Asymmetric Shapley-Shubik Power Index

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Abstract

The Shapley-Shubik(S-S) and Banzhaf indices measure voters' weighted marginal contribution or weighted numbers of swingings in bringing losing coalitions to winning. However a single result may mean winning to some voters, but losing to some others. Therefore voters can also contribute in successfully blocking a bill. The paper extends the "swinging" and "pivoting" schemes to the case of blocking bills and define the asymmetric Banzhaf and S-S indices, respectively. We further value the participation of the non-swinging players by crediting them with their Shapley values in the formed winning or blocking coalitions, instead of their marginal contribution. This leads to an equivalence to the asymmetric S-S index. The two traditional indices, in which symmetry conditions are assumed, are special cases of our new definitions. Moreover the asymmetric S-S index with the Banzhaf's condition is also the ordinary S-S index.

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1 The Asymmetric Banzhaf Index

Let $\Gamma(N, \mathcal{W}, \mathcal{P})$ denote a finite simple game (or voting game) with $N = \overline{12\cdots n}$ being the set of players (or voters) $1, 2, \cdots, n$. The $\mathcal{W}$ denotes the set of *winning* coalitions such that: (a) the empty coalition $\emptyset$ never wins; (b) the grand coalition N always wins; (c) any superset of a winning coalition also wins. Those coalitions which are not winning are called *losing*. The simple game sets the winning rule for a bill. It sets the losing rule as well. The $\mathcal{P}$ specifies the probability distribution of the potential coalition of players voting for the bill, or the potential coalition of players voting against the bill. Simple games have been used in various voting bodies. In this paper, we shall use lower-case italic letters or numerals for players, italic capitals for coalitions, script capitals for sets of coalitions. Set subtraction will be indicated by " \ ". We shall employ the vinculum in naming the elements of a set, thus "$\overline{abc}$" for "$\{a, b, c\}$". The number of elements of a finite set X is denoted $|X|$. For the set of coalitions $\mathcal{S}$, we denote by $\mathcal{S}^+$ the set of all supersets of elements of $\mathcal{S}$. Let the characteristic function $v : 2^N \to \{0, 1\}$ defined by

$$v(T) = \begin{cases} 1, & \text{if } T \in \mathcal{W}; \\ 0, & \text{otherwise.} \end{cases}$$

Thus T gets 1 credit if the bill is passed by T, or 0 otherwise. We say player i is a *dictator* (or *dummy*) if $v(T \cup \bar{i}) - v(T \setminus \bar{i}) = 1$ (or 0 respectively) for any $T \subseteq N$. Player i is a *veto player* if $v(T \setminus \bar{i}) = 0$ for any $T \subseteq N$. And the game $\Gamma(N, \mathcal{W})$ is *decisive* if $v(T) + v(N \setminus T) = 1$ for any $T \subseteq N$.

There are two major power indices measuring players' relative strength in the voting body. If $T \subseteq N$, we say player i is a *swinger* of T, or i swings in T, if $T \in \mathcal{W}$ but $T \setminus \bar{i} \notin \mathcal{W}$. Thus i swings in T if and only if $v(T) - v(T \setminus \bar{i}) = 1$. The Banzhaf index $b_i[\Gamma]$ counts the coalitions in which player i swings (see Banzhaf 1965),

$$b_i[\Gamma] \stackrel{\text{def}}{=} \frac{1}{2^{n-1}} \sum_{T \subseteq N} [v(T) - v(T \setminus \bar{i})].$$

However the Banzhaf indices do not sum up to 1 or another constant. The Shapley-Shubik(S-S) index $\phi_i[\Gamma]$, on the other hand, applies the value concept to the simple game (see Shapley 1953; Shapley and Shubik 1954). To review the idea, we quote them:

> There is a group of individuals all willing to vote for some bill. They vote in order. As soon as a majority has voted for it, it is declared passed, and the member who voted last is given credit for having passed it. Let us choose the voting order of the members randomly. Then we may compute the frequency with which an individual ...... is *pivotal.* This later number serves to give us our index. It measures the number of times that the action of the individual actually changes the state of affairs.

As paraphrased above, all players are assumed to vote for (YES) the bill, and they vote in order, say one ordering of the players having the form

$$i_1 \to i_2 \to \cdots \to i_k \to \cdots \to i_n. \tag{1}$$

In this ordering, there is a unique k such that $\overline{i_1 i_2 \cdots i_{k-1}} \notin \mathcal{W}$ but $\overline{i_1 i_2 \cdots i_{k-1} i_k} \in \mathcal{W}$. The player i_k is called *pivotal* in the ordering (1). There are $n!$ orderings of players in total. The S-S index $\phi_i[\Gamma]$ is defined as the fraction of the orderings in which i is pivotal. Thus the S-S indices sum up to 1. It has the expression

$$\phi_i[\Gamma] = \sum_{T \subseteq N} \frac{(|T|-1)!(n-|T|)!}{n!} [v(T) - v(T \setminus \bar{i})]. \tag{2}$$

Therefore both indices are weighted marginal contribution in forming winning coalitions: $v(T) - v(T \setminus \bar{i})$. In other words, both indices are weighted numbers of swinging positions in $\Gamma(N, \mathcal{W})$. The weights are $1/2^{n-1}$ and $(|T|-1)!(n-|T|)!/n!$, respectively. More generally when the probability(Prob.) distribution $\mathcal{P}$ is regarded, player i's expected number of swingings in $\Gamma(N, \mathcal{W})$

is

$$\sum_{T \subseteq N} \text{Prob}(\mathbf{S} = T|\mathcal{P})[v(T) - v(T \setminus \bar{i})]$$

where $\mathbf{S}$ denotes the random coalition of players who vote for the bill.

However there are two major drawbacks in applying the Shapley value to measuring power indices. First it pre-supposes that voters are willing to form the grand agreement. As we know in most votings, however, the voters are not required to vote for a debating bill. There is generally no mechanism to enforce the grand coalition. Otherwise such a voting is neither costless nor necessary. Secondly, the strategic interactions and strategic behavior of voters are ignored.

In game theory, each player is assumed to achieve his maximum share of utility and the characteristic function $v(T)$ denotes the maximum utility T can achieve. In this setting, any coalition T then tries to maximize the function $v(T)$. With the super-additivity on v, i.e. $v(T) + v(S) \leq v(S \cup T)$ for any $S \cap T = \emptyset$, the Shapley value's Efficiency Axiom assumes that the grand coalition N should be achieved before $v(N)$ is fairly distributed. Accordingly for the voting game, the assumption that all players vote YES to the bill is essential in applying the S-S index properly. Say, if some players preceding i_k vote NO to the bill in the ordering (1), then i_k may not be pivotal. Therefore the assumption is not generally true in voting bodies. As a vote is an instrument to resolve a debating bill, some individuals want to achieve the blocked result $(v(\mathbf{S}) = 0)$ while some others the passed result $(v(\mathbf{S}) = 1)$. Hence v is not a utility function in the voting game at all.

To address the second drawback, we should analyze the potential cooperative structure in the voting body. For an example of five voters from five political parties (say Conservative, Democracy, Liberal, Republic, and Communism) in a committee, their potential cooperation on a generic vote is not evenly distributed. Instead, for example, the Conservative voter is more likely to vote with the Republic and against the Liberal. It is customary to apply the probability theory to address the uncertainty issue in cooperation perspectives. We use $\mathcal{P}$ to specify the probability distribution of $\mathcal{S}$, the coalition of voters who vote for the bill.

