On Authority Distributions in Organizations: Controls[1]

Xingwei Hu and Lloyd S. Shapley

Department of Mathematics, University of California
Los Angeles, CA 90095 – 1555

E-mail: xhu@math.ucla.edu

A member's command game gives the direct boss and approval relations between him and other members in his organization. In a complicated organization, "commands" can be passed or implemented through command channels. This implies a global authority topology in the organization. This paper will investigate three related forms of command channels and their contexts. In the stricter form, a coalition has the complete power over a player. This form can be found in the military, a slavery as well as private property right. The other two equivalent forms analyze the sharing power over commonly-owned property. They can explain such issues as the controlling power, democracy in sharing sovereign or indivisible common property. *Journal of Economic Literature* Classification Numbers: D23, D72, L22, M14.

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1. INTRODUCTION

It is quite universal that, especially in large organizations, commands can be implemented through the command channels such that a coalition can "command" some players *indirectly*. We call such an *indirect command* "control" to distinguish it from the definitions in command games (Hu and Shapley, 2001). When we talk about a command game, we should believe that it is issue based and a specific issue is concerned. While a boss can command an employee in his firm, he can not command the employee when the employee is voting in the presidential election. A player in an organization may also be in many other organizations. For example, consider a resident of Los Angeles: he is an employee in a financial firm, a resident of the state of California and a citizen of the United State. So when analyzing a player's controlling structure, we should have in mind what issue is concerned with the player and with the organization. This includes the citizenship, property right, rank in the organization, election or the eligibility of being elected, etc. Therefore experience with applications should bring a better sense of how to verbalize the mathematical construct defined here and suggest ways to improve or extend the definitions.

As stated in Hu and Shapley(2001), a player in the abstract organization may mean a piece of physical or financial asset as well as an employee. If, say, the player is an indivisible piece of property, we could come up the question: is the property is privately held or publicly held? If publicly held, the controlling power over the player then can be used to measure the property right or ownership right among the members. For example, consider a shareholder's right over a plant in a public company. On the other hand, the command power in the player's command game and the authority distribution in the organization quantify the management's power of manipulating the property, directly or indirectly respectively. As a commonly shared property, a computer in a public library may prompt any instruction by its users. However a private computer (say, having a password) can only be accessed by its owner who has the complete power on the computer. In terms of command games, if player i is a privately held property by S, then S has the dictatorial power on i and $N \setminus S$ are dummies in i's command game. If i is publicly shared by S (of course S commands i), then there may exist another T such that T also commands i and $S \not\subseteq T$.

Some special property, such as sovereign in a nation, is shared by all players in the organization and can not be divisible. In this case, any change of the property needs all players' participation, together with some decision rule. We shall formally address these problems in this paper.

We first note that different organizations may have different forms of "control". In the military, the control may be necessarily strict enough to avoid possible "uprise" (i.e. the controlled members by an player can control the player) or possible impropriety (i.e. two disjoint coalitions give controversial commands to the same member). In a nonprofit club or association, the controls merely respect the importance of each member's participation and social values (such as human right, voting right). The form of control should respect the mission of the organization, the tasks at hand, the issues to be addressed, and the internal structure of the organization, among others. On the other hand, commands may generally be "improper" in which two disjoint coalitions can both be winning in a member's command game. For example, a private may have to obey direct orders from his captain and his colonel as well as from his sergeant. In this case, a global control can then be needed to resolve the conflicts by exercising the power over the disjoint coalitions.

Let S be a coalition in the organization N. Intuitively, the control function, say $\gamma(S)$, is meant to be the set of members that the coalition S can control. In defining $\gamma(S)$ we must recognize, on the one hand, the possibility of indirect control – e.g., members outside S being co-opted to join with S in bossing other outsiders, and on the other hand, the possibility that some members of S may not have full control over their own actions and require approval. Furthermore, for S to implement a control, S should be able to manipulate his own resources and facilities (such as weapons, computers, subordinates, secretaries, assistants, personal money, etc.), say $\sigma(S)$, which are exclusively operated by S. In words, other disjoint coalitions must cooperate with S to exercise the powers on $\sigma(S)$. We say that S has the *complete power* on $\sigma(S)$. Of course, in defining $\sigma(S)$, we should also recognize the possibility of the passage of complete power through channels. The passage of complete power then requires a sequence of coalitions to be reached.

As S has the complete power on $\sigma(S)$, it can use $\sigma(S)$ as its private army to implement controls over other players. An equivalent form of $\gamma(\cdot)$ is called the *command scope* $\delta(\cdot)$ which deals with the multi-step commands, i.e. the commandable members by S join with S to command other members.

The forms $\delta(\cdot)$ and $\sigma(\cdot)$ are studied in §2 and §3 respectively. The control function $\gamma(\cdot)$ and control game are studied in §4. We shall use the same notations and definitions as those in Hu and Shapley(2001).

2. MULTI-STEP COMMANDS

Let i be a generic member in the organization N. He must obey the commands of any coalition in $\mathcal{W}_i$. From another point of view, a coalition may command many players in the organization. For any coalition $S \subseteq N$, we define the *command function* $\omega(S)$ as the set of all members that are commandable by S:

$$\omega(S) \stackrel{\text{def}}{=} \{i \in N : S \in \mathcal{W}_i\}.$$

Equivalently, we can recover the function $\mathcal{W}_{(\cdot)}$ (with domain N) from the "inverse" relation $\mathcal{W}_i = \{S \subseteq N : i \in \omega(S)\}$. In turn we can also recover the boss sets and approval sets: $\mathcal{B}_i = \{S \in \mathcal{N}_i | S \in \mathcal{W}_i\} = \mathcal{W}_i \cap \mathcal{N}_i$ and $\mathcal{A}_i = \{S \in \mathcal{N}_i | S \cup \bar{i} \in \mathcal{W}_i \text{ but } S \notin \mathcal{W}_i\}$. By the fact $\phi \subset \mathcal{W}_i = \mathcal{W}_i^+ \subset \mathcal{N}$, we can show Proposition 2.1.

PROPOSITION 2.1. *1. $\omega(\emptyset) = \emptyset$; 2. $\omega(N) = N$; 3. the command function $\omega(\cdot)$ is weakly increasing.*

The one-step command function is given by $\omega(S)$. Nevertheless $\omega(S)$ can join with S to command a larger coalition $\omega(S \cup \omega(S))$, and so on. In general, we recursively define the command chain for $\forall S \subseteq N$ by

$$\begin{cases} \delta_0(S) & \stackrel{\text{def}}{=} \emptyset, \\ \delta_{t+1}(S) & \stackrel{\text{def}}{=} \omega(S \cup \delta_t(S)), \qquad t = 0, 1, 2, \cdots. \end{cases}$$

PROPOSITION 2.2. *There is a nonnegative $t^* \leq n$ such that the command chain increases strictly up to the term $\delta_{t^*}(S)$ and is constant thereafter.*

Proof. Clearly if $\delta_1(S) = \emptyset$, then $\delta_t(S) = \emptyset$ for all $t \geq 1$ and hence $t^* = 0$. Otherwise by the monotonicity of $\omega(\cdot)$, $\delta_2(S) = \omega(S \cup \delta_1(S)) \supseteq \omega(S) = \delta_1(S)$. If we assume that $\delta_t(S) \supseteq \delta_{t-1}(S)$ for some $t \geq 1$, then $\delta_{t+1}(S) = \omega(S \cup \delta_t(S)) \supseteq \omega(S \cup \delta_{t-1}(S)) = \delta_t(S)$. Therefore by induction

$\delta_t(S)$ is weakly increasing in t. On the other hand if $\delta_t(S) = \delta_{t-1}(S)$, then

$$\delta_{t+1}(S) = \omega(S \cup \delta_t(S)) = \omega(S \cup \delta_{t-1}(S)) = \delta_t(S) = \delta_{t-1}(S),$$

and it can be further shown by induction that $\delta_k(S) = \delta_{t-1}(S)$ for all $k \geq t$.

