

# On Authority Distributions in Organizations : Equilibrium<sup>1</sup>

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One assumption in the Shapley-Shubik power index is that there is no interaction nor influence among the voting members. This paper will apply the command structure of Shapley(1994) to model members' interaction relations by simple games. An equilibrium authority distribution is then formulated by the power-in-power-out mechanism. It turns out to have much similarity to the invariant measure of a Markov chain and therefore some similar interpretations are followed for the new setting. In some sense, one's authority distribution quantifies his general administrative power in the organization and his long-run influence on all members. We provide a few applications in conflict resolution, college and journal ranking, and organizational choice. *Journal of Economic Literature* Classification Numbers: C68, C71, D58, D72.

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## 1. PRELIMINARY

The paper treats an organization as an ensemble of simple games: one “command game” for each member in the organization. The general “authority distribution” would be based on the internal structure in the long run, instead of some specific external tasks.

### 1.1. Notations

The conventional notation for simple games was introduced by von Neumann and Morgenstern(1947) and extended by Shapley(1962). As far as our present subject is concerned, we shall distinguish three levels of Boolean abstraction : lower-case italic letters or numerals for *individuals*, italic capitals for *sets of individuals*, and script capitals for *sets of sets of individuals* – thus  $c \in C \in \mathcal{C}$ . The empty set of individuals, “ $\emptyset$ ”, is different from the empty set of sets, “ $\phi$ ”. “ $\subseteq$ ” is used for set inclusion and “ $\subset$ ” for strict set inclusion, and set subtraction will be indicated by “ $\setminus$ ” (read “without”). In naming the elements of a set, we shall often employ the vinculum, thus “ $\overline{137}$ ” for “ $\{1, 3, 7\}$ ”. The number of elements of a finite set  $X$  is denoted  $|X|$ .

$N$  will conventionally denote the set of all individuals or members in an organization, and  $\mathcal{N}$  will denote its power set, i.e., the set of all subsets (*coalitions*) of  $N$ . If  $i$  is any element of  $N$ , then “ $N_i$ ” will denote  $N \setminus \overline{i}$  and “ $\mathcal{N}_i$ ” will denote the power set of  $N_i$ . We denote by  $\mathcal{S}^+$  the set of

all *supersets* of elements of  $\mathcal{S}$ , by  $\mathcal{S}^\cup$  (or  $\mathcal{S}^\cap$ ) the *union* (or *intersection*, respectively) of all elements of  $\mathcal{S}$ , and by  $\mathcal{S}^m$  the smallest  $\mathcal{T} \subseteq \mathcal{S}$  such that  $\mathcal{T}^+ = \mathcal{S}^+$ , or equivalently, the set of all  $R \in \mathcal{S}$  that are minimal, i.e., such that no  $T \in \mathcal{S}$  satisfies  $T \subset R$ .

## 1.2. Simple Games

While a *simple game* on  $N$  is often represented as the ordered pair  $(N, \mathcal{W})$ , we denote it by the symbol  $\Gamma(N, \mathcal{W})$  where

$$\phi \subset \mathcal{W} = \mathcal{W}^+ \subset \mathcal{N}. \quad (1)$$

The first strict inclusion in this definition tells us that (given  $\mathcal{W} = \mathcal{W}^+$ )  $N$  is always in  $\mathcal{W}$  and the second tells us that  $\emptyset$  is never in  $\mathcal{W}$ . Hence in a simple game,  $\mathcal{W}^m \neq \phi$ . In the absence of any more specific interpretation, the coalitions in  $\mathcal{W}$  will be called *winning* and those in  $\mathcal{N} \setminus \mathcal{W}$  will be called *losing*. By the equality in Eq.(1), every superset of a winning coalition is winning while every subset of a losing coalition is losing. Thus in defining a specific simple game it suffices to list just  $\mathcal{W}^m$ , the set of *minimal winning coalitions*. A player  $i$  is called *essential* if  $i \in \mathcal{W}^{m\cup}$ ; otherwise he is a *dummy*. We call him a *dictator* if  $\bar{i} = \mathcal{W}^{m\cup}$ .

### 1.3. Shapley-Shubik Power Indices

For any ordering  $i_1 i_2 \cdots i_n$  of all members in the simple game  $\Gamma(N, \mathcal{W})$ , we consider the increasing sequence of coalitions

$$\emptyset, \overline{i_1}, \overline{i_1 i_2}, \overline{i_1 i_2 i_3}, \dots, \overline{i_1 i_2 \cdots i_n} = N.$$

In this sequence, there exists a unique number  $t$  such that  $\overline{i_1 i_2 \cdots i_{t-1}}$  is losing while  $\overline{i_1 i_2 \cdots i_{t-1} i_t}$  is winning; we say that  $i_t$  *pivots* in the sequence. Now consider all possible ways to order  $N$ . There are  $n!$  of them. The *Shapley-Shubik (S-S) power index* of a player in the simple game is defined as the fraction of orderings in which he pivots. As already shown (Shapley, 1953; Shapley and Shubik, 1954), his power index may be defined as his probability of pivoting when the members are uniformly randomly ordered.

It is easy to see that the S-S power indices of a simple game are nonnegative and sum to 1. Dictators, and only dictators, have power 1; dummies, and only dummies, have power 0. Players who appear symmetrically in  $\mathcal{W}$  have equal power indices, and it is not hard to see that if “ $j$  outweighs  $i$ ” in the sense that replacing  $i$  with  $j$  never hurts a winning coalition, then  $j$ ’s power index is necessarily not less than  $i$ ’s. If the game is the winning rule for voting a bill and the players satisfy certain symmetry conditions, then player  $i$ ’s S-S index is the chance of the two “critical” situations: 1. the bill is passed unless he votes against it; 2. the bill is blocked unless he votes for it. In either of the critical situations, player  $i$ ’s vote (either YES

or NO) determines the result of the bill. Two symmetry conditions (SC-I and SC-II) are stated in Hu(2001). Let  $\mathbf{S}$  be the coalition of the players who will vote for (YES) the bill. In general, it is a random coalition to the voting body before the bill is voted; otherwise the bill is not necessary to vote (since the result is known). Clearly  $N \setminus \mathbf{S}$  consists of the players who vote against (NO) the bill. The symmetry conditions can be stated as:

- SC-I: the players have symmetric attitude toward the bill. That is, the probability of  $\mathbf{S} = T$  depends only on the size of  $T$ .
- SC-II: the size of  $\mathbf{S}$  is uniformly distributed on  $\{0, 1, \dots, n\}$ . Therefore, the probability of  $|\mathbf{S}| = k$  is  $\frac{1}{n+1}$  for any  $k \in \{0, 1, \dots, n\}$ .

In the absence of any more specific information about  $\mathbf{S}$ , we shall assume the two symmetry conditions when a vote is concerned in the present paper. Actually the two conditions are not essential. As remarked in §6, we may relax them to obtain the *asymmetric authority distribution*. As stated before, one assumption in the definition of the S-S index is that every ordering has the same probability  $\frac{1}{n!}$ . As already shown (Owen, 1971; Shapley, 1977; Owen and Shapley, 1989; Cheng, 1994), this assumption is relaxed in a generalization, called the *non-symmetric Shapley-Owen power index*, where the voters are endowed with different ideologies. Another assumption is that the probabilities are independent, i.e., no interaction exists among the voting members. In most real democratic societies, however, there is

some degree of political organization, whereby voters influence each other according to the distribution of authority throughout the organization; we will address this influence in the present paper.

A local organizational topology is set up for each member in the form of a simple game in §2. In §3, an equilibrium authority distribution is derived for organizations in which each member has some, either direct or indirect, impact on others. §4 discusses the ultimate influence between players. A few related applications, such as journal ranking and organizational choice, are addressed in §5. Omitted proofs can be found in the theory of Markov chains.