We have so far set up the winning rule $\Gamma(N, \mathcal{W})$ from the point of "passing the bill" and classified the players by $\mathbf{S}$ or $N \backslash \mathbf{S}$. However, we can also equivalently re-define "winning" by that "the bill is not passed" or "the bill is blocked" and re-define its "winning" rule Γ^*. A coalition T in Γ^* then has the players who vote against (or NO) the bill. A "winning" coalition T in Γ^* then makes "passing the bill" impossible, or *blocks* forming any winning coalition in Γ. Therefore it is "winning" in Γ^* if and only if it has nonempty intersection with all winning coalitions in Γ. Γ^* is also a simple game, denoted $\Gamma^*(N, \mathcal{W}^*)$ where $\mathcal{W}^* \stackrel{\text{def}}{=} \{Z | Z \cap T \neq \emptyset, \forall T \in \mathcal{W}\} = \{N \backslash T | T \notin \mathcal{W}\}$. The game $\Gamma^*(N, \mathcal{W}^*)$ is called the *dual* of $\Gamma(N, \mathcal{W})$. Clearly the dual's dual is itself: $\Gamma^{**} = \Gamma$. The characteristic function of Γ^* is $v^*(T) \stackrel{\text{def}}{=} 1 - v(N \setminus T)$. It is 1 if $T \in \mathcal{W}^*$ and 0 otherwise. Let $\mathbf{S}^* = N \setminus \mathbf{S}$, the random coalition of players who vote NO to the bill. Player i's expected number of swingings in Γ^* is then

$$\sum_{Z \subseteq N} \text{Prob}(\mathbf{S}^* = Z | \mathcal{P})[v^*(Z) - v^*(Z \setminus \bar{i})] \stackrel{T=N \backslash Z}{=} \sum_{T \subseteq N} \text{Prob}(\mathbf{S} = T | \mathcal{P})[v(T \cup \bar{i}) - v(T)]. \tag{3}$$

Now player i's total contribution comes from two aspects. If $i \in \mathbf{S} \in \mathcal{W}$, he contributes in successfully passing the bill. On the other hand, if $i \notin \mathbf{S} \notin \mathcal{W}$ or $i \in \mathbf{S}^* \in \mathcal{W}^*$, he contributes in successfully blocking the bill. There are two more cases he gets not credit: 1. $i \in \mathbf{S} \notin \mathcal{W}$; 2. $i \notin \mathbf{S} \in \mathcal{W}$. In the first case, he votes for the bill but the bill is blocked; in the second case he votes against the bill but the bill is passed. In either case, he is not on the side of the successful voters. From the voter's point of view, he is winning if he votes YES and $\mathbf{S} \in \mathcal{W}$. He is also winning if he votes NO and $\mathbf{S}^* \in \mathcal{W}^*$. In this sense, we credit him if and only if he is winning.

We say that player i is a *d-swinging* (double swinging) player of T if either i is a swinging of T in the game $\Gamma(N, \mathcal{W})$ or he is a swinging player of T in the dual game $\Gamma^*(N, \mathcal{W}^*)$. The asymmetric Banzhaf index $b_i[\Gamma, \mathcal{P}]$ is defined by player i's expected number of d-swingings,

$$b_i[\Gamma, \mathcal{P}] \stackrel{\text{def}}{=} \sum_{T \subseteq N} \text{Prob}(\mathbf{S} = T | \mathcal{P})[v(T) - v(T \setminus \bar{i})] + \sum_{T \subseteq N} \text{Prob}(\mathbf{S} = T | \mathcal{P})[v(T \cup \bar{i}) - v(T)]. \tag{4}$$

It has a simple form $b_i[\Gamma, \mathcal{P}] \stackrel{\text{def}}{=} \sum_{T \subseteq N} \text{Prob}(\mathbf{S} = T|\mathcal{P})[v(T \cup \bar{i}) - v(T \setminus \bar{i})]$. When the information about the voting body's cooperative structure and the personal preference toward the vote is hidden or hard to infer, the Banzhaf and S-S indices assume uniformity among the potential $\mathbf{S}$. We list a few symmetry conditions(S-C) for the information about $\mathbf{S}$:

- (S-C1): $\mathbf{S}$ is uniformly distributed on 2^N;
- (S-C2): $|\mathbf{S}|$ is uniformly distributed on $\{0, 1, 2, \cdots, n\}$;
- (S-C3): given $|\mathbf{S}| = k$, $\mathbf{S}$ is uniformly distributed on $\{T \subseteq N||T| = k\}$.

The last condition assumes that the partial information, given the size of $\mathbf{S}$ is known, is fairly even. If all players vote independently and they all have the indifference preference between YES or NO, then they may just cast their votes by fairly casual randomness, i.e. voting YES or NO with probability .5. In this case, it is likely that the $\mathbf{S} = T$ has the probability $1/2^n$ for any $T \subseteq N$. This is the condition (S-C1).

Proposition 1.1 *1. If $\mathcal{P}$ satisfies (S-C1), then $b_i[\Gamma, \mathcal{P}] = b_i[\Gamma]$. 2. If $\mathcal{P}$ satisfies (S-C2) and (S-C3), then $b_i[\Gamma, \mathcal{P}] = \phi_i[\Gamma]$.*

The proof is in the Appendix (§4). In term of pivoting scheme, we may technically remove the assumption of grand coalition as follows: in the ordering (1), it is necessary for the players preceding i_k to vote YES, but the information about the votes of $\overline{i_{k+1} \cdots i_n}$ is censored or truncated. The worst case is of course that the players of $\overline{i_{k+1} \cdots i_n}$ all vote against the bill. But this does not change player i_k's pivotal status. In this sense, we can order the players this way: the players voting YES always precede the players voting NO. Therefore in an ordering of all players, the first $|\mathbf{S}|$ players are $\mathbf{S}$ and the rest $\mathbf{S}^*$. As of any ordering like (1), if $\mathbf{S} \in \mathcal{W}$ then there exists a unique k such that $\overline{i_1 i_2 \cdots i_{k-1}} \notin \mathcal{W}$ but $\overline{i_1 i_2 \cdots i_{k-1} i_k} \in \mathcal{W}$; otherwise if $\mathbf{S} \notin \mathcal{W}$ then there exists a unique k such that $N \setminus \overline{i_n i_{n-1} \cdots i_{k+1}} \in \mathcal{W}$ but $N \setminus \overline{i_n i_{n-1} \cdots i_{k+1} i_k} \notin \mathcal{W}$.

We then credit the unique i_k in the ordering of players. This scheme is a variant of the double-pivoting scheme defined in §2.

2 The Double Pivoting Scheme

Let $\mathbf{U}_i$ be player i's vote which takes the value of either 1 (YES) or 0 (NO). As a result, $\mathbf{S} = \{i \in N | \mathbf{U}_i = 1\}$. We extend the S-S index by the "double pivoting" scheme. For the ordering (1) of players, we list their votes, call a *sequence of votes*, in the same order as that of the players,

$$\mathbf{U}_{i_1} \to \mathbf{U}_{i_2} \to \cdots \to \mathbf{U}_{i_{k-1}} \to \mathbf{U}_{i_k} \to \mathbf{U}_{i_{k+1}} \to \cdots \to \mathbf{U}_{i_n}. \tag{5}$$

In this sequence of votes, there exists a unique player i_k, called *d-pivotal*, such that he satisfies one of the two exclusive conditions:

- Type I: $\mathbf{S}_Y \stackrel{\text{def}}{=} \{i \in \overline{i_1 i_2 \cdots i_k} | \mathbf{U}_i = 1\} \in \mathcal{W}$ but $\mathbf{S}_Y \setminus \overline{i_k} \notin \mathcal{W}$;
- Type II: $\mathbf{S}_N \stackrel{\text{def}}{=} \{i \in \overline{i_1 i_2 \cdots i_k} | \mathbf{U}_i = 0\} \in \mathcal{W}^*$ but $\mathbf{S}_N \setminus \overline{i_k} \notin \mathcal{W}^*$.

He votes YES in Type I or NO in Type II. In either case, i_k concludes the vote regardless of the votes of the players $\overline{i_{k+1} \cdots i_n}$. In Type I the bill is passed; it is blocked in Type II. In this sequence, we credit the pivotal player i_k with the contribution 1.

Now for the fixed ordering of players, the sequence of votes is random and it has the same distribution as that of $\mathbf{S}$. We take the expected contribution for each ordering of players. When all $n!$ orderings of players are considered, we define voter i's asymmetric S-S power index $\phi_i[\Gamma, \mathcal{P}]$ by his total expected contribution divided by $n!$. The following example illustrates the idea of double pivoting.

