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A NUMERICAL METHOD FOR SOLVING VARIABLE COEFFICIENT ELLIPTIC EQUATION WITH INTERFACES

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ABSTRACT. A new 2nd order accurate numerical method is proposed for solving the variable coefficient elliptic equation in the presence of interfaces where the variable coefficients, the source term, and hence the solution itself and its derivatives may be discontinuous. Jump conditions at interface are prescribed. The boundary and the interface are only required to be Lipschitz continuous instead of smooth, and the interface is allowed to intersect with the boundary. The method is derived from a weak formulation of the variable coefficient elliptic equation [12]. Numerical experiments show that the method is 2nd order accurate in L^∞ norm.

1. INTRODUCTION

The "immersed boundary" method [16, 17] uses a numerical approximation of δ-function which smears out the solution on a thin finite band around the interface. In [19], the "immersed boundary" method was combined with the level set method resulting in a first order numerical method that is simple to implement even in multiple spatial dimensions. However, for both methods, the numerical smearing at the interface forces continuity in solution at the interface regardless of the interface condition $[u] = a$, where a might not be zero.

In [13, 14], a fast method for solving Laplace's equations on irregular regions with smooth boundaries was introduced. By using Fredholm integral equation of the second kind, solutions can be extended to a rectangular region. Since solutions are harmonic, Fredholm integral equations can be used again to capture the jump conditions in solution and its normal derivative, $[u] \neq 0$ and $[u_n] = 0$. Then these jump conditions are used to evaluate discrete Laplacian, and then fast Poisson solver on a regular region can be applied with 2nd or higher order accuracy.

In [7], the "immersed interface" method was presented for solving elliptic equation with smooth source term and $[u] = 0$ and $[\beta u_n] = 0$. The method achieves a second order accuracy by incorporating the interface conditions into the finite difference stencil in a way that preserves continuities in both solution and its co-normal derivative. The corresponding linear system is sparse but not symmetric or positive definite. A fast iterative method [8] conjunctured with "immersed interface" method has been developed for constant coefficient problems with interface conditions $[u] \neq 0$ and $[\beta u_n] \neq 0$, and achieves 2nd order accuracy.

In [1], a finite element method was developed for solving such a problem with the interface conditions $[u] = 0$ and $[\beta u_n] \neq 0$. Interfaces are aligned with cell boundaries. The 2nd order accuracy was obtained in an energy norm. Nearly the 2nd order of accuracy was obtained in L^2 norm.

In [10], another finite element method was developed for solving the problem with the interface conditions $[u] = 0$ and $[\beta u_n] = 0$. Cartesian grids are used, and then associated uniform triangulations are added on. Interfaces are not necessarily aligned with cell boundaries. Numerical evidence shows that its conforming version achieves 2nd order accuracy in L^∞ norm, and higher than first order for its non-conforming version.

The boundary condition capturing method [11] uses the Ghost Fluid Method (GFM) [2] to capture the boundary conditions. The Ghost Fluid Method (GFM) is robust and simple to implement, so is the resulting

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boundary condition capturing method. The boundary condition capturing method is implemented using a standard finite difference discretization on a Cartesian grid, making it simple to apply in multi-dimensions, including three spatial dimensions. Furthermore, the coefficient matrix of the associated linear system is the standard symmetric positive definite matrix for the variable coefficient Poisson equation in the absence of interfaces allowing for straightforward application of standard "black box" solvers. The convergence proof of the method is provided in [12]. In [12], a weak formulation of the problem was studied. The boundary condition capturing method can be obtained from discretizing the weak formulation. The convergence proof follows naturally. The boundary condition capturing method can solve the elliptic equation with interface conditions $[u] \neq 0$ and $[\beta u_n] \neq 0$ in multi-dimensions (including 2 dimensions and 3 dimensions), however the boundary condition capturing method is only first order accurate.

In this paper, inspired by the boundary condition capturing method [11] and the weak formulation derived in [12], we extend the weak formulation to include the case that the boundary and the interface are only required to be Lipschitz continuous instead of smooth, and the interface is allowed to intersect with the boundary. We then propose a numerical method by discretizing the weak formulation. Our method is capable of solving the elliptic equation with variable coefficients and interface conditions $[u] \neq 0$ and $[\beta u_n] \neq 0$, and is capable of dealing with the case that the boundary and the interface are only Lipschitz continuous and the interface intersects with boundary. Numerical experiments show that our method is 2nd order accurate in L^∞ norm if the interface is smooth or if the non-smoothness of the interfaces is properly handled.

2. Equations and Weak Formulation

Consider an open bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$. Let Γ be a Lipschitz continuous interface of co-dimension $n-1$, represented by the zero level-set of a continuous but piece-wise smooth function $\phi(x)$, which is a signed distance function of the interface locally [18]. We assume that ϕ divides Ω into disjoint subdomains, $\Omega^- = \{\phi < 0\}$ and $\Omega^+ = \{\phi > 0\}$. Thus, we may write $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$. The unit normal vector of the interface is $n = \nabla\phi/|\nabla\phi|$, for $\phi(x) = 0$, pointing from Ω^- to Ω^+.

We seek solutions of the variable coefficient elliptic equation away from the interface given by

$$\nabla \cdot (\beta(x)\nabla u(x)) = f(x), \quad x \in \Omega \setminus \Gamma, \tag{2.1a}$$

in which $x = (x_1, \ldots, x_n)$ denotes the spatial variables and ∇ is the gradient operator. The coefficient $\beta(x)$ is assumed to be a positive definite, symmetric $n \times n$ matrix, the components of which are continuously differentiable on the closure of each disjoint subdomain, Ω^- and Ω^+, but they may be discontinuous across the interface Γ. It follows that there are positive constants $m < M$ with $m\ \mathbf{I} \leq \beta(x) \leq M\ \mathbf{I}$, where $\mathbf{I}$ stands for the $n \times n$ identity matrix. We suppose that on the interface, β assumes the limiting values from within Ω^-. The right-hand side $f(x)$ is assumed to lie in $L^2(\Omega)$.

Given functions a and b along the interface Γ, we prescribe the jump conditions

$$\begin{cases} [u]_\Gamma(x) \equiv u^+(x) - u^-(x) = a(x), \\ [(\beta u)_n]_\Gamma(x) \equiv (\beta u)_n^+(x) - (\beta u)_n^-(x) = b(x), \end{cases} \quad x \in \Gamma. \tag{2.1b}$$

Note that $(\beta u)_n = n \cdot \beta\nabla u$, and the "$\pm$" subscripts refer to limits taken from within the subdomains $\Omega^\pm$.

Finally, we prescribe boundary conditions

$$u(x) = g(x), \quad x \in \partial\Omega, \tag{2.1c}$$

for a given function g on the boundary $\partial\Omega$.

In [12], a weak formulation of the problem has been obtained in the case that the boundary $\partial\Omega$ and the interface Γ are smooth, and the interface Γ does not intersect with the boundary $\partial\Omega$. Here we extend the weak formulation a bit to include the case that the boundary $\partial\Omega$ and the interface Γ are only Lipschitz continuous instead of smooth, and the interface Γ is allowed to intersect with the boundary $\partial\Omega$.

We are going to use the usual Sobolev spaces $H_0^1(\Omega)$ and $H^1(\Omega)$. We use the usual inner product for $H^1(\Omega)$. For $H_0^1(\Omega)$, instead of the usual inner product we choose one which is better suited to our problem:

$$B[u,v] = \int_{\Omega^+} \beta\nabla u\cdot\nabla v + \int_{\Omega^-} \beta\nabla u\cdot\nabla v. \tag{2.2}$$

This induces a norm on $H_0^1(\Omega)$ which is equivalent to the usual one, thanks to the Poincaré inequality and the uniform bounds for the coefficient matrix.

Let R_Γ and $R_{\partial\Omega}$ denote the restriction operators from $H^1(\Omega)$ to $L^2(\Gamma)$ and $L^2(\partial\Omega)$, respectively. Such restrictions are well defined, because the boundary and the interface are required to be Lipschitz continuous, [4] and [15]. Throughout this section, we shall always assume that our interface data a and b are the restrictions of functions $\widetilde{a}$ and $\widetilde{b}\in H^1(\Omega)$, respectively:

$$a = R_\Gamma(\widetilde{a}) \quad\text{and}\quad b = R_\Gamma(\widetilde{b}). \tag{2.3}$$

We shall always assume that our boundary data g can be obtained by the following way

$$g = R_{\partial\Omega}(\widetilde{c}) - R_{\partial\Omega}(\widetilde{a})\chi(\partial\Omega^-\cap\partial\Omega) \quad\text{for some}\quad \widetilde{c}\in H^1(\Omega), \tag{2.4}$$

where $\partial\Omega^-$ is the boundary of Ω^- and $\chi(\cdot)$ is the characteristic function. This (2.4) could be thought as a compatibility condition between a and g. To simplify the notation, from now on we will drop the tildes.

We will construct a unique solution of the problem in the class

$$H(a,c) = \{u : u - c + a\,\chi(\overline{\Omega^-}) \in H_0^1(\Omega)\},$$

where $\overline{\Omega^-}$ is the closure of Ω^-. If $u\in H(a,c)$ then

$$[u]_\Gamma = a$$

Note that $H_0^1(\Omega)$ can be identified with $H(0,0)$.