On the contrary, if t^* does not exist, then we have $|\delta_0(S)| < |\delta_1(S)| < \cdots$. This implies $|\delta_{n+1}(S)| \geq 1 + |\delta_n(S)| \geq 2 + |\delta_{n-1}(S)| \geq \cdots \geq n+1$ which contradicts to $\delta_{n+1}(S) \subseteq N$. The contradiction proves the existence of t^*.

Finally since $|\delta_{t+1}(S)| > |\delta_t(S)|$ for all $0 \leq t < t^*$, we have

$$|\delta_{t^*}(S)| \geq 1 + |\delta_{t^*-1}(S)| \geq 2 + |\delta_{t^*-2}(S)| \geq \cdots \geq t^* + |\delta_0(S)| = t^*.$$

Therefore $t^* \leq |\delta_{t^*}(S)| \leq n$. ▮

Now we can define the *command scope* $\delta(S)$ of $S \subseteq N$ by any of

$$\delta(S) \overset{\text{def}}{=} \delta_{t^*}(S), \quad \delta_n(S), \quad \lim_{t\to\infty} \delta_t(S), \quad \cup_{t=0}^{\infty}\delta_t(S) \quad \text{or} \quad \cup_{t=0}^{n} \delta_t(S).$$

The following example (Figure 1) will serve to illustrate the command chains and other sequences in the paper. In this example, we let $N = \overline{0123456789}$ with the command games defined by

$$\mathcal{W}_0^m = \{\overline{012}, \overline{013}, \overline{023}, \overline{123}\},$$

$$\mathcal{W}_1^m = \{\overline{0}, \overline{1456}\}, \qquad \mathcal{W}_2^m = \{\overline{02}\},$$

$$\mathcal{W}_3^m = \{\overline{07}, \overline{08}, \overline{378}\}, \qquad \mathcal{W}_4^m = \mathcal{W}_5^m = \mathcal{W}_6^m = \{\overline{1}\},$$

$$\mathcal{W}_7^m = \mathcal{W}_8^m = \{\overline{3}\}, \qquad \mathcal{W}_9^m = \{\overline{7}\}.$$

$$\begin{array}{ccccccccc}
 & & 4 & & & & & & \\
 & & \uparrow & & & & & & \\
5 & \leftarrow & 1 & \leftarrow & 0 & \rightarrow & 3 & \rightarrow & 8 \\
 & & \downarrow & & | & & \downarrow & & \\
 & & 6 & & 2 & & 7 & \rightarrow & 9
\end{array}$$

FIG. 1. An Example of Command Chains with $N = \overline{0123456789}$

Table I displays the command chains for a representative sample of coalitions in the organization. Note that adding $\overline{56}$ to $\overline{234}$ does not expand $\overline{234}$'s command scope nor shorten the steps to command $\overline{3789}$.

TABLE I.

Sample command chains in the organization of Figure 1.

S	$\overline{234}$	$\overline{1246}$	$\overline{012468}$	$\overline{23456}$	$\overline{123568}$	$\overline{0234578}$
$\delta_1(S)$	$\overline{78}$	$\overline{456}$	$\overline{0123456}$	$\overline{78}$	$\overline{045678}$	$\overline{012378}$
$\delta_2(S)$	$\overline{3789}$	$\overline{1456}$	$\overline{012345678}$	$\overline{3789}$	$\overline{0123456789}$	$\overline{0123456789}$
$\delta_3(S)$	$''$	$''$	$\overline{0123456789}$	$''$	$''$	$''$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\delta(S)$	$\overline{3789}$	$\overline{1456}$	N	$\overline{3789}$	N	N

The induction argument can also lead to Proposition 2.3 which claims that $\delta(S) \subseteq \delta(T)$ if $S \subseteq T$. For example, the citizens of the United States have larger command scope than that of the state of California, which in turn have larger command scope than that of the county of Los Angeles, and so on.

PROPOSITION 2.3. *1.* $\delta(\emptyset) = \emptyset$; *2.* $\delta(N) = N$; *3. the command scope function* $\delta : 2^N \to 2^N$ *is weakly increasing.*

Proof. The first two parts are trivial. To show the third part, for any $S \subseteq T$, $\delta_1(S) = \omega(S) \subseteq \omega(T) = \delta_1(T)$. If we assume that $\delta_k(S) \subseteq \delta_k(T)$ for some $k \geq 1$, then $\delta_{k+1}(S) = \omega(S \cup \delta_k(S)) \subseteq \omega(T \cup \delta_k(T)) = \delta_{k+1}(T)$. By induction, $\delta_t(S) \subseteq \delta_t(T)$ for all $t \geq 1$. We let $t \to \infty$ and complete the proof. ∎

PROPOSITION 2.4. *For any* $S \subseteq N$, $\delta(S) = \omega(S \cup \delta(S)) = \delta(S \cup \delta(S))$.

Proof. First

$$\delta(S) = \delta_{n+1}(S) = \omega(S \cup \delta_n(S)) = \omega(S \cup \delta(S)) = \delta_1(S \cup \delta(S)).$$

Secondly if we assume that $\delta(S) = \delta_k(S \cup \delta(S))$ for some $k \geq 1$, then

$$\begin{aligned} \delta_{k+1}(S \cup \delta(S)) &= \omega(S \cup \delta(S) \cup \delta_k(S \cup \delta(S))) \\ &= \omega(S \cup \delta(S) \cup \delta(S)) = \omega(S \cup \delta(S)) = \delta(S). \end{aligned}$$

By induction, $\delta_k(S \cup \delta(S)) = \delta(S)$ for all $k \geq 1$. We let $k \to \infty$ and complete the proof. ▮

COROLLARY 2.1.

(i) If $R \subseteq \delta(S)$, then $\delta(S \cup R) = \delta(S)$.

(ii) If $R \subseteq S$ and $S \setminus R \subseteq \delta(R)$, then $\delta(R) = \delta(S)$.

(iii) $\delta(\delta(S)) \subseteq \delta(S)$.

(iv) $\delta(\cdot)$ and $\omega(\cdot)$ have the same fixed points.

Proof. First if $R \subseteq \delta(S)$, then $\delta(S) \subseteq \delta(S \cup R) \subseteq \delta(S \cup \delta(S)) = \delta(S)$. Next if $R \subseteq S$ and $S \setminus R \subseteq \delta(R)$, then $\delta(R) = \delta(R \cup (S \setminus R)) = \delta(S)$. Third, $\delta(\delta(S)) \subseteq \delta(S \cup \delta(S)) = \delta(S)$. Finally to prove (iv), if $\omega(S) = S$, then $\delta_1(S) = S$ and so $\delta_2(S) = \omega(S \cup \delta_1(S)) = \omega(S) = S$. By induction, we conclude that $\delta_k(S) = S$ for any $k \geq 1$ and hence $\delta(S) = S$. On the other hand if $\delta(S) = S$, then $\omega(S) = \omega(S \cup \delta(S)) = \delta(S) = S$. ▮

Corollary 2.1(i) simply states that admitting any members from $\delta(S)$ to S does not expand its command scope. However, admitting members from $\delta(S)$ may help S command $\delta(S)$ in fewer number of steps. For example, it takes t^* steps for S to command $\delta(S)$; but it takes only one step for $S \cup \delta(S)$ to command $\delta(S)$ since $\omega(S \cup \delta(S)) = \delta(S)$. Corollary 2.1(ii) claims that if one part, R, of S commands the other part, $S \setminus R$, of S in finite steps, then S has the same command scope as part R. However, in general, $\delta(\delta(S)) \neq \delta(S)$. In the last example, $\delta(\overline{7}) = \overline{9}$, but $\delta(\delta(\overline{7})) = \delta(\overline{9}) = \emptyset$. Finally, N and $\emptyset$ are two trivial fixed points of $\delta(\cdot)$.