Example 1:

Suppose that in the legislative system $N = \overline{x y_1 y_2 y_3 y_4 y_5}$ the winning rule for some bill requires the support of the President x and the majority of the house representatives $\overline{y_1 \cdots y_5}$.

For the ordering $y_1 \to y_2 \to y_3 \to x \to y_4 \to y_5$, y_4 is d-pivotal in the sequences:

$$Y \to Y \to N \to Y \to Y \to *, \quad Y \to N \to Y \to Y \to Y \to *, \quad N \to Y \to Y \to Y \to Y \to *,$$

$$Y \to N \to N \to Y \to N \to *, \quad N \to Y \to N \to Y \to N \to *, \quad N \to N \to Y \to Y \to N \to *,$$

where $*$'s are y_5's votes. These are YES's or NO's, but they do not change the pivotal status of y_4. The bill is passed in the first three sequences and blocked in the last three.

In the above definition, the orderings of players are fixed as $\mathbf{S}$ varies over its distribution $\mathcal{P}$. We can analyze the asymmetric S-S index from an alternative way. Given $\mathbf{S} = T \in \mathcal{W}$ and that i gets credit in Type I. Let us count all orderings of players in which i is d-pivotal. In any of these orderings, i must swing in some subset Z of T such that all players in $Z \setminus \bar{i}$ precede i in the ordering, i precedes all players in $T \setminus Z$, and any player in $N \setminus T$ could precede i. Therefore the total number of these orderings are

$$\begin{aligned} & \textstyle\sum_{Z \subseteq T, i \text{ swings in } Z} \sum_{l=0}^{n-|T|} (|Z|-1+l)!(|T|-|Z|+n-|T|-l)! \binom{n-|T|}{l} \\ = & \textstyle\sum_{i \in Z \subseteq T} [v(Z) - v(Z \setminus \bar{i})] \sum_{l=0}^{n-|T|} \frac{(|Z|+l-1)!(n-|Z|-l)!(n-|T|)!}{l!(n-|T|-l)!} \end{aligned}$$

where l is the number of players in $N \setminus T$ who precede player i in such a ordering. Similarly we can count the total orderings of Type II in which i is d-pivotal. As $\mathbf{S}$ varies, we have the formula for $\phi_i[\Gamma, \mathcal{P}]$,

$$\begin{aligned} & \tfrac{1}{n!} \textstyle\sum_{T \in \mathcal{W}} \text{Prob}(\mathbf{S} = T|\mathcal{P}) \sum_{Z \subseteq T} [v(Z) - v(Z \setminus \bar{i})] \sum_{l=0}^{n-|T|} \frac{(|Z|+l-1)!(n-|Z|-l)!(n-|T|)!}{l!(n-|T|-l)!} \\ & + \tfrac{1}{n!} \textstyle\sum_{T \in \mathcal{W}^*} \text{Prob}(\mathbf{S}^* = T|\mathcal{P}) \sum_{Z \subseteq T} [v^*(Z) - v^*(Z \setminus \bar{i})] \sum_{l=0}^{n-|T|} \frac{(|Z|+l-1)!(n-|Z|-l)!(n-|T|)!}{l!(n-|T|-l)!}. \end{aligned} \tag{6}$$

We can apply the following lemma to simplify (6).

Lemma 2.1 $\sum_{l=0}^{n-t} \binom{j+l-1}{j-1} \binom{n-j-l}{t-j} = \binom{n}{t}$ *for* $\forall 0 < j \leq t \leq n$.

Proposition 2.1

$$\begin{aligned} \phi_i[\Gamma, \mathcal{P}] &= \textstyle\sum_{T \in \mathcal{W}} Prob(\mathbf{S} = T|\mathcal{P}) \sum_{Z \subseteq T} [v(Z) - v(Z \setminus \bar{i})] \Delta(|T|, |Z|) \\ & \quad + \textstyle\sum_{Z \notin \mathcal{W}} Prob(\mathbf{S} = Z|\mathcal{P}) \sum_{T \supseteq Z} [v(T \cup \bar{i}) - v(T)] \Delta(n - |Z|, n - |T|) \end{aligned}$$

where $\Delta(t,z) \stackrel{def}{=} \frac{(z-1)!(t-z)!}{t!}$ *for* $\forall 0 < z \leq t$.

Corollary 2.1 *If* $Prob(\mathbf{S} = N|\mathcal{P}) = 1$ *or* $Prob(\mathbf{S}^* = N|\mathcal{P}) = 1$*, then* $\phi_i[\Gamma, \mathcal{P}] = \phi_i[\Gamma]$.

In contrast with the quotation in §1, we have a more realistic story with the asymmetric S-S index. First the voters cast their ballots of votes (either YES or NO) according to the one-man-one-vote rule. Then the ballots are collected and well randomly shuffled such that any ordering of the ballots is equitably probable. Now we count the ballots one-by-one until a confirmative result, either the bill is successfully passed or successfully blocked, can be announced. The last ballot which concludes the counting process is credited by the asymmetric S-S index.

Proposition 2.2 *1.* $\phi_i[\Gamma, \mathcal{P}] \geq 0$*; 2. if* i *is a dummy, then* $\phi_i[\Gamma, \mathcal{P}] = 0$*; 3. if* i *is a dictator, then* $\phi_i[\Gamma, \mathcal{P}] = 1$.

However $\phi_i[\Gamma, \mathcal{P}] = 0$ (or 1) does not necessarily imply that player i is a dummy (or dictator, respectively). In the example with $N = \overline{123}, \mathcal{W} = \{\overline{1}, \overline{23}\}^+$ and $\text{Prob}(\mathbf{S} = \overline{1}) = 1$, we have $\phi_1[\Gamma, \mathcal{P}] = 1$ and $\phi_2[\Gamma, \mathcal{P}] = \phi_3[\Gamma, \mathcal{P}] = 0$. But player 1 is not a dictator; neither player 2 nor player 3 is a dummy. By Proposition 2.1, the index $\phi_i[\Gamma, \mathcal{P}]$ is actually a composite of weighted marginal contribution. In particular if we apply the symmetry conditions in Proposition 1.1, our composite method leads to the S-S index, as stated in Proposition 2.3. As a remark, there exist other bill-based or cooperation-based conditions, instead of these two uniformity ones, for the probability distribution $\mathcal{P}$. In a multi-party voting, for example, partial cooperations are more likely formed within each party. The formation of $\mathbf{S}$ would relate to the collective interests to the bill.

Proposition 2.3 *1. If* $\mathcal{P}$ *satisfies (S-C1), then* $\phi_i[\Gamma, \mathcal{P}] = \phi_i[\Gamma]$*. 2. If* $\mathcal{P}$ *satisfies (S-C2) and (S-C3), then* $\phi_i[\Gamma, \mathcal{P}] = \phi_i[\Gamma]$.

Example 2:

Let $N = \overline{123}$ and $\mathcal{W} = \{\overline{12}, \overline{13}, \overline{23}\}^+$. We further assume that player 1 will vote YES for sure, player 2 NO for sure and player 3 votes YES with probability p_3 $(0 < p_3 < 1)$. Therefore in this voting body, the event $\mathbf{S} = \overline{12}$ or $\mathbf{S} = \overline{123}$ will never occur. The possible perspectives are $\mathbf{S} = \overline{13}$ or $\mathbf{S} = \overline{23}$. Player 3 is the target of players 1 and 2 when they seek support or cooperation. In this game, player 3's vote will determine the result. But his vote alone can not win. His vote, combined with that of player 1 or 2, will win or block the vote. For the ordering $1 \to 2 \to 3$, there are eight possible sequences of votes: Y→Y→Y, Y→Y→N, Y→N→Y, Y→N→N, N→Y→Y, N→Y→N, N→N→Y, N→N→N. But only two sequences have positive probability: Y→N→Y (with probability p_3) and Y→N→N (with probability $1 - p_3$). The first sequence has the same probability as that of $\mathbf{S} = \overline{13}$ while the second has the same probability distribution as that of $\mathbf{S} = \overline{23}$. In both sequences, player 3 gets 1 credit each. So the expected credit in the ordering is $(0, 0, 1)$. We list all orderings of players and the respective expected credits in Table I and then sum up the expected credits and divide it by 3! to obtain the asymmetric S-S indices

$$\begin{aligned} & \tfrac{1}{3!}[(0,0,1) + (0,0,1) + (p_3, 1-p_3, 0) + (0, 1-p_3, p_3) + (p_3, 0, 1-p_3) + (p_3, 1-p_3, 0)] \\ = \; & (\tfrac{p_3}{2}, \tfrac{1-p_3}{2}, .5). \end{aligned}$$

