UCLA
COMPUTATIONAL AND APPLIED MATHEMATICS

Minimizers of Cost-Functions Involving Non-Smooth Data-Fidelity
Terms. Application to the Processing of Outliers

Mila Nikolova

July 2002

CAM Report 02-40

Minimizers of cost-functions involving non-smooth data-fidelity terms. Application to the processing of outliers

Mila Nikolova

CNRS URA820–ENST Dpt. TSI, ENST, 46 rue Barrault, 75013 Paris, France (`nikolova@enst.fr`).

Abstract

We present a theoretical study of the recovery of an unknown vector $x \in \mathrm{R}^p$ (a signal, an image) from noisy data $y \in \mathrm{R}^q$ by minimizing with respect to x a regularized cost-function $\mathcal{F}(x, y) = \Psi(x, y) + \alpha\Phi(x)$, where Ψ is a data-fidelity term, Φ is a smooth regularization term and $\alpha > 0$ is a parameter. Typically, $\Psi(x, y) = \|Ax - y\|^2$ where A is a linear operator. The data-fidelity terms Ψ involved in regularized cost-functions are generally smooth functions; only a few papers make an exception and they consider restricted situations. Non-smooth data-fidelity terms are avoided in image processing. In spite of this, we consider both smooth and non-smooth data-fidelity terms. Our ambition is to catch essential features exhibited by the local minimizers of regularized cost-functions in relation with the smoothness of the data-fidelity term.

In order to fix the context of our study, we consider $\Psi(x, y) = \sum_i \psi(a_i^T x - y_i)$, where a_i^T are the rows of A and ψ is C^m on $\mathrm{R} \setminus \{0\}$. We show that if $\psi'(0^-) < \psi'(0^+)$, typical data y give rise to local minimizers $\hat{x}$ of $\mathcal{F}(., y)$ which fit exactly a certain number of the data entries: there is a possibly large set $\hat{h}$ of indexes such that $a_i^T \hat{x} = y_i$ for every $i \in \hat{h}$. In contrast, if ψ is smooth on R, for almost every y, the local minimizers of $\mathcal{F}(., y)$ do not fit any entry of y. Thus, the possibility that a local minimizer fits some data entries is due to the non-smoothness of the data-fidelity term. This is a strong mathematical property which is useful in practice. By way of application, we construct a cost-function allowing aberrant data (outliers) to be detected and to be selectively smoothed. Our numerical experiments advocate the use of non-smooth data-fidelity terms in regularized cost-functions for special purposes in image and signal processing.

Key words: inverse problems, MAP estimation, non-smooth analysis, perturbation analysis, proximal analysis, reconstruction, regularization, stabilization, outliers, total variation, variational methods

1 Introduction

We consider the general problem where a sought vector (*e.g.* an image, a signal) $\hat{x} \in \mathrm{R}^p$ is obtained from noisy data $y \in \mathrm{R}^q$ by minimizing a regularized cost-function $\mathcal{F} : \mathrm{R}^p \times \mathrm{R}^q \to \mathrm{R}$ of the form:

$$\mathcal{F}(x, y) = \Psi(x, y) + \alpha\Phi(x), \tag{1}$$

where typically $\Psi : \mathrm{R}^p \times \mathrm{R}^q \to \mathrm{R}$ is a data-fidelity term and $\Phi : \mathrm{R}^p \to \mathrm{R}$ is a regularization function, with $\alpha > 0$ a parameter. In many applications, the relation between x and y is modelled by $y_i = a_i^T x + n_i$ for $i = 1, \ldots, q$, where $a_i^T : \mathrm{R}^p \to \mathrm{R}$ are linear operators and n_i accounts for perturbations. We focus on such situations and assume that a_i^T, $i = 1, \ldots, q$ are known. The relevant data-fidelity term assumes the form

$$\Psi(x, y) = \sum_{i=1}^{q} \psi_i(a_i^T x - y_i) \tag{2}$$

where $\psi_i : \mathrm{R} \to \mathrm{R}$, $i = 1, \ldots, q$, are continuous functions which decrease on $(-\infty, 0]$ and increase on $[0, +\infty)$. Usually, $\psi_i = \psi$, for all i. A pretty general choice is $\psi(t) = |t|^\rho$ which yields [31, 4]

$$\Psi(x, y) = \sum_{i=1}^{q} |a_i^T x - y_i|^\rho. \tag{3}$$

Let $A \in \mathrm{R}^{q \times p}$ be the matrix whose rows are a_i^T for $i = 1, \ldots, q$. This matrix can be ill-posed, or singular, or invertible. Most often, $\Psi(x, y) = \|Ax - y\|^2$, that is $\psi(t) = t^2$. Such data-fidelity terms are currently used in de-noising, in de-blurring and in numerous inverse problems [37, 35, 13, 33, 1, 14, 38]. In a statistical framework, Ψ accounts for both the distortion and the noise intervening between the original x and the device recording the data y. The above quadratic form of Ψ corresponds to white Gaussian noise $\{n_i\}$. Recall that many papers are dedicated to the minimization of $\Psi(., y)$ *alone* and of the form (3), *i.e.* $\mathcal{F} = \Psi$, mainly for $\psi(t) = t^2$ [22], in some cases for $\psi(t) = |t|$ [8], but functions $\psi(t) = |t|^\rho$ for different values for ρ in the range $(0, \infty]$ have also been considered [31, 30]. Specific data-fidelity terms arise in applications such as emission and transmission computed tomography, X-ray radiography, eddy-currents evaluation and many others [23, 20, 34, 10]. In general, for every y, the data-fidelity term $\Psi(., y)$ is a function which is smooth, and usually convex. The introduction of non-smooth data fidelity terms in regularized cost-functions (1) remains very unusual. Only a few papers make an exception; let us cite [2, 3] where Ψ corresponds to $\psi(t) = |t|$ and $a_i = 1$, $\forall i$. Non-smooth data-fidelity terms Ψ are avoided in image processing, for instance. In spite of this, we analyze the effects produced by both smooth and non-smooth data-fidelity terms Ψ. In the latter case we suppose that $\{\psi_i\}$ *are any functions which are* C^m*-smooth on* $\mathrm{R} \setminus \{0\}$, $m \geq 2$, *whereas at zero they admit finite side derivatives which satisfy* $\psi_i'(0^-) < \psi_i'(0^+)$.

The regularization term Φ usually takes the form

$$\Phi(x) = \sum_{i=1}^{r} \varphi(\|G_i^T x\|), \tag{4}$$

where $G_i^T : \mathrm{R}^p \to \mathrm{R}^s$, for $s \in \mathrm{N}^*$, are linear operators, *e.g.* operators yielding the differences between neighboring samples, $\|.\|$ stands for a norm on R^s and $\varphi : \mathrm{R} \to \mathrm{R}$ is a potential function. In a Bayesian estimation framework, Φ is the prior energy of the unknown x modelled using a Markov random field [6, 17, 24]. Several customarily used potential functions φ are [20, 29, 21, 33, 9, 7, 39, 36]

$$
\begin{array}{lll}
\mathrm{L}^\nu & \varphi(t) = |t|^\nu, \ 1 \leq \nu \leq 2, \\
\text{Lorentzian} & \varphi(t) = \nu t^2 / (1 + \nu t^2), \\
\text{Concave} & \varphi(t) = \nu|t| / (1 + \nu|t|), \\
\text{Gaussian} & \varphi(t) = 1 - \exp(-\nu t^2), \\
\text{Huber} & \varphi(t) = t^2 \text{ if } |t| \leq \nu, \ \ \varphi(t) = \nu(\nu + 2|t - \nu|) \text{ if } |t| > \nu, \\
\text{Mean-field} & \varphi(t) = -\log\left(\exp(-\nu t^2) + 1\right),
\end{array}
\tag{5}
$$

where $\nu > 0$ is a parameter. Being convex and differentiable, the function L^ν for $1 < \nu \leq 2$ is preferred in many applications requiring intensive computation [9, 10]. In our paper, Φ *in (1) is any* C^m*-smooth function*, with $m \geq 2$.