Definition 2.1. A function $u\in H(a,c)$ is a weak solution of (2.1a), (2.1b), (2.1c) if $v = u - c + a\,\chi(\overline{\Omega^-})$ satisfies

$$-B[v,\psi] = F(\psi), \tag{2.5a}$$

for all $\psi\in H_0^1(\Omega)$, where

$$B[u,v] = \int_{\Omega^+} \beta\nabla u\cdot\nabla v + \int_{\Omega^-} \beta\nabla u\cdot\nabla v. \tag{2.5b}$$

and

$$F(\psi) = \int_{\Omega} f\psi + \int_{\Omega} \beta\nabla c\cdot\nabla\psi - \int_{\Omega^-} \beta\nabla a\cdot\nabla\psi + \int_{\Gamma} b\psi. \tag{2.5c}$$

A classical solution of (2.1a), (2.1b), (2.1c) is necessarily a weak solution, because the boundary $\partial\Omega$ and the interface Γ are Lipschitz continuous and hence the divergence theorem is valid, see [4].

Theorem 2.1. *If $f\in L^2(\Omega)$, and a, b, and $c\in H^1(\Omega)$, then there exists a unique weak solution of (2.5a), (2.5b), (2.5c) in $H(a,c)$.*

Proof. The left-hand side (2.5b) of (2.5a) is a bounded bilinear form on $H_0^1(\Omega)$ and the right-hand side (2.5c) of (2.5a) is a bounded linear functional on $H_0^1(\Omega)$. By the Riesz representation theorem, there exists a unique $v\in H_0^1(\Omega)$ (so $u = v + c - a\chi(\overline{\Omega^-})\in H(a,c)$) such that $-B[v,\psi] = F(\psi)$, for all $\psi\in H_0^1(\Omega)$. □

3. Numerical Method

For easy of discussion in this section, and accuracy testing in the next section, we assume that a and b are smooth on the closure of Ω. We also assume that g, β and f are smooth on the closure of each Ω^+ and Ω^-, but they may be discontinuous across the interface Γ. Also $g + a\chi(\overline{\Omega^-})$ is smooth on the closure of Ω i.e. a and g are compatible (2.4).

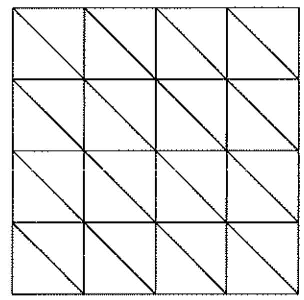

FIGURE 3.1. a uniform triangulation

3.1. **Grid and Interpolation.** For simplicity, we restrict ourselves to the special case of a rectangular domain $\Omega = [x_W, x_E] \times [y_S, y_N]$ in the plane, and β is scalar. Given positive integers I and J, set $\Delta x = (x_E - x_W)/I$ and $\Delta y = (y_N - y_S)/J$. We define a uniform grid $\{(x_i, y_j)\} = \{(x_W + i\Delta x, y_S + j\Delta y)\}$ for $i = 0, \cdots, I$ and $j = 0, \cdots, J$. Each (x_i, y_j) is called a grid point. A grid point is called a boundary point if $i = 0, I$ or $j = 0, J$; otherwise an interior point. The grid size is defined as $h = \max(\Delta x, \Delta y) > 0$. The radio $\Delta x/\Delta y$ is fixed when the grid size h goes to zero.

Two sets of grid functions are needed and denoted by

$$H^{1,h} = \{\omega^h = (\omega_{i,j}) : 0 \le i \le I, 0 \le j \le J\}, \tag{3.1a}$$

and

$$H_0^{1,h} = \{\omega^h = (\omega_{i,j}) \in H^{1,h} : \quad \omega_{i,j} = 0 \text{ if } , i = 0, I, \text{ or } j = 0, J\}. \tag{3.1b}$$

For each rectangular region $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, we cut it into two pieces of right triangular regions: one is bounded by $x = x_i$, $y = y_j$ and $y = \frac{y_{j+1} - y_j}{x_i - x_{i+1}}(x - x_{i+1}) + y_j$, and the other is bounded by $x = x_{i+1}$, $y = y_{j+1}$ and $y = \frac{y_{j+1} - y_j}{x_i - x_{i+1}}(x - x_{i+1}) + y_j$. Collect all those triangular regions, we obtain a uniform triangulation $\mathbf{T}^h$: $\Omega = \bigcup_{K \in \mathbf{T}^h} K$. See Figure 3.1. We can also choose the hypotenuse to be $y = \frac{y_{j+1} - y_j}{x_{i+1} - x_i}(x - x_i) + y_j$, and get another uniform triangulation from the same Cartesian grid. There is no conceptual difference of our method on these two triangulations.

If $\phi(x_i, y_j) \le 0$, we count the grid point (x_i, y_j) as in Ω^-; otherwise in Ω^+. We call an edge (an edge of a triangle in the triangulation) an interface edge if two of its vertices (vertices of triangles in the triangulation) belong to different subdomains; and a regular edge if two of its vertices belong to the same subdomain, either Ω^+ or Ω^-.

We call a cell K an interface cell if its vertices belong to different subdomains, and clearly the interface goes through the interface cell K. In the interface cell, we write $K = K^+ \bigcup K^-$. K^+ and K^- are separated by a straight line segment, denoted by Γ_K^h. Two end points of the line segment Γ_K^h are located on the interface Γ. The vertices of K^+ are located in $\Omega^+ \bigcup \Gamma$ and the vertices of K^- are located in $\Omega^- \bigcup \Gamma$. K^+ and K^- are approximations of the regions of $K \cap \Omega^+$ and $K \cap \Omega^-$ respectively. We call a cell K a regular cell if all its vertices belong to the same subdomain, either Ω^+ or Ω^-. For a regular cell, we also write $K = K^+ \bigcup K^-$, where $K^- = \{\}$ (empty set) if all vertices of K are in Ω^+, and $K^+ = \{\}$ (empty set) if all vertices of K are in Ω^-. Clearly $\Gamma_K^h = \{\}$(empty set) in a regular cell, and K^+ and K^- are approximations of the regions $K \cap \Omega^+$ and $K \cap \Omega^-$ respectively. We use $|K^+|$ and $|K^-|$ to represent their area of K^+ and K^- respectively.

Two extension operators are needed. The first one is $T^h : H_0^{1,h} \to H_0^1(\Omega)$. For any $\psi^h \in H_0^{1,h}$, $T^h(\psi^h)$ is a standard continuous piece-wise linear function, which is a linear function in every triangular cell and $T^h(\psi^h)$ matches ψ^h on grid points. Clearly such function set, denoted by $\overline{H_0^{1,h}}$, is a finite dimensional subspace of $H_0^1(\Omega)$.

The second extension operator U^h is constructed as follows. For any $u^h \in H^{1,h}$ with $u^h = g^h$ at boundary points, $U^h(u^h)$ is a piece-wise linear function and matches u^h on grid points. It is a linear function in each regular cell, just like the first extension operator $U^h(u^h) = T^h(u^h)$ in regular cell. In each interface cell, it consists of two pieces of linear functions, one is on K^+ and the other is on K^-. The location of its discontinuity in the interface cell is the straight line segment Γ_K^h. Note that two end points of the line segment are located on the interface Γ, and hence the interface condition $[u] = a$ could be and is enforced exactly at these two end points. In each interface cell, the interface condition $[\beta \nabla u \cdot n] = b$ is enforced with the value b at the middle point of these two end points. Clearly such a function is not continuous in general, neither the set of such functions a linear space. We denote the set of such functions as $\overline{H_{a,c}^{1,h}}$, which should be thought as an approximation of the solution class $H(a, c)$ with the restriction of $[\beta u_n] = b$. Similar versions of such extension can be found in the literature [11], [10]. In order to use this extension, we need the following lemma.

Lemma 3.1. *On any triangle $K \in \mathbf{T}^h$ and for any given $u^h \in H^{1,h}$ with $u^h = g^h$ on the boundary points, $U^h(u^h)$ can be constructed uniquely.*

The proof is provided in the Appendix.

3.2. **Discrete Weak Formulation.** Under the current setting, we discretize the weak formulation (2.5a,2.5b,2.5c) as follows:

Method 3.1. Find a discrete function $u^h \in H^{1,h}$ such that

$$v^h = u^h - c^h + a^h \chi^h \in H_0^{1,h} \tag{3.2a}$$

and

$$-B^h[v^h, \psi^h] = F(\psi^h), \tag{3.2b}$$

for all $\psi^h \in H_0^h$, where

$$\begin{aligned} B^h[v^h, \psi^h] = \sum_{K \in \mathbf{T}^h} & \left(\int_{K^+} \beta \nabla (U^h(u^h) - c) \cdot \nabla T^h(\psi^h) \right. \\ & \left. + \int_{K^-} \beta \nabla (U^h(u^h) - c + a) \cdot \nabla T^h(\psi^h) \right), \end{aligned} \tag{3.2c}$$

and

$$\begin{aligned} F^h(\psi^h) &= \sum_{K \in \mathbf{T}^h} \int_{K^+ \bigcup K^-} f T^h(\psi^h) + \sum_{K \in \mathbf{T}^h} \int_{\Gamma_K^h} b T^h(\psi^h) \\ &+ \sum_{K \in \mathbf{T}^h} \int_{K^+ \bigcup K^-} \beta \nabla c \cdot \nabla T^h(\psi^h) - \sum_{K \in \mathbf{T}^h} \int_{K^-} \beta \nabla a \cdot \nabla T^h(\psi^h). \end{aligned} \tag{3.2d}$$

Here $c^h \in H^{1,h}$ and matches with c at grid points, $a^h \in H^{1,h}$ and matches with a at grid points, and $\chi^h \in H^{1,h}$ and matches with $\chi(\overline{\Omega^-})$ at grid points. Note that (3.2a) implies $u^h = g^h$ at boundary points.