In this example, player 3 gets half of the power in the simple majority game as players 1 and 2 compete each other.

ordering	expected credit	ordering	expected credit
$1 \to 2 \to 3$	$(0, 0, 1)$	$1 \to 3 \to 2$	$(0, 1-p_3, p_3)$
$2 \to 1 \to 3$	$(0, 0, 1)$	$2 \to 3 \to 1$	$(p_3, 0, 1-p_3)$
$3 \to 1 \to 2$	$(p_3, 1-p_3, 0)$	$3 \to 2 \to 1$	$(p_3, 1-p_3, 0)$

Table I: The Orderings and Expected Credits in Example 2

Example 3:

To think and tabulate the index by the alternative way, we let $N = \overline{123}$, $\mathcal{W} = \{\overline{12}, \overline{13}\}^+$. The pivotal players are indicated in Table II for all orderings and all perspective $\mathbf{S}$. With the assigned or estimated probability distribution of $\mathbf{S}$ in the table, we can obtain the credits for each $\mathbf{S}$. Dividing the summed expected credits by 3!, we have the asymmetric S-S indices:

$$\begin{aligned} & \tfrac{1}{3!}[(.2, .05, .05) + (0, .3, .3) + (.9, 0, 0) + (.6, 0, 0) + (.45, .45, 0) + (.6, 0, .6) \\ + \quad & (1.2, 0, 0) + (.2, .05, .05) = (4.15, .85, 1)/6 = (\tfrac{83}{120}, \tfrac{17}{120}, \tfrac{20}{120}). \end{aligned}$$

To contrast, the asymmetric Banzhaf indices are

$$(.15 + .1 + .15 + .2 + .2 + .05, .1 + .15, .1 + .2) = (\frac{17}{20}, \frac{5}{20}, \frac{6}{20})$$

and the S-S indices are $(\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$. We compare these indices as follows:

$b_i[\Gamma]$:	$(\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$;	$\phi_i[\Gamma]$:	$(\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$;
$b_i[\Gamma, \mathcal{P}]$:	$(\frac{17}{20}, \frac{5}{20}, \frac{6}{20})$;	$\phi_i[\Gamma, \mathcal{P}], \mu_i[\Gamma, \mathcal{P}]$:	$(\frac{83}{120}, \frac{17}{120}, \frac{20}{120})$.

$\mathbf{S}$	$\emptyset$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{12}$	$\overline{13}$	$\overline{23}$	$\overline{123}$
probability	.05	.1	.15	.1	.15	.2	.2	.05
d-swingers		2, 3	1	1	1, 2	1, 3	1	1
$1 \to 2 \to 3$	1	3	1	1	2	3	1	2
$1 \to 3 \to 2$	1	2	1	1	2	3	1	3
$2 \to 1 \to 3$	1	3	1	1	1	3	1	1
$2 \to 3 \to 1$	3	3	1	1	1	1	1	1
$3 \to 1 \to 2$	1	2	1	1	2	1	1	1
$3 \to 2 \to 1$	2	2	1	1	1	1	1	1
credits	(.2,.05,.05)	(,.3,.3)	(.9, ,)	(.6, ,)	(.45,.45,)	(.6, ,.6)	(1.2, ,)	(.2,.05,.05)

Table II: The d-swinging and d-pivotal players in Example 3: $\Gamma(\overline{123}, \{\overline{12}, \overline{13}\}^+)$

3 Structural Analysis of S

This section addresses the asymmetric S-S index from a different point of view. The formula of the indices $\phi_i[\Gamma, \mathcal{P}]$ and $b_i[\Gamma, \mathcal{P}]$ do not credit the non-swinging players in $\mathbf{S}$. Suppose that $\mathbf{S} = T \in \mathcal{W}$ with probability 1. We believe that it would be unfair to credit 1 of contribution to each swinger in T as those in $b_i[\Gamma]$ and $b_i[\Gamma, \mathcal{P}]$. First, the set of swingers in T may not be a winning coalition and it needs the participation of some or all non-swinging players in T. Secondly, the total credits are not evenly distributed to different winning coalitions. If there are more than one swinger in T, then at least two credits are distributed to the swingers in T but T itself gets only 1 credit (i.e. $v(T) = 1$). As a consequence, neither $\sum_{i=1}^{n} b_i[\Gamma]$ nor $\sum_{i=1}^{n} b_i[\Gamma, \mathcal{P}]$ is a constant. Thirdly as we observe in the vast majority of voting games (such as the two-candidate US presidential elections), a typical result of voting is not won or lost by a margin of one vote but with some level of leading (i.e. by many votes). As a result, there exists no swinger in a typical voting result and no one gets credit in the indices although many players make great effort to make the coalitions winning or blocking. One question is that who should get credit if the vote is passed or blocked. Of course, it is unfair to evenly distributed the contribution to all players in T; an obvious reason is that T may contain dummies who should get no credit at all. We shall answer the question by the structural analysis of $\mathbf{S}$ or $\mathbf{S}^*$, instead of the marginal contributions.

Let us give an example to illustrate the unfairness. Let $n = \overline{12345}$ and $\mathcal{W} = \{\overline{123}, \overline{124}, \overline{134}, \overline{145}, \overline{2345}\}^+$. Three credits of contribution are given to the coalition $\overline{123}$ while only one to the coalition $\overline{1234}$. In $\overline{1234}$, the participation of $\overline{234}$ is not recognized in their indices although player 1 can not win by himself in the game. Finally, no one is credited in forming the grand winning coalition N. We first propose a simple approach to address the issue. Although 1 credit of contribution in the ordering (1) is given to the pivotal player i_k alone, it can be equitably distributed to all swingers of $S = \overline{i_1 i_2 \cdots i_k}$ without changing the S-S indices. Formally, we let

S' denote the set of swingers of $S = \overline{i_1 \cdots i_k}$ in (1). In the ordering, we credit $1/|S'|$ for each member of S' and 0 for all others. Let $\phi'_i[\Gamma]$ defined by the total credit, divided by $n!$, which player i receives in all $n!$ orderings of players.

Proposition 3.1 $\phi'_i[\Gamma] = \phi_i[\Gamma]$.

We now address the value approach. Given $\mathbf{S} = T \in \mathcal{W}$, all players in T vote YES to the bill and all players in $N \setminus T$ vote NO. To distribute the contribution of winning the bill by T, we note that a swinger of T should receive more credit than a non-swinging player of T. We also note that dummies should receive no credit at all. In addition, the players of $N \setminus T$ should receive no credit for winning the bill since they vote against the bill. Now we define the *local game* $\Gamma(N, \mathcal{W}_T)$ with the winning coalitions $\mathcal{W}_T = \{Z \subseteq N | Z \cap T \in \mathcal{W}\}$ and thus its characteristic function is $v(Z \cap T)$ for $\forall Z \subseteq N$. It is a well-defined simple game and the players of $N \setminus T$ are dummies in the local game. As all players of T vote YES, a fair allocation of the contribution is of course the S-S index in the local game $\Gamma(N, \mathcal{W}_T)$. So the players of $N \setminus T$ get no credit in winning the vote by T. As the swingers in T make up the veto players in the local game, any swinger gets a same amount of credit, which is larger than that of any non-swinging player of T. The total credit in winning the bill by T is 1. In the example with $n = \overline{12345}$ and $\mathcal{W} = \{\overline{123}, \overline{124}, \overline{134}, \overline{145}, \overline{2345}\}^+$, if $\mathbf{S} = T = \overline{1234}$, then $\mathcal{W}_T$ contains $\overline{123}$, $\overline{124}$, $\overline{134}$ and their supersets. Player 1 is a veto player in the local game. The fair contribution of the winning result $\overline{1234}$ is $(\frac{3}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0)$, instead of crediting 1 to player 1 only. To apply the S-S index properly in $\Gamma(N, \mathcal{W}_T)$, we may fictitiously assume that all players of $N \setminus T$ also vote "YES" and so we obtain a grand coalition in voting for the bill. But these fictitious "YES"'s by $N \setminus T$ are not recognized since these players are dummies in the local game $\Gamma(N, \mathcal{W}_T)$. Another way is to limit the player set to T as illustrated in the following proposition.