The visual aspect of a minimizer of a cost-function is determined on the one hand by the data, on the other hand by the shape of the cost-function. Our ambition is to catch essential features expressed by the local minimizers of cost-functions of the form (1)-(2) in relation with the smoothness of the data-fidelity term Ψ. Note that all our results hold for local minimizers, and hence for global minimizers as well; so we systematically

speak of local minimizers. There is a striking distinction in the behavior of the local minimizers relevant to smooth and to non-smooth data-fidelity terms. It concerns the possibility to fit *exactly* a certain number of the data entries, *i.e.* that for y given, a local minimizer $\hat{x}$ of $\mathcal{F}(., y)$ satisfies $a_i^T \hat{x} = y_i$ for some, or even for many, indexes i (§ 2). Intuitively, one is unlikely to obtain such minimizers, especially when data are noisy. *Our main result states that for $\mathcal{F}$ of the form (1)-(2), with Ψ non-smooth as specified, typical data y give rise to local minimizers $\hat{x}$ which fit a certain number of the data entries, i.e. there is a nonempty set $\hat{h}$ of indexes such that $a_i^T \hat{x} = y_i$ for every $i \in \hat{h}$* (§ 3 and § 4). This effect is due to the non-differentiability of Ψ since it cannot occur when $\mathcal{F}$ is differentiable (§ 5). The obtained result is a strong mathematical property which can be used in different ways. Based on it, we construct a cost-function allowing aberrant data (outliers) to be detected and to be selectively smoothed from signals, or from images, or from noisy data, while preserving efficiently all the non-aberrant entries (§ 7). This is illustrated using numerical experiments.

Readers may associate cost-functions where Ψ is non-smooth (*e.g.* $\psi(t) = |t|$) with cost-functions where Φ is smooth and Φ is non-smooth, *e.g.* $\Psi(x, y) = \|Ax - y\|^2$ and $\varphi(t) = |t|$ in (4), as in total variation methods [33, 1, 14, 12]. Since the latter methods arouse an increasing interest in the area of image and signal restoration, we compare in §6 non-smooth regularization to the cost-functions considered in this paper. To this end, we use some previous results [26, 27] (§ 6) and illustrate the strikingly different visual effects they produce (§ 7).

2   The problem of an exact fit for some data entries

We shall use the symbol $\|.\|$ to denote the ℓ_2-norm of vectors. Next, we denote by N^* the positive integers and $\mathrm{R}_+ = \{t \in \mathrm{R} : t \geq 0\}$. The letter S will systematically denote the centered, unit sphere in R^n, say $S := \{x \in \mathrm{R}^n : \|x\| = 1\}$, for whatever dimension n appropriate to the context. For $x \in \mathrm{R}^n$ and $\rho > 0$, we put $B(x, \rho) := \{x' \in \mathrm{R}^n : \|x' - x\| < \rho\}$. For any $i = 1, \ldots, n$ the letter e_i represents the ith vector of the canonical basis of R^n ($e_i[i] = 1$ and $e_i[j] = 0, \forall j \neq i$). The closure of a set N will be denoted $\overline{N}$. For a subspace T, its orthogonal complement is denoted $T^\perp$. If $f : \mathrm{R}^p \times \mathrm{R}^q \to \mathrm{R}$ depends on two variables, its kth differential with respect to the jth variable is denoted $D_j^k f$. The notation $f \in \mathcal{C}^m(N)$ means that the function f is $\mathcal{C}^m$-smooth on the set N. For a discrete, finite set $h \subset \{1, \ldots, n\}$, with $n \in \mathrm{N}^*$, the symbol $\#h$ stands for cardinality and h^c for complementary. Next we introduce a set-valued function which is constantly evoked in what follows.

Definition 1 *Let $\mathcal{H}$ be the function which with every $x \in \mathrm{R}^p$ and $y \in \mathrm{R}^q$ associates the following set:*

$$(x, y) \quad \to \quad \mathcal{H}(x, y) = \left\{ i \in \{1, \ldots, q\} : a_i^T x = y_i \right\}. \tag{6}$$

Given y and a local minimizer $\hat{x}$ of $\mathcal{F}(., y)$, the set of all data entries which are fitted exactly by $\hat{x}$ reads $\hat{h} := \mathcal{H}(\hat{x}, y)$. Furthermore, with every $h \subseteq \{1, \ldots, q\}$ we associate the following sets:

$$(h, y) \to \quad \Theta_h(y) \quad := \{x \in \mathrm{R}^p : a_i^T x = y_i, \forall i \in h \text{ and } a_i^T x \neq y_i, \forall i \in h^c\}, \tag{7}$$

$$h \quad \to \quad T_h \quad := \{u \in \mathrm{R}^p : a_i^T u = 0, \forall i \in h\}, \tag{8}$$

$$h \quad \to \quad M_h \quad := \{(x, y) \in \mathrm{R}^p \times \mathrm{R}^q : a_i^T x = y_i, \forall i \in h \text{ and } a_i^T x \neq y_i, \forall i \in h^c\}. \tag{9}$$

Note that for every y and $h \neq \emptyset$, the sets $\Theta_h(y)$ and M_h are composed of a finite number of connected components whereas their closures $\overline{\Theta_h(y)}$ and $\overline{M_h}$, respectively, are affine subspaces. The family of all Θ_h,

when h ranges over all the subsets of $\{1, \ldots, q\}$, forms a partition of R^p. Observe that for $y \in \mathrm{R}^q$ fixed, $\{x \in \mathrm{R}^p : (x, y) \in M_h\} = \Theta_h(y)$. Notice also the equivalences:

$$\mathcal{H}(x', y') = h \iff x' \in \Theta_h(y') \iff (x', y') \in M_h. \tag{10}$$

The theory in this paper is developed by analyzing how the local minimizers of every $\mathcal{F}(., y)$ behave under small variations of the data y. We thus consider local minimizer functions.

Definition 2 *Let $f : \mathrm{R}^p \times \mathrm{R}^q \to \mathrm{R}$ and $N \subseteq \mathrm{R}^q$. The family $f(., N) := \{f(., y) : y \in N\}$ is said to admit a local minimizer function $\mathcal{X} : N \to \mathrm{R}^p$ if for any $y \in N$ the function $f(., y)$ has a strict local minimum at $\mathcal{X}(y)$.*

The next lemma addresses local minimizer functions relevant to smooth cost-functions.