## 2. LOCAL TOPOLOGY OF AUTHORITY

In an organization, some members may have a certain degree of discretionary power; some may even be “free agents”, i.e., dictators in their own “command games”, accountable to no one else. Others may be merely “cogs in the machinery”, i.e., dummies in their own command games. Let us develop these ideas formally.

### 2.1. Boss and Approval Sets

Let  $i$  denote a generic member of the organization  $N = \overline{12 \cdots n}$  (with  $n > 1$  finite). In general there will be certain other individuals or more generally, sets of other individuals, that  $i$  must obey, regardless of his own

judgment or desires. We call them *boss sets* and denote them collectively by  $\mathcal{B}_i$ . Thus, if there is an individual  $b$  who can “boss”  $i$ , this is indicated by  $\bar{b} \in \mathcal{B}_i$ , not by  $b \in \mathcal{B}_i$ . Clearly, the empty coalition  $\emptyset$  should never be a boss set ( $\emptyset \notin \mathcal{B}_i$ ). Given any boss set  $S \in \mathcal{B}_i$ , we shall assume that all its supersets in  $\mathcal{N}_i$  are also boss sets. That is,

$$\phi \subseteq \mathcal{B}_i = \mathcal{B}_i^+ \cap \mathcal{N}_i \subset \mathcal{N}_i, \quad \text{all } i \in N, \quad (2)$$

which may be contrasted with Eq.(1). If  $i$  is not bossable, then  $\mathcal{B}_i = \phi$ , or  $\mathcal{N}_i \notin \mathcal{B}_i$ .

We shall also associate with each  $i \in N$  another collection  $\mathcal{A}_i$  of coalitions, called *approval sets*, in  $\mathcal{N}_i$ . The consent of any one of these approval sets is sufficient to allow  $i$  to act, if he wishes. But it is not able to force him to act. So approval sets are not boss sets, and hence  $\mathcal{A}_i \cap \mathcal{B}_i = \phi$ . On the other hand, if  $\emptyset \notin \mathcal{A}_i$ , then  $i$  can not act alone without an authorization from an approval set in  $\mathcal{A}_i$  or an order from a boss set in  $\mathcal{B}_i$ . Imitating Eq.(2), if  $S$  is an approval set, then all its supersets in  $\mathcal{N}_i \setminus \mathcal{B}_i$  are also assumed approval sets.

The collection of approval sets  $\mathcal{A}_i$  is a guide to the amount of *personal discretion* that  $i$  enjoys, if any. At one extreme, if  $\mathcal{A}_i = \mathcal{N}_i$  then  $i$  is called a *free agent*. He needs no approval (since  $\emptyset \in \mathcal{A}_i$ ), and no one can boss him (since  $\mathcal{B}_i = \phi$ ). We shall denote the set of free agents by  $F$ . At the other extreme, if  $\mathcal{A}_i = \phi$ , then  $i$  has no discretionary power, and we

shall call him a *cog*. For an intermediate example of “partial discretion”, merely consider a corporation president who is bossable by a  $\frac{2}{3}$  majority of the board of directors but is allowed to follow his own judgment as long as he has the support of a simple majority from the board. In the U.S. Legislature, the president’s proposal can be vetoed by  $\frac{2}{3}$  of the senators and  $\frac{2}{3}$  of the representatives.

We let  $\mathcal{Z}_i = \mathcal{A}_i \cup \mathcal{B}_i$ , the set of all approval and boss sets of  $i$ . However if  $\mathcal{Z}_i = \phi$ , then  $i$  is a cog, but not bossable. The cog takes no action always! We set the convention  $\mathcal{Z}_i \supset \phi$  for any  $i \in N$  such that the organization does not contain any non-bossable cog. Therefore if no coalition can boss  $i$  (i.e.  $\mathcal{B}_i = \phi$ ), then either  $i$  is a free agent (i.e.  $\emptyset \in \mathcal{A}_i$ ) or he needs at least an authorization from some non-empty approval set. On the other hand,  $N_i$  bosses  $i$  if he is a cog.

LEMMA 2.1.  $\mathcal{Z}_i = \mathcal{Z}_i^+ \cap \mathcal{N}_i \subseteq \mathcal{N}_i$ .

*Proof.* It suffices to show that  $\mathcal{Z}_i \supseteq \mathcal{Z}_i^+ \cap \mathcal{N}_i$ . If  $S \in \mathcal{Z}_i^+ \cap \mathcal{N}_i$ , then there exists a minimal coalition  $T \in \mathcal{N}_i$  such that  $T \subseteq S$  and  $T \in \mathcal{Z}_i$ . Now we have three cases: (I) if  $T \in \mathcal{B}_i$ , then  $S \in \mathcal{B}_i$  by Eq.(2) and so  $S \in \mathcal{Z}_i$ ; (II) if  $T \in \mathcal{A}_i$  and  $S \notin \mathcal{B}_i$ , i.e.  $T$  is an approval set and  $S$  is a superset of  $T$  in  $\mathcal{N}_i \setminus \mathcal{B}_i$ , then  $S \in \mathcal{A}_i$  and  $S \in \mathcal{Z}_i$ ; (III) if  $T \in \mathcal{A}_i$  and  $S \in \mathcal{B}_i$ , it is trivial that  $S \in \mathcal{Z}_i$ . Therefore  $S \in \mathcal{Z}_i$  in all cases and we complete the proof. ■



## 2.2. Command Games

For a coalition  $S \subseteq N$  to “command”  $i \in N$ , either  $S \setminus \bar{i}$  can boss  $i$  or  $i$  can agree to follow  $S \setminus \bar{i}$ 's collective suggestion. Formally we define  $i$ 's *command sets* as

$$\mathcal{W}_i \stackrel{\text{def}}{=} \mathcal{B}_i \cup \{S \cup \bar{i} : S \in \mathcal{Z}_i\}. \quad (3)$$

It is easy to see that  $\{S \cup \bar{i} : S \in \mathcal{Z}_i\} = \mathcal{Z}_i^+ \setminus \mathcal{Z}_i$  and  $\mathcal{B}_i^+ = \mathcal{B}_i \cup \{S \cup \bar{i} : S \in \mathcal{B}_i\}$  from Lemma 2.1 and Eq.(2). Therefore we have the equivalence:  $\mathcal{W}_i = \mathcal{B}_i \cup [\mathcal{Z}_i^+ \setminus \mathcal{Z}_i] = \mathcal{B}_i^+ \cup \{S \cup \bar{i} : S \in \mathcal{A}_i\}$ .

PROPOSITION 2.1.  $\mathcal{Z}_i^+ \supseteq \mathcal{W}_i \supseteq \mathcal{B}_i^+$  and  $\mathcal{W}_i = \mathcal{W}_i^+$ .

*Proof.* First of all,  $\mathcal{W}_i = \mathcal{B}_i \cup [\mathcal{Z}_i^+ \setminus \mathcal{Z}_i] \subseteq \mathcal{Z}_i \cup [\mathcal{Z}_i^+ \setminus \mathcal{Z}_i] = \mathcal{Z}_i^+$ . Secondly,  $\mathcal{W}_i \supseteq \mathcal{B}_i^+$  is from the above equivalent definition. To prove  $\mathcal{W}_i = \mathcal{W}_i^+$ , for  $\forall T \in \mathcal{W}_i^+$ , there exists a minimal coalition  $S \in \mathcal{W}_i$  such that  $S \subseteq T$ . If  $S \in \mathcal{B}_i$  then  $S \subseteq T \setminus \bar{i} \in \mathcal{B}_i$  and hence  $T \in \mathcal{W}_i$ . Or if  $i \in S$  and  $S \setminus \bar{i} \in \mathcal{Z}_i$ , then  $i \in T$  and by Lemma 2.1 (together with  $S \setminus \bar{i} \subseteq T \setminus \bar{i}$ ),  $T \setminus \bar{i} \in \mathcal{Z}_i$ .