We define the function $\eta : 2^N \to 2^N$ such that

$$\eta(S) = \omega(S \cup \eta(S)), \quad \text{for } \forall S \subseteq N. \tag{1}$$

Note that the solution to Eq.(1) is not unique. The constant function $\eta(\cdot) = N$ is a trivial solution. For a non-trivial solution, consider

$$\begin{cases} \eta(\emptyset) &= \emptyset; \\ \eta(S) &= N, \quad \forall S \neq \emptyset. \end{cases}$$

PROPOSITION 2.5. *The command scope $\delta(\cdot)$ is the minimal solution to Eq.(1), i.e. $\delta(S) \subseteq \eta(S)$ for any $S \subseteq N$.*

Proof. By Proposition 2.4, $\delta(\cdot)$ is a solution to Eq.(1). Next, for any solution $\eta(\cdot)$ of Eq.(1), $\delta_0(S) = \emptyset \subseteq \eta(S)$ and $\delta_1(S) = \omega(S) \subseteq \omega(S \cup \eta(S)) = \eta(S)$. If we assume that $\delta_k(S) \subseteq \eta(S)$ for some $k \geq 1$, then $\delta_{k+1}(S) = \omega(S \cup \delta_k(S)) \subseteq \omega(S \cup \eta(S)) = \eta(S)$. By induction, $\delta_t(S) \subseteq \eta(S)$ for all $t \geq 0$.

Therefore $\delta(S) = \delta_n(S) \subseteq \eta(S)$. ∎

Observe that in Table I $\delta(\overline{012468}) = \delta(\overline{123568}) = \delta(\overline{0234578}) = N$. More generally, some policy, such as the national defense in a country or membership fee in a club, may involve the participation of all members in the organization. Otherwise, whoever does not obey the policy should be removed from the organization. So if a coalition has N as its command scope, then the coalition could act as the implementer of the policy. Formally we let $\mathcal{K}$ be the set of those coalitions which have the command scope N, i.e., $\mathcal{K} \stackrel{\text{def}}{=} \{S \subseteq N | \delta(S) = N\}$. Clearly, the set of free agents F is necessarily in $\mathcal{K}^{\cap}$.

PROPOSITION 2.6. *$\Gamma(N, \mathcal{K})$ is a well-defined simple game.*

Proof. By Proposition 2.3, $N \in \mathcal{K}$ and $\emptyset \notin \mathcal{K}$. Secondly if $S \in \mathcal{K}$ and $S \subseteq T$, then by the same proposition, $T \in \mathcal{K}$. This concludes the proof. ∎

In some sense, $\mathcal{K}$ describes the origins of the organizational power, no matter how many steps are needed to command the organization or all players in the organization. Therefore when the property right of the organization is concerned, the Shapley-Shubik power index for the game $\Gamma(N, \mathcal{K})$ specifies how much power one has on the entire organization. For example, an elected democratic government derives its just powers from the consent of the governed (all players), no matter how many elections are needed to institute the government. In a direct democracy, the number of command steps is just 1. In a slavery, however, a slave should always follow his lord (as well as his lord's wife or designated agents, etc); his lord can decide what the slave should do and what the slave should not do. So a slave is a dummy in his own command game. Without any further reference to the degree of freedom, we call an organization *democratic* if every member $i \in N$ is a veto player in his own decision-making game $\Gamma(N, \mathcal{W}_i)$. That is, $i \in \mathcal{W}_i^{\cap}$ or $\mathcal{B}_i = \phi$ for all $i \in N$ in a democratic organization N. The organization of Figure 1 is not democratic: players 7 and 8 are player 3's slaves; player 9 is player 7's slave.

PROPOSITION 2.7. (EGALITARIAN) *If N is democratic, then $\mathcal{K} = \{N\}$; and so any player's Shapley-Shubik power index is just $\frac{1}{n}$ in $\Gamma(N, \mathcal{K})$.*

Proof. (by contradiction) Assume that $S \in \mathcal{K}$ and $S \neq N$.

Pick any $i \notin S$, then $i \notin \omega(S) = \delta_1(S)$ by $i \in \mathcal{W}_i^{\cap}$. If $i \notin \delta_k(S)$ for some $k \geq 1$, then $i \notin \omega(S \cup \delta_k(S)) = \delta_{k+1}(S)$ by the same reason. By induction, $i \notin \delta_k(S)$ for all $k \geq 1$. Thus $i \notin \delta(S)$. This contradicts the assumption $S \in \mathcal{K}$. ▮

The condition $i \in \mathcal{W}_i^{\cap}$ just says that any other player or group of players can not violate i's own rights in his command game. These inalienable rights may be life, property, speech, or election of governance. When an election is concerned, his command game $\Gamma(N, \mathcal{W}_i)$ determines his vote (either YES or NO). If the organization is democratic, then the players of N_i can not decide his vote ($N_i \notin \mathcal{W}_i$). For any possible action imposed by N_i, i has the veto power. However, his choice may be subject to some limitations and approvals by other members. A democratic government therefore obtains governing power not only from the supporters but also from all the governed. In this sense, the government is based on the consent of all people. As a consequence, any re-organization should reflect the origin of powers. In 1960s, the U.S. Supreme Court set the one-person-one-vote principle as one of the standards of constitutional fairness in elections at the state and local levels.

However, we should say that the authority distribution measures the administrative power. When running the governance, an officer, say the president, generally has more administrative power than a general voter. We should also mention that the participation of all players does not necessarily imply the grand agreement by all players. Instead, as the players have diversified interests, it is unlikely to reach a grand coalition. To set a policy and implement the policy over all members, it generally requires some basic rules (or laws), such as simple majority rule, for the players to make collective decisions or policies in the organization. When a policy is made, any player should obey the collective decision regardless of what vote he has made in making the policy. In the corporate controls, these basic rules are generally made by the owners, shareholders or the Board of Directors who are the representatives of the owners.

3. TOTALLY-CONTROLLED COALITIONS

We recognize that without necessary subordinates and army, even a dictator can not fully control his organization nor implement his full control. By the same argument, a Chief Executive Office is just a puppet if he has no necessary resources of secretaries, computers, and money. We shall value the presence of those "subordinates" in this section. As we know, for a coalition to have the dictatorial power on an player, any disjoint coalition is necessarily dummies in the player's command game. Again, the dictatorial power can also be passed.

PROPOSITION 3.1. *Given the power transition matrix P in the organization, we define $\sigma_t(S)$ for any $S \subseteq N$ by*

$$\begin{cases} \sigma_0(S) & \stackrel{def}{=} \emptyset, \\ \sigma_{t+1}(S) & \stackrel{def}{=} \{i | \sum_{j \in S \cup \sigma_t(S)} P(i,j) = 1\}, \ \forall\ t = 0, 1, 2, \cdots. \end{cases}$$

Then the sequence $\{\sigma_t(S)\}$ is increasing strictly up to some $t^* \leq n$ and is constant thereafter.