Lemma 1 *Let $\mathcal{F} : \mathrm{R}^p \times \mathrm{R}^q$ be a $\mathcal{C}^m$-function with $m \geq 2$. For $y \in \mathrm{R}^q$, assume that $\hat{x} \in \mathrm{R}^p$ is such that $D_1 \mathcal{F}(\hat{x}, y) = 0$ and $D_1^2 \mathcal{F}(\hat{x}, y)$ is positive definite.*

Then there exist a neighborhood $N \subset \mathrm{R}^q$ containing y and a $\mathcal{C}^{m-1}$-function $\mathcal{X} : N \to \mathrm{R}^p$ such that for every $y' \in N$ we have $D_1 \mathcal{F}(\mathcal{X}(y'), y') = 0$ and $D_1^2 \mathcal{F}(\mathcal{X}(y'), y')$ is positive definite. In particular, $\hat{x} = \mathcal{X}(y)$.

Equivalently, $\mathcal{X} : N \to \mathrm{R}^p$ is a local minimizer function relevant to $\mathcal{F}(., N)$ such that $D_1^2 \mathcal{F}(\mathcal{X}(y'), y')$ is positive definite for every $y' \in N$.

Proof. Being a local minimizer of $\mathcal{F}(., y)$, $\hat{x}$ satisfies $D_1 \mathcal{F}(\hat{x}, y) = 0$. We focus on the equation $D_1 \mathcal{F}(x', y') = 0$ in the vicinity of $(\hat{x}, y)$ and notice that $D_1^2 \mathcal{F}(\hat{x}, y)$ determines an isomorphism from R^p to itself. From the implicit functions theorem [5], there exist $\rho_1 > 0$ and a unique $\mathcal{C}^{m-1}$-function $\mathcal{X} : B(y, \rho_1) \to \mathrm{R}^p$ such that $D_1 \mathcal{F}(\mathcal{X}(y'), y') = 0$, $\forall y' \in B(y, \rho_1)$. Furthermore, since $y' \to \det D_1^2 \mathcal{F}(\mathcal{X}(y'), y')$ is continuous and $\det D_1^2 \mathcal{F}(\hat{x}, y) > 0$, there is $\rho_2 \in (0, \rho_1]$ such that $\det D_1^2 \mathcal{F}(\mathcal{X}(y'), y') > 0$, $\forall y' \in B(y, \rho_2)$. $\diamond$

Remark 1 (on the conditions required in Lemma 1.) The minimizers of $\mathcal{C}^m$-functions of the form

$$\mathcal{F}(x, y) = \|Ax - y\|^2 + \alpha \Phi(x)$$

are extensively studied in [16]. It is shown there that if $\mathrm{rank} A = p$, and under some assumptions ensuring that $\mathcal{F}(., y)$ admits local minimizers for every $y \in \mathrm{R}^q$, the data domain R^q contains a subset N whose interior is dense in R^q, such that for every $y \in N$, every local minimizer $\hat{x}$ of the corresponding $\mathcal{F}(., y)$ is strict and $D_1^2 \mathcal{F}(\hat{x}, y)$ is positive definite. Reciprocally, all data leading to minimizers at which the conditions of Lemma 1 fail, belong to a closed negligible subset of R^q: the chance of acquiring data placed in such subsets is null. $\diamond$

The central question of this paper is how the shape of a cost-function $\mathcal{F}$ favors, or inhibits, the possibility that a local minimizer $\hat{x}$ of $\mathcal{F}(., y)$, for $y \in \mathrm{R}^q$, fits a certain number of the entries of this same y, $i.e.$ that the set $\hat{h} := \mathcal{H}(\hat{x}, y)$ is nonempty. It will appear that this possibility is closely related to the smoothness of Ψ. Recall some facts about non-smooth functions [32].

Definition 3 *Let $E_0 \subseteq \mathrm{R}^p$ be an affine subspace and E be the relevant vector space. Consider a function $f : E_0 \to \mathrm{R}$, and let $x \in E_0$ and $u \in E$. The function f admits a one-sided derivative at x in the direction of*

4

$u \neq 0$, denoted by $\delta g(x)(u)$, if the following (possibly infinite) limit exists:

$$\delta f(x)(u) := \lim_{t \downarrow 0} \frac{f(x+tu) - f(x)}{t}.$$

If $u = 0$, put $\delta f(x)(0) = 0$.

The downward pointing arrow above means that $t \in R_+$ converges to zero by positive values. If f is differentiable at x, then $\delta f(x)(u) = Df(x).u$. If $f : R \to R$, we have $\delta f(x)(1) = f'(x^+)$. The left-sided derivative of f at x for u is $-\delta f(x)(-u)$. In the following, $\delta_1 \mathcal{F}$ will address one-sided derivatives of $\mathcal{F}$ with respect to its first argument.

3   Cost-functions with non-smooth data-fidelity terms

In § 3 and § 4 we focus on cost-functions which read

$$\mathcal{F}(x,y) \quad = \quad \Psi(x,y) + \alpha\Phi(x,y), \tag{11}$$

$$\Psi(x,y) \quad = \quad \sum_{i=1}^{q} \psi(a_i^T x - y_i), \tag{12}$$

where $\psi : R \to R$ is C^m on $R \setminus \{0\}$, with $m \geq 2$, whereas at zero it admits finite side derivatives satisfying $\psi'(0^-) < \psi'(0^+)$. The term $\Phi : R^p \times R^q \to R$ is any C^m-function. This formulation allows us to address data-fidelity terms composed of a non-smooth function Ψ and of a smooth function $\tilde{\Psi}$, since we can write down $\Phi(x,y) = \tilde{\Psi}(x,y) + \tilde{\Phi}(x)$ with $\tilde{\Phi}$ a regularization term. _E.g._, we can have $\Phi(x,y) = \sum_i \left(\phi_i(B_i^T x - y_{q_i}) + \varphi_i(G_i^T x) \right)$, where $\phi_i : R^{q_i} \to R$ and $\varphi_i : R^{p_i} \to R$ are C^m-functions, $y_{q_i} \in R^{q_i}$ are data, and $B_i \in R^{q_i \times p}$ and $G_i \in R^{p_i \times p}$, with $p_i \in N^*$ and $q_i \in N^*$.

Remark 2 The results presented in § 3 and § 4 are developed for Ψ of the form (12), that is $\psi_i = \psi$ for all i, but we should emphasize that they remain true for Ψ of the form (2), provided that all ψ_i, for $i = 1, \ldots, q$, have finite side derivatives at zero satisfying $\psi_i'(0^-) < \psi_i'(0^+)$. The proofs are straightforward to extend to this situation but at the expense of complicated notations which risk clouding the presentation.

We start by providing a sufficient condition for a strict local minimum.

Proposition 1 _For $y \in R^q$, let $\mathcal{F}(.,y) : R^p \to R$ be of the form (11)-(12) where $\Phi \in C^m(R^p \times R^q)$ for $m \geq 1$ and $\psi \in C^m(R \setminus \{0\})$ satisfies $-\infty < \psi'(0^-) < \psi'(0^+) < +\infty$. Let $\hat{x} \in R^p$ be such that_

1. _the restricted function $\mathcal{F}|_{\overline{\Theta_{\hat{h}}(y)}}(.,y) : \overline{\Theta_{\hat{h}}(y)} \to R$ reaches a strict local minimum at $\hat{x}$;_

2. _$\delta_1 \mathcal{F}(\hat{x}, y)(v) > 0, \forall v \in T_{\hat{h}}^{\perp} \cap S$,_

where $\hat{h} := \mathcal{H}(\hat{x}, y)$, $\Theta_{\hat{h}}(y)$ and $T_{\hat{h}}$ are determined according to (6), (7) and (8), respectively.