Proposition 3.2 *For* $\forall i \in T \in \mathcal{W}$, *i's S-S indices in* $\Gamma(N, \mathcal{W}_T)$ *and* $\Gamma(T, \{Z \in \mathcal{W} | Z \subseteq T\})$ *are* *equal.*

Similarly given $\mathbf{S}^* = T \in \mathcal{W}^*$, we let $\mathcal{W}_T^* \stackrel{\text{def}}{=} \{Z \subseteq N | Z \cap T \in \mathcal{W}^*\}$. Then the local game $\Gamma^*(N, \mathcal{W}_T^*)$ of $\Gamma^*(N, \mathcal{W}^*)$ is also a simple game. As before, the fair allocation of the contribution for blocking the bill by T should be the S-S index in the local game $\Gamma^*(N, \mathcal{W}_T^*)$. In the example with $n = \overline{12345}$ and $\mathcal{W} = \{\overline{123}, \overline{124}, \overline{134}, \overline{145}, \overline{2345}\}^+$, if $\mathbf{S} = \overline{15}$, then the bill is blocked and $\mathbf{S}^* = \overline{234}$. As $\mathcal{W}_{\overline{234}}^* = \{\overline{24}, \overline{34}\}^+$, the fair contribution for blocking the bill by $\overline{234}$ is then $(0, \frac{1}{6}, \frac{1}{6}, \frac{4}{6}, 0)$. However, the game $\Gamma^*(N, \mathcal{W}_T^*)$ is not necessarily the dual of $\Gamma(N, \mathcal{W}_T)$ for all $T \in \mathcal{W} \cap \mathcal{W}^*$. For example, in $N = \overline{123}$ with $\mathcal{W} = \{\overline{12}, \overline{13}\}^+$ and $\mathcal{W}^* = \{\overline{1}, \overline{23}\}^+$, $\mathcal{W}_{\overline{12}} = \{\overline{12}\}^+$ but $\mathcal{W}_{\overline{12}}^* = \{\overline{1}\}^+$.

Corollary 3.1 *For $\forall i \in T \in \mathcal{W}^*$, i's S-S indices in $\Gamma^*(N, \mathcal{W}_T^*)$ and $\Gamma^*(T, \{Z \in \mathcal{W}^* | Z \subseteq T\})$ are equal.*

Let $\phi_i[\Gamma, \mathcal{W}_T]$ denote player i's S-S index in the game $\Gamma(N, \mathcal{W}_T)$ and $\phi_i[\Gamma^*, \mathcal{W}_T^*]$ his S-S index in the game $\Gamma^*(N, \mathcal{W}_T^*)$. Extending (3) and (4), we now define the index $\mu_i[\Gamma, \mathcal{P}]$ by player i's total expected contribution in both Γ and Γ^*,

$$\mu_i[\Gamma, \mathcal{P}] \stackrel{\text{def}}{=} \sum_{T \in \mathcal{W}} \text{Prob}(\mathbf{S} = T | \mathcal{P}) \phi_i[\Gamma, \mathcal{W}_T] + \sum_{T \in \mathcal{W}^*} \text{Prob}(\mathbf{S}^* = T | \mathcal{P}) \phi_i[\Gamma^*, \mathcal{W}_T^*]. \tag{7}$$

Proposition 3.3 *For $\forall i \in N$, $\mu_i[\Gamma, \mathcal{P}] = \phi_i[\Gamma, \mathcal{P}]$.*

Thus we have two equivalent interpretations for the asymmetric S-S indices: one from the double pivoting scheme, the other from the value concept and the double swinging scheme. As the asymmetric S-S index takes the cooperation perspective into account, it has some special features. One of those is that a veto player does not have to have more power than another player. For the example of $\Gamma(\overline{123}, \{\overline{12}, \overline{13}\}^+)$ with $\text{Prob}(\mathbf{S} = \overline{1}) = .9$ and $\text{Prob}(\mathbf{S} = \overline{23}) = .1$, player 1 is a veto player, but players 2 and 3 form a cooperation to against 1. Clearly $\phi_1[\Gamma, \mathcal{P}] = .1 < .45 = \phi_2[\Gamma, \mathcal{P}] = \phi_3[\Gamma, \mathcal{P}]$. In the United Nations Security Council, USA and China have quite different power distribution although they both are veto players.

Proposition 3.4 *1.* $\sum_{i=1}^{n} \phi_i[\Gamma, \mathcal{P}] = 1$*; 2. if* $\Gamma(N, \mathcal{W})$ *is decisive and* $\phi_i[\Gamma, \mathcal{P}] = 1$*, then* i *is the dictator of the game.*

The condition of decisiveness part is necessary. For example, in the game $\Gamma(\overline{123}, \{\overline{123}\})$ with Prob$(\mathbf{S} = \overline{12}) = 1$, we have $\phi_3[\Gamma, \mathcal{P}] = 1$ but player 3 is not a dictator. The game is not decisive since $v(\overline{1}) + v(\overline{23}) = 0 \neq 1$.

Example 3 Revisit:

Here $N = \overline{123}$, $\mathcal{W} = \{\overline{12}, \overline{13}\}^+$ and $\mathcal{W}^* = \{\overline{1}, \overline{23}\}^+$. In Table III, we calculate the S-S indices for all potential $\Gamma(N, \mathcal{W}_T)$ and $\Gamma^*(N, \mathcal{W}_T^*)$. The $\mu_i[\Gamma, \mathcal{P}]$'s are

$$\begin{aligned} & .05(\tfrac{4}{6}, \tfrac{1}{6}, \tfrac{1}{6}) + .1(0, \tfrac{3}{6}, \tfrac{3}{6}) + .15(\tfrac{6}{6}, 0, 0) + .1(\tfrac{6}{6}, 0, 0) \\ + \quad & .15(\tfrac{3}{6}, \tfrac{3}{6}, 0) + .2(\tfrac{3}{6}, 0, \tfrac{3}{6}) + .2(\tfrac{6}{6}, 0, 0) + .05(\tfrac{4}{6}, \tfrac{1}{6}, \tfrac{1}{6}) = (\tfrac{83}{120}, \tfrac{17}{120}, \tfrac{20}{120}) \end{aligned}$$

which are identical to $\phi_i[\Gamma, \mathcal{P}]$'s in Example 3.

S	$\emptyset$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{12}$	$\overline{13}$	$\overline{23}$	$\overline{123}$
Probability	.05	.1	.15	.1	.15	.2	.2	.05
T	$\overline{123}$	$\overline{23}$	$\overline{13}$	$\overline{12}$	$\overline{12}$	$\overline{13}$	$\overline{1}$	$\overline{123}$
$\mathcal{W}_T$					$\{\overline{12}\}^+$	$\{\overline{13}\}^+$		$\{\overline{12}, \overline{13}\}^+$
$\mathcal{W}_T^*$	$\{\overline{1}, \overline{23}\}^+$	$\{\overline{23}\}^+$	$\{\overline{1}\}^+$	$\{\overline{1}\}^+$			$\{\overline{1}\}^+$	
Contribution	$(\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$	$(0, \frac{3}{6}, \frac{3}{6})$	$(\frac{6}{6}, 0, 0)$	$(\frac{6}{6}, 0, 0)$	$(\frac{3}{6}, \frac{3}{6}, 0)$	$(\frac{3}{6}, 0, \frac{3}{6})$	$(\frac{6}{6}, 0, 0)$	$(\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$

Table III: The Contribution by Value in Example 3: $\Gamma(\overline{123}, \{\overline{12}, \overline{13}\}^+)$