Proof. If $i \in \sigma_1(S)$, $\sum_{j \in S} P(i,j) = 1$. Then $\sum_{j \in S \cup \sigma_1(S)} P(i,j) = 1$. Therefore $i \in \sigma_2(S)$ and $\sigma_1(S) \subseteq \sigma_2(S)$. Next if we assume that $\sigma_{t-1}(S) \subseteq \sigma_t(S)$ for some $t \geq 2$, then for $\forall i \in \sigma_t(S)$, we have $\sum_{j \in S \cup \sigma_{t-1}(S)} P(i,j) = 1$. This implies by $\sigma_{t-1}(S) \subseteq \sigma_t(S)$ that $\sum_{j \in S \cup \sigma_t(S)} P(i,j) = 1$. Therefore $i \in \sigma_{t+1}(S)$ and $\sigma_t(S) \subseteq \sigma_{t+1}(S)$. By induction, $\sigma_t(S)$ is weakly increasing in t. Next if $\sigma_{t+1}(S) = \sigma_t(S)$, then

$$\begin{aligned}\sigma_{t+2}(S) &= \{i| \textstyle\sum_{j\in S\cup\sigma_{t+1}(S)} P(i,j) = 1\} \\ &= \{i| \textstyle\sum_{j\in S\cup\sigma_t(S)} P(i,j) = 1\} = \sigma_{t+1}(S) = \sigma_t(S).\end{aligned}$$

So $\sigma_k(S)$ is constant for all $k \geq t$ by induction. The existence of t^* and $t^* \leq n$ can be shown by the same argument in the proof of Proposition 2.2. ▮

If there exist command games for all members, there is an alternative way to define the sequence $\sigma_t(S)$

$$\begin{cases} \sigma_0(S) & \stackrel{\text{def}}{=} \emptyset, \\ \sigma_{t+1}(S) & \stackrel{\text{def}}{=} \{i|S \cup \sigma_t(S) \supseteq \mathcal{W}_i^{m\cup}\},\ t = 0, 1, 2, \cdots. \end{cases}$$

By this, $S \cup \sigma_t(S)$ has the dictatorial power on $\sigma_{t+1}(S)$. Now we define the *totally-controlled coalition* of S by $\sigma(S) \stackrel{\text{def}}{=} \lim_{t\to\infty} \sigma_t(S)$ or $\sigma(S) \stackrel{\text{def}}{=} \sigma_n(S)$ in the sense that S has the *complete power* over $\sigma(S)$, regardless of how many steps S may take to reach every member in $\sigma(S)$. For example, a senator may have the complete power on his secretaries who, together with the senator, may have the complete power over others. In the organization of Figure 1, $\sigma(\overline{3}) = \overline{789}$ and $\sigma(\overline{1}) = \overline{456}$; but $\sigma(\overline{012468}) = \overline{12456} \neq N = \delta(\overline{012468})$. Clearly $F \cap S \subseteq \sigma(S)$.

PROPOSITION 3.2. *$\sigma : 2^N \to 2^N$ is weakly increasing.*

Proof. For any $S \subseteq T$, $\sum_{j \in S} P(i,j) = 1$ implies $\sum_{j \in T} P(i,j) = 1$. Therefore $\sigma_1(S) \subseteq \sigma_1(T)$. If we assume that $\sigma_k(S) \subseteq \sigma_k(T)$ for some $k \geq 1$, then $\sum_{j \in S \cup \sigma_k(S)} P(i,j) = 1$ implies $\sum_{j \in T \cup \sigma_k(T)} P(i,j) = 1$ and thus $\sigma_{k+1}(S) \subseteq \sigma_{k+1}(T)$. By induction, $\sigma_t(S) \subseteq \sigma_t(T)$ for all $t \geq 0$. Letting $t \to \infty$ concludes the proof. ∎

While S can command $\delta(S)$ in some finite steps, S has the complete power over $\sigma(S)$. To help understand the concept, we let $N = \overline{123}$ with $\mathcal{W}_1^m = \mathcal{W}_2^m = \{\overline{1}\}$ and $\mathcal{W}_3^m = \{\overline{1}, \overline{2}\}$. In this example, both 2 and 3 are 1's slaves. But 2 can be regarded as 1's agent to manage 3 or run the slavery over 3. Although 1 and 2 have the same command power over 3, their roles are different. When they have conflicting commands over 3, 1 has the residual power since he can dictates 2. Therefore 2's command power comes from his lord 1. In the game $\Gamma(\overline{123}, \{S | 3 \in \sigma(S)\})$, player 1 is the dictator and player 3 is his private property.

Proposition 3.1 provides a way to figure out $\sigma(S)$ from the power transition matrix P: 1) we let $\sigma = \emptyset$ and the $n \times 1$ vector V be the sum of all j^{th} columns for $j \in S$; 2) if there exists $i \notin \sigma$ but $V_i = 1$, then (a) add i to σ and (b) add the i^{th} column of P to V if $i \notin S$; 3) repeat the last step until $V_i < 1$ for all $i \in N \setminus \sigma$. All these steps can be tabulated or programmed. We list the programming pseudo code in the following algorithm.

```
[ Algorithm : σ(S) ]
σ =: ∅;
for ∀i ∈ N do
    V[i] =: 0;
    for ∀j ∈ S do
        V[i] =: V[i] + P(i, j);
    end for
end for
while (∃ j ∉ σ but V[j] = 1) do
    σ := σ + j̄;
    if j ∉ S then
        for ∀i ∈ N do
            V[i] =: V[i] + P(i, j);
        end for
    end if
end while
return σ
```

To illustrate the tabulation method, we assume that $N = \overline{12345678}$, $S = \overline{156}$, and the power transition matrix P is tabulated in Table II. We process our method (in the order of $\Delta_1, \sigma_1, \Delta_2, \sigma_2, \cdots$) until $\Delta_k \subseteq S \cup \sigma_{k-1}$.

Since $S = \overline{156}$, we sum the columns for members 1, 5 and 6 in P to obtain the column $\sum_{i \in S}$. Now in the column $\sum_{i \in S}$, member 2 has value equal to 1; so we add member 2 to $\Delta_1 = \overline{2} \notin S$ and obtain $\sigma_1 = \sigma_0 \cup \Delta_1 = \overline{2}$. Next we sum the columns of member 2 and $\sum_{i \in S}$ to obtain the column of $\sum_{i \in S \cup \sigma_1}$. From this column, we observe that members 5 and 7 have the value 1 and so we have $\Delta_2 = \overline{57} \notin S \cup \sigma_1$. Now $\sigma_2 = \sigma_1 \cup \Delta_2 = \overline{257}$. Next we sum the column $\sum_{i \in S \cup \sigma_1}$ with the column of member 7 ($5 \in S$ and its column is already summed in $\sum_{i \in S}$) to obtain the column $\sum_{i \in S \cup \sigma_2}$. In the column, member 1 has the value 1 but $1 \in S \cup \sigma_2$. Therefore we stop here with $\sigma(S) = \sigma_3 = \sigma_2 \cup \Delta_3 = \overline{1257}$.

PROPOSITION 3.3. *$\sigma(S \cup \sigma(S)) = \sigma(S)$.*

Proof. By the monotonicity of $\sigma(\cdot)$, $\sigma(S) \subseteq \sigma(S \cup \sigma(S))$. It suffices to show the other part. First for $\forall i \in \sigma_1(S \cup \sigma(S))$, $\sum_{j \in S \cup \sigma(S)} P(i,j) = 1$ implies $\sum_{j \in S \cup \sigma_n(S)} P(i,j) = 1$. Therefore $i \in \sigma_{n+1}(S) = \sigma(S)$ and it follows that $\sigma_1(S \cup \sigma(S)) \subseteq \sigma(S)$. Next if we assume that $\sigma_k(S \cup \sigma(S)) \subseteq \sigma(S)$ for some $k \geq 1$, then for $\forall i \in \sigma_{k+1}(S \cup \sigma(S))$, $\sum_{j \in S \cup \sigma(S) \cup \sigma_k(S \cup \sigma(S))} P(i,j) = 1$ implies $\sum_{j \in S \cup \sigma(S)} P(i,j) = 1$, so $\sum_{j \in S \cup \sigma_n(S)} P(i,j) = 1$ and $i \in \sigma_{n+1}(S) = \sigma(S)$. This again shows that $\sigma_{k+1}(S \cup \sigma(S)) \subseteq \sigma(S)$. By induction, $\sigma_t(S \cup \sigma(S)) \subseteq \sigma(S)$ for any $t \geq 1$. Letting $t \to \infty$ proves $\sigma(S \cup \sigma(S)) \subseteq \sigma(S)$. ▮

TABLE II.