Therefore  $T \in \mathcal{W}_i$ . ■

It is not hard to see that the following statements (i)–(iv) are equivalent: (i)  $i$  is a cog; (ii)  $\mathcal{Z}_i^+ = \mathcal{W}_i = \mathcal{B}_i^+$ ; (iii)  $\mathcal{W}_i = \mathcal{B}_i^+$ ; (iv)  $\mathcal{Z}_i^+ = \mathcal{W}_i$ . We can also show that  $\mathcal{W}_i \setminus \mathcal{Z}_i = \mathcal{W}_i \setminus \mathcal{B}_i = \{\bar{i} \cup T : T \in \mathcal{Z}_i\}$ , and that  $\mathcal{Z}_i \setminus \mathcal{W}_i = \mathcal{Z}_i^+ \setminus \mathcal{W}_i = \mathcal{Z}_i \setminus \mathcal{B}_i = \mathcal{A}_i$ , the personal discretion.

As  $\mathcal{Z}_i \neq \emptyset$  and  $\emptyset \notin \mathcal{B}_i$ , Proposition 2.1 and Eq.(3) imply that  $G_i \stackrel{\text{def}}{=} \Gamma(N, \mathcal{W}_i)$  is a well defined simple game. We shall call  $G_i$  the *command game* for  $i$ . The ensemble of all command games  $G \stackrel{\text{def}}{=} \{G_i : i \in N\}$  completely specifies the authority structure or “constitution” of the organization. In particular, if member  $i$  is a free agent, then his command game is just  $\Gamma(N, \{i\}^+)$ ; if member  $j$  is a cog, then his command game is  $\Gamma(N, \mathcal{B}_j^+)$ . However, this does not imply that a cog is not an essential player in other members’ command games, nor that a free agent is a dictator or is even essential in other members’ command games.

The Confucian model of society has a simple organizational form. According to the Confucianism, there are many principles a society should follow, among them:

- (P1) the man follows the king;
- (P2) the son follows his father;
- (P3) the wife follows her husband;
- (P4) the king should respect his people.

This authority structure has circular commands with different degrees of discretion. In most of the Chinese history, this orderly organization worked properly. Violations got punished. This also applies for (P4) because the people would exercise their command by overthrowing the king if his evils are insufferable to the vast majority, say 90%, of the governed.

### 3. AUTHORITY DISTRIBUTION

If member  $i$  has some power in member  $j$ 's command game  $G_j$  and  $j$  has some power in  $i$ 's command game  $G_i$  at the same time, then how can we define a fair distribution of “power” throughout the organization? We shall call it “authority distribution” to distinguish it from other uses of the word “power”. A Markov-chain structure is introduced.

#### 3.1. Authority Equilibrium

An *authority distribution*  $\pi : N \rightarrow [0, 1]$  over an organization  $(N, G)$  satisfies  $\pi_i \geq 0$  for any  $i \in N$  and  $\sum_{i \in N} \pi_i = 1$ . For convenience, we let  $\pi$  be a  $1 \times n$  row vector. Denote by  $P(i, j)$  member  $j$ 's S-S power index in member  $i$ 's command game  $G_i$ . And we call the  $n \times n$  matrix  $P = [P(i, j)]_{i, j=1}^n$  the *power transition matrix* of the organization. If  $P(i, j) > 0$ , then some of member  $i$ 's “power”, if he has, transfers to member  $j$ .  $P(i, i)$  measures  $i$ 's personal discretion  $\mathcal{A}_i$ . Clearly the power transition matrix  $P$  is a *stochastic matrix*, with nonnegative entries and each row sums to 1.

The Shapley value was first derived from three fairness axioms for superadditive games. As far as simple games are concerned, Dubey(1975) modifies one of the axioms and proves the existence and uniqueness of the S-S power index without reference to general (non-simple) games. To state these axioms, we let  $\mathcal{C}(N)$  denote the set of all simple games on  $N$ , and we use the function  $v : 2^N \rightarrow \{0, 1\}$  to indicate an element  $v \in \mathcal{C}(N)$ .

That is,  $v(S) = 1$  if and only if  $S$  is winning. Moreover for simple games  $v, w \in \mathcal{C}(N)$ , the operations  $\vee$  and  $\wedge$  are defined by

$$(v \vee w)(S) \stackrel{\text{def}}{=} \max(v(S), w(S)), \quad (v \wedge w)(S) \stackrel{\text{def}}{=} \min(v(S), w(S)).$$

As already shown (Dubey, 1975), there exists a unique function  $\varphi : \mathcal{C}(N) \rightarrow [0, 1]^n$  such that  $\varphi$  satisfies the axioms (A1) – (A3):

(A1) For any  $v \in \mathcal{C}(N)$ ,  $\sum_{j \in N} \varphi_j(v) = 1$  and  $\varphi_j(v) = 0$  if  $j$  is a dummy in  $v$ ;

(A2) if members  $i$  and  $j$  have symmetric roles in  $v \in \mathcal{C}(N)$ , then  $\varphi_i(v) = \varphi_j(v)$ ;

(A3)  $\varphi(v \vee w) + \varphi(v \wedge w) = \varphi(v) + \varphi(w)$  for any  $v, w \in \mathcal{C}(N)$ ;

and  $\varphi_i(v)$  is  $i$ 's S-S power index in the simple game  $v \in \mathcal{C}(N)$ . In an organization, we assume that member  $i$ 's authority is derived from his “command powers” in all members’ command games, which assume the Axioms (A1)–(A3). So  $i$ 's command power in  $\mathcal{W}_j$  is just  $P(j, i)$ . We also note, in reality, that one’s command power over those with higher authorities makes more contribution to his authority than his command power over those with lower authorities, other things equal. For example, the computers in our society have no authority even though they work well; however a manager has more authority than a programmer, not because the manager knows the computers better, but because he has more command powers over the programmer than that the programmer has over the

manager. Formally, we propose that an authority distribution  $\pi$  should follow Axiom (A4).

$$(A4) \pi_i = \sum_{j \in N} \pi_j P(j, i) \text{ for } \forall i \in N, \text{ or in matrix, } \pi = \pi P.$$

We call  $\pi = \pi P$  the *authority equilibrium equation*. The existence of  $\pi$  is well known from the Markovian theory. In a naive sense of “counterbalance equilibrium”,  $\pi_j P(j, i)$  is the authority flowing from  $j$  to  $i$ . So  $\pi_i$  is the sum of those flowing into  $i$ :  $\pi_i = \sum_{j \in N} \pi_j P(j, i)$ . However  $\pi_i P(i, j) \neq \pi_j P(j, i)$

in general. For example, let  $N = \overline{123}$  where  $\mathcal{W}_1 = \{\overline{12}, \overline{13}, \overline{23}\}^+$ ,  $\mathcal{W}_2 = \{\overline{21}, \overline{23}\}^+$  and  $\mathcal{W}_3 = \{\overline{23}\}^+$ . Then  $P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$  and  $\pi = (\frac{1}{7}, \frac{4}{7}, \frac{2}{7})$ ; but

$$\pi_1 P(1, 2) \neq \pi_2 P(2, 1).$$

We should remark that our present interpretation of the “command games” does not envisage “commands” as being instructions that must be absolutely obeyed, as in a military or legal setting. Rather, perhaps, we might here call them “recommendation”, “influence” or “advising” games, although we have up to this point excluded the so-called “improper” simple games, in which two disjoint coalitions can both be winning.

However, if there is no clear command-game setting in the organization, it is not hard to formulate the power transition matrix  $P$  to designate how much the members “influence each other directly”, either by prestige,

moral, or else, on personal decision making. For example, a 10-year-old child's direct decision may be determined by his parents' guidance with probability .3, by peer children with probability .4, and his own interest with probability .3, although there may be no specific command game on the child. The direct person-to-person impact  $P(i, j)$  can also be estimated by linear regressions given lots of historical data. The restriction is that  $P$  should be a stochastic matrix.