4 Appendix: Proofs

Proof of Proposition 1.1

1. As 2^N has 2^n elements, $\text{Prob}(\mathbf{S} = T|\mathcal{P}) = \frac{1}{2^n}$ for $\forall T \subseteq N$. By (4),

$$\begin{aligned} b_i[\Gamma, \mathcal{P}] &= \textstyle\sum_{i\in T} \frac{1}{2^n}[v(T) - v(T\setminus \bar{i})] + \sum_{i\notin Z} \frac{1}{2^n}[v(Z\cup \bar{i}) - v(Z)] \\ &\overset{T=Z\cup\bar{i}}{=} \textstyle\frac{1}{2^n}\sum_{i\in T}[v(T) - v(T\setminus \bar{i})] + \frac{1}{2^n}\sum_{i\in T}[v(T) - v(T\setminus \bar{i})] = b_i[\Gamma]. \end{aligned}$$

2. By the Bayesian formula, we have that

$$\begin{aligned} \text{Prob}(\mathbf{S} = T|\mathcal{P}) &= \text{Prob}(|\mathbf{S}| = |T||\mathcal{P})\text{Prob}(\mathbf{S} = T||\mathbf{S}| = |T|, \mathcal{P}) \\ &= \frac{1}{n+1} \div \begin{pmatrix} n \\ |T| \end{pmatrix} = \frac{(|T|)!(n-|T|)!}{(n+1)!}. \end{aligned}$$

By (4),

$$\begin{aligned} b_i[\Gamma, \mathcal{P}] &= \textstyle\sum_{i\in T} \frac{(|T|)!(n-|T|)!}{(n+1)!}[v(T) - v(T\setminus \bar{i})] + \sum_{i\notin T} \frac{(|T|)!(n-|T|)!}{(n+1)!}[v(T\cup \bar{i}) - v(T)] \\ &\overset{Z=T\cup\bar{i}}{=} \textstyle\sum_{i\in T} \frac{(|T|)!(n-|T|)!}{(n+1)!}[v(T) - v(T\setminus \bar{i})] + \sum_{i\in Z} \frac{(|Z|-1)!(n-|Z|+1)!}{(n+1)!}[v(Z) - v(Z\setminus \bar{i})] \\ &\overset{T=Z}{=} \textstyle\sum_{i\in Z} \frac{(|Z|)!(n-|Z|)!+(|Z|-1)!(n-|Z|+1)!}{(n+1)!}[v(Z) - v(Z\setminus \bar{i})] = \phi_i[\Gamma]. \end{aligned}$$

Proof of Lemma 2.1

It is equivalent to $\sum_{l=0}^{s} \begin{pmatrix} j+l-1 \\ l \end{pmatrix} \begin{pmatrix} n-j-l \\ s-l \end{pmatrix} = \begin{pmatrix} n \\ s \end{pmatrix}$ with $s = n - t$. We observe the fact: if $\{a_i | 1 \le i \le s, a_1 \le a_2 \le \cdots \le a_s\}$ (called a multiple s-subset) is chosen from $\{1, 2, \cdots, n\}$ such that repetition is allowed, then there are $\begin{pmatrix} n+s-1 \\ s \end{pmatrix}$ choices. The number of choices can be obtained this way: if $z_i = a_i + i - 1$, then we actually choose $z_i, \cdots, z_s$ from $\{1, 2, \cdots, n+s-1\}$ without repetition.

Now we consider all multiple s-subsets, say A's, of $\{1, 2, \cdots, n-s+1\}$. There are $\begin{pmatrix} n \\ s \end{pmatrix}$ A's. On the other hand, for any j with $1 \le j \le n-s = t$, we partition $\{1, 2, \cdots, n-s+1\}$ into two nonempty parts: $\{1, 2, \cdots, j\}$ and $\{j+1, j+2, \cdots, n-s+1\}$. For any A, if $|A \cap \{1, 2, \cdots, j\}| = l$, then $|A \cap \{j+1, j+2, \cdots, n-s+1\}| = s-l$. The number of A's with l elements chosen from the first part is then $\begin{pmatrix} j+l-1 \\ l \end{pmatrix} \begin{pmatrix} (n-s+1-j)+(s-l)-1 \\ s-l \end{pmatrix}$. The sum over l concludes the proof, $\sum_{l=0}^{s} \begin{pmatrix} j+l-1 \\ l \end{pmatrix} \begin{pmatrix} n-j-l \\ s-l \end{pmatrix} = \begin{pmatrix} n \\ s \end{pmatrix}$.

Proof of Proposition 2.1

By Lemma 2.1 with $j = |Z|$ and $t = |T|$,

$$\sum_{l=0}^{n-|T|} \frac{(|Z|+l-1)!}{l!(|Z|-1)!} \frac{(n-|Z|-l)!}{(|T|-|Z|)!(n-l-|T|)!} = \frac{n!}{(|T|)!(n-|T|)!}.$$

Therefore we have

$$\sum_{l=0}^{n-|T|} \frac{(|Z|+l-1)!(n-|Z|-l)!(n-|T|)!}{n!l!(n-|T|-l)!} = \frac{(|Z|-1)!(|T|-|Z|)!}{(|T|)!} = \Delta(|T|,|Z|)$$

which, together with (6), concludes that

$$\begin{aligned} \phi_i[\Gamma,\mathcal{P}] &= \textstyle\sum_{T\in\mathcal{W}} \mathrm{Prob}(\mathbf{S}=T|,\mathcal{P}) \sum_{Z\subseteq T}[v(Z)-v(Z\setminus\bar{i})]\Delta(|T|,|Z|) \\ &\quad + \textstyle\sum_{T\in\mathcal{W}^*} \mathrm{Prob}(\mathbf{S}^*=T|\mathcal{P}) \sum_{Z\subseteq T}[v^*(Z)-v^*(Z\setminus\bar{i})]\Delta(|T|,|Z|). \end{aligned} \tag{8}$$

From $v^*(Z)-v^*(Z\setminus\bar{i}) = v(N\setminus Z\cup\bar{i}) - v(N\setminus Z)$, the second part of (8) is then

$$\begin{aligned} & \textstyle\sum_{T\in\mathcal{W}^*} \mathrm{Prob}(\mathbf{S}^*=T) \sum_{i\in Z\subseteq T} \Delta(|T|,|Z|)[v^*(Z)-v^*(Z\setminus\bar{i})] \\ = \quad & \textstyle\sum_{T\in\mathcal{W}^*} \mathrm{Prob}(\mathbf{S}^*=T) \sum_{i\in Z\subseteq T} \frac{(|Z|-1)!(|T|-|Z|)!}{(|T|)!}[v(N\setminus Z\cup\bar{i}) - v(N\setminus Z)] \\ \overset{K=N\setminus Z}{=} \quad & \textstyle\sum_{T\in\mathcal{W}^*} \mathrm{Prob}(\mathbf{S}^*=T) \sum_{i\notin K, N\setminus K\subseteq T} \frac{(n-|K|-1)!(|T|-n+|K|)!}{(|T|)!}[v(K\cup\bar{i})-v(K)] \\ \overset{Z=N\setminus T}{=} \quad & \textstyle\sum_{Z\notin\mathcal{W}} \mathrm{Prob}(\mathbf{S}=Z) \sum_{i\notin K, Z\subseteq K} \frac{(n-|K|-1)!(|K|-|Z|)!}{(n-|Z|)!}[v(K\cup\bar{i})-v(K)] \\ \overset{T=K}{=} \quad & \textstyle\sum_{Z\notin\mathcal{W}} \mathrm{Prob}(\mathbf{S}=Z) \sum_{T\supseteq Z}[v(T\cup\bar{i})-v(T)]\Delta(n-|Z|, n-|T|). \end{aligned}$$

Proof of Corollary 2.1

If $\mathbf{S}=N$ a.s., then the second summation in Proposition 2.1 is just 0; there is only one nonzero item in the first summation, $\phi_i[\Gamma,\mathcal{P}] = \sum_{Z\subseteq N}[v(Z)-v(Z\setminus\bar{i})]\Delta(n,|Z|) = \phi_i[\Gamma]$. If $\mathbf{S}^*=N$ a.s.,then the first summation in Proposition 2.1 is just 0; the second has only one nonzero item,

$$\begin{aligned} \phi_i[\Gamma,\mathcal{P}] \quad &= \quad \textstyle\sum_{T\supseteq\emptyset}[v(T\cup\bar{i})-v(T)]\Delta(n,n-|T|) = \sum_{i\notin T}[v(T\cup\bar{i})-v(T)]\frac{(n-|T|-1)!(|T|)!}{n!} \\ &\overset{Z=T\cup\bar{i}}{=} \quad \textstyle\sum_{Z\subseteq N}[v(Z)-v(Z\setminus\bar{i})]\frac{(|Z|-1)!(n-|Z|)!}{n!} = \phi_i[\Gamma]. \end{aligned}$$

Proof of Proposition 2.2

Parts 1. and 2. are simply from Proposition 2.1.