Tabulation Approach for $\sigma(S)$ with $S = \overline{156}$

	1	2	3	4	5	6	7	8	$\sum_{i \in S}$	$\sum_{i \in S \cup \sigma_1}$	$\sum_{i \in S \cup \sigma_2}$
1	.1	.3	0	0	.3	0	.3	0	.4	.7	1
2	.5	0	0	0	.3	.2	0	0	1	–	–
3	0	.1	.1	.5	0	0	.1	.2	0	.1	.2
4	0	0	.1	.2	0	0	0	.7	0	0	0
5	.1	.4	0	0	.2	.3	0	0	.6	1	–
6	.1	.1	.2	.2	0	.1	.3	0	.2	.3	.6
7	0	.1	0	0	.9	0	0	0	.9	1	–
8	0	0	0	0	.1	0	.1	.8	.1	.1	.2
Δ_k									$\overline{2}$	$\overline{57}$	$\overline{1}$
σ_k									$\overline{2}$	$\overline{257}$	$\overline{1257}$

However $\sigma(S_1)$ and $\sigma(S_2)$ are not necessarily exclusive given $S_1 \cap S_2 = \emptyset$. For example, consider the hierarchic structure: $\mathcal{W}_0^{m\cup} = \mathcal{W}_1^{m\cup} = \mathcal{W}_2^{m\cup} = \overline{0}$, $\mathcal{W}_3^{m\cup} = \mathcal{W}_4^{m\cup} = \mathcal{W}_5^{m\cup} = \overline{12}$, $\mathcal{W}_6^{m\cup} = \mathcal{W}_7^{m\cup} = \mathcal{W}_8^{m\cup} = \mathcal{W}_9^{m\cup} = \overline{345}$. The disjoint coalitions $\overline{0}$, $\overline{12}$ and $\overline{345}$ all have the complete power on $\overline{6789}$. In another organization $N = \overline{123}$ with $\mathcal{W}_1^m = \{\overline{2}\}$, $\mathcal{W}_2^m = \{\overline{3}\}$ and $\mathcal{W}_3^m = \{\overline{1}\}$, $\sigma(S) = N$ for $\forall S \neq \emptyset$. In this quite unrealistic example, each member is a cog, but none is a yesman. In a concrete organization, money is generally a cog. However it is essential in all command games except itself's. In general, physical assets, financial assets and slaves are also cogs.

PROPOSITION 3.4. *If $i \notin S$ but $i \in \sigma(S)$, then i is necessarily a cog.*

Proof. There exists a $t \geq 1$ such that $i \in \sigma_t(S)$ but $i \notin \sigma_{t-1}(S)$. Now $\sum_{j \in S \cup \sigma_{t-1}(S)} P(i,j) = 1$ implies $P(i,i) = 0$. Therefore i is cog. ▮

COROLLARY 3.1. *If S and T are two disjoint coalitions and $i \in \sigma(S) \cap \sigma(T)$, then i is a cog.*

From the above analysis, $i \in \sigma(S)$ if and only if $\mathcal{W}_i^{m\cup} \subseteq S \cup \sigma(S)$. We consider the simple game $\Gamma(N, \{Z | i \in \sigma(Z)\})$. If S and T are two disjoint minimal winning coalitions in the game, we conjecture that either $S \subseteq \sigma(T)$ or $T \subseteq \sigma(S)$. To be more focused on the current topics, we shall not go to the details about the relations of S, $\sigma(S)$, T and $\sigma(T)$. Instead, we recall that (Hu and Shapley, 2001) a coalition S is called *closed* if player j is a dummy in player i's command game for any $i \in S$ and $j \in N \setminus S$. Therefore $P(i,j) = 0$. It can be further shown that $\sum_{j \in N \setminus S} P^k(i,j) = 0$ for any $i \in S$ and any $k \geq 1$. Closed coalitions are generalization to the irreducible classes and subsets of F. As we see, the union or intersection of any two closed coalitions is also closed.

Proposition 3.5. *(i) If* $\sum_{j\in S} P^t(i,j) = 1$, *then* $i \in \sigma_t(S)$; *(ii) if* S *is closed, then* $i \in \sigma_t(S)$ *implies* $\sum_{j\in S} P^k(i,j) = 1$ *for any* $k \geq t$.

Proof. (By induction on t)

(i) It is trivial when $t = 1$. If we assume that it is true for some $t \geq 1$, i.e., $\sum_{j\in S} P^t(i,j) = 1$ implies $i \in \sigma_t(S)$, then given $1 = \sum_{j\in S} P^{t+1}(i,j)$,

$$\begin{aligned} 1 &= \sum_{j\in S}\sum_{z\in N} P(i,z)P^t(z,j) \\ &= \sum_{z\in N} P(i,z)\sum_{j\in S} P^t(z,j) \leq \sum_{z\in N} P(i,z)1 = 1 \end{aligned} \tag{2}$$

where "=" holds in Eq.(2) if and only if $\sum_{j\in S} P^t(z,j) = 1$ for all $z \in \{z|P(i,z) > 0\}$. That is, for $\forall z \in \{z|P(i,z) > 0\}$, $z \in \sigma_t(S)$ by our induction assumption. Therefore $\sum_{j\in\sigma_t(S)} P(i,j) = 1$ and so we have that $\sum_{j\in S\cup\sigma_t(S)} P(i,j) = 1$ and $i \in \sigma_{t+1}(S)$. The proof is complete by induction.

(ii) If $\sum_{j\in S} P^t(i,j) = 1$, then for any $k \geq t$,

$$\begin{aligned} \sum_{j\in S} P^k(i,j) &= \sum_{z\in N}\sum_{j\in S} P^t(i,z)P^{k-t}(z,j) \\ &= \sum_{z\in S} P^t(i,z)\sum_{j\in S} P^{k-t}(z,j) = \sum_{z\in S} P^t(i,z)1 = 1. \end{aligned}$$

That is, $\sum_{j\in S} P^t(i,j) = 1$ implies $\sum_{j\in S} P^k(i,j) = 1$ for all $k \geq t$. Therefore, it suffices to show that $i \in \sigma_t(S)$ implies $\sum_{j\in S} P^t(i,j) = 1$.