The method 3.1 can be simplified as

Method 3.2. Find a discrete function $u^h \in H^{1,h}$ such that

$$u^h = g^h \text{ on boundary points} \tag{3.3a}$$

and for all $\psi^h \in H_0^{1,h}$,

$$\begin{aligned} & - \sum_{K \in \mathbf{T}^h} \left(\int_{K^+} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi^h) + \int_{K^-} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi^h) \right) \\ & = \sum_{K \in \mathbf{T}^h} \left(\int_{K^+ \bigcup K^-} f T^h(\psi^h) + \int_{\Gamma_K^h} b T^h(\psi^h) \right). \end{aligned} \tag{3.3b}$$

We know that $\overline{H_0^{1,h}}$ is a finite dimensional subspace of $H_0^1(\Omega)$. We construct its base according to grid points. Let For $m = 1, \cdots, I-1$ and $n = 1, \cdots, J-1$, Let $\psi_{m,n}^h = \{\delta_{i,m}\delta_{j,n} : i = 0, \cdots, I, j = 0, \cdots, J\} \in H_0^h(\Omega)$ where

$$\delta_{i,m} = \begin{cases} 1, & i = m, \\ 0, & i \neq m. \end{cases}$$

Clearly $\{T^h(\psi_{m,n}^h) : m = 1, \cdots, I-1, n = 1, \cdots, J-1\}$ is a base of $\overline{H_0^{1,h}}$. Hence our method 3.2 can be rewritten as:

Method 3.3. Find a discrete function $u^h = \{u_{i,j}\} \in H^{1,h}$ such that

$$u_{i,j} = g_{i,j} \quad \text{if } i = 0, I, \text{ or } j = 0, J \tag{3.4a}$$

and for $m = 1, \cdots, I-1$ and $n = 1, \cdots, J-1$,

$$\begin{aligned} &-\sum_{K \in \mathbf{T}^h} \left(\int_{K+} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi_{m,n}^h) + \int_{K^-} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi_{m,n}^h) \right) \\ &= \sum_{K \in \mathbf{T}^h} \left(\int_{K^+ \bigcup K^-} f T^h(\psi_{m,n}^h) + \int_{\Gamma_K^h} b T^h(\psi_{m,n}^h) \right). \end{aligned} \tag{3.4b}$$

Remark 1. The discrete weak formulation can be extended to multi-dimensions and irregular triangulations as follows:

Method 3.4. Find a discrete function $u^h \in H^{1,h}$ such that

$$u^h = g^h \text{ on boundary points} \tag{3.5a}$$

and for all $\psi^h \in H_0^{1,h}$,

$$\begin{aligned} &-\sum_{K \in \mathbf{T}^h} \left(\int_{K+} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi^h) + \int_{K^-} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi^h) \right) \\ &= \sum_{K \in \mathbf{T}^h} \left(\int_{K^+ \bigcup K^-} f T^h(\psi^h) + \int_{\Gamma_K^h} b T^h(\psi^h) \right). \end{aligned} \tag{3.5b}$$

where $\mathbf{T}^h$ is any "triangulations" in multi-dimensions, and $H^{1,h}$ and $H_0^{1,h}$ are proper grid function spaces, and U^h and T^h are corresponding extensions in multi-dimensions.

4. Numerical Experiments

The interface might hit grid points, which may cause inaccuracy in dealing with a situation of zero over zero. To avoid this, we set $\phi(x_i, y_j) = -\epsilon$, if $|\phi(x_i, y_j)| < \epsilon$ ($= 10^{-8}\Delta x$ in all our calculations).

For simplicity reason, in each triangular cell K, we set β^+ to be the β value at the center of K^+, and β^- the β value at the center of K^-. They approximate $\frac{1}{|K^+|}\int_{K^+} \beta$ and $\frac{1}{|K^-|}\int_{K^-} \beta$ within enough accuracy.

In each triangle K, for $\int_{K^+} f\psi$, we first cut K^+ into two triangles if K^+ is not a triangle, then on each triangle, we use a 2nd order accurate numerical quadrature to evaluate the integration of $f\psi$ on the triangle; similarly for $\int_{K^-} f\psi$.

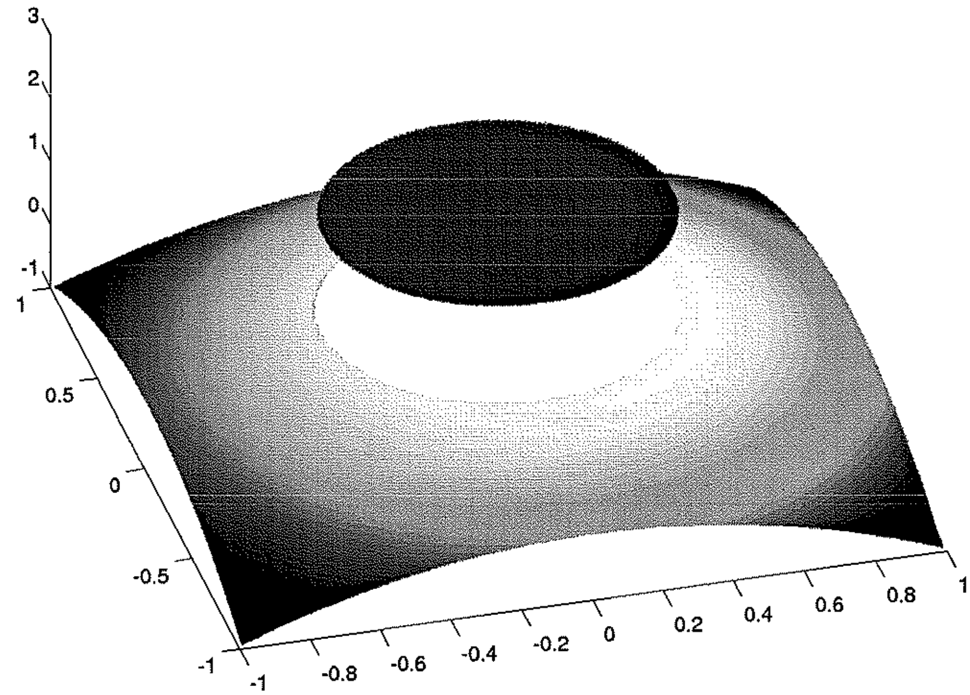

FIGURE 4.1. $\nabla \cdot (\beta \nabla u) = f(x, y)$, f piece-wise smooth but not constant, β piece-wise smooth but not constant, $[u] \neq$ *constant*, $[\beta u_n] \neq$ *constant*, and $\beta^+/\beta^- \approx 1/1000$.

$\Delta x = \Delta y$	L^∞-error in U	order	L^∞-error in ∇U	order	Δx, Δy	L^∞-error in U	order	L^∞-error in ∇U	order
$\frac{2}{40}$	5.54e-1		3.92e-0		$\frac{2}{41}, \frac{2}{39}$	4.91e-1		3.92e-0	
$\frac{2}{80}$	1.45e-1	1.93	1.34e-0	1.55	$\frac{2}{81}, \frac{2}{79}$	1.37e-1	1.84	1.57e-0	1.32
$\frac{2}{160}$	3.19e-2	2.18	6.43e-1	1.06	$\frac{2}{161}, \frac{2}{159}$	3.84e-2	1.83	5.59e-1	1.49
$\frac{2}{320}$	8.94e-3	1.84	2.84e-1	1.18	$\frac{2}{321}, \frac{2}{319}$	9.04e-3	2.09	2.98e-1	0.91

TABLE 4.1.

4.1. **Smooth Interface. Example 1**: Consider $\nabla \cdot (\beta \nabla u) = f(x, y)$ on $[-1, 1] \times [-1, 1]$. The interface is a circle $x^2 + y^2 = 0.5^2$ and is described by the zero level-set of the level-set function $\phi = x^2 + y^2 - 0.25$. The interface Γ cuts the domain Ω into Ω^-, where $\phi \leq 0$, and Ω^+, where $\phi > 0$. The unit normal is $\vec{n} = (x/\sqrt{x^2 + y^2}, y/\sqrt{x^2 + y^2})$. On Ω^-, we let $u(x, y) = x^2 + y^2 + 2$, $\beta^- = 1000(xy + 3)$ and $f(x, y) = 8000xy + 12000$. On Ω^+, we let $u(x, y) = 1 - x^2 - y^2$, $\beta^+ = x^2 - y^2 + 3$ and $f(x, y) = 8y^2 - 8x^2 - 12$. Appropriate Dirichlet boundary condition is used. The jump conditions are $[u] = -1 - 2x^2 - 2y^2$ and $[\beta u_n] = -2x(x^2 - y^2 + 1000xy + 3003)n_1 - 2y(x^2 - y^2 + 1000xy + 3003)n_2$. Figure 4.1 shows the numerical solution of our method with step size $\Delta x = \Delta y = \frac{2}{320}$. Note that all data are variables and $\beta^+/\beta^- \approx 1/1000$. Table 4.1 shows that our method achieves 2nd order accuracy in L^∞ for solutions and first order accuracy in L^∞ norm for the gradient of solutions away from the interface on two different sets of grids.