3. If i is a dictator, then $v(T) - v(T \setminus \bar{i}) = v(Z \cup \bar{i}) - v(Z) = 1$ for all T and Z such that $i \in T$ and $i \notin Z$. Now by Proposition 2.1,

$$\begin{aligned}
\phi_i[\Gamma, \mathcal{P}] &= \textstyle\sum_{T \in \mathcal{W}} \mathrm{Prob}(\mathbf{S} = T) \sum_{i \in Z \subseteq T} \frac{(|Z|-1)!(|T|-|Z|)!}{(|T|)!} 1 \\
&\quad + \textstyle\sum_{T \notin \mathcal{W}} \mathrm{Prob}(\mathbf{S} = T) \sum_{i \notin Z \supseteq T} \frac{(n-|Z|-1)!(|Z|-|T|)!}{(n-|T|)!} 1 \\
&\overset{z=|Z|}{=} \textstyle\sum_{T \in \mathcal{W}} \mathrm{Prob}(\mathbf{S} = T) \sum_{z=1}^{|T|} \frac{(z-1)!(|T|-z)!}{(|T|)!} \frac{(|T|-1)!}{(z-1)!(|T|-z)!} \\
&\quad + \textstyle\sum_{T \notin \mathcal{W}} \mathrm{Prob}(\mathbf{S} = T) \sum_{z=|T|}^{n-1} \frac{(n-z-1)!(z-|T|)!}{(n-|T|)!} \frac{(n-|T|-1)!}{(z-|T|)!(n-z-1)!} \\
&= \textstyle\sum_{T \in \mathcal{W}} \mathrm{Prob}(\mathbf{S} = T) + \sum_{T \notin \mathcal{W}} \mathrm{Prob}(\mathbf{S} = T) = 1.
\end{aligned}$$

Proof of Proposition 2.3

1. By the proof of Proposition 1.1, $\mathrm{Prob}(\mathbf{S} = T|\mathcal{P}) = \frac{1}{2^n}$ for any $T \subseteq N$. For $\forall i \in N$, the first summation in Proposition 2.1 is

$$\begin{aligned}
&\textstyle\frac{1}{2^n} \sum_{i \in T \in \mathcal{W}} \sum_{Z \subseteq T} \frac{(|Z|-1)!(|T|-|Z|)!}{(|T|)!} [v(Z) - v(Z \setminus \bar{i})] \\
&= \textstyle\frac{1}{2^n} \sum_{i \in Z} (|Z|-1)! [v(Z) - v(Z \setminus \bar{i})] \sum_{T \supseteq Z} \frac{(|T|-|Z|)!}{(|T|)!} \\
&\overset{t=|T|}{=} \textstyle\frac{1}{2^n} \sum_{i \in Z} (|Z|-1)! [v(Z) - v(Z \setminus \bar{i})] \sum_{t=|Z|}^{n} \frac{(t-|Z|)!}{t!} \begin{pmatrix} n-|Z| \\ t-|Z| \end{pmatrix} \\
&= \textstyle\frac{1}{2^n} \sum_{i \in Z} (|Z|-1)!(n-|Z|)! [v(Z) - v(Z \setminus \bar{i})] \sum_{t=|Z|}^{n} \frac{1}{t!(n-t)!}.
\end{aligned}$$

The second summation in Proposition 2.1 is

$$\begin{aligned}
&\textstyle\frac{1}{2^n} \sum_{Z \notin \mathcal{W}} \sum_{Z \subseteq T, i \notin T} \frac{(n-|T|-1)!(|T|-|Z|)!}{(n-|Z|)!} [v(T \cup \bar{i}) - v(T)] \\
&= \textstyle\frac{1}{2^n} \sum_{i \notin T} (n-|T|-1)! [v(T \cup \bar{i}) - v(T)] \sum_{Z \subseteq T} \frac{(|T|-|Z|)!}{(n-|Z|)!} \\
&\overset{t=|Z|}{=} \textstyle\frac{1}{2^n} \sum_{i \notin T} (n-|T|-1)! [v(T \cup \bar{i}) - v(T)] \sum_{t=0}^{|T|} \frac{(|T|-t)!}{(n-t)!} \begin{pmatrix} |T| \\ t \end{pmatrix} \\
&\overset{Z=T \cup \bar{i}}{=} \textstyle\frac{1}{2^n} \sum_{i \in Z} (n-|Z|)!(|Z|-1)! [v(Z) - v(Z \setminus \bar{i})] \sum_{t=0}^{|Z|-1} \frac{1}{t!(n-t)!}.
\end{aligned}$$

The identity $\sum_{t=|Z|}^{n} \frac{1}{t!(n-t)!} + \sum_{t=0}^{|Z|-1} \frac{1}{t!(n-t)!} = \frac{2^n}{n!}$ completes the proof of the first part.

2. By the proof of Proposition 1.1, $\mathrm{Prob}(\mathbf{S} = T|\mathcal{P}) = \frac{(|T|)!(n-|T|)!}{(n+1)!}$. For $\forall i \in N$, the first

summation in Proposition 2.1 is

$$\begin{aligned}
& \textstyle\sum_{i\in T\in\mathcal{W}} \frac{(|T|)!(n-|T|)!}{(n+1)!} \sum_{Z\subseteq T} \frac{(|Z|-1)!(|T|-|Z|)!}{(|T|)!}[v(Z)-v(Z\setminus\bar{i})] \\
= \quad & \textstyle\sum_{i\in Z} \frac{(|Z|-1)!}{(n+1)!}[v(Z)-v(Z\setminus\bar{i})] \sum_{T\supseteq Z}(n-|T|)!(|T|-|Z|)! \\
\overset{t=|T|}{=} \quad & \textstyle\sum_{i\in Z} \frac{(|Z|-1)!}{(n+1)!}[v(Z)-v(Z\setminus\bar{i})] \sum_{t=|Z|}^{n}(n-t)!(t-|Z|)! \begin{pmatrix} n-|Z| \\ t-|Z| \end{pmatrix} \\
= \quad & \textstyle\sum_{i\in Z} \frac{(|Z|-1)!(n-|Z|+1)!}{(n+1)!}[v(Z)-v(Z\setminus\bar{i})].
\end{aligned}$$

The second summation in Proposition 2.1 is

$$\begin{aligned}
& \textstyle\sum_{Z\notin\mathcal{W}} \frac{(n-|Z|)!(|Z|)!}{(n+1)!} \sum_{Z\subseteq T, i\notin T} \frac{(n-|T|-1)!(|T|-|Z|)!}{(n-|Z|)!}[v(T\cup\bar{i})-v(T)] \\
= \quad & \textstyle\sum_{i\notin T} \frac{(n-|T|-1)!}{(n+1)!}[v(T\cup\bar{i})-v(T)] \sum_{Z\subseteq T}(|T|-|Z|)!(|Z|)! \\
\overset{z=|Z|}{=} \quad & \textstyle\sum_{i\notin T} \frac{(n-|T|-1)!}{(n+1)!}[v(T\cup\bar{i})-v(T)] \sum_{z=0}^{|T|}(|T|-z)!z! \begin{pmatrix} |T| \\ z \end{pmatrix} \\
\overset{Z=T\cup\bar{i}}{=} \quad & \textstyle\sum_{i\in Z} \frac{(n-|Z|)!(|Z|)!}{(n+1)!}[v(Z)-v(Z\setminus\bar{i})].
\end{aligned}$$