It is trivial when $t = 1$. If we assume this is true for some $t \geq 1$ (that is, $P^t(z,j) = 0, \forall z \in \sigma_t(S), j \notin S$), then for any $i \in \sigma_{t+1}(S)$,

$\sum_{j\in S\cup\sigma_t(S)} P(i,j) = 1$ and so $P(i,j) = 0$ for any $j \notin S \cup \sigma_t(S)$. Now,

$$\begin{aligned} 1 &= \sum_{j\in N} P^{t+1}(i,j) = \sum_{j\in S} P^{t+1}(i,j) + \sum_{j\notin S} P^{t+1}(i,j) \\ &= \sum_{j\in S} P^{t+1}(i,j) + \sum_{j\notin S} \sum_{z\in N} P(i,z)P^t(z,j) \\ &= \sum_{j\in S} P^{t+1}(i,j) + \sum_{j\notin S} \sum_{z\in S\cup\sigma_t(S)} P(i,z)P^t(z,j) \\ &= \sum_{j\in S} P^{t+1}(i,j) + \sum_{j\notin S} \sum_{z\in\sigma_t(S)\setminus S} P(i,z)P^t(z,j) \\ &= \sum_{j\in S} P^{t+1}(i,j) + \sum_{j\notin S} \sum_{z\in\sigma_t(S)\setminus S} P(i,z)0 = \sum_{j\in S} P^{t+1}(i,j). \end{aligned}$$

By induction, $i \in \sigma_t(S)$ implies $\sum_{j\in S} P^t(i,j) = 1$ for all $t \geq 1$. ▮

However, part (ii) is not valid if S is not closed. For example, consider the organization with $N = \overline{123}$, $\mathcal{W}_1^m = \{\overline{2}\}$, $\mathcal{W}_2^m = \{\overline{3}\}$ and $\mathcal{W}_3^m = \{\overline{1}\}$. In this organization, $1 \in \sigma_3(\overline{2})$ but $P^t(1,2) \neq 1$ for all $t \geq 3$.

Corollary 3.2. *If S is closed, $\sigma(S) = \{i | \sum_{j\in S} P^n(i,j) = 1\}$.*

In general, the coalition $\sigma(S)$ is essentially the "private holding" of S in the organization. In this sense, S can implement his task with the help of $\sigma(S)$ if ever needed.

Proposition 3.6. *For any $S \subseteq N$, $\sigma(S) \subseteq \delta(S)$.*

Proof. We first note that $\sum_{j\in S} P(i,j) = 1$ necessarily implies $S \in \mathcal{W}_i$. Thus, $\sigma_1(S) \subseteq \delta_1(S)$. If we assume that $\sigma_k(S) \subseteq \delta_k(S)$ for some $k \geq 1$, then for any $i \in \sigma_{k+1}(S)$, $\sum_{j\in S\cup\sigma_k(S)} P(i,j) = 1$ implies $S \cup \sigma_k(S) \in$

$\mathcal{W}_i$. Therefore $i \in \omega(S \cup \sigma_k(S)) \subseteq \omega(S \cup \delta_k(S)) = \delta_{k+1}(S)$. By induction, $\sigma_t(S) \subseteq \delta_t(S)$ for any $t \geq 1$. Letting $t \to \infty$ concludes the proof. ▌

4. CONTROL GAMES

For any $i \in N$, he must obey the orders of any boss set in $\mathcal{B}_i$. If he is not a free agent nor he receives any order from a boss set, then his proposed action needs some authorization from an approval set in $\mathcal{A}_i$. However, if i is a free agent, then no coalition can boss him and his action needs no approval from others. A free agent can decide, by himself, what he wants to do, how he would do and when he will finish his job, with no objection from other members in his organization. We can imagine that there exists no free agent in the vast majority of organizations. For example, simply consider a public company. A shareholder is not a free agent since the market value and profit, among others, are essential players in his command game. The management is not a free agent since the set of shareholders is his boss.

An alternative way of speaking about $\mathcal{B}_i$ is the *boss function.* For any $S \subseteq N$, we define its boss function as

$$\beta(S) \stackrel{\text{def}}{=} \{i \in N : S \setminus \bar{i} \in \mathcal{B}_i\}. \tag{3}$$

In words, $\beta(S)$ is the set of all players who must obey any order issued by S. In general $S \cap \beta(S) \neq \emptyset$. For example, if $S \cap \beta(S) = \emptyset$ and $\beta(S) \neq \emptyset$, we take $S' = S \cup \bar{i}$ for some $i \in \beta(S)$, then $i \in S' \cap \beta(S')$. Conversely, given β, we can recover the boss sets from $\mathcal{B}_i = \{S \subseteq N_i : i \in \beta(S)\}$. Thus $\beta(\cdot) : \mathcal{N} \to \mathcal{N}$ is in a certain sense a kind of inverse to the "function" $\mathcal{B}_{(\cdot)} : N \to 2^{\mathcal{N}}$ and conversely. Furthermore, we note that $\beta(\cdot)$ is weakly increasing. For convenience, we let $\mathcal{Z}_i = \mathcal{A}_i \cup \mathcal{B}_i$ and associate another function $\alpha(S)$ by

$$\alpha(S) \stackrel{\text{def}}{=} \{i \in N : S \setminus \bar{i} \in \mathcal{Z}_i\}.$$

Clearly $\beta(S) \subseteq \alpha(S)$ for $\forall S \subseteq N$. In words, $\alpha(S) \setminus \beta(S)$ is the set of all players whose proposed action can be authorized by S. Given $\alpha(\cdot)$ and $\mathcal{B}_i$, we can recover $\mathcal{A}_i = \{S \subseteq N_i : i \in \alpha(S)\} \setminus \mathcal{B}_i$. And similarly, $\alpha(\cdot)$ is a weakly increasing function by the fact (Hu and Shapley, 2001) that $\mathcal{Z}_i = \mathcal{Z}_i^+ \cap \mathcal{N}_i$.

PROPOSITION 4.1. *For any $S \subseteq N$, $\beta(S) \subseteq \omega(S) \subseteq \alpha(S)$.*

Proof. This is from

$$\begin{aligned}
\beta(S) &= \{i|S \setminus \bar{i} \in \mathcal{B}_i\} = \{i|S \in \mathcal{B}_i \text{ or } S \in \{\bar{i} \cup T|T \in \mathcal{B}_i\}\ \} \\
&\subseteq \{i|S \in \mathcal{B}_i \text{ or } S \in \{\bar{i} \cup T|T \in \mathcal{B}_i \cup \mathcal{A}_i\}\ \} \\
&\subseteq \{i|S \in \mathcal{B}_i \cup \mathcal{A}_i \text{ or } S \in \{\bar{i} \cup T|T \in \mathcal{B}_i \cup \mathcal{A}_i\}\ \} \\
&= \{i|S \setminus \bar{i} \in \mathcal{B}_i \cup \mathcal{A}_i\} = \alpha(S)
\end{aligned}$$

and $\{i|S \in \mathcal{B}_i \text{ or } S \in \{\bar{i} \cup T|T \in \mathcal{B}_i \cup \mathcal{A}_i\}\} = \{i|S \in \mathcal{W}_i\} = \omega(S)$. ∎

COROLLARY 4.1. *For any* $S \subseteq N$, $\omega(S) = \beta(S) \cup [S \cap \alpha(S)]$.

Proof. By the above proof,

$$\begin{aligned}
\omega(S) &= \{i|S \in \mathcal{B}_i \text{ or } S \in \{\bar{i} \cup T|T \in \mathcal{B}_i \cup \mathcal{A}_i\}\} \\
&= \{i|S \setminus \bar{i} \in \mathcal{B}_i \text{ or } S \in \{\bar{i} \cup T|T \in \mathcal{B}_i \cup \mathcal{A}_i\}\} \\
&= \{i|S \setminus \bar{i} \in \mathcal{B}_i\} \cup \{i|S \in \{\bar{i} \cup T|T \in \mathcal{B}_i \cup \mathcal{A}_i\}\} \\
&= \beta(S) \cup \{i|i \in S \text{ and } S \setminus \bar{i} \in \mathcal{B}_i \cup \mathcal{A}_i\} \\
&= \beta(S) \cup [\{i|i \in S\} \cap \{i|S \setminus \bar{i} \in \mathcal{B}_i \cup \mathcal{A}_i\}] = \beta(S) \cup [S \cap \alpha(S)].
\end{aligned}$$

∎

As S has the complete power over the totally-controlled coalition $\sigma(S)$, it can use $\sigma(S)$ to implement its indirect commands. In the following, we will study the command scope $\delta(S)$ from the view point of implementation.