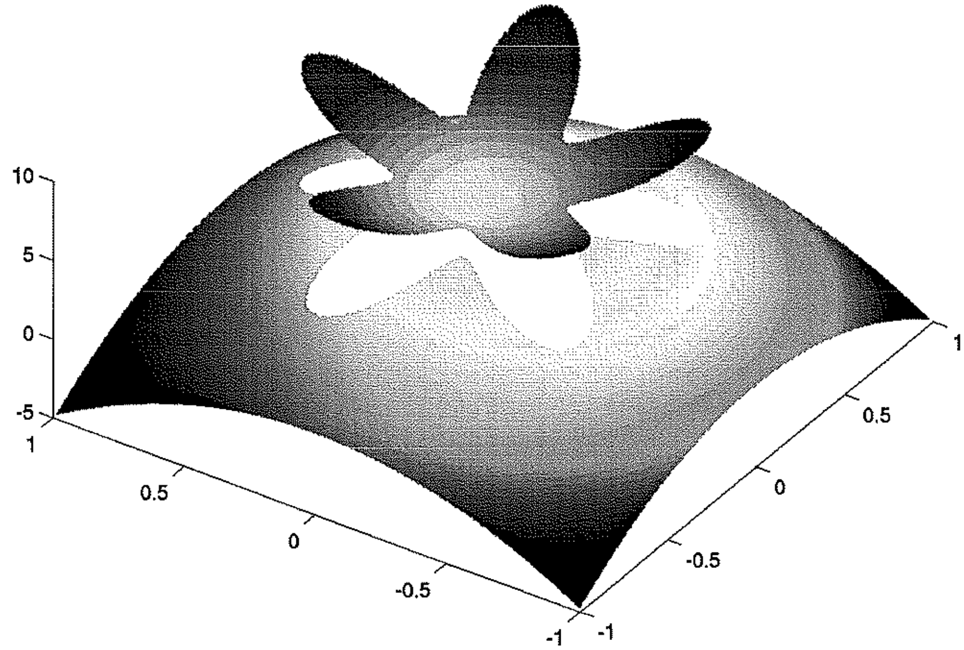

FIGURE 4.2. $\nabla \cdot (\beta \nabla u) = f(x,y)$, f piece-wise smooth but not constant, β piece-wise smooth but not constant, $[u] \neq$ *constant*, $[\beta u_n] \neq$ *constant*.

$\Delta x = \Delta y$	L^∞-error in U	order	L^∞-error in ∇U	order	Δx, Δy	L^∞-error in U	order	L^∞-error in ∇U	order
$\frac{2}{40}$	3.18e-2		1.48e-1		$\frac{2}{41}, \frac{2}{39}$	2.75e-2		2.73e-1	
$\frac{2}{80}$	9.02e-3	1.82	1.32e-1	0.17	$\frac{2}{81}, \frac{2}{79}$	9.28e-3	1.57	1.61e-1	0.76
$\frac{2}{160}$	2.08e-3	2.12	4.79e-2	1.46	$\frac{2}{161}, \frac{2}{159}$	2.54e-3	1.87	4.19e-2	1.94
$\frac{2}{320}$	5.55e-4	1.91	2.05e-2	1.22	$\frac{2}{321}, \frac{2}{319}$	5.67e-4	2.16	2.72e-2	0.62

TABLE 4.2.

Example 2: Consider $\nabla \cdot (\beta \nabla u) = f(x,y)$ on $[-1,1] \times [-1,1]$. The interface is a curve $(x(\theta), y(\theta))$ where $x(\theta) = .02\sqrt{5} + (.5 + .2\sin(5\theta))\cos(\theta)$, $y(\theta) = .02\sqrt{5} + (.5 + .2\sin(5\theta))\sin(\theta)$, $\theta \in [0, 2\pi)$. The interface is described by the zero level-set of the level-set function $\phi = (x - .02\sqrt{5})^2 + (y - .02\sqrt{5})^2 - (.5 + .2\sin(5\theta))^2$, where $\theta = \tan^{-1}(\frac{y-.02\sqrt{5}}{x-.02\sqrt{5}})$. The interface cuts the domain Ω into Ω^-, where $\phi \leq 0$, and Ω^+, where $\phi > 0$. The unit normal is $\vec{n} = (n_1, n_2) = \nabla\phi / |\nabla\phi|$. On Ω^-, we let $u(x,y) = 7x^2 + 7y^2 + 1$, $\beta^- = (x^2 - y^2 + 3)/7$, $f(x,y) = -8y^2 + 8x^2 + 12$. On Ω^+, we let $u(x,y) = 5 - 5x^2 - 5y^2$, $\beta^+ = (xy + 2)/5$ and $f(x,y) = -8xy - 8$. Appropriate Dirichlet boundary condition is used. The jump conditions are $[u] = 4 - 12x^2 - 12y^2$ and $[\beta u_n] = (2xy^2 - 2x^2y - 2x^3 - 10x)n_1 + (2y^3 - 2x^2y - 2xy^2 - 10y)n_2$. Figure 4.2 shows the numerical solution of our method with step size $\Delta x = \Delta y = \frac{2}{320}$. The difficulties of this problem is that all data are variables instead of constants, and the geometry of the interface is more complicate than the example 1. Table 4.2 shows that our method achieves 2nd order accuracy in L^∞ norm for solutions and about 1st order accuracy in L^∞ for the gradients of the solutions away from the interface on two different sets of grids.

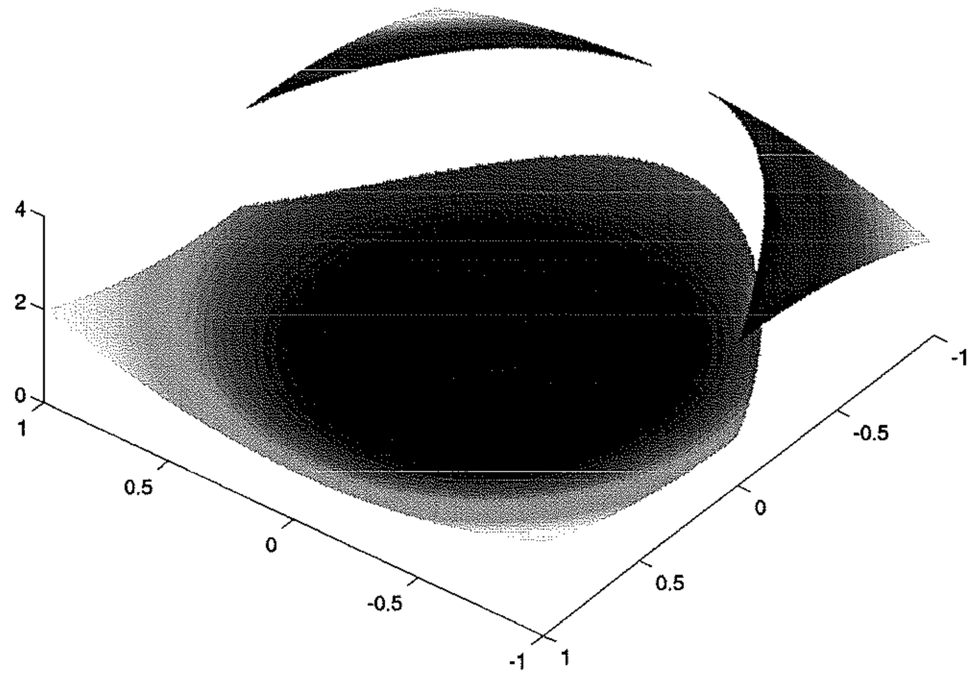

FIGURE 4.3. $\nabla \cdot (\beta \nabla u) = f(x,y)$, f piece-wise smooth, but not constant, β piece-wise smooth, but not constant, $[u] \neq$ *constant*, $[\beta u_n] \neq$ *constant*, Γ is smooth but is tangential to $\partial\Omega$ and intersects with $\partial\Omega$.

$\Delta x = \Delta y$	L^∞-error in U	order	L^∞-error in ∇U	order	Δx, Δy	L^∞-error in U	order	L^∞-error in ∇U	order
$\frac{2}{40}$	4.88e-4		2.70e-3		$\frac{2}{41}, \frac{2}{39}$	5.05e-4		2.95e-3	
$\frac{2}{80}$	1.25e-4	1.96	1.11e-3	1.28	$\frac{2}{81}, \frac{2}{79}$	1.35e-4	1.90	1.20e-3	1.30
$\frac{2}{160}$	3.40e-5	1.88	7.02e-4	0.66	$\frac{2}{161}, \frac{2}{159}$	3.51e-5	1.94	5.88e-4	1.03
$\frac{2}{320}$	8.99e-6	1.92	4.59e-4	0.61	$\frac{2}{321}, \frac{2}{319}$	9.05e-6	1.96	2.89e-4	1.02

TABLE 4.3.