A concrete organization also holds financial and physical assets, besides its human resource. To include these bossable members into an organization, we introduce the concept of *yesman*. A member  $i$  is called a *yesman* if  $i$  is a dummy in all command games. Equivalently, the  $i^{th}$  column in  $P$  is a zero vector. A yesman, such as a slave, shows no sign of personal opinion until his bosses have given him some instructions or hints of the direction. A typical yesman is a computer, which follows all instructions from its users. A secretary sometimes is a yesman, when he is only responsible to his boss sets and takes no positions in all command games. By (A4), the existence of yesmanship in an organization does not affect the authority distribution; and hence we may remove them when calculating the authority distribution. However, yesmen may have some capability, and cost as well, in some tasks with the same capability/cost structure as that of other members; so we can treat the organization's capability/cost in a uniform way. Clearly, a yesman is a cog, but a cog is not necessarily a yesman. On

the other hand, a *poo-h-bah*, who holds many public or private posts should be in a high rank of authority or of great influence, as indicated in (A4).

### 3.2. Administrative Power

In some sense,  $\pi_i$  is player  $i$ 's probability of being critical in organizational decisions in the long run. This is our second interpretation for the authority distribution. Say, there is a bill to vote in the organization and we assume that each member has to vote either YES or NO. Then the winning rule for the bill can be formulated as a simple game  $\Gamma(N, \mathcal{V})$  on  $N$ . As we see in many organizations, different bills may have different voting rules, even within the same organization.

Let  $q$  be a probability distribution to pick a player from the organization where  $q(i)$  equals to player  $i$ 's S-S index in the voting game  $\Gamma(N, \mathcal{V})$ . One could propose a “process” interpretation for the transition probabilities  $P$ . In this interpretation, the r.v.  $X_t$  denotes the player who is “influential” to the bill in period  $t$ . At time  $t = 0$ , a player is chosen to be the initial influential player according to the probability distribution  $q$ . If  $X_0 = i_0$ , then player  $i_0$  is influential in the sense that the organizational decision is made in period  $t = 0$  according to the voting game  $\Gamma(N, \mathcal{V})$ . Given  $X_0 = i_0$ , the influential player in period  $t = 1$  will be chosen according to  $i_0$ 's command game. That is, conditional on  $X_0 = i_0$ , the influential player in period  $t = 1$  will be player  $j$  with probability  $P(i_0, j)$ . Hence,

$P(i_0, j) = \text{Prob}(X_1 = j | X_0 = i_0)$  can be interpreted as the “one-step” influence of player  $j$  on player  $i_0$ . More generally,  $P^m(i, j)$  is interpreted as the  $m$ -step influence of player  $j$  on player  $i$ . Given the initial distribution  $q$ , it follows that  $\sum_{i \in N} P^m(i, j)q(i)$  is interpreted simply as player  $j$ 's  $m$ -step influence on the bill. In the irreducible and aperiodic case (see §3.3), it is well known that  $\lim_{m \rightarrow \infty} \sum_{i \in N} P^m(i, j)q(i) = \pi_j$  for all initial distribution  $q$ . In other words, player  $j$ 's long-run influence on the bill exists and equals to  $\pi_j$ . This long-run influence does not depend on the specific bill to vote or its winning rule  $\Gamma(N, \mathcal{V})$ . Furthermore, if we take  $q(i) = 1$  for any fixed  $i \in N$ , then  $\pi_j = \lim_{m \rightarrow \infty} P^m(i, j)$ . This justifies the interpretation for  $\pi_j$  as the long-run influence of player  $j$  on other players. We shall address this point in §4.

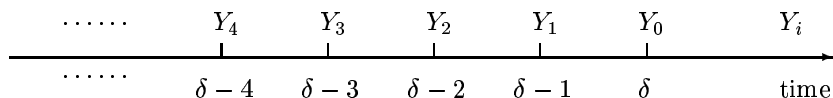
If we reverse the above process, we could obtain another interactive process. Let the bill be proposed in period  $t = 0$  and let the proposed bill be formally voted in period  $t = \delta$  ( $\delta \gg 0$ ). If the players are not well organized and they vote independently (i.e. all players are free agents), then player  $i$ 's probability of being critical in the voting game  $\Gamma(N, \mathcal{V})$  equals to his S-S index in the game (given the conditions SC-I and SC-II are satisfied). In general, before the period  $t = \delta$ , every member can exercise his power in others' command games to alter or re-confirm their preference of choices, if he can. And these “exercises of power” make a stochastic point process during the time periods  $\{0, 1, 2, \dots, \delta\}$ . We want to find out



how frequently each member is critical in the interactive decision-making process. In the absence of any specific pattern of randomness except the two symmetry conditions, we can figure out an embedded and truncated Markov chain  $\{Y_0, Y_1, Y_2, \dots, Y_\delta\}$  where the r.v.  $Y_t$  is the critical player in period  $\delta - t$ .

Without loss of generality, we assume that  $Y_0$  is chosen from  $N$  with a probability distribution  $q'$ . Therefore, the result of the bill will be determined by  $Y_0$ 's critical vote in period  $\delta$ : the bill is passed if  $Y_0$  votes YES or the bill is blocked if  $Y_0$  votes NO. If the organization knows that  $i_0$  will be the critical player in voting the bill, then all players will exercise their powers in  $i_0$ 's command game in period  $\delta - 1$ . More formally in period  $\delta - 1$ , given  $Y_0 = i_0$ , there would be a vote to determine  $i_0$ 's choice (YES or NO) of period  $\delta$ . The vote has the winning rule  $\mathcal{W}_{i_0}$ . Of course,  $i_0$  has to obey the decision by his command game and in period  $\delta$  he will take the choice, which is determined by his command game in period  $\delta - 1$ . Mathematically, given  $Y_0 = i_0$ ,  $i_0$ 's choice in period  $\delta$  is determined by his command game of period  $\delta - 1$ . If the command game decides the choice YES (or NO) for  $i_0$ , then  $i_0$  must vote YES (or NO respectively) in period  $\delta$ . Now in  $i_0$ 's command game, member  $i_1$  is critical with probability  $P(i_0, i_1)$ . We take  $Y_1$  to be the critical member in the vote of period  $\delta - 1$ . Then  $Y_1$  has the conditional probability distribution:  $\text{Prob}(Y_1 = i_1 | Y_0 = i_0) = P(i_0, i_1)$ . So if  $Y_1$  votes YES (or No) in period  $\delta - 1$ , then  $Y_0$  has to vote YES (or

NO, respectively) in period  $\delta$  and therefore the bill is passed (or blocked, respectively). That is,  $Y_1$  is the real critical member in the last two periods  $\delta$  and  $\delta - 1$ . Once again, given that  $i_1$  is critical in period  $\delta - 1$ ,  $i_1$ 's critical decision is subject to  $Y_2$ 's critical vote in the command game  $\Gamma(N, \mathcal{W}_{i_1})$  of period  $\delta - 2$  and  $Y_2$  has the conditional probability distribution:  $\text{Prob}(Y_2 = i_2 | Y_1 = i_1) = P(i_1, i_2)$ . Therefore  $Y_2$  is the real critical member in the last three periods  $\delta - 2$ ,  $\delta - 1$  and  $\delta$ . We continue this process and let  $\delta \rightarrow \infty$ ; then we obtain a homogenous Markov chain  $\{Y_0, Y_1, \dots\}$  (Figure 1), with probability transition matrix  $P$ .



**FIG. 1.** Interpretation of authority by Markov chains

If the process is irreducible and aperiodic,  $\pi_i$  is the frequency of player  $i$ 's presence in the chain. On the other hand, the distribution  $\pi$  is independent of the specific bill to vote and its respective voting rule; it is also independent of the initial probability distribution  $q'$  by which  $Y_0$  is chosen.