PROPOSITION 4.2. *For* $\forall S \subseteq N$ *and* $t = 1, 2, \cdots$, *we let*

$$\begin{cases}
\gamma_0(S) \overset{def}{=} \sigma(S), \\
\gamma_t(S) \overset{def}{=} \sigma(S) \cup \beta(S \cup \gamma_{t-1}(S)) \cup [S \cap \alpha(S \cup \gamma_{t-1}(S))].
\end{cases}$$

Then there is a nonnegative $t^* \leq n-1$ *such that the control sequence increases strictly up to the term* $\gamma_{t^*}(S)$ *and is constant thereafter.*

The proposition can be proven by induction and the monotonicity of $\beta(\cdot)$ and $\alpha(\cdot)$. The *control function* $\gamma(S)$ is defined by $\gamma(S) \overset{\text{def}}{=} \lim_{t\to\infty} \gamma_t(S)$. Clearly $\gamma(N) = N$, $\gamma(\emptyset) = \emptyset$. $\gamma(S) = \emptyset$ if and only if $\sigma(S) = \emptyset$. It is also easy to see that $\gamma(S) = \sigma(S)$ if $\alpha(S \cup \sigma(S)) = \emptyset$. Given $\gamma_{k-1}(S)$ in the control chain, on one hand, $\gamma_{k-1}(S)$ with S can boss some new members of $\beta(S \cup \gamma_{k-1}(S)) \setminus \gamma_{k-1}(S)$. On the other hand, some members in S may not have full control on their own action and obtain full authorization from $S \cup \gamma_{k-1}(S)$. $\gamma_k(S)$ admits these new members.

PROPOSITION 4.3. *For any $S \subseteq N$, $\gamma(S) = \delta(S)$.*

Proof. By Proposition 3.6, $\gamma_0(S) = \sigma(S) \subseteq \delta(S)$. If we assume that $\gamma_k(S) \subseteq \delta(S)$ for some $k \geq 0$, then by Corollary 4.1 and Proposition 2.4,

$$\begin{aligned}
\gamma_{k+1}(S) &= \sigma(S) \cup \beta(S \cup \gamma_k(S)) \cup (S \cap \alpha(S \cup \gamma_k(S))) \\
&\subseteq \delta(S) \cup \beta(S \cup \delta(S)) \cup ((S \cup \delta(S)) \cap \alpha(S \cup \delta(S))) \\
&= \delta(S) \cup \omega(S \cup \delta(S)) = \delta(S) \cup \delta(S) = \delta(S).
\end{aligned}$$

By induction, $\gamma_k(S) \subseteq \delta(S)$ for all $k \geq 0$. We let $k \to \infty$ and obtain that $\gamma(S) \subseteq \delta(S)$.

To see the second part, note first that $\delta_0(S) = \emptyset \subseteq \gamma_0(S)$. If we assume that $\delta_k(S) \subseteq \gamma_k(S)$ for some $k \geq 0$, then by Corollary 4.1,

$$\begin{aligned}
\delta_{k+1}(S) &= \omega(S \cup \delta_k(S)) = \beta(S \cup \delta_k(S)) \cup [(S \cup \delta_k(S)) \cap \alpha(S \cup \delta_k(S))] \\
&\subseteq \beta(S \cup \gamma_k(S)) \cup [(S \cup \gamma_k(S)) \cap \alpha(S \cup \gamma_k(S))] \\
&\subseteq \beta(S \cup \gamma_k(S)) \cup [S \cap \alpha(S \cup \gamma_k(S))] \cup \gamma_k(S) \\
&\subseteq \sigma(S) \cup \beta(S \cup \gamma_k(S)) \cup [S \cap \alpha(S \cup \gamma_k(S))] \cup \gamma_k(S) \\
&= \gamma_{k+1}(S) \cup \gamma_k(S) = \gamma_{k+1}(S).
\end{aligned}$$

By induction, $\delta_t(S) \subseteq \gamma_t(S)$ for any $t \geq 0$. Letting $t \to \infty$ concludes that

$\delta(S) \subseteq \gamma(S)$. ▮

The fact $\delta(S) = \gamma(S)$ and $\delta_t(S) \subseteq \gamma_t(S)$ for any $t \geq 0$ in the above proof has proven Corollary 4.2.

COROLLARY 4.2. *For any $S \subseteq N$, its control sequence reaches $\gamma(S)$ in fewer or equal number of steps than its command chain reaches $\delta(S)$.*

In contrast to Table I, Table III displays the control sequences for the same representative sample of coalitions in the organization of Figure 1, It is interesting that in the first, second and fourth coalitions, we have neither $S \subseteq \gamma(S)$ nor $\gamma(S) \subseteq S$.

For each $i \in N$, we define the set of *controlling coalitions* for i by

$$\mathcal{C}_i \stackrel{\text{def}}{=} \{S \subseteq N : i \in \gamma(S)\}.$$

Clearly $\Gamma(N, \mathcal{C}_i)$ is a simple game for all $i \in N$. Accordingly, we define the *control game* for each $i \in N$ to be $H_i = \Gamma(N, \mathcal{C}_i)$. The symbol "$H$" will denote the ensemble of the control games: $H = \{H_i : i \in N\}$. Obviously a yesman is necessarily a dummy in all control games. However a cog is not necessarily a dummy in his control game.

PROPOSITION 4.4. *$\mathcal{K}$ is the intersection of $\mathcal{C}_i$'s, $\mathcal{K} = \cap_{i=1}^n \mathcal{C}_i$.*

TABLE III.

Sample control sequences in the organization of Figure 1.

S	$\overline{234}$	$\overline{1246}$	$\overline{012468}$	$\overline{23456}$	$\overline{123568}$	$\overline{0234578}$
$\gamma_0(S)$	$\overline{789}$	$\overline{456}$	$\overline{12456}$	$\overline{789}$	$\overline{456789}$	$\overline{23789}$
$\gamma_1(S)$	$\overline{3789}$	$\overline{1456}$	$\overline{0123456}$	$\overline{3789}$	$\overline{013456789}$	$\overline{0123789}$
$\gamma_2(S)$	"	"	$\overline{012345678}$	"	$\overline{0123456789}$	$\overline{0123456789}$
$\gamma_3(S)$	"	"	$\overline{0123456789}$	"	"	"
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$\gamma(S)$	$\overline{3789}$	$\overline{1456}$	N	$\overline{3789}$	N	N

We list the minimal controlling coalitions for the organization in Figure 1 and deduce them later:

$$\mathcal{C}_0^m = \{\overline{02}, \overline{03}, \overline{07}, \overline{08}, \overline{123}\}, \quad \mathcal{C}_1^m = \{\overline{0}, \overline{1}\}, \quad \mathcal{C}_2^m = \{\overline{02}, \overline{123}\},$$

$$\mathcal{C}_3^m = \{\overline{07}, \overline{08}, \overline{3}\}, \quad \mathcal{C}_4^m = \{\overline{0}, \overline{1}\}, \quad \mathcal{C}_5^m = \{\overline{0}, \overline{1}\},$$

$$\mathcal{C}_6^m = \{\overline{0}, \overline{1}\}, \quad \mathcal{C}_7^m = \{\overline{07}, \overline{08}, \overline{3}\}, \quad \mathcal{C}_8^m = \{\overline{07}, \overline{08}, \overline{3}\},$$

$$\mathcal{C}_9^m = \{\overline{08}, \overline{3}, \overline{7}\}.$$

Note that members 4, 5, 6 and 9 are dummies in all control games. It is interesting that $\overline{456}$ are essential players in 1's command game.