Example 3: Consider $\nabla \cdot (\beta \nabla u) = f(x,y)$ on $\Omega = [-1,1] \times [-1,1]$. The interface is a parabola and is described by the zero level-set of the level-set function $\phi = x^2 - y - 1$. The domain is cut by the interface Γ into Ω^-, where $\phi \leq 0$, and Ω^+, where $\phi > 0$. Note that the interface is tangential to the boundary $\partial\Omega$ at $(0,1)$ point, and it intersects with the boundary $\partial\Omega$ at $(-1,0)$ and $(1,0)$ at certain nonzero angles. The unit normal vector of the interface is $\vec{n} = (\frac{2x}{\sqrt{1+4x^2}}, \frac{-1}{\sqrt{1+4x^2}})$. On Ω^-, we let $u(x,y) = x^2 + y^2$, $\beta^- = x^2 - y^2 + 3$ and $f(x,y) = -8y^2 + 8x^2 + 12$. On Ω^+, we let $u(x,y) = 4 - x^2 - y^2$, $\beta^+ = xy + 2$ and $f(x,y) = -8xy - 8$. Appropriate Dirichlet boundary conditions are used. The jump conditions are $[u] = 4 - 12x^2 - 12y^2$ and $[\beta u_n] = (2xy^2 - 2x^2y - 2x^3 - 10x)n_1 + (2y^3 - 2x^2y - 2xy^2 - 10y)n_2$. Figure 4.3 shows the numerical solution of our method with step size $\Delta x = \Delta y = \frac{2}{320}$. Other than the difficulty of all variables, the interface is tangential to the boundary and intersects with the boundary at certain nonzero angles. Figure 4.3 shows the numerical solution of our method with step size $\Delta x = \Delta y = \frac{2}{320}$. Table 4.3 shows that our method still achieves 2nd order accuracy in L^∞ norm in solutions and about 1st order accuracy in L^∞ norm in the gradients of the solutions on two different sets of grids.

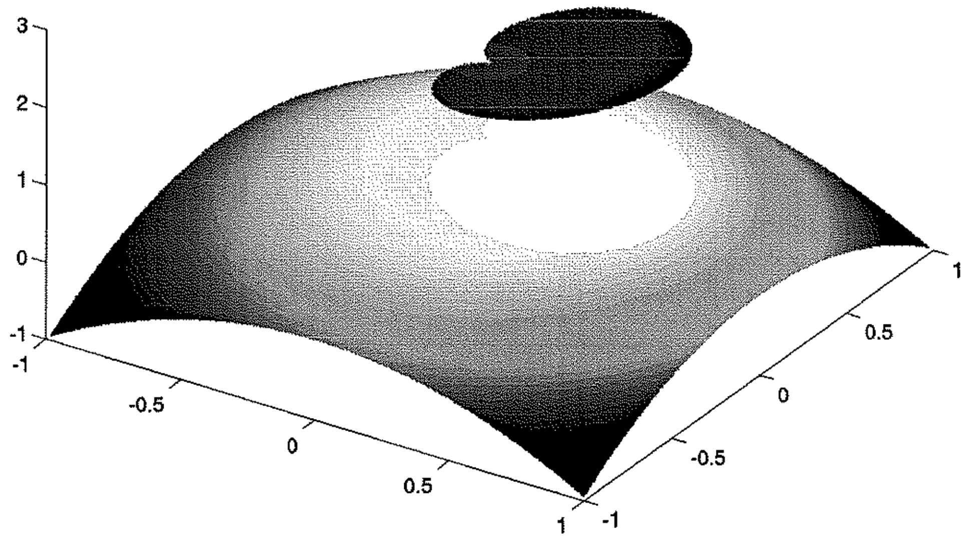

FIGURE 4.4. $\nabla \cdot (\beta \nabla u) = f(x, y)$, f piece-wise smooth, but not constant, β piece-wise smooth, but not constant, $[u] \neq$ *constant*, $[\beta u_n] \neq$ *constant*, Γ is smooth except at the cusp point$(-0.25, 0)$.

$\Delta x = \Delta y$	L^∞-error in U	order	Δx, Δy	L^∞-error in U	order
$\frac{2}{40}$	4.08e-3		$\frac{2}{41}, \frac{2}{39}$	3.60e-3	
$\frac{2}{80}$	7.90e-4	2.36	$\frac{2}{81}, \frac{2}{79}$	9.85e-4	1.87
$\frac{2}{160}$	2.14e-4	1.88	$\frac{2}{161}, \frac{2}{159}$	2.96e-4	1.73
$\frac{2}{320}$	1.48e-4	0.53	$\frac{2}{321}, \frac{2}{319}$	6.77e-5	2.13
$\frac{2}{640}$	2.29e-5	2.70	$\frac{2}{641}, \frac{2}{639}$	2.43e-5	1.48

TABLE 4.4.

4.2. **Non-smooth Interface.** Recall that the interface is approximated by a straight line segment Γ_K^h in each interface cell K. That is accurate enough to achieve 2nd order accuracy for our method if there is only one smooth interface in K. In the numerical experiments below we observe that, for interface with kinks and/or cusps, our method is convergent; and if two legs of the kinks are located in different triangular cells and there is only one interface goes through any cell, our method is 2nd order accurate in L^∞ norm.

Example 4: Consider $\nabla \cdot (\beta \nabla u) = f(x, y)$ on $\Omega = [-1, 1] \times [-1, 1]$. The interface is a cardioid and defined by the zero level-set of the level-set function $\phi = (3(x^2 + y^2) - x)^2 - x^2 - y^2$. The domain is cut by the interface into Ω^-, where $\phi \leq 0$, and Ω^+, where $\phi > 0$. Note that the interface are smooth except at cusp point $(-0.25, 0)$. On Ω^-, we let $u(x, y) = x^2 + y^2$, $\beta^- = x^2 - y^2 + 3$ and $f(x, y) = -8y^2 + 8x^2 + 12$. On Ω^+, we let $u(x, y) = 4 - x^2 - y^2$, $\beta^+ = xy + 3$ and $f(x, y) = -8xy - 8$. Appropriate Dirichlet boundary condition is used. The jump conditions are $[u] = 4 - 12x^2 - 12y^2$ and $[\beta u_n] = (2xy^2 - 2x^2y - 2x^3 - 10x)n_1 + (2y^3 - 2x^2y - 2xy^2 - 10y)n_2$. Figure 4.4 shows the numerical solution of our method with step size $\Delta x = \Delta y = \frac{2}{320}$. The difficulty of this problem is that the interface Γ has a cusp at $(-0.25, 0)$, which is not Lipschitz continuous. The numerical accuracy tests seems to suggest that 2nd order accurate convergence in L^∞ norm for solutions on two different sets of grids, see Table 4.4.

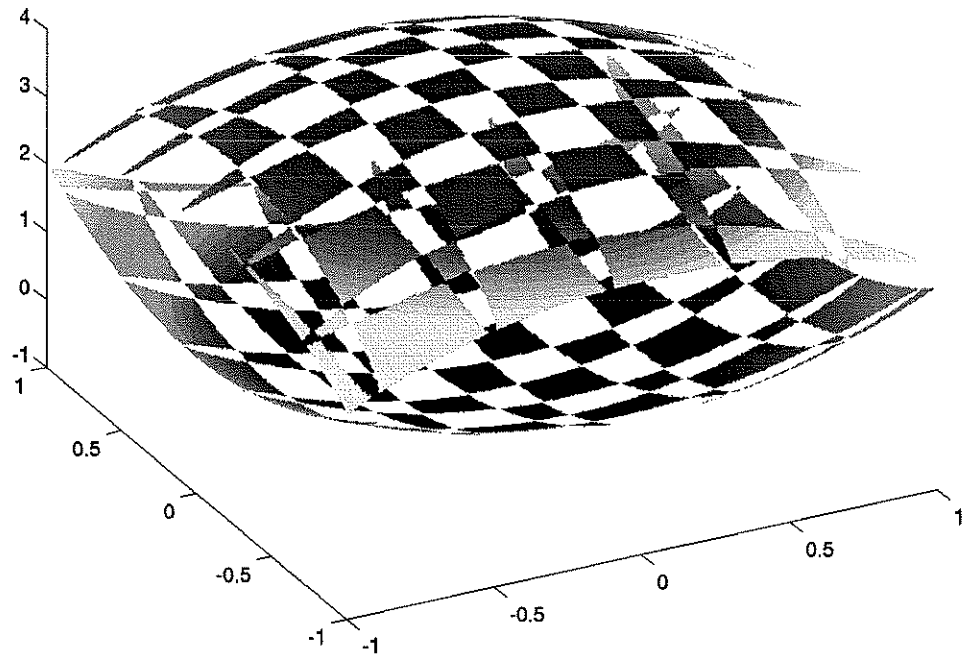

FIGURE 4.5. $\nabla \cdot (\beta \nabla u) = f(x,y)$, f piece-wise smooth, but not constant, β piece-wise smooth, but not constant, $[u] \neq constant$, $[\beta u_n] \neq constant$, Γ has many discontinuities, touches and intersects with the boundary.

$\Delta x = \Delta y$	L^∞-error in U	order	Δx, Δy	L^∞-error in U	order
$\frac{2}{40}$	2.38e-1		$\frac{2}{41}, \frac{2}{39}$	1.24e-1	
$\frac{2}{80}$	7.88e-2	1.59	$\frac{2}{81}, \frac{2}{79}$	6.75e-2	0.88
$\frac{2}{160}$	5.43e-2	0.54	$\frac{2}{161}, \frac{2}{159}$	4.56e-2	0.57
$\frac{2}{320}$	2.57e-2	1.08	$\frac{2}{321}, \frac{2}{319}$	2.25e-2	1.02

TABLE 4.5.