In the above senses, we say that  $\pi_i$  is player  $i$ 's "general administrative power" in the organization.

### 3.3. Asymptotic Properties

The Markov-chain setting of authority distribution shall lead to some similar implications. For any nonnegative integer  $k$ , we denote  $P^k$  the  $k^{\text{th}}$  power of  $P$  with the convention  $P^0 = I$ , the  $n \times n$  identity matrix. We let  $P^k(j, i)$  be the entry at the  $j^{\text{th}}$  row and the  $i^{\text{th}}$  column of the matrix  $P^k$ , and call  $i$  *influences*  $j$  (or *influences*  $j$  *directly*) if  $P^k(j, i) > 0$  (or  $P(j, i) > 0$ , respectively) for some integer  $k \geq 0$ . We denote this as  $j \longrightarrow i$  or  $i \longleftarrow j$ . Otherwise,  $j \not\longrightarrow i$  or  $i \not\longleftarrow j$ . Although “ $\longrightarrow$ ” or “ $\longleftarrow$ ” is reflexive and transitive; it is not symmetric in general. Two members  $i$  and  $j$  are said to *communicate* if they influence each other, denoted “ $i \longleftrightarrow j$ ”. Communication “ $\longleftrightarrow$ ” is symmetric. Communication defines an equivalence relationship over the members: two members that communicate with each other are said to be in the same equivalence class. Therefore, any two classes are either disjoint or identical. We say that an organization(or a coalition) is *irreducible* if any two members in the organization(or coalition, respectively) communicate with each other. Otherwise, it's *reducible*. In a naive sense, one influences all members, and vice versa, in an irreducible organization or coalition. Moreover, if  $i \longrightarrow j$  and  $j \longleftrightarrow k$ , then  $i \longrightarrow k$ ; if  $i \longleftrightarrow j$  and  $j \longrightarrow k$ , then  $i \longrightarrow k$ . This is, influence is a class property as well. This implies that there is no free agent in an irreducible organization since no one else influences a free agent except himself.

The organization with  $N = \overline{123}$ ,  $\mathcal{W}_1 = \{\overline{12}, \overline{13}, \overline{23}\}^+$ ,  $\mathcal{W}_2 = \{\overline{21}, \overline{23}\}^+$  and  $\mathcal{W}_3 = \{\overline{23}\}^+$  is irreducible. The organization with  $N = \overline{123456}$ ,  $\mathcal{W}_1 = \{\overline{12}, \overline{13}, \overline{15}\}^+$ ,  $\mathcal{W}_2 = \{\overline{12}, \overline{24}\}^+$ ,  $\mathcal{W}_3 = \{\overline{34}\}^+$ ,  $\mathcal{W}_4 = \{\overline{3}, \overline{4}\}^+$ ,  $\mathcal{W}_5 = \{\overline{6}\}^+$  and  $\mathcal{W}_6 = \{\overline{5}\}^+$ , is reducible. Here  $\{3, 4\}$  is an irreducible class while  $\{5, 6\}$  is another irreducible class. However neither member 1 nor member 2 belongs to any irreducible class. We also note that there exists no influence between the two irreducible classes. In general, the members in an organization  $N$  can be classified, in a unique manner, into non-overlapping coalitions  $T, C_1, C_2, \dots, C_k$  such that each  $C_i$  is an irreducible class and

$$T = \{i \in N \mid i \longrightarrow j \text{ but } i \not\leftarrow j \text{ for some } j \in N\}.$$

We call the members of  $T$  *authority-transient*. Each  $C_i$  has no impact on other irreducible classes, and vice versa. So  $C_i$  is actually “*independent*” of all other irreducible classes. For our further convenience, we shall assume that the organization has the above decomposition and therefore  $P$  has the form (after some possible re-arrangements) of

$$P = \begin{bmatrix} P_T & Q_1 & \cdots & Q_k \\ & P_{C_1} & & \\ & & \ddots & \\ & & & P_{C_k} \end{bmatrix} \quad (4)$$

where  $P_S$  is the restriction of  $P$  to  $S \subseteq N$ . We let  $m = |T|$ . It is easy to see that  $\lim_{t \rightarrow \infty} P_T^t$  is a zero matrix. This implies that: (i)  $I - P_T$  is invertible;

(ii) members in  $T$  have no authority, i.e.,  $\pi_i = 0$  for all  $i \in T$ ; and (iii) for any organization  $N$ , there exists at least one irreducible class.

Influences can also be illustrated by directed graphs. Let the set of vertices  $V = N$  and the set of (directed) edges  $E = \{(i, j) \in N \times N | P(i, j) > 0\}$ . A path from  $u$  to  $v$  is a linked list of vertices  $u = u_0, u_1, u_2, \dots, u_t = v$  such that  $(u_i, u_{i+1}) \in E$ . In the terminology of graph,  $i \longrightarrow j$  if and only if there is a path from  $i$  to  $j$ . Therefore an organization is irreducible if and only if there exists a path from  $i$  to  $j$  for  $\forall i, j \in N$ .

Member  $i$  is said to have *period*  $d$  if  $P^t(i, i) = 0$  whenever  $t > 0$  is not divisible by  $d$  and  $d$  is the greatest integer with this property. A member with period 1 is said to be *aperiodic*. Let  $d(i)$  denote the period of  $i$ . Periodicity is also a class property : if  $i \longleftrightarrow j$ , then  $d(i) = d(j)$ . For an example, let  $\mathcal{W}_i = \{\overline{i+1, i-1}\}^+$  for  $i = 2, \dots, n-1$ ,  $\mathcal{W}_1 = \{\overline{n, 2}\}^+$  and  $\mathcal{W}_n = \{\overline{n-1, 1}\}^+$ . Then each member has period 2 if  $n$  is even or 1 if  $n$  is odd. From the theory of Markov Chains, for an irreducible organization with period  $d$ , we can decompose  $N$  as a union of disjoint nonempty coalitions:  $N = D_1 \cup D_2 \cup \dots \cup D_d$  such that  $D_i$  influences  $D_{i+1}$  directly (we use the modular notation  $d+1 = 1$ ). That is,  $P(j, k) > 0$  for some  $j \in D_i$  and  $k \in D_{i+1}$ . Therefore in an irreducible organization with period  $d$ , we let

$$S_i \stackrel{\text{def}}{=} \{j \in N | P^{td}(i, j) > 0 \text{ for some } t > 0\}$$

for  $\forall i \in N$ . Then  $j \in S_i$  if and only if  $i \in S_j$ . We list a few asymptotic behaviors here.

PROPOSITION 3.1.

(i) *If  $i$  has period  $d(i)$  and  $P^s(j, i) > 0$ , then  $P^{s+td(i)}(j, i) > 0$  for all  $t$  sufficiently large.*

(ii) *In an irreducible organization with period  $d$ ,  $\lim_{t \rightarrow \infty} P^{td}(i, j)$  exists for  $\forall i \in S_j$ ; furthermore there is a unique method to assign an authority distribution  $\pi$  to the authority equilibrium equation  $\pi = \pi P$ , that is,  $\pi_j = \frac{1}{d} \lim_{t \rightarrow \infty} P^{td}(i, j)$  for  $\forall i \in S_j$ ; and given any initial distribution  $\pi^{(0)}$  and define  $\pi^{(t+1)} = \pi^{(t)} P$ , then  $\pi_j = \frac{1}{d} \lim_{t \rightarrow \infty} (\pi^{(t+d)})_j$ , the  $j^{\text{th}}$  component of the vector  $\pi^{(t+d)}$ . Moreover,  $\sum_{j \in S_i} \pi_j = \frac{1}{d}$  for  $\forall i \in N$ . The convergence is linear.*

(iii) *In an irreducible organization,  $\pi_i > 0$  for  $\forall i \in N$ .*

Proposition 3.1(i) says that if  $i$  influences  $j$  sometime in the past, then  $i$  will influence  $j$  periodically after some future. Proposition 3.1 (ii) can be used to approximate the authority distribution by the numerical iterations. However, when there are two or more irreducible classes in the organization, the solution to the authority equilibrium equation  $\pi = \pi P$  is not unique. More generally, for the organization with the decomposition Eq.(4),  $\{\pi \in [0, 1]^n | \pi = \pi P, \sum_{i \in N} \pi_i = 1\}$  is a  $k - 1$  dimensional convex

hull in  $[0, 1]^n$ . One remedy is to assign a specific *authority quota* to each class. Actually, if we assign class  $C_i$  with authority quota  $q_i$  such that  $0 \leq q_i \leq 1$  and  $\sum_{i=1}^k q_i = 1$ , then there exists a unique authority distribution to solve the authority equilibrium equation  $\pi = \pi P$  and the quota restriction  $\sum_{j \in C_i} \pi_j = q_i$  for all  $1 \leq i \leq k$ . Another remedy is to consider the *internal authority distribution* within each class only.

