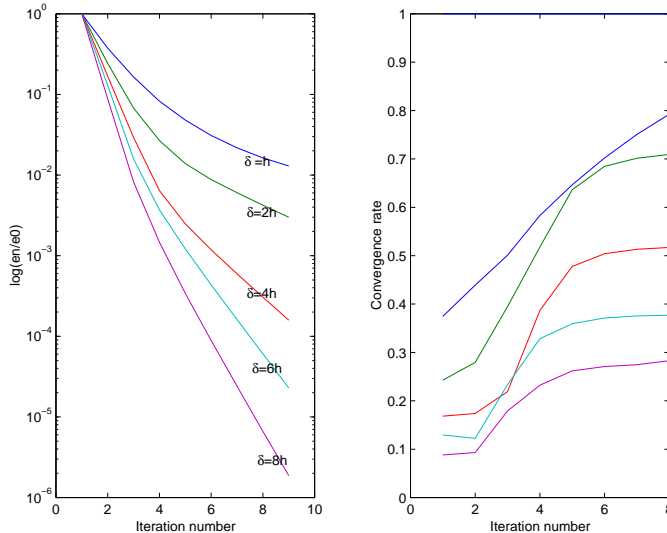


Figure 1: Convergence for the two-level method for decomposition (19) with different overlaps, $h = 4/128$, and $H = 4/8$.

Figure 1 shows the convergence rate for Algorithm 2 with different overlapping

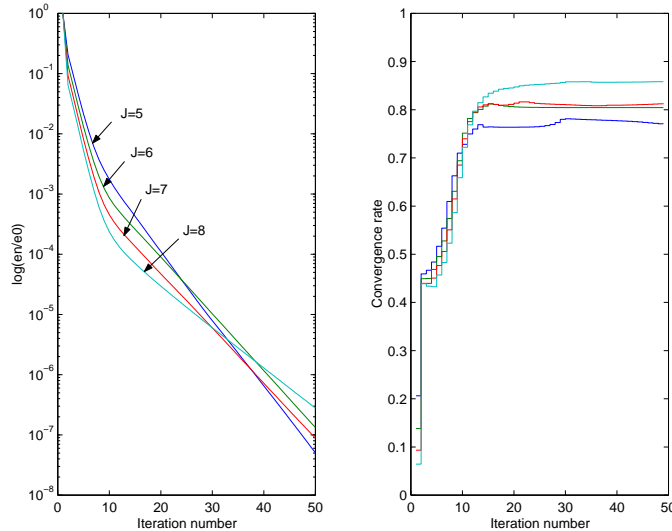


Figure 2: Convergence for the multigrid method

sizes for decomposition (19). Figure 2 shows the convergence rate for Algorithm 2 with the multigrid method for decomposition (21) and J indicates the number of levels. In the figures en is the H^1 -error between the computed solution and the true finite element solution and $e0$ is the initial error. $\log(en/e0)$ is used for one of the subfigures. The convergence rate is faster in the beginning and then approaches a constant after some iterations.

References

- [1] A. Brandt and C. W. Cryer. Multigrid algorithms for the solution of linear complementary problems arising from free boundary problems. *SIAM J. Sci. Comput.*, 4:655–684, 1983.
- [2] T. F. Chan and I. Sharapov. Subspace correction multi-level methods for elliptic eigenvalue problems. *Numer. Linear Algebra Appl.*, 9(1):1–20, 2002.
- [3] W. Hackbusch and H. D. Mittelmann. On multilevel methods for variational inequalities. *Numer. Math.*, 42:65–76, 1983.
- [4] R. H. W. Hoppe. Multigrid algorithms for variational inequalities. *SIAM J. Numer. Anal.*, 24(5):1046–1065, 1987.
- [5] R. H. W. Hoppe and R. Kornhuber. Adaptive multilevel methods for obstacle problems. *SIAM J. Numer. Anal.*, 31:301–323, 1994.
- [6] R. Kornhuber. Monotone multigrid methods for elliptic variational inequalities I. *Numer. Math.*, 69:167–184, 1994.
- [7] R. Kornhuber. Monotone iterations for elliptic variational inequalities, II. *Numer. Math.*, 72:481–499, 1996.
- [8] J. Mandel. A multilevel iterative method for symmetric, positive definite linear complementary problems. *Appl. Math. Optim.*, 11:77–95, 1984.

- [9] I. A. Sharapov. Multilevel subspace correction for large scale optimization problems. Technical Report CAM-97-31, UCLA, Computational and Applied Mathematics, 1997.
- [10] X.-C. Tai. Parallel function and space decomposition methods. In P. Neittaanmäki, editor, *Finite element methods, fifty years of the Courant element, Lecture notes in pure and applied mathematics*, volume 164, pages 421–432. Marcel Dekker inc., 1994. Available online at <http://www.mi.uib.no/~tai>.
- [11] X.-C. Tai. Parallel function decomposition methods and numerical applications. In J. Wang, M. B. Allen, B. M. Chen, and T. Mathew, editors, *Iterative methods in scientific computation*, volume 4 of *Series in computational and applied mathematics*, pages 231–238. IMACS, 1998.
- [12] X.-C. Tai. Convergence rate analysis of domain decomposition methods for obstacle problems. *East-West J. Numer. Math.*, 9(3):233–252, 2001.
- [13] X.-C. Tai. Rate of convergence for some constraint decomposition methods for nonlinear variational inequalities. *Numer. Math.*, accepted and to appear. Available online at <http://www.mi.uib.no/~tai>, report-150, Department of Mathematics, University of Bergen.
- [14] X.-C. Tai and M. Espedal. Applications of a space decomposition method to linear and nonlinear elliptic problems. *Numer. Math. for Part. Diff. Equat.*, 14:717–737, 1998.
- [15] X.-C. Tai and M. Espedal. Rate of convergence of some space decomposition method for linear and nonlinear elliptic problems. *SIAM J. Numer. Anal.*, 35:1558–1570, 1998.
- [16] X.-C. Tai, B. ove Heimsund, and J. Xu. Rate of convergence for parallel subspace correction methods for nonlinear variational inequalities. In *Proceedings of the 13th International Conference on Domain Decomposition Methods in Lyon, France*.
- [17] X.-C. Tai and P. Tseng. Convergence rate analysis of an asynchronous space decomposition method for convex minimization. *Math. Comput.*, 2001.
- [18] X.-C. Tai and J.-C. Xu. Global and uniform convergence of subspace correction methods for some convex optimization problems. *Math. Comput.*, 71:105–124, 2001.
- [19] J. Xu. Iteration methods by space decomposition and subspace correction. *SIAM Rev.*, 34:581–613, 1992.