We combine the two results,

$$\phi_i[\Gamma,\mathcal{P}] = \sum_{i\in Z} \frac{(|Z|-1)!(n-|Z|+1)!+(n-|Z|)!(|Z|)!}{(n+1)!}[v(Z)-v(Z\setminus\bar{i})] = \phi_i[\Gamma].$$

Proof of Proposition 3.1

For $\forall j \in S'$, if we switch the position of j with that of the pivotal player i_k in (1), then j will pivot in the new ordering and i_k will receive the same amount $1/|S'|$ of credit for the index $\phi'[\Gamma]$. However if we switch the position of i_k with that of any $j \in S \setminus S'$, then j is not pivotal in the new ordering. Now for any $T \in \mathcal{W}$ such that i swings in T, we order T such that the last player is a swinger of T. There are $|T'|(|T|-1)!$ of them. In each of those orderings, i receives credit $1/|T'| = [v(T)-v(T\setminus\bar{i})]/|T'|$. Combined with the $(n-|T|)!$ orderings of $N\setminus T$, the total credit player i receives in all orderings is then

$$\phi'_i[\Gamma] = \frac{1}{n!}\sum_{i\in T\in\mathcal{W}} |T'|(|T|-1)!(n-|T|)!\frac{[v(T)-v(T\setminus\bar{i})]}{|T'|} = \phi_i[\Gamma].$$

Proof of Proposition 3.2

We first prove the following statement: if k is a dummy in $\Gamma(N, \mathcal{W})$ and removed from the game, then for $\forall i \in N \setminus \overline{k}$, i's S-S indices are equal in the games $\Gamma(N, \mathcal{W})$ and $\Gamma(N \setminus \overline{k}, \{T \setminus \overline{k} | T \in \mathcal{W}\})$, the game after player k is removed. To see this,

$$\begin{aligned}
& \phi_i[\Gamma] = \sum_{T \subseteq N} \tfrac{(|T|-1)!(n-|T|)!}{n!} [v(T) - v(T \setminus \overline{i})] \\
= \; & \sum_{k \in T \subseteq N} \tfrac{(|T|-1)!(n-|T|)!}{n!} [v(T) - v(T \setminus \overline{i})] + \sum_{k \notin T \subseteq N} \tfrac{(|T|-1)!(n-|T|)!}{n!} [v(T) - v(T \setminus \overline{i})] \\
\overset{Z=T\setminus\overline{k}}{=} \; & \sum_{Z \subseteq N \setminus \overline{k}} \tfrac{(|Z|)!(n-1-|Z|)!}{n!} [v(Z) - v(Z \setminus \overline{i})] + \sum_{T \subseteq N \setminus \overline{k}} \tfrac{(|T|-1)!(n-|T|)!}{n!} [v(T) - v(T \setminus \overline{i})] \\
\overset{T=Z}{=} \; & \sum_{Z \subseteq N \setminus \overline{k}} \tfrac{(|Z|-1)!((n-1)-|Z|)!}{(n-1)!} [v(Z) - v(Z \setminus \overline{i})]
\end{aligned}$$

which is player i's S-S index in the game $\Gamma(N \setminus \overline{k}, \{T \setminus \overline{k} | T \in \mathcal{W}\})$.

Finally we successively remove the players of $N \setminus T$ from the game $\Gamma(N, \mathcal{W}_T)$ and obtain the game $\Gamma(T, \{Z \in \mathcal{W} | Z \subseteq T\})$. The fair contribution distribution is unchanged by the above statement.

Proof of Proposition 3.3

By Proposition 3.2 and Corollary 3.1, for $\forall i \in T$,

$$\begin{aligned}
\phi_i[\Gamma, \mathcal{W}_T] &= \sum_{Z \subseteq T} \tfrac{(|Z|-1)!(|T|-|Z|)!}{(|T|)!} [v(Z) - v(Z \setminus \overline{i})], \\
\phi_i[\Gamma^*, \mathcal{W}^*_T] &= \sum_{Z \subseteq T} \tfrac{(|Z|-1)!(|T|-|Z|)!}{(|T|)!} [v^*(Z) - v^*(Z \setminus \overline{i})].
\end{aligned}$$

We plug the above identities in (7) and (8) to conclude the proof.

Proof of Proposition 3.4

1. By (7) and Proposition 3.3,

$$\begin{aligned}
& \sum_{i=1}^n \phi_i[\Gamma, \mathcal{P}] = \sum_{i=1}^n \mu_i[\Gamma, \mathcal{P}] \\
= \; & \sum_{T \in \mathcal{W}} \text{Prob}(\mathbf{S} = T | \mathcal{P}) \sum_{i=1}^n \phi_i[\Gamma, \mathcal{W}_T] + \sum_{T \in \mathcal{W}^*} \text{Prob}(\mathbf{S}^* = T | \mathcal{P}) \sum_{i=1}^n \phi_i[\Gamma^*, \mathcal{W}^*_T] \\
= \; & \sum_{T \in \mathcal{W}} \text{Prob}(\mathbf{S} = T | \mathcal{P}) + \sum_{T \in \{N \setminus Z | Z \notin \mathcal{W}\}} \text{Prob}(\mathbf{S} = N \setminus T | \mathcal{P}) \\
\overset{Z=N\setminus T}{=} \; & \sum_{T \in \mathcal{W}} \text{Prob}(\mathbf{S} = T | \mathcal{P}) + \sum_{Z \notin \mathcal{W}} \text{Prob}(\mathbf{S} = Z | \mathcal{P}) = 1.
\end{aligned}$$

2. We observe a basic fact about the dictatorship: i is a dictator in $\Gamma(N, \mathcal{W})$ if and only if $\phi_i[\Gamma] = 1$. If there exists $T \in \mathcal{W}$ such that $\text{Prob}(\mathbf{S} = T) > 0$, then by (7) it is necessary to have $\phi_i[\Gamma, \mathcal{W}_T] = 1$. Therefore i is the dictator in the local game $\Gamma(N, \mathcal{W}_T)$ and thus $v(\overline{i}) = 1$. Since

the game is decisive, $v(N \setminus \bar{i}) = 0$. Therefore $v(T \cup \bar{i}) - v(T \setminus \bar{i}) = 1$ for any T and thus i is the dictator. Otherwise there exists $T \notin \mathcal{W}$ such that Prob($\mathbf{S} = T$) > 0. By Proposition 2.1, we must have

$$\sum_{Z \supseteq T, i \notin Z} \frac{(n-|Z|-1)!(|Z|-|T|)!}{(n-|T|)!}[v(Z \cup \bar{i}) - v(Z)] = 1.$$

The identity $\sum_{Z \supseteq T, i \notin Z} \frac{(n-|Z|-1)!(|Z|-|T|)!}{(n-|T|)!} \overset{z=|Z|}{=} \sum_{z=|T|}^{n-1} \frac{(n-z-1)!(z-|T|)!}{(n-|T|)!} \begin{pmatrix} n-|T|-1 \\ z-|T| \end{pmatrix} = 1$ forces that $v(Z \cup \bar{i}) - v(Z)] = 1$ for any $Z \subseteq N$ such that $i \notin Z \supseteq T$. We take $Z = N \setminus \bar{i}$ and obtain that $v(N \setminus \bar{i}) = 0$. By the same argument as that in the second part, $v(\bar{i}) = 1$ and player i is the dictator of the game.

References

[1] Banzhaf, John F. 1965. "Weighted Voting Doesn't Work: a Mathematical Analysis." *Rutgers Law Review* **19**:317-343.

[2] Shapley, Lloyd S. 1953. "A Value for n-Person Games." *Annals of Mathematics Studies* **28**:307-317.

[3] Shapley, Lloyd S. and Martin Shubik. 1954. "A Method for Evaluating the Distribution of Power in a Committee System." *American Political Science Review* **48**:787-792.