### 3.4. Authority within a Class

Let  $C$  be an irreducible class and  $P_C$  be the restriction of  $P$  to  $C$ . Formally, we can define the *internal authority distribution* for members within the class  $C$ , denoted  $\pi^C$ , by

$$\pi^C = \pi^C P_C, \quad \sum_{i \in C} \pi_i^C = 1, \quad \pi_i^C \geq 0, \quad \forall i \in C. \quad (5)$$

PROPOSITION 3.2. (i)  $\pi^C$  exists uniquely; (ii) if  $C$  has period  $d$  and  $j \in C$ , then  $\lim_{t \rightarrow \infty} P^{td}(i, j) = d\pi_j^C$  for  $\forall i \in S_j$ .

Each member in an organization is directly concerned with only a fraction of all the official orders, requisitions, etc., that flow through the organization, so the power transition matrix  $P$  will likely be quite sparse – i.e., have lots of zeros. By (A4), the existence of  $T$  does not affect the authority distribution. So we may remove  $T$  from  $N$  before we calculate the authority distribution. This reduction is useful when a large organi-

zation has a large portion of authority-transient members. For example,

given  $P = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ , after we remove the authority-transient mem-

bers  $T = \overline{12}$ , we have  $P_{N \setminus T} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ . So  $(\pi_3, \pi_4) = (\pi_3, \pi_4)P_{N \setminus T}$ , together with  $\pi_1 = \pi_2 = 0$ , gives that  $\pi_3 = \frac{3}{7}, \pi_4 = \frac{4}{7}$ .

As a remark, given a command game  $G_i$ , we can compute its S-S power indices uniquely. But the inverse is not true. For example, in  $N = \overline{12345}$ , both sets of minimal winning coalitions  $\mathcal{W}_1^m = \{\overline{123}, \overline{145}\}$  and  $\mathcal{W}_1^m = \{\overline{124}, \overline{135}\}$  have the same S-S power index vector  $(\frac{7}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15}, \frac{2}{15})$ . Moreover, if  $\sum_{j \in S} P(i, j) = 1$  for some individual  $i$  and coalition  $S$ , then  $S$  necessarily commands  $i$ . However, the inverse is false. In the above example,  $\overline{1234}$  commands 1; but  $\sum_{j=1,2,3,4} P(1, j) = \frac{13}{15} < 1$ .

#### 4. ULTIMATE INFLUENCE BY AUTHORITY

The S-S power index  $P(i, j)$  measures  $j$ 's direct influence on  $i$ . For the "ultimate influence" or "ultimate power", accordingly, is related to  $\lim_{t \rightarrow \infty} P^t(i, j)$ , if it exists. By Proposition 3.1,  $\pi_i = \lim_{t \rightarrow \infty} P^t(j, i)$  if the organization is irreducible and aperiodic; and the limit is independent of the choice of  $j$ . This is our third interpretation of  $\pi_i$  as  $i$ 's "uniform

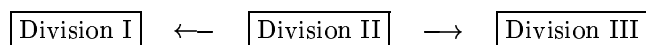


ultimate influence” to any other members in the organization. Proposition 3.2 provides the ultimate influence among members within the same irreducible class.

In other situations, we notice the less practical importance when both members are authority-transient or when both members belong to different irreducible classes. In these two cases, the ultimate influences between the two members are zeros. In this section, we will go to details for the case of non-transient over transient members.

#### 4.1. Limit Influence of a Class

For any  $i \in T$ , there may be some  $j \in T$  such that  $j$  influences  $i$ . But since  $j$  has no authority in the organization, his influences on  $i$  should come from the members in  $N \setminus T$ . In some sense,  $j$  acts as “messenger” only, passing the influences from  $N \setminus T$  to  $i$ . This argument can also be generalized to the case of influence between sub-organizations. In Figure 2, both Division I and III have influence on Division II; however the later has no impact on Division I or III. One question is their relative importance on II and by how much.



**FIG. 2.** An example of authority-transient sub-organizations

Given an irreducible class  $C$  and any  $i \in T$ , we propose an authority accumulative sequence  $\{\mu_i^C(t)\}_{t=0}^\infty$  defined by

$$\mu_i^C(t+1) = \sum_{j \in T} P(i, j) \mu_j^C(t) + \sum_{j \in C} P(i, j)$$

with the initial condition  $\mu_j^C(0) = 0$  for  $\forall j \in T$ . Intuitively  $\mu_i^C(t+1)$  is the authority absorption in  $C$  from  $i$  at or before the  $t+1^{st}$  step:  $\sum_{j \in C} P(i, j)$  is the “authority” absorbed directly by  $C$  from  $i$  while  $\sum_{j \in T} P(i, j) \mu_j^C(t)$  is the “authority” absorbed indirectly by  $C$  from  $i$  before the  $t+1^{st}$  step. Therefore the ultimate authority absorption  $\lim_{t \rightarrow \infty} \mu_i^C(t)$ , if it exists, measures the ultimate influence by  $C$  on member  $i$ .

PROPOSITION 4.1. *The limit  $\mu_i^C = \lim_{t \rightarrow \infty} \mu_i^C(t)$  exists and it is the minimal non-negative solution of the system of inhomogeneous equations*

$$\mu_i^C = \sum_{j \in T} P(i, j) \mu_j^C + \sum_{j \in C} P(i, j). \quad (6)$$

*Proof.* For any  $i \in T$ ,  $\mu_i^C(t)$  is non-decreasing in  $t$  (by induction), but it remains bounded by 1. So  $\lim_{t \rightarrow \infty} \mu_i^C(t)$  exists. The limit obviously satisfies Eq.(6). Conversely, if  $\{\mu_i^C\}$  is any non-negative solution of Eq.(6), we have  $\mu_i^C \geq \sum_{j \in C} P(i, j) = \mu_i^C(1)$ . By induction  $\mu_i^C \geq \mu_i^C(t)$  for all  $t \geq 1$ , and so the limits  $\lim_{t \rightarrow \infty} \{\mu_i^C(t)\}$  represents a minimal non-negative solution. ■

COROLLARY 4.1. *Given  $T, C_1, C_2, \dots, C_k$  in  $N$  as of Eq.(4), for any  $i \in T$  and any irreducible class  $C$ ,*

$$(i) \mu_i^C(t) = \sum_{j \in C} P^t(i, j);$$

$$(ii) \sum_{j=1}^k \mu_i^{C_j} = 1, \text{ the irreducible classes have total ultimate influence}$$

over  $T$ ;

$$(iii) \text{ if } T = \overline{12 \cdots m}, \text{ then } \begin{pmatrix} \mu_1^C \\ \vdots \\ \mu_m^C \end{pmatrix} = (I - P_T)^{-1} \begin{pmatrix} \sum_{t \in C} P(1, t) \\ \vdots \\ \sum_{t \in C} P(m, t) \end{pmatrix}.$$