Then $\mathcal{F}(.,y)$ reaches a strict local minimum at $\hat{x}$.

Proof.   The result is a tautology if $\hat{h} = \emptyset$ since then $\Theta_{\hat{h}}(y) = R^p$. So consider that $\hat{h}$ is nonempty. First of all, we put $\mathcal{F}$ into a more convenient form. Define

$$\tilde{\psi}(t) := \psi(t) - \frac{t}{2} \left(\psi'(0^-) + \psi'(0^+) \right) - \psi(0). \tag{13}$$

5

Now we have

$$\tilde{\psi}'(0^+) = -\tilde{\psi}'(0^-) > 0 \quad \text{and} \quad \tilde{\psi}(0) = 0, \tag{14}$$

which will allow important simplifications. By means of $\tilde{\psi}$, the cost-function $\mathcal{F}$ assumes the form

$$\begin{aligned}
\mathcal{F}(x,y) &= \tilde{\Psi}(x,y) + \tilde{\Phi}(x,y), \tag{15}\\
\text{where} \quad \tilde{\Psi}(x,y) &= \sum_{i=1}^{q} \tilde{\psi}(a_i^T x - y_i)\\
\text{and} \quad \tilde{\Phi}(x,y) &= \sum_{i=1}^{q} \frac{\psi'(0^-) + \psi'(0^+)}{2}(a_i^T x - y_i) + q\psi(0) + \alpha\Phi(x,y).
\end{aligned}$$

Both $\tilde{\Psi}$ and $\tilde{\Phi}$ satisfy the assumptions about Ψ and Φ, respectively. Henceforth, we deal with the formulation of $\mathcal{F}$ given in (15). For notational convenience, we systematically write ψ for $\tilde{\psi}$, Ψ for $\tilde{\Psi}$ and Φ for $\tilde{\Phi}$.

Let us consider the altitude increment of $\mathcal{F}(.,y)$ at $\hat{x}$ in the direction of an arbitrary $u \in S$

$$\mathcal{F}(\hat{x} + tu, y) - \mathcal{F}(\hat{x}, y) \quad \text{for } t \in \mathrm{R}_+.$$

In order to avoid misunderstandings, u_0 will denote a vector of $T_{\hat{h}}$ and $u_\perp$ a vector of $T_{\hat{h}}^\perp$. Using the fact that every $u \in S$ has a unique decomposition into

$$u = u_0 + u_\perp \quad \text{with} \quad u_0 \in T_{\hat{h}} \cap \overline{B(0,1)} \quad \text{and} \quad u_\perp \in T_{\hat{h}}^\perp \cap \overline{B(0,1)}, \tag{16}$$

we decompose the altitude increment of $\mathcal{F}(.,y)$ accordingly:

$$\begin{aligned}
\mathcal{F}(\hat{x} + tu, y) - \mathcal{F}(\hat{x}, y) &= \mathcal{F}(\hat{x} + tu_0 + tu_\perp, y) - \mathcal{F}(\hat{x} + tu_0, y) \tag{17}\\
&+ \mathcal{F}(\hat{x} + tu_0, y) - \mathcal{F}(\hat{x}, y). \tag{18}
\end{aligned}$$

The term on the right side of (17) is analyzed with the aid of assumption 2. In order to calculate the side derivative $\delta_1\mathcal{F}(\hat{x}, y)$, we decompose $\mathcal{F}$ into

$$\begin{aligned}
\mathcal{F}(x',y') &= \Psi_{\hat{h}}(x',y') + \mathcal{F}_{\hat{h}}(x',y'), \tag{19}\\
\text{where} \quad \Psi_{\hat{h}}(x',y') &:= \sum_{i \in \hat{h}} \psi(a_i^T x' - y_i')\\
\text{and} \quad \mathcal{F}_{\hat{h}}(x',y') &= \sum_{i \in \hat{h}^c} \psi(a_i^T x - y_i') + \alpha\Phi(x',y').
\end{aligned}$$

This decomposition is recurrently used in the following.

Remark 3 The function $\mathcal{F}_{\hat{h}}$ is $\mathcal{C}^m$ on a neighborhood of $(\hat{x}, y)$ which contains $B(\hat{x}, \sigma) \times B(y, \sigma)$ for

$$\sigma := \frac{1}{2(\|a\|_\infty + 1)} \min_{i \in \hat{h}^c} |a_i^T \hat{x} - y_i|, \tag{20}$$

$$\|a\|_\infty := \max_{i=1}^{q} \|a_i\|. \tag{21}$$

Indeed, for every $(x', y') \in B(\hat{x}, \sigma) \times B(y, \sigma)$ we have

$$\begin{aligned}
i \in \hat{h}^c \quad \Rightarrow \quad |a_i^T x' - y_i'| &= \left|(a_i^T \hat{x} - y_i) + a_i^T(x' - \hat{x}) + (y_i - y_i')\right| \tag{22}\\
&\geq \left|a_i^T \hat{x} - y_i\right| - \left|a_i^T(x' - \hat{x})\right| - |y_i - y_i'|\\
&\geq \min_{i \in \hat{h}^c} |a_i^T \hat{x} - y_i| - \|a\|_\infty \sigma - \sigma = (\|a\|_\infty + 1)\sigma > 0,
\end{aligned}$$

since clearly $\|a\|_\infty > 0$ and $\sigma > 0$. ◇

In contrast, $\Psi_{\hat{h}}$ is non-smooth at $(\hat{x}, y)$. Using Definition 3 we calculate that for every $u \in \mathbf{R}^p$

$$\delta_1 \mathcal{F}(x, y)(u) = \delta_1 \Psi_{\hat{h}}(\hat{x}, y)(u) + D\mathcal{F}_{\hat{h}}(\hat{x}, y).u \tag{23}$$

$$\text{where} \quad \delta_1 \Psi_{\hat{h}}(\hat{x}, y)(u) = \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u|, \tag{24}$$

since $\delta\psi(a_i^T \hat{x} - y_i)(u) = \lim_{t \downarrow 0} \psi(t a_i^T u)/t = \psi'(0^+)|a_i^T u|$, for every $i \in \hat{h}$, which result accounts for (14). Notice that $\delta_1 \Psi_{\hat{h}}(\hat{x}, y)(u) = \delta_1 \Psi_{\hat{h}}(\hat{x}, y)(-u) \geq 0$, for every $u \in \mathbf{R}^p$. Applying assumption 2 to both $u_\perp \in T_{\hat{h}}^\perp$ and $-u_\perp$ yields

$$|D\mathcal{F}_{\hat{h}}(\hat{x}, y).u_\perp| < \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u_\perp|, \quad \forall u_\perp \in T_{\hat{h}}^\perp. \tag{25}$$

Consider now the function

$$f : T_{\hat{h}}^\perp \cap S \quad \to \quad \mathbf{R},$$

$$u_\perp \quad \to \quad f(u_\perp) := \frac{|D\mathcal{F}_{\hat{h}}(\hat{x}, y).u_\perp|}{\psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u_\perp|}.$$