Example 5: Consider $\nabla \cdot (\beta \nabla u) = f(x,y)$ on $\Omega = [-1,1] \times [-1,1]$. The interface is defined by the zero level-set of the level-set function $\phi = (\sin(5\pi x) - y)(-\sin(5\pi y) - x)$. The domain is cut by the interface into Ω^-, where $\phi \leq 0$, and Ω^+, where $\phi > 0$. Note that there are many kinks on the interface, the interface intersects with $\partial\Omega$, and there are more than one piece of Ω^- and Ω^+. The unit normal vector of the interface is $\vec{n} = \nabla\phi / |\nabla\phi|$. On Ω^-, we let $u(x,y) = x^2 + y^2$, $\beta^- = x^2 - y^2 + 3$ and $f(x,y) = -8y^2 + 8x^2 + 12$. On Ω^+, we let $u(x,y) = 4 - x^2 - y^2$, $\beta^+ = xy + 2$ and $f(x,y) = -8xy - 8$. Appropriate Dirichlet boundary condition is used. The jump conditions are $[u] = 4 - 12x^2 - 12y^2$ and $[\beta u_n] = (2xy^2 - 2x^2y - 2x^3 - 10x)n_1 + (2y^3 - 2x^2y - 2xy^2 - 10y)n_2$. The interface has many kinks and some of their coordinates are irrational. There is no easy way to put the pairs of all kinks into different triangular cells. That is the reason why our method is only 1st order accurate, see Table 4.5. Also see Figure 4.5 for the numerical solution of our method with step size $\Delta x = \Delta y = \frac{2}{320}$.

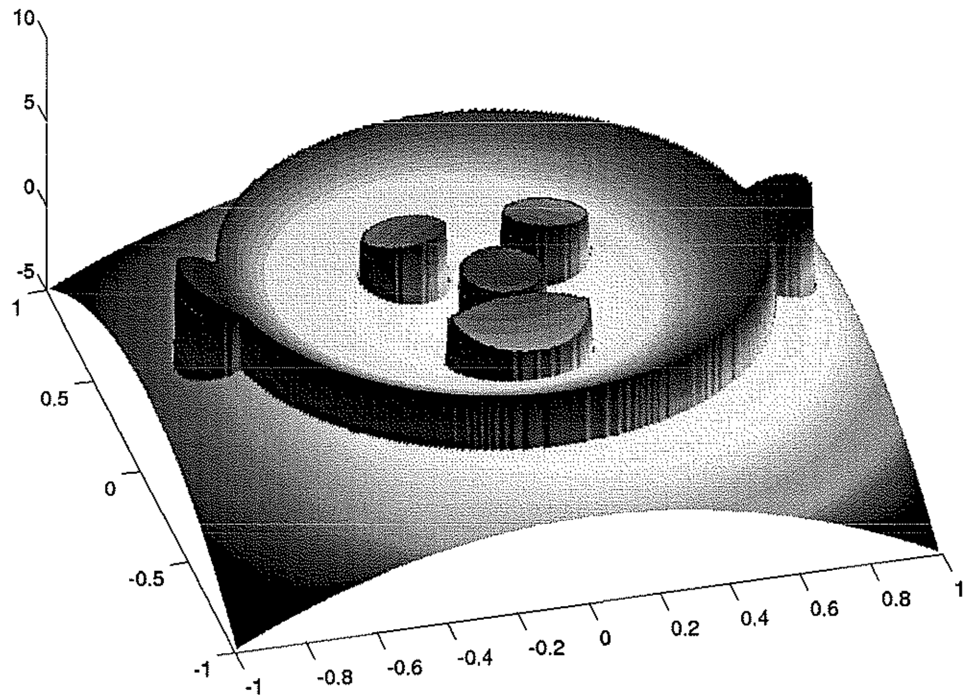

FIGURE 4.6. $\nabla \cdot (\beta \nabla u) = f(x,y)$, f piece-wise smooth, but not constant, β piece-wise smooth, but not constant, $[u] \neq$ *constant*, $[\beta u_n] \neq$ *constant*, Γ is smooth except at the corners of ears, the corners of mouth, and the touching point of the nose and the mouth.

$\Delta x = \Delta y$	L^∞-error in U	order	L^∞-error in ∇U	order	Δx, Δy	L^∞-error in U	order
$\frac{2}{40}$	6.06e-2		3.09e-1		$\frac{2}{41}, \frac{2}{39}$	3.12e-1	
$\frac{2}{80}$	1.64e-2	1.89	1.16e-1	1.41	$\frac{2}{81}, \frac{2}{79}$	2.60e-1	0.26
$\frac{2}{160}$	4.34e-3	1.92	4.71e-2	1.30	$\frac{2}{161}, \frac{2}{159}$	2.07e-1	0.33
$\frac{2}{320}$	1.15e-3	1.92	1.81e-2	1.40	$\frac{2}{321}, \frac{2}{319}$	1.60e-1	0.37

TABLE 4.6.

Example 6: Consider $\nabla \cdot (\beta \nabla u) = f(x,y)$ on $\Omega = [-1,1] \times [-1,1]$. The interface is defined by the zero level-set of the level-set function $\phi = \max(\min(\phi_1, \phi_2, \phi_3), \phi_4, \phi_5, \phi_6, \min(\phi_7, \phi_8))$, where $\phi_1 = \sqrt{x^2 + y^2 - 0.75^2 - 0.15^2}$, $\phi_2 = (x-0.75)^2 + y^2 - 0.15^2$, $\phi_3 = (x+0.75)^2 + y^2 - 0.15^2$, $\phi_4 = -\frac{0.1}{0.12}(x-0.2)^2 - \frac{0.12}{0.1}(y-0.22)^2 + 0.12 \cdot 0.1$, $\phi_5 = -\frac{0.1}{0.12}(x+0.2)^2 - \frac{0.12}{0.1}(y-0.22)^2 + 0.12 \cdot 0.1$, $\phi_6 = -x^2 - (y+0.08)^2 + 0.12^2$, $\phi_7 = -x^2 - (y+0.625)^2 + 0.425^2$ and $\phi_8 = -x^2 - (y+0.25)^2 + 0.2^2$. The domain is cut by the interface into Ω^-, where $\phi \leq 0$, and Ω^+, where $\phi > 0$. Note that the interfaces have kinks around ears and mouth, and the mouth and the nose are tangential at point $(0, -0.2)$, see Figure 4.6. On Ω^-, we let $u(x,y) = 7x^2 + 7y^2 + 1$, $\beta^- = (x^2 - y^2 + 3)/7$, $f(x,y) = -8y^2 + 8x^2 + 12$. On Ω^+, we let $u(x,y) = 5 - 5x^2 - 5y^2$, $\beta^+ = (xy + 2)/5$ and $f(x,y) = -8xy - 8$. Appropriate Dirichlet boundary condition is used. The jump conditions are $[u] = 4 - 12x^2 - 12y^2$ and $[\beta u_n] = (2xy^2 - 2x^2y - 2x^3 - 10x)n_1 + (2y^3 - 2x^2y - 2xy^2 - 10y)n_2$. Figure 4.6 shows the numerical solution with step size $\Delta x = \Delta y = \frac{2}{320}$. Table 4.6 shows the results of numerical accuracy tests on two different sets of grids. For the 1st set of the grids, all kinks and the touch point $(0, -0.2)$ are on grid points, and the pairs of legs of all kinks and touching point are located in different triangular cells. Our method achieve 2nd order accuracy in L^∞ norm for solutions and more than 1st order accuracy for the gradients of the solutions away from the interfaces. For the 2nd set of the grids, the pairs of legs of kinks and the touching point are in some triangular cells. Our method losses 2nd order accuracy but is still convergent in L^∞ norm for solutions.