*Proof.* Clearly part (i) holds true for  $t = 1$ . Now if we assume that it's true for some  $t \geq 1$ , then

$$\begin{aligned} & \sum_{j \in C} P^{t+1}(i, j) \\ &= \sum_{j \in C} \left[ \sum_{s \in T} P(i, s) P^t(s, j) + \sum_{s \in C} P(i, s) P^t(s, j) \right] \\ &= \sum_{s \in T} P(i, s) \sum_{j \in C} P^t(s, j) + \sum_{s \in C} P(i, s) \sum_{j \in C} P^t(s, j) \\ &= \sum_{s \in T} P(i, s) \mu_s^C(t) + \sum_{s \in C} P(i, s) 1 = \mu_i^C(t+1). \end{aligned}$$

By the principle of induction, the statement (i) is true for all  $t \geq 1$ . Part (ii) is from (i) and that  $\lim_{t \rightarrow \infty} P^t(i, j) = 0$  for  $\forall i, j \in T$ . Part (iii) is from

Eq.(6). ■

#### 4.2. Redistribution of Ultimate Influence

The next proposition illustrates how to re-distribute the irreducible class  $C$ 's total ultimate influence  $\mu_i^C$  on  $i \in T$  among the members of  $C$ .

PROPOSITION 4.2. *Given any  $i \in T$  and any aperiodic irreducible class  $C$ ,  $\lim_{t \rightarrow \infty} P^t(i, j) = \pi_j^C \mu_i^C$  for  $\forall j \in C$ .*

*Proof.* As stated before,  $\lim_{t \rightarrow \infty} P^t(i, s) = 0$  for any  $i, s \in T$ . Now,

$$\begin{aligned}
& \lim_{t \rightarrow \infty} P^{t+1}(i, j) \\
&= \lim_{t \rightarrow \infty} \sum_{s \in T} P^t(i, s)P(s, j) + \lim_{t \rightarrow \infty} \sum_{s \in C} P^t(i, s)P(s, j) \\
&= \sum_{s \in T} \lim_{t \rightarrow \infty} P^t(i, s)P(s, j) + \sum_{s \in C} \lim_{t \rightarrow \infty} P^t(i, s)P(s, j) \\
&= \sum_{s \in T} 0P(s, j) + \sum_{s \in C} \lim_{t \rightarrow \infty} P^t(i, s)P(s, j) \\
&= \sum_{s \in C} (\lim_{t \rightarrow \infty} P^t(i, s))P(s, j).
\end{aligned}$$

We solve the system of equations, contrasted with Eq.(5),

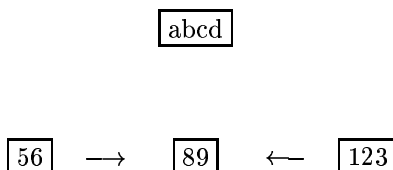
$$\begin{cases} \lim_{t \rightarrow \infty} P^t(i, j) = \sum_{s \in C} (\lim_{t \rightarrow \infty} P^t(i, s))P(s, j), & \forall j \in C, \\ \sum_{j \in C} \lim_{t \rightarrow \infty} P^t(i, j) = \mu_i^C \end{cases}$$

to get  $\lim_{t \rightarrow \infty} P^t(i, j) = \pi_j^C \mu_i^C$ . ■

COROLLARY 4.2. *In an aperiodic organization,  $\lim_{t \rightarrow \infty} P^t$  exists.*

In contrast to authority-transient coalitions or sub-organizations, a closed coalition  $S$  keeps all authority inside, although it may influence the outside  $N \setminus S$ . In Figure 3, the coalition  $\overline{89}$  influences  $\overline{123}$  and  $\overline{56}$ . But none of the coalitions  $\overline{123}$ ,  $\overline{56}$  or  $\overline{abcde}$  influences  $\overline{89}$ .

We say a coalition  $S$  is *authority-closed* if no authority flows out of  $S$ , i.e.  $P(i, j) = 0$  for all  $i \in S$  and  $j \in N \setminus S$ . However authority may flow from  $N \setminus S$  into  $S$ . Examples of closed coalitions include  $N$ ,  $F$ ,  $\emptyset$ , etc. Clearly, irreducible classes are closed. Furthermore, if  $S_1$  and  $S_2$  are closed, then



**FIG. 3.** A closed coalition,  $\overline{89}$ , keeps all authority inside

$S_1 \cup S_2$  and  $S_1 \cap S_2$  are also closed. When  $S$  is closed, then any authority flowing into  $S$  will be preserved inside  $S$  forever:  $P^t(i, j) = 0$ , for  $\forall i \in S$ ,  $j \notin S$  and  $t \geq 0$ . To see the authority-garnering mechanism, we have that  $\sum_{j \in S} P^t(i, j)$  is increasing in  $t$  for any closed coalition  $S$ . As a consequence, the “total authority” absorbed by a closed  $S$  from  $i$ ,  $\lim_{t \rightarrow \infty} \sum_{j \in S} P^t(i, j)$  exists.

## 5. APPLICATIONS

In addition to its internal interactions and conflict resolutions, a well organized group  $N$  can take external tasks or evaluate external issues as well as it can be controlled or affected by its outsiders. All these aspects would be related to its authority structure or its authority distribution.

### 5.1. Organizational Choice

This subsection works as an application of authority distributions to a simple form of conflict resolutions. Let  $p_i^{(t)} \in (-\infty, \infty)$  be member  $i$ 's

(quantitative) opinion on some issue at time  $t$  where  $t = 0, 1, \dots$ . In the organization, his opinion is assumed to be linearly adopted by all members according to his command powers. That is, his opinion at time  $t + 1$  is

$$p_i^{(t+1)} = \sum_{j \in N} P(i, j) p_j^{(t)}, \quad \forall i \in N.$$

If we let the column vector  $\mathbf{p}^{(t)} = (p_1^{(t)}, p_2^{(t)}, \dots, p_n^{(t)})$ , then  $\mathbf{p}^{(t)} = P\mathbf{p}^{(t-1)} = \dots = P^t \mathbf{p}^{(0)}$ .

PROPOSITION 5.1. *In an aperiodic organization with the decomposition Eq.(4), the limit  $\mathbf{p} = \lim_{t \rightarrow \infty} \mathbf{p}^{(t)}$  exists. For any  $j \in C_i$ ,  $p_j = \sum_{s \in C_i} \pi_s^{C_i} p_s^{(0)}$ . The limit opinion for  $T$  is given by  $\mathbf{p}_T = (I - P_T)^{-1} \sum_{i=1}^k Q_i \mathbf{p}_{C_i}$ .*

For any  $i, j \in T$ , although  $p_j^{(0)}$  may influence  $p_i^{(t)}$  for some finite  $t$ , the influence vanishes as  $t \rightarrow \infty$ . The limit  $p_i$  for any  $i \in T$  is independent of the initial choice of  $p_j^{(0)}$  for all  $j \in T$ . However, Proposition 5.1 does not hold in periodic cases. For example, let  $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ , then we have  $\lim_{t \rightarrow \infty} \mathbf{p}^{(2t+1)} = \mathbf{p}^{(1)}$ ; but  $\lim_{t \rightarrow \infty} \mathbf{p}^{(2t)} = \mathbf{p}^{(0)}$ .