Since for every $u_\perp \in T_{\hat{h}}^\perp \cap S$, there is at least one index $i \in \hat{h}$ such that $a_i^T u_\perp \neq 0$, this function is well defined and continuous. If $u_\perp \to D\mathcal{F}_{\hat{h}}(\hat{x}, y).u_\perp$ is not identically null on $T_{\hat{h}}^\perp$, put

$$c_0 := \sup_{u_\perp \in T_{\hat{h}}^\perp \cap S} f(u_\perp). \tag{26}$$

Since $T_{\hat{h}}^\perp \cap S$ is compact, f reaches the maximum value c_0. By (25) we see that $0 < c_0 < 1$. If $D\mathcal{F}_{\hat{h}}(\hat{x}, y).u_\perp = 0$, $\forall u_\perp \in T_{\hat{h}}^\perp$, we put $c_0 := 1/2$. In both cases,

$$|D\mathcal{F}_{\hat{h}}(\hat{x}, y).u_\perp| \leq c_0 \, \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u_\perp|, \quad \forall u_\perp \in T_{\hat{h}}^\perp. \tag{27}$$

Using (19), the right side of (17) takes the form

$$\mathcal{F}(\hat{x} + tu_0 + tu_\perp, y) - \mathcal{F}(\hat{x} + tu_0, y) = \Psi_{\hat{h}}(\hat{x} + tu_0 + tu_\perp, y) - \Psi_{\hat{h}}(\hat{x} + tu_0, y) \tag{28}$$

$$+ \mathcal{F}_{\hat{h}}(\hat{x} + tu_0 + tu_\perp, y) - \mathcal{F}_{\hat{h}}(\hat{x} + tu_0, y). \tag{29}$$

First, we focus on the right side of (28). From the definition of $\hat{h}$ and (16),

$$\Psi_{\hat{h}}(\hat{x} + tu_0, y) = 0,$$

$$\Psi_{\hat{h}}(\hat{x} + tu_0 + tu_\perp, y) = \sum_{i \in \hat{h}} \psi\left(a_i^T(\hat{x} + tu_\perp + tu_0) - y_i\right) = \sum_{i \in \hat{h}} \psi(t a_i^T u_\perp).$$

Applying Definition 3 to $\psi'(0^+)$ shows that there is $\eta_0 \in (0, \sigma]$ such that

$$\frac{\psi(t)}{t} \geq \psi'(0^+) - \frac{1 - c_0}{2} \psi'(0^+), \quad \forall t \in (0, \|a\|_\infty \eta_0),$$

since $(1 - c_0)/2 \in (0, 1)$. On the other hand, $|a_i^T u| \leq \|a_i\| \|u\| \leq \|a\|_\infty$, $\forall u \in \overline{B(0,1)}$ and $\forall i \in \{1, \ldots, q\}$. Then

$$t \in (0, \eta_0) \quad \Rightarrow \quad \psi(t a_i^T u_\perp) \geq \frac{c_0 + 1}{2} \psi'(0^+) \, t \, |a_i^T u_\perp|, \quad \forall u_\perp \in T_{\hat{h}}^\perp \cap \overline{B(0,1)}.$$

7

Hence, taking $t \in (0, \eta_0)$ ensures that $\forall u \in S$, decomposed into $u = u_0 + u_\perp$ as in (16), we have

$$\Psi_{\hat{h}}(\hat{x} + tu_0 + tu_\perp, y) \geq \frac{c_0 + 1}{2} t \, \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u_\perp|. \tag{30}$$

Second, we consider (29). Define the constants

$$c_1 := \min_{u_\perp \in T_{\hat{h}}^\perp \cap S} \sum_{i \in \hat{h}} |a_i^T u_\perp|, \tag{31}$$

$$c_2 := c_1 \psi'(0^+) \frac{1 - c_0}{4}, \tag{32}$$

and notice that (31) implies

$$\sum_{i \in \hat{h}} |a_i^T u_\perp| \geq c_1 \|u_\perp\|, \quad \forall u_\perp \in T_{\hat{h}}^\perp. \tag{33}$$

Since $\mathcal{F}_{\hat{h}}(., y) \in \mathcal{C}^1 (B(\hat{x}, \sigma))$ [Remark 3], the mean-value theorem [5] shows that for every $u \in S$ and for every $t \in [0, \sigma)$ there exists $\theta \in (0, 1)$ such that

$$\mathcal{F}_{\hat{h}}(\hat{x} + tu_0 + tu_\perp, y) - \mathcal{F}_{\hat{h}}(\hat{x} + tu_0, y) = t D_1 \mathcal{F}_{\hat{h}}(\hat{x} + tu_0 + \theta tu_\perp, y).u_\perp, \tag{34}$$

where $u = u_0 + u_\perp$ is decomposed as in (16). Moreover, there is $\eta_1 \in (0, \eta_0)$ such that for every $t \in (0, \eta_1)$,

$$\left| D_1 \mathcal{F}_{\hat{h}}(\hat{x} + tu_0 + \theta tu_\perp, y).u_\perp - D_1 \mathcal{F}_{\hat{h}}(\hat{x}, y).u_\perp \right| \leq c_2 \|u_\perp\|, \quad \forall u \in S, \quad \forall \theta \in (0, 1),$$

and hence

$$\left| D_1 \mathcal{F}_{\hat{h}}(\hat{x} + tu_0 + \theta tu_\perp, y).u_\perp \right| \leq \left| D_1 \mathcal{F}_{\hat{h}}(\hat{x}, y).u_\perp \right| + c_2 \|u_\perp\|, \quad \forall u \in S, \quad \forall \theta \in (0, 1), \tag{35}$$

where again u is put into the form (16). Starting with (28)-(29), we derive

$$\mathcal{F}(\hat{x} + tu_0 + tu_\perp, y) - \mathcal{F}(\hat{x} + tu_0, y) \tag{36}$$

$$\geq \frac{c_0 + 1}{2} t \, \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u_\perp| - t \left| D_1 \mathcal{F}_{\hat{h}}(\hat{x} + tu_0 + \theta tu_\perp, y).u_\perp \right| \qquad \text{[by (30) and (34)]}$$

$$\geq \frac{c_0 + 1}{2} t \, \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u_\perp| - t \left| D_1 \mathcal{F}_{\hat{h}}(\hat{x}, y).u_\perp \right| - tc_2 \|u_\perp\| \qquad \text{[by (35)]}$$

$$\geq \frac{1 - c_0}{2} t \, \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u_\perp| - tc_2 \|u_\perp\| \qquad \text{[by (27)]}$$

$$\geq \frac{1 - c_0}{2} \psi'(0^+) tc_1 \|u_\perp\| - tc_2 \|u_\perp\| \qquad \text{[by (33)]}$$

$$= \frac{1 - c_0}{4} \psi'(0^+) tc_1 \|u_\perp\|. \qquad \text{[by (32)]} \tag{37}$$

Consequently,

$$t \in (0, \eta_1) \quad \Rightarrow \quad \mathcal{F}(\hat{x} + tu_0 + tu_\perp, y) - \mathcal{F}(\hat{x} + tu_0, y) > 0, \quad \forall u \in S \text{ with } u_\perp \neq 0. \tag{38}$$