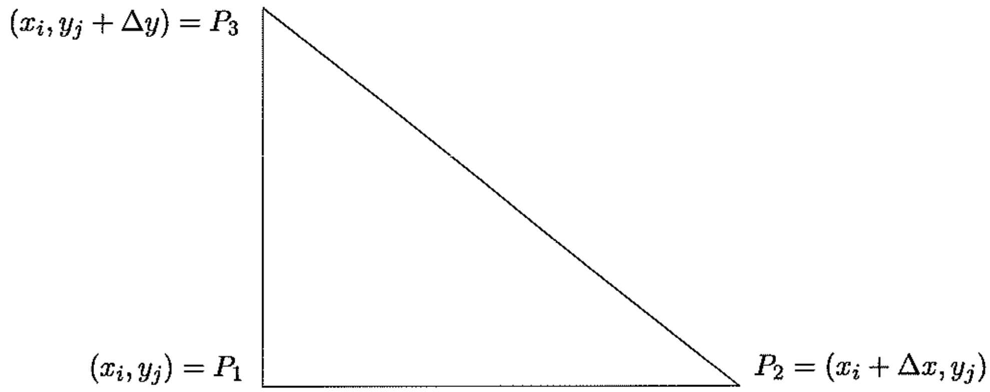

FIGURE 5.1. Case 0: the regular cell

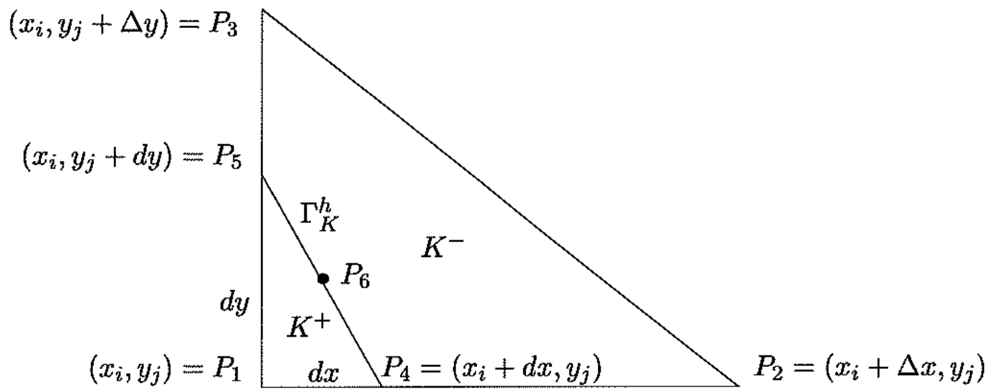

FIGURE 5.2. Case 1: the interface cutting through two legs of a triangle

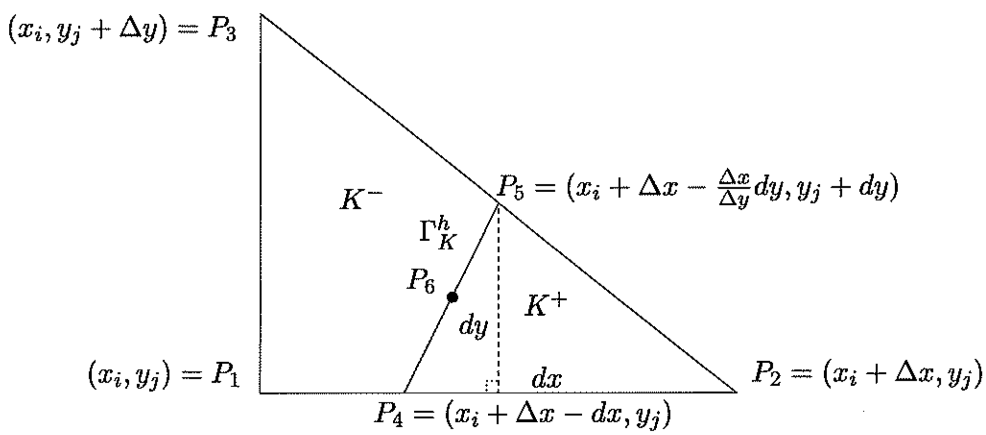

FIGURE 5.3. Case 2: the interface cutting through a leg and a hypotenuse of a triangle

5. APPENDIX: PROOF OF LEMMA 3.1

There are three typical cases for $U^h(u^h)$.
Case 0: If K is a regular triangle, see Figure 5.1. $U^h(u^h) = T^h(u^h)$ i.e.

$$U^h(u^h) = u(P_1) + \frac{u(P_2) - u(P_1)}{\Delta x}(x - x_i) + \frac{u(P_3) - u(P_1)}{\Delta y}(y - y_j). \tag{5.1}$$

Case 1: If K is an interface triangle, and the interface Γ cutting through two legs of K, see Figure 5.2. We have

$$U^h(u^h) = \begin{cases} u(P_1) + u_x^+(x - x_i) + u_y^+(y - y_j), & (x,y) \in K^+, \\ u(P_2) + u_x^-(x - x_i - \Delta x) + (\frac{u(P_3)-u(P_2)}{\Delta y} + \frac{\Delta x}{\Delta y} u_x^-)(y - y_j), & (x,y) \in K^-. \end{cases} \tag{5.2a}$$

Here $u_y^- = \frac{u(P_3)-u(P_2)}{\Delta y} + \frac{\Delta x}{\Delta y} u_x^-$. Three interface conditions are enforced as follows:

$$\begin{cases} dx\, u_x^+ & +(\Delta x - dx)\, u_x^- & & = r_1 \\ & (\Delta x - \frac{\Delta x}{\Delta y} dy)\, u_x^- & +dy\, u_y^+ & = r_2 \\ -dy\beta^+\, u_x^+ & +(dy + \frac{\Delta x}{\Delta y} dx)\beta^-\, u_x^- & -dx\beta^+\, u_y^+ & = r_3 \end{cases} \tag{5.2b}$$

where $r_1 = u(P_2) - u(P_1) + a(P_4)$, $r_2 = u(P_2) - u(P_1) + \frac{u(P_3)-u(P_2)}{\Delta y} dy + a(P_5)$, $r_3 = -\beta^- \frac{u(P_3)-u(P_2)}{\Delta y} dx + b(P_6)\sqrt{dx^2 + dy^2}$. Here $\beta^+ = \frac{1}{|K^+|} \int_{K^+} \beta$ and $\beta^- = \frac{1}{|K^-|} \int_{K^-} \beta$ are the averages of β in K^+ and K^- regions (In our numerical experiments, we take β^+ to be the β at the center of K^+, and β^- the β at the center of K^-).

Let

$$A = \begin{bmatrix} dx & (\Delta x - dx) & 0 \\ 0 & \Delta x - \frac{\Delta x}{\Delta y} dy & dy \\ -dy\beta^+ & (dy + \frac{\Delta x}{\Delta y} dx)\beta^- & -dx\beta^+ \end{bmatrix}, \quad A_1 = \begin{bmatrix} r_1 & (\Delta x - dx) & 0 \\ r_2 & \Delta x - \frac{\Delta x}{\Delta y} dy & dy \\ r_3 & (dy + \frac{\Delta x}{\Delta y} dx)\beta^- & -dx\beta^+ \end{bmatrix},$$
$$A_2 = \begin{bmatrix} dx & r_1 & 0 \\ 0 & r_2 & dy \\ -dy\beta^+ & r_3 & -dx\beta^+ \end{bmatrix}, \quad A_3 = \begin{bmatrix} dx & (\Delta x - dx) & r_1 \\ 0 & \Delta x - \frac{\Delta x}{\Delta y} dy & r_2 \\ -dy\beta^+ & (dy + \frac{\Delta x}{\Delta y} dx)\beta^- & r_3 \end{bmatrix}. \tag{5.2c}$$

Clearly

$$\begin{aligned} u_x^+ &= det(A_1)/det(A), & u_x^- &= det(A_2)/det(A), \\ u_y^+ &= det(A_3)/det(A), & u_y^- &= \frac{u(P_3)-u(P_2)}{\Delta y} + \frac{\Delta x}{\Delta y} u_x^-. \end{aligned} \tag{5.2d}$$

Note that the matrix A consists of information of grid, interface and coefficients β, and is independent from u^h, a or b. Also note that the determinants of matrices A_1, A_2 and A_3 are linear functions of u^h, a and b. Hence they could be rewritten in the forms of

$$\begin{aligned} u_x^+ &= c_{x,2}^+ \tfrac{u(P_2)-u(P_1)}{\Delta x} + c_{x,3}^+ \tfrac{u(P_3)-u(P_1)}{\Delta y} + c_{x,4}^+ a(P_4) + c_{x,5}^+ a(P_5) + c_{x,6}^+ b(P_6), \\ u_x^- &= c_{x,2}^- \tfrac{u(P_2)-u(P_1)}{\Delta x} + c_{x,3}^- \tfrac{u(P_3)-u(P_1)}{\Delta y} + c_{x,4}^- a(P_4) + c_{x,5}^- a(P_5) + c_{x,6}^- b(P_6), \\ u_y^+ &= c_{y,2}^+ \tfrac{u(P_2)-u(P_1)}{\Delta x} + c_{y,3}^+ \tfrac{u(P_3)-u(P_1)}{\Delta y} + c_{y,4}^+ a(P_4) + c_{y,5}^+ a(P_5) + c_{y,6}^+ b(P_6), \\ u_y^- &= c_{y,2}^- \tfrac{u(P_2)-u(P_1)}{\Delta x} + c_{y,3}^- \tfrac{u(P_3)-u(P_1)}{\Delta y} + c_{y,4}^- a(P_4) + c_{y,5}^- a(P_5) + c_{y,6}^- b(P_6). \end{aligned} \tag{5.2e}$$

Lemma 5.1. *All coefficients c in (5.2e) are finite and independent from u^h, a and b.*

Proof. From above discussion, it is easy to see that all coefficients c are independent from u^h, a and b. Below we prove that $c_{x,3}^+$ is finite. The proofs for the other coefficients are similar.

$$c_{x,3}^+ = \frac{-(\beta^+ - \beta^-)\,(\Delta x - dx)\,dx\,dy}{\beta^+\left((\Delta x - dx)\,dy^2 + \frac{\Delta x}{\Delta y}(\Delta y - dy)\,dx^2\right) + \beta^-\left(\frac{\Delta x}{\Delta y} dx^2\,dy + dy^2 dx\right)}. \tag{5.2f}$$

It could be thought as a function of dx and dy. It is smooth on $[0, \Delta x] \times [0, \Delta y]$ except one point $(dx, dy) = (0, 0)$. It is easy to see that if $dx = 0$ and $dy \neq 0$, or $dx \neq 0$ and $dy = 0$, $c_{x,3}^+ = 0$. Now denote $k = dy/dx \in (0, +\infty)$, and rewrite it as

$$c_{x,3}^+ = \frac{-(\beta^+ - \beta^-)\,(\Delta x - dx)k}{\beta^+\left((\Delta x - dx)k^2 + \frac{\Delta x}{\Delta y}(\Delta y - k\,dx)\right) + \beta^-\left(\frac{\Delta x}{\Delta y}\,dx\,k + dx\,k^2\right)}. \tag{5.2g}$$