As an application to the above control games, we consider a task τ which has the implementers of $\overline{689}$. To control the task, we then have to control the three implementers and so we may have a task control game defined by the product of H_6, H_8 and H_9. The task control game has the winning coalitions: $\mathcal{C}_6 \cap \mathcal{C}_8 \cap \mathcal{C}_9 = \{\overline{03}, \overline{07}, \overline{08}, \overline{13}\}^+$.

To determine a simple game, it suffices to specify its minimal winning coalitions. We here illustrate an algorithm to deduce the minimal controlling coalitions $\mathcal{C}_i^m$'s from the minimal command coalitions $\mathcal{W}_i^m$'s by the example. First we note that

$$\delta(\overline{0}) = \overline{1456},\ \delta(\overline{1}) = \overline{456},\ \delta(\overline{3}) = \overline{789},\ \delta(\overline{7}) = \overline{9},$$
$$\delta(\overline{2}) = \delta(\overline{4}) = \delta(\overline{5}) = \delta(\overline{6}) = \delta(\overline{8}) = \delta(\overline{9}) = \emptyset.$$

In light of Corollary 2.1, we may first simplify the minimal coalitions in command games such that no part of a coalition controls the other part of the same coalition. Although the step is unnecessary, it shortens the steps for the sequences to convergence. Therefore, $\overline{1456}$ in $\mathcal{W}_1^m$ can be simplified to $\overline{1}$ since $\overline{1}$ controls $\overline{456}$. $\overline{378}$ in $\mathcal{W}_3^m$ can be simplified to $\overline{3}$ since $\overline{78} \subseteq \delta(\overline{3})$. We should maintain the minimality and remove any proper supersets. Denote the simplified results by $\mathcal{C}_i^m(1)$'s,

$$\mathcal{C}_0^m(1) = \{\overline{02}, \overline{03}, \overline{123}\},\ \mathcal{C}_1^m(1) = \{\overline{0}, \overline{1}\},\qquad \mathcal{C}_2^m(1) = \{\overline{02}\},$$
$$\mathcal{C}_3^m(1) = \{\overline{07}, \overline{08}, \overline{3}\},\quad \mathcal{C}_4^m(1) = \mathcal{C}_5^m(1) = \mathcal{C}_6^m(1) = \{\overline{1}\},$$
$$\mathcal{C}_7^m(1) = \mathcal{C}_8^m(1) = \{\overline{3}\},\ \mathcal{C}_9^m(1) = \{\overline{7}\}.$$

Generally we denote the results after t steps by $\mathcal{C}_i^m(t)$'s. Given $\mathcal{C}_i^m(t)$'s, a coalition can control i by commanding a coalition in $\mathcal{C}_i^m(t)$. Therefore we add to $\mathcal{C}_i^m(t+1)$ those coalitions (after simplified) which command the coalitions of $\mathcal{C}_i^m(t)$. For example, $\mathcal{C}_0^m(2)$ contains the minimal coalitions of

$$\mathcal{C}_0^m(1) \cup \{\overline{0} \times \mathcal{C}_2^m(1)\} \cup \{\overline{0} \times \mathcal{C}_3^m(1)\} \cup \{\mathcal{C}_1^m(1) \times \mathcal{C}_2^m(1) \times \mathcal{C}_3^m(1)\},$$

and $\mathcal{C}_1^m(2)$ contains the minimal coalitions of $\mathcal{C}_0^m(1) \cup \mathcal{C}_1^m(1)$, and so on. After simplification to their minimality, we have

$$\mathcal{C}_0^m(2) = \{\overline{02}, \overline{03}, \overline{07}, \overline{08}, \overline{123}\}, \ \mathcal{C}_1^m(2) = \{\overline{0}, \overline{1}\}, \quad \mathcal{C}_2^m(2) = \{\overline{02}, \overline{123}\},$$
$$\mathcal{C}_3^m(2) = \{\overline{07}, \overline{08}, \overline{3}\}, \quad \mathcal{C}_4^m(2) = \mathcal{C}_5^m(2) = \mathcal{C}_6^m(2) = \{\overline{0}, \overline{1}\},$$
$$\mathcal{C}_7^m(2) = \mathcal{C}_8^m(2) = \{\overline{07}, \overline{08}, \overline{3}\}, \ \mathcal{C}_9^m(2) = \{\overline{3}, \overline{7}\}.$$

We continue the process until the system of sequences is stable, i.e. no change after one more step. The results after 3 steps are

$$\mathcal{C}_0^m(3) = \{\overline{02}, \overline{03}, \overline{07}, \overline{08}, \overline{123}\}, \ \mathcal{C}_1^m(3) = \{\overline{0}, \overline{1}\}, \quad \mathcal{C}_2^m(3) = \{\overline{02}, \overline{123}\},$$
$$\mathcal{C}_3^m(3) = \{\overline{07}, \overline{08}, \overline{3}\}, \quad \mathcal{C}_4^m(3) = \mathcal{W}_5^m(3) = \mathcal{C}_6^m(3) = \{\overline{0}, \overline{1}\},$$
$$\mathcal{C}_7^m(3) = \mathcal{C}_8^m(3) = \{\overline{07}, \overline{08}, \overline{3}\}, \ \mathcal{C}_9^m(3) = \{\overline{08}, \overline{3}, \overline{7}\}.$$

They are stable now and therefore $\mathcal{C}_i^m = \mathcal{C}_i^m(3)$ for any $i \in N$.

5. CONCLUSIONS

This paper has set up a framework for the organizational control where a group of players can implement indirect commands over other players. Without further specific reference to the players, we have shown the controls over either privately held assets or publicly shared assets. To put it very simply, consider the owners of a public company control the asset through the management. As a consequence, if we focus on more concrete contexts, the organizational controls can be related to the existing theories, such as the incentive problem, the free-rider problem and agency problem in the theory of firm.

There is a link between the paper and the related work (Hu and Shapley, 2001). The control function provides a global form of command structure. One could quantify player j's "controlling power" over i by j's Shapley-Shubik power index in the control game $\Gamma(N, \mathcal{C}_i)$. The controlling power, in political sense, acts as the residual power when conflicting commands occur. In economic sense, it acts as the property right. Unlike the authority distribution, a coalition takes finite steps to control its property. We have provided several algorithms to implement the global commands in addition to the definitions. It is interesting to relate the results to several specific issues, such as one-person-one-vote principle and task control.

From a theoretical point of views, there are several related issues worth mentioning. First, we should remark that we can also define the control games by other forms of "control function". The selected form should respect the existing tasks, traditions, and explain the authority structure in the organization as well. To achieve this, we believe that empirical research and existing theories have provided us plenty of resource and evidence. To integrate the framework with the empirical evidence seems a challenging issue.

Secondly, to keep the sequence analysis tractable, we have always assumed that every player obeys his command game strictly. Notice, however, that the set-up has not taken the player's behavior into account. For example, the top management in a public company has much more private information about the company than the owners. The management could hide, for their own interests, the private information from the owners and as a consequence, the owners in turn may have no prompt control over the management without knowing the private information. Of course, to gather the information is not costless. One could incorporate such features in our model and investigate their rich contexts and interpretations.

Thirdly, in our analysis, the command and control games are assumed simple games. Hence, any outsider can not interfere with the internal affairs. However any concrete organization has to be subject to the changes of outside, such as environment, investment and political policies, etc. And any organization is organized to take some external tasks. The incorporation of the tasks with the organization is related to organizational design, organization reform, efficiency, and other problems in the theory of organization.

Finally, an organization is a dynamic body instead of just a static structure. The promotion or layoff of a player is then a restructure of the organization. To keep the sequence analysis valid for long term, it is useful to relate the results of the paper to the basic principles of the organization, such as the Constitution of a nation, or the Charter of a corporation.

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