From assumption 1, there exists $\eta_2 \in (0, \eta_1]$ such that

$$t \in (0, \eta_2) \quad \Rightarrow \quad \mathcal{F}(\hat{x} + tu_0, y) - \mathcal{F}(\hat{x}, y) > 0, \quad \forall u_0 \in T_{\hat{h}} \cap \overline{B(0, 1)} \setminus \{0\}. \tag{39}$$

If $u_0 = 0$, then (38) holds since $\|u_\perp\| = 1$, whereas if $u_\perp = 0$, then (39) is true since $\|u_0\| = 1$. Introducing (38) and (39) into (17)-(18) shows that if $t \in (0, \eta_2)$, then $\mathcal{F}(\hat{x} + tu, y) - \mathcal{F}(\hat{x}, y) > 0$ for every $u \in S$. ◇

8

Remark 4 The conditions required in Proposition 1 are pretty weak. Indeed, if an arbitrary function $\mathcal{F}(.,y) : \mathrm{R}^p \to \mathrm{R}$ has a strict minimum at $\hat{x}$, then assumption 1 is trivially true and necessarily $\delta_1 \mathcal{F}(\hat{x},y)(v) \geq 0$, $\forall v \in T_{\hat{h}}^\perp \cap S$ [32]. In comparison, assumption 2 only requires that the latter inequality be strict.

Observe that the above sufficient condition for strict minimum concerns the behavior of $\mathcal{F}(.,y)$ on two orthogonal subspaces *separately*. This occurs because of the non-smoothness of ψ.

4 Minimizers which fit exactly some data entries

The theorem below states the main contribution of this work.

Theorem 1 *Consider $\mathcal{F}$ as given in (11)-(12) where $\Phi \in \mathcal{C}^m(\mathrm{R}^p \times \mathrm{R}^q)$ for $m \geq 2$ and $\psi \in \mathcal{C}^m(\mathrm{R} \setminus \{0\})$ has finite side derivatives at zero such that $\psi'(0^-) < \psi'(0^+)$. Given $y \in \mathrm{R}^q$ and $\hat{x} \in \mathrm{R}^p$, let $\hat{h} := \mathcal{H}(\hat{x},y)$, $\Theta_{\hat{h}}(y)$ and $T_{\hat{h}}$ be obtained by (6), (7) and (8), respectively. Suppose the following:*

1. *the set $\{a_i : i \in \hat{h}\}$ is independent;*

2. *for every $u \in T_{\hat{h}} \cap S$ we have $D_1 \left(\mathcal{F}|_{\overline{\Theta_{\hat{h}}(y)}} \right)(\hat{x},y).u = 0$ and $D_1^2 \left(\mathcal{F}|_{\overline{\Theta_{\hat{h}}(y)}} \right)(\hat{x},y)(u,u) > 0$;*

3. *for every $u \in T_{\hat{h}}^\perp \cap S$ we have $\delta_1 \mathcal{F}(\hat{x},y)(u) > 0$.*

Then there are a neighborhood $N \subset \mathrm{R}^q$ containing y and a $\mathcal{C}^{m-1}$ local minimizer function $\mathcal{X} : N \to \mathrm{R}^p$ relevant to $\mathcal{F}(.,N)$ [Definition 2], yielding in particular $\hat{x} = \mathcal{X}(y)$, so that for every $y' \in N$,

$$
\begin{aligned}
a_i^T \mathcal{X}(y') = y_i' &\quad \text{if} \quad i \in \hat{h}, \\
a_i^T \mathcal{X}(y') \neq y_i' &\quad \text{if} \quad i \in \hat{h}^c.
\end{aligned}
\tag{40}
$$

The latter means that $\mathcal{H}(\mathcal{X}(y'),y') = \hat{h}$ is constant on N.

Proof. If $\hat{h} = \emptyset$, then $\Theta_{\hat{h}}(y') = \mathrm{R}^p$, $\forall y'$. Applying Lemma 1 shows the existence of $\tilde{N} \subset \mathrm{R}^q$ and of a $\mathcal{C}^{m-1}$ local minimizer function $\mathcal{X}$ relevant to $\mathcal{F}(.,\tilde{N})$. By the continuity of $\mathcal{X}$, there is $N \subset \tilde{N}$ where (40) holds, in which case it is reduced to $a_i^T \mathcal{X}(y') \neq y_i'$, $\forall i \in \{1,\ldots,q\}$.

In the following we consider that $\hat{h}$ is nonempty. As in the proof of Proposition 1, we use the formulation of $\mathcal{F}$ given in (13)-(15), and write ψ for $\tilde{\psi}$ and Φ for $\tilde{\Phi}$. This proof is based on two lemmas given next.

Lemma 2 *Let assumptions 1 and 2 of Theorem 1 be satisfied. Then there exist $\nu > 0$ and a $\mathcal{C}^{m-1}$-function $\mathcal{X} : B(y,\nu) \to \mathrm{R}^p$ so that for every $y' \in B(y,\nu)$, the point $\hat{x}' := \mathcal{X}(y')$ belongs to $\Theta_{\hat{h}}(y')$ and satisfies*

$$
D_1 \left(\mathcal{F}|_{\overline{\Theta_{\hat{h}}(y')}} \right)(\hat{x}',y').u = 0 \quad \text{and} \quad D_1^2 \left(\mathcal{F}|_{\overline{\Theta_{\hat{h}}(y')}} \right)(\hat{x}',y')(u,u) > 0, \quad \forall u \in T_{\hat{h}} \setminus \{0\}.
\tag{41}
$$

In particular, $\hat{x} = \mathcal{X}(y)$.

Proof. We start by giving some comments about the restricted functions in (41).

9

Remark 5 For σ as in (20), the inequality reached in (22) shows that $\forall (x', y') \in B(\hat{x}, \sigma) \times B(y, \sigma)$ we have $\mathcal{H}(x', y') \subseteq \hat{h}$. On the other hand, if $x' \in \overline{\Theta_{\hat{h}}(y')}$, then $\mathcal{H}(x', y') \supseteq \hat{h}$. If we put

$$B_{\hat{h}}\left((\hat{x}, y), \sigma\right) := \left(B(\hat{x}, \sigma) \times B(y, \sigma)\right) \cap \overline{M_{\hat{h}}}, \tag{42}$$

where $M_{\hat{h}}$ is given in (9), we have

$$(x', y') \in B_{\hat{h}}\left((\hat{x}, y), \sigma\right) \quad \Rightarrow \quad \mathcal{H}(x', y') = \hat{h},$$

and $B_{\hat{h}}\left((\hat{x}, y), \sigma\right) \subset M_{\hat{h}}$. By (7) and (10), for every $(x', y') \in M_{\hat{h}}$ we find $\Psi_{\hat{h}}(x', y') = 0$ and hence $\mathcal{F}|_{\overline{\Theta_{\hat{h}}(y')}}(x', y') = \mathcal{F}_{\hat{h}}|_{\overline{\Theta_{\hat{h}}(y')}}(x', y')$. Since $\mathcal{F}_{\hat{h}} \in \mathcal{C}^m\left(B(\hat{x}, \sigma) \times B(y, \sigma)\right)$ [Remark 3], we get

$$\mathcal{F}|_{\overline{\Theta_{\hat{h}}(y')}}(x', y') \in \mathcal{C}^m\left(B_{\hat{h}}\left((\hat{x}, y), \sigma\right)\right) \quad \text{and} \quad \mathcal{F}|_{\overline{\Theta_{\hat{h}}(y')}}(x', y') = \mathcal{F}_{\hat{h}}(x', y'), \quad \forall (x', y') \in B_{\hat{h}}\left((\hat{x}, y), \sigma\right).$$