Proposition 5.1 provides a simple way to resolve the conflicts within an aperiodic organization. For example, consider an irreducible and aperiodic organization  $N = \overline{1234}$  with authority distribution  $\pi = (.3, .1, .4, .2)$ . On the issue of the relative utility of one quantity of good  $x$  to one quantity of good  $y$ , they disagree with each other. Member 1, 2, 3 and 4 initially

believe the relative utility is 1.3, 1.4, 1.5, 1.6, respectively. If the linear adoption scheme applies, the limit opinion would be  $1.3 * .3 + 1.4 * .1 + 1.5 * .4 + 1.6 * .2 = 1.45$ , which could act as the organizational opinion after long-time discussions or evolutions. For another example, consider an irreducible and aperiodic organization  $N$  which wants to admit a few, say  $t$ , new employees from  $m$  ( $m \gg t$ ) applicants. To decide whom to be admitted, we first put the initial evaluations in a  $n \times m$  matrix  $B$  where  $B(i, j)$  is member  $i$ 's initial evaluation on applicant  $j$ . So the  $i^{th}$  row is member  $i$ 's initial evaluations on all applicants and the  $j^{th}$  column is the initial evaluations on applicant  $j$  by all members in the organization. In some real situations, the initial evaluations may be integers between 1 to 9, or on other scale. If the linear adoption scheme works,

$$\begin{aligned} \lim_{k \rightarrow \infty} P^k B &= (\pi_1 \mathbf{1}, \pi_2 \mathbf{1}, \dots, \pi_n \mathbf{1}) B \\ &= (\mathbf{1} \sum_{i=1}^n \pi_i B(i, 1), \dots, \mathbf{1} \sum_{i=1}^n \pi_i B(i, m)) \end{aligned}$$

where  $\mathbf{1}$  is the column vector with 1's for all entries. Therefore applicant  $j$ 's eventual or final evaluation (after infinite iterations or discussions) is  $\sum_{i=1}^n \pi_i B(i, j)$ . Then we may give the offers to those who have top  $t$  final evaluations.

## 5.2. Ranking by Bilateral Impact

To apply the authority distribution to the situations without command games, we analyze a few ranking problems when the data of directly bilateral impact are provided.

### College Ranking by Applicants' Acceptance

Suppose that there are large numbers of college applicants to apply the colleges  $C_1, C_2, \dots, C_n$ . Each applicant files multiple applications. Each college then offers some of its applicants admissions and rejects all others. Now some applicants may get no offer from any college; the other then get one offer or multiple offers. An applicant with multiple offers will decide which college to go to and reject all other colleges which make offers to him. Of all applicants who apply to and receive offers from  $C_i$ , we let  $P(i, j)$  be the proportion of those applicants who decide to go to college  $C_j$ . Such applicants of course apply to and receive offers from  $C_j$  as well. For example, if  $C_1$  admits 10000, of which 8000 decide to register with  $C_1$  and 1500 decide to register with  $C_2$ , then  $P(1, 1) = \frac{8000}{10000} = .8$  and  $P(1, 2) = \frac{1500}{10000} = .15$ .

To rank the colleges by the acceptance rates of the applicants to whom offers were made, we can apply the authority distribution associated with the matrix  $P$ . We provide an example with the colleges B, H, M, P, S, Y and O (others). We name them with  $C_1$  through  $C_7$  in that order. Assume



the acceptance rates have the following matrix  $P$

	B	H	M	P	S	Y	O
B	.4	.1	.1	.1	.1	.1	.1
H	.05	.9	.01	.01	.01	.01	.01
M	.1	.2	.6	.0	.05	.0	.05
P	.05	.1	.05	.5	.05	.05	.2
S	.1	.2	.05	.05	.45	.05	.1
Y	.05	.1	.05	.05	.05	.6	.1
O	.05	.05	.05	.05	.05	.05	.7

The so called “authority distribution” can be regarded as the measure of relative attractiveness of the colleges from the applicants’ point of view.

The solution of  $\pi = \pi P$  is

$$\pi = (.0868289, .530694, .073586, .0535171, .0551895, .0654097, .134775).$$

Therefore we rank the colleges as: H(1), B(2), M(3), Y(4), S(5) and P(6).

If we allow some level, say .01, of trust, we may believe that M(.073586) ties with Y(.0654097) in the rank 3<sup>rd</sup> and S(.0535171) ties with P(.0551895) in the rank 5<sup>th</sup>.

### Journal Rankings by Citations

Let  $J_1, J_2, \dots, J_{n-1}$  be  $n - 1$  journals in a scientific field, and  $J_n$  be the collection of all other journals. We technically treat  $J_n$  as a single journal. For any journal  $J_i$ , each issue contains many papers, and each paper has

its list of references or citations. A paper in  $J_j$  can be cited in another paper in  $J_i$  as a reference. Of all papers cited by  $J_i$  (repetition counted), we let  $P(i, j)$  be the proportion of those papers which are published on  $J_j$ . So  $P$  measures the direct impact between any two journals and  $P(i, i)$  is the self-citation rate for  $J_i$ . The authority distribution for  $\pi = \pi P$  would quantify the long-term influence of each journal in the group of journals and can be used to rank these journals.

### Planning of a Freeway System

A few small towns believe that building a freeway system would be to their common benefit. Say, they plan to build freeways  $F_1, F_2, \dots, F_{n-1}$ . We let  $F_n$  be the existing traffic channels of car, truck and bus. We assume that all the potential freeways have the same length. Otherwise we can make up the assumption by dividing long freeways into smaller segments and rename them all. The freeways with higher traffic intensity should be built with more driving lanes and so receive more investments. Of all the traffic flow on  $F_i$ , we let  $P(i, j)$  be the (estimated) proportion of the traffic flowing into  $F_j$ . Then the authority distribution  $\pi$  satisfying  $\pi = \pi P$  will measure the relative traffic intensity on each  $F_i$  and can be used in the investment allocation.

A similar issue can be found in designing an Internet or Intranet system.

## 6. CONCLUSIONS

In the classical approach to the power distribution for a specific voting problem or a specific voting rule, some set-theoretical methods (simple games) assume only two results, either “winning” or “losing”. The S-S index quantifies each member’s probability to pivot in an ordering of members. However, an organization is not simply a voting body confronted with a specific voting problem. Our internal structure of “boss” and “approval” is introduced to avoid specific voting problems.

The command games help in providing a conceptual framework for members’ specific positions within the organization. Next, the quantitative authority distribution can be used to explain members’ general administrative power. It is derived from a political counterbalance equilibrium. A member’s authority is from others while his authority is redistributed in his command game. To imitate the decision-makings and interactions from the time when a bill is first proposed to the time when it is formally voted, we construct a time-reversed stochastic process of critical players. Given that the critical player in period  $t$  is known, the conditional probability of picking the critical player in period  $t - 1$  obeys the Markovian property. Hence the authority distribution is the frequency of being critical in the long-run decision-making process. In addition, it does not depend on the specific issue to vote when certain conditions, such as irreducibility and aperiodicity, are satisfied. Finally the distribution can also measure the

ultimate influence between members, more precisely between a member and his irreducible organization or sub-organization. In two applications, we use the distribution to resolve internal conflicts and rank journals and colleges.

From a theoretical point of views, there are several related issues worth mentioning. First, to keep the Markovian analysis tractable, we have focused on its interpretation by a decision-making process and its asymptotic behavior. Notice, however, that the set-up is not restrictive for the issues at hand since the theory of Markov Chains has rich features of periodicity, convergence, non-homogeneity, generalizations, etc. One could incorporate such features in our model and investigate their rich contexts and interpretations. Secondly, in our analysis, the power transition matrix will always be a stochastic matrix. Hence, any outsider can not interfere with the internal affairs. Hu(2000) has studied some simple forms of campaign strategies to affect an organization's voting by outsiders, based on the same authority structure. Thirdly, any organization is organized to take some tasks. In the related work of Hu and Shapley(2001), we incorporate the tasks into some generalization, called "control game", of command games by the compounding mechanism. This issue is related to designs, efficiency, and other problems in the theory of organization. Finally, we have constantly mentioned the symmetry conditions (SC-I and SC-II) for the S-S index to quantify a player's chance of being critical. If we replace the S-S indices of

$P$  with their respective asymmetric S-S indices, then we would formulate the *asymmetric authority distribution*. In this case, personal preferences on the issue to vote, or other specific factors, are generally taken into account.

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