Let dx goes to zero,

$$\lim_{dx\to 0, dy=kdx} c^+_{x,3} = \frac{-(\beta^+ - \beta^-)k}{\beta^+(k^2+1)}, \tag{5.2h}$$

hence the limit is bounded for any $k \in (0, +\infty)$,

$$\lim_{dx\to 0, dy=kdx} |c^+_{x,3}| \le |\frac{-\beta^+ + \beta^-}{\beta^+}|. \tag{5.2i}$$

Therefore $|c^+_{x,3}|$ is bounded, for any $(dx, dy) \in [0, \Delta x] \times [0, \Delta y]$. □

Case 2: If K is an interface triangle, and the interface Γ cutting through one leg and the hypotenuse of K, see Figure 5.3. We have

$$U^h(u^h) = \begin{cases} u(P_2) + u^+_x(x - x_i - \Delta x) + u^+_y(y - y_j), & (x,y) \in K^+, \\ u(P_1) + u^-_x(x - x_i) + \frac{u(P_3)-u(P_1)}{\Delta y}(y - y_j), & (x,y) \in K^-, \end{cases} \tag{5.3a}$$

where $u^-_y = \frac{u(P_3)-u(P_1)}{\Delta y}$. Three interface conditions are enforced as follows:

$$\begin{cases} (-dx)\, u^+_x & +(dx - \Delta x)\, u^-_x & & = r_1, \\ -\frac{\Delta x}{\Delta y}dy\, u^+_x & +(\frac{\Delta x}{\Delta y}dy - \Delta x)\, u^-_x & +dy\, u^+_y & = r_2 \\ dy\beta^+\, u^+_x & -dy\beta^-\, u^-_x & +(\frac{\Delta x}{\Delta y}dy - dx)\beta^+\, u^+_y & = r_3 \end{cases} \tag{5.3b}$$

where $r_1 = u(P_1) - u(P_2) + a(P_4)$, $r_2 = u(P_1) - u(P_2) + \frac{u(P_3)-u(P_1)}{\Delta y}dy + a(P_5)$, and $r_3 = \beta^- \frac{u(P_3)-u(P_1)}{\Delta y}(\frac{\Delta x}{\Delta y}dy - dx) + b(P_6)\sqrt{dy^2 + (\frac{\Delta x}{\Delta y}dy - dx)^2}$.

Let

(5.3c)

$$A = \begin{bmatrix} -dx & (dx - \Delta x) & 0 \\ -\frac{\Delta x}{\Delta y}dy & (\frac{\Delta x}{\Delta y}dy - \Delta x) & dy \\ dy\beta^+ & -dy\beta^- & (\frac{\Delta x}{\Delta y}dy - dx)\beta^+ \end{bmatrix}, \quad A_1 = \begin{bmatrix} r_1 & (dx - \Delta x) & 0 \\ r_2 & (\frac{\Delta x}{\Delta y}dy - \Delta x) & dy \\ r_3 & -dy\beta^- & (\frac{\Delta x}{\Delta y}dy - dx)\beta^+ \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -dx & r_1 & 0 \\ -\frac{\Delta x}{\Delta y}dy & r_2 & dy \\ dy\beta^+ & r_3 & (\frac{\Delta x}{\Delta y}dy - dx)\beta^+ \end{bmatrix}, \quad A_3 = \begin{bmatrix} -dx & (dx - \Delta x) & r_1 \\ -\frac{\Delta x}{\Delta y}dy & (\frac{\Delta x}{\Delta y}dy - \Delta x) & r_2 \\ dy\beta^+ & -dy\beta^- & r_3 \end{bmatrix}.$$

Clearly

$$\begin{aligned} u^+_x &= det(A_1)/det(A), & u^-_x &= det(A_2)/det(A), \\ u^+_y &= det(A_3)/det(A), & u^-_y &= \frac{u(P_3)-u(P_1)}{\Delta y}. \end{aligned} \tag{5.3d}$$

Same as in Case 1, the matrix A consists of information of grid, interface and coefficients β, and is independent from u^h, a or b. The determinants of matrices A_1, A_2 and A_3 are linear functions of u^h, a and b. Hence they could be rewritten in the forms of

$$\begin{aligned} u^+_x &= d^+_{x,2}\frac{u(P_2)-u(P_1)}{\Delta x} + d^+_{x,3}\frac{u(P_3)-u(P_1)}{\Delta y} + d^+_{x,4}a(P_4) + d^+_{x,5}a(P_5) + d^+_{x,6}b(P_6), \\ u^-_x &= d^-_{x,2}\frac{u(P_2)-u(P_1)}{\Delta x} + d^-_{x,3}\frac{u(P_3)-u(P_1)}{\Delta y} + d^-_{x,4}a(P_4) + d^-_{x,5}a(P_5) + d^-_{x,6}b(P_6), \\ u^+_y &= d^+_{y,2}\frac{u(P_2)-u(P_1)}{\Delta x} + d^+_{y,3}\frac{u(P_3)-u(P_1)}{\Delta y} + d^+_{y,4}a(P_4) + d^+_{y,5}a(P_5) + d^+_{y,6}b(P_6), \\ u^-_y &= d^-_{y,2}\frac{u(P_2)-u(P_1)}{\Delta x} + d^-_{y,3}\frac{u(P_3)-u(P_1)}{\Delta y} + d^-_{y,4}a(P_4) + d^-_{y,5}a(P_5) + d^-_{y,6}b(P_6). \end{aligned} \tag{5.3e}$$

Lemma 5.2. *All coefficients d in (5.3e) are finite and independent from u^h, a and b.*

Proof. The proof is the same as the proof of Lemma 5.1, and is omitted here. □

From above discussion, we complete the proof of Lemma 3.1 and all coefficients c and d are independent from u^h, a and b.

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REFERENCES

[1] Zhiming Chen and Jun Zou, *Finite Element methods and Their Convergence for Elliptic and Parabolic Interface Problems*, Numerische Mathematik, vol. 79, pp. 175-202 (1998).

[2] Fedkiw, R., Aslam, T., Merriman, B., and Osher, S., *A Non-Oscillatory Eulerian Approach to Interfaces in Multimaterial Flows (The Ghost Fluid Method)*, J. Computational Physics, vol. 152, n. 2, 457-492 (1999).

[3] Fedkiw, R., and Liu, X.-D., *The Ghost Fluid Method for Viscous Flows, Progress in Numerical Solutions of Partial Differential Equations*, Arcachon, France, edited by M. Hafez, July 1998.

[4] Grisvard, P. *Elliptic problems in Nonsmooth Domains – Monographs and Studis in Mathematics*, Pitman Advanced Publishing Program. ISSN 0743-0329, (1985).

[5] Hou, T., Li, Z., Osher, S., Zhao, H., *A Hybrid Method for Moving Interface Problems with Application to the Hele-Shaw Flow*, Journal of Comput. Phys., vol. 134, pp. 236-252 (1997).

[6] Kang, M., Fedkiw, R., and Liu. X.-D., *A Boundary Condition Capturing Mathod for Multiphase Incompressible Flow*, Journal of Comput. Phys. (submitted).

[7] LeVeque, R.J. and Li, Z., *The Immersed Interface Method for Elliptic Equations with Discontinuous Coefficients and Singular Sources*, SIAM J. Numer. Anal., vol. 31, 1019 (1994).

[8] Li, Z., *A Fast Iterative Algorithm for Elliptic Interface Problems*, SIAM J. Numer. Anal., vol. 35, no. 1, pp. 230-254, (1998).

[9] Li, Z., *A Note on Immersed Interface Method for for Three Dimensional Elliptic Equations*, Computers Math. Appl. 35, 9-17, 1996.

[10] Li, Z., T. Lin, and X. Wu, *New Cartesian grid methods for interface problems using the finite element formulation*, Preprint.

[11] Xu-Dong Liu, Ronald P. Fedkiw and Myungjoo Kang *A Boundary Condition Capturing Method for Poisson's Equation on Irregular Domains* Journal of Computational Physics, vol. 160, no 1, pp.151-178, (2000).

[12] Liu, X.-D. and Sideris, T., *Convergence of The Ghost Fluid Method for Elliptic Equations with Interfaces*, (submitted).

[13] Mayo, A., *The Fast Solution of Poisson's and the Biharmonic Equations in Irregular Domains*, SIAM J. Numer. Anal., vol. 21, no. 2, pp. 285-299 (1984).

[14] Mayo, A., *Fast High Order Accurate Solutions of Laplace's Equation on Irregular Domains*, SIAM J. Sci. Stat. Comput., vol. 6, no. 1, pp. 144-157 (1985).

[15] Necas, J., *Introduction to the Theory of Nonlinear Elliptic Equations*, Teubner-Texte zur Mathematik. Band 52, ISSN 0138-502X, (1983).

[16] Peskin, C., *Numerical Analysis of Blood Flow in the Heart*, Journal of Comput. Phys., vol. 25, pp. 220-252 (1977).

[17] Peskin, C. and Printz, B., *Improved Volume Conservation in the Computation of Flows with Immersed Elastic Boundaries*, J. Comput. Phys., vol. 105, pp. 33-46 (1993).

[18] Osher, S. and Sethian, J.A., *Fronts Propagating with Curvature Dependent Speed: Algorithms Based on Hamilton-Jacobi Formulations*, Journal of Comput. Phys., vol. 79, n. 1, pp. 12-49, (1988).

[19] Sussman, M., Smereka, P. and Osher, S., *A level set approach for computing solutions to incompressible two-phase flow*, J. Comput. Phys., vol. 114, pp. 146-154 (1994).

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