$\diamond$

We now pursue the proof of the lemma. Let the indexes contained in $\hat{h}$ read $\hat{h} = \{j_1, \ldots, j_{\#\hat{h}}\}$. Let $I_{\hat{h}}$ be the $\#\hat{h} \times q$ matrix with entries $I_{\hat{h}}[i, j_i] = 1$ for $i = 1, \ldots, \#\hat{h}$, the remaining entries being null. Thus $y_{\hat{h}} := I_{\hat{h}} y \in \mathrm{R}^{\#\hat{h}}$ is composed of only those entries of y whose indexes are in $\hat{h}$. Similarly, put $A_{\hat{h}} := I_{\hat{h}} A$, then $A_{\hat{h}} \in \mathrm{R}^{\#\hat{h} \times p}$ and $A_{\hat{h}} \hat{x} = y_{\hat{h}}$. With these notations, $\overline{M_{\hat{h}}} = \left\{(x', y') \in \mathrm{R}^p \times \mathrm{R}^q : A_{\hat{h}} x' - I_{\hat{h}} y' = 0\right\}$. By assumption 1, $\mathrm{rank} A_{\hat{h}} = \#\hat{h}$. Then for every y' we have the following dimensions: $\dim \overline{\Theta_{\hat{h}}(y')} = \dim T_{\hat{h}} = p - \#\hat{h}$ while $\dim \overline{M_{\hat{h}}} = p - \#\hat{h} + q$. Recalling that $A_{\hat{h}} A_{\hat{h}}^T$ is invertible, put

$$P_{\hat{h}} := A_{\hat{h}}^T \left(A_{\hat{h}} A_{\hat{h}}^T\right)^{-1} I_{\hat{h}}. \tag{43}$$

Let $C_{\hat{h}} : T_{\hat{h}} \to \mathrm{R}^{p - \#\hat{h}}$ be an isomorphism. The affine mapping below

$$\begin{aligned} \Gamma : \quad \overline{M_{\hat{h}}} &\to \mathrm{R}^{p - \#\hat{h}}, \\ (x', y') &\to \Gamma(x', y') = C_{\hat{h}}\left(x' - \hat{x} - P_{\hat{h}}(y' - y)\right) \end{aligned} \tag{44}$$

is well defined for every $y' \in \mathrm{R}^q$ since on the one hand $\hat{x} + P_{\hat{h}}\left(y' - y\right)$ is the orthogonal projection[1] of $\hat{x}$ onto $\overline{\Theta_{\hat{h}}(y')}$, whereas $x' \in \overline{\Theta_{\hat{h}}(y')}$ by (10). Consider also the following conjugate mapping

$$\begin{aligned} \Gamma^\dagger : \quad \mathrm{R}^{p - \#\hat{h}} \times \mathrm{R}^q &\to \overline{\Theta_{\hat{h}}(y')}, \\ (z, y') &\to \Gamma^\dagger(z, y') = C_{\hat{h}}^{-1} z + \hat{x} + P_{\hat{h}}(y' - y), \end{aligned} \tag{45}$$

which is also well defined. Let

$$\nu_0 := \frac{\sigma}{2} \min\left\{1, \left(\sup_{z \in S} \|C_{\hat{h}}^{-1} z\| + \sup_{y' \in S} \|P_{\hat{h}} y'\|\right)^{-1}\right\}. \tag{46}$$

---

[1]The orthogonal projection of $\hat{x}$ onto $\overline{\Theta_{\hat{h}}(y')}$, denoted by $\hat{x}_{y'}$, is unique and is determined by solving the problem:

$$\text{minimize } \|\hat{x}_{y'} - \hat{x}\| \quad \text{subject to} \quad \hat{x}_{y'} \in \overline{\Theta_{\hat{h}}(y')}.$$

The latter constraint also reads $A_{\hat{h}} \hat{x}_{y'} = y'_{\hat{h}}$ if we denote $y'_{\hat{h}} = I_{\hat{h}} y'$. It is easily calculated that the solution to this problem reads

$$\hat{x}_{y'} = \hat{x} - A_{\hat{h}}^T \left(A_{\hat{h}} A_{\hat{h}}^T\right)^{-1} \left(A_{\hat{h}} \hat{x} - y'_{\hat{h}}\right).$$

Recalling that $A_{\hat{h}} \hat{x} = I_{\hat{h}} y$ from the definition of $\hat{h}$, we obtain that $\hat{x}_{y'} = \hat{x} + P_{\hat{h}}\left(y' - y\right)$.

10

Clearly, $0 < \nu_0 < \sigma$. It is worth noticing that

$$\Gamma^\dagger(z, y') \in \overline{\Theta_{\hat{h}}(y')} \cap B(\hat{x}, \sigma) \subset \Theta_{\hat{h}}(y'), \quad \forall (z, y') \in B(0, \nu_0) \times B(y, \nu_0), \tag{47}$$

since on the one hand (45) shows that $\Gamma^\dagger(z, y') \in \overline{\Theta_{\hat{h}}(y')}$, on the other hand,

$$\|\Gamma^\dagger(z, y') - \hat{x}\| \le \|C_{\hat{h}}^{-1}\| \, \|z\| + \|P_{\hat{h}}\| \, \|y' - y\| \le (\|C_{\hat{h}}^{-1}\| + \|P_{\hat{h}}\|) \, \nu_0 < \sigma.$$

Introduce now the function

$$\begin{aligned} \mathcal{G} : \mathrm{R}^{p - \#\hat{h}} \times \mathrm{R}^q &\rightarrow \mathrm{R}, \\ (z, y') &\rightarrow \mathcal{G}(z, y') := \mathcal{F}_{\hat{h}}\left(\Gamma^\dagger(z, y'), \, y'\right). \end{aligned} \tag{48}$$

Since for every $y' \in \mathrm{R}^q$ we have

$$z = \Gamma(x', y') \quad \Leftrightarrow \quad x' = \Gamma^\dagger(z, y'),$$

then

$$\mathcal{G}\left(\Gamma(x', y'), y'\right) = \mathcal{F}_{\hat{h}}(x', y') = \mathcal{F}|_{\overline{\Theta_{\hat{h}}(y')}}(x', y'), \quad \forall (x', y') \in B_{\hat{h}}\left((\hat{x}, y), \sigma\right),$$

where the last equality comes from Remark 5. Now for every $(x', y') \in B_{\hat{h}}\left((\hat{x}, y), \sigma\right)$, the derivatives of $\mathcal{F}|_{\overline{\Theta_{\hat{h}}(y')}}$, mentioned in (41), can be calculated in terms of $\mathcal{G}$ and Γ:

$$D_1\left(\mathcal{F}|_{\overline{\Theta_{\hat{h}}(y')}}\right)(x', y').u_0 = D_1\mathcal{G}\left(\Gamma(x', y'), y'\right).C_{\hat{h}}u_0, \qquad \forall u_0 \in T_{\hat{h}}, \tag{49}$$

$$D_1^2\left(\mathcal{F}|_{\overline{\Theta_{\hat{h}}(y)}}\right)(x', y')(u_0, u_0) = D_1^2\mathcal{G}\left(\Gamma(x', y'), y'\right).\left(C_{\hat{h}}u_0, C_{\hat{h}}u_0\right), \quad \forall u_0 \in T_{\hat{h}}. \tag{50}$$

Since $C_{\hat{h}}$ is an isomorphism, $D_1\Gamma(x', y').u_0 = C_{\hat{h}}.u_0 \ne 0$ for every $u_0 \in T_{\hat{h}} \setminus \{0\}$ whereas $C_{\hat{h}}.T_{\hat{h}} = \mathrm{R}^{p - \#\hat{h}}$. Then assumption 2, combined with the fact that $\Gamma(\hat{x}, y) = 0$ by construction, yield

$$\begin{aligned} D_1\mathcal{G}(0, y) &= 0, \\ D_1^2\mathcal{G}(0, y)(u, u) &> 0, \quad \forall u \in \mathrm{R}^{p - \#\hat{h}} \setminus \{0\}. \end{aligned}$$

By Lemma 1, there exist $\nu \in (0, \nu_0]$ and a unique $\mathcal{C}^{m-1}$-function $\mathcal{Z} : B(y, \nu) \to B(0, \nu_0)$, such that

$$D_1\mathcal{G}\left(\mathcal{Z}(y'), \, y'\right) = 0 \quad \text{and} \quad D_1^2\mathcal{G}\left(\mathcal{Z}(y'), \, y'\right) \text{ is positive definite}, \quad \forall y' \in B(y, \nu), \tag{51}$$

with in particular $\mathcal{Z}(y) = 0$. Next we express the derivatives in (51) in terms of $\mathcal{F}_{\hat{h}}$ and $\Gamma^\dagger$. From (47) and Remark 5 it follows that $\mathcal{F}_{\hat{h}}$ is $\mathcal{C}^m$ at every $\left(\Gamma^\dagger(z, y'), y'\right)$ relevant to $(z, y') \in B(0, \nu_0) \times B(y, \nu)$ in which case (48) gives rise to

$$D_1\mathcal{G}(z, y').u = D_1\mathcal{F}_{\hat{h}}(\Gamma^\dagger(z, y'), y').C_{\hat{h}}^{-1}u, \tag{52}$$

$$D_1^2\mathcal{G}(z, y')(u, u) = D_1^2\mathcal{F}_{\hat{h}}(\Gamma^\dagger(z, y'), y')\left(C_{\hat{h}}^{-1}u, C_{\hat{h}}^{-1}u\right). \tag{53}$$

Put

$$\mathcal{X}(y') := \Gamma^\dagger\left(\mathcal{Z}(y'), y'\right), \quad \forall y' \in B(y, \nu), \tag{54}$$

11

and notice that $\mathcal{X}(y') \in \Theta_{\hat{h}}(y')$. Then (51) implies that for every $y' \in B(y, \nu)$,

$$D_1 \mathcal{F}_{\hat{h}}(\mathcal{X}(y'), y').C_{\hat{h}}^{-1} u = 0, \qquad \forall u \in \mathbf{R}^{p-\#\hat{h}},$$

$$D_1^2 \mathcal{F}_{\hat{h}}(\mathcal{X}(y'), y') \left(C_{\hat{h}}^{-1} u, C_{\hat{h}}^{-1} u \right) > 0, \qquad \forall u \in \mathbf{R}^{p-\#\hat{h}} \setminus \{0\}.$$

Since $C_{\hat{h}}^{-1} u \neq 0$, $\forall u \in \mathbf{R}^{p-\#\hat{h}} \setminus \{0\}$ and $C_{\hat{h}}^{-1}.\mathbf{R}^{p-\#\hat{h}} = T_{\hat{h}}$, it follows that for every $y' \in B(y, \nu)$,

$$D_1 \mathcal{F}_{\hat{h}} (\mathcal{X}(y'), \ y').u_0 = 0 \quad \text{and} \quad D_1^2 \mathcal{F}_{\hat{h}}(\mathcal{X}(y'), y').(u_0, u_0) > 0, \quad \forall u_0 \in T_{\hat{h}} \setminus \{0\}.$$

Applying again Remark 5 allows us to write down that if $y' \in B(y, \nu)$,

$$D_1 \left(\mathcal{F}|_{\overline{\Theta_{\hat{h}}(y')}} \right) (\mathcal{X}(y'), \ y').u_0 = 0 \quad \text{and} \quad D_1^2 \left(\mathcal{F}|_{\overline{\Theta_{\hat{h}}(y')}} \right) (\mathcal{X}(y'), y')(u_0, u_0) > 0, \quad \forall u_0 \in T_{\hat{h}} \setminus \{0\}.$$

The proof of Lemma 2 is complete. $\diamond$

The next lemma addresses assumption 3.

Lemma 3 *Given $\hat{x} \in \mathbf{R}^p$ and $y \in \mathbf{R}^q$, let $\hat{h} = \mathcal{H}(\hat{x}, y) \neq \emptyset$. Let assumption 3 of Theorem 1 hold. Then there exists $\mu > 0$ such that*

$$y' \in B(\hat{x}, \mu) \quad \text{and} \quad x' \in \Theta_{\hat{h}}(y') \cap B(\hat{x}, \mu) \quad \Rightarrow \quad \delta_1 \mathcal{F}(x', y')(u_\perp) > 0, \quad \forall u_\perp \in T_{\hat{h}}^\perp \cap S. \tag{55}$$

Proof. We decompose $\mathcal{F}$ according to (19). Let σ and $B_{\hat{h}}((\hat{x}, y), \sigma)$ be defined according to (20) and (42), respectively. Remark 5 applies on $B_{\hat{h}}((\hat{x}, y), \sigma)$ and, similarly to (23)-(24), for every $(x', y') \in B_{\hat{h}}((\hat{x}, y), \sigma)$ we have

$$\delta_1 \mathcal{F}(x', y')(u) = \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u| + D_1 \mathcal{F}_{\hat{h}}(x', y').u, \quad \forall u \in \mathbf{R}^p. \tag{56}$$

By the continuity of $D_1 \mathcal{F}_{\hat{h}}$, there is $\mu \in (0, \sigma]$ such that for every $(x', y') \in B_{\hat{h}}((\hat{x}, y), \mu)$,

$$\left| D_1 \mathcal{F}_{\hat{h}}(x', y').u_\perp - D_1 \mathcal{F}_{\hat{h}}(\hat{x}, y).u_\perp \right| \leq \frac{1 - c_0}{2} \psi'(0^+) c_1 \|u_\perp\|, \quad \forall u_\perp \in T_{\hat{h}}^\perp, \tag{57}$$

where $c_0 \in (0, 1)$ and $c_1 > 0$ are the constants given in (26) and (31), respectively. We derive the following inequality chain which holds for all $(x', y') \in B_{\hat{h}}((\hat{x}, y), \mu)$ and for all $u_\perp \in T_{\hat{h}}^\perp$:

$$
\begin{aligned}
& \left| D_1 \mathcal{F}_{\hat{h}}(x', y').u_\perp \right| \\
\leq \ & \left| D_1 \mathcal{F}_{\hat{h}}(\hat{x}, y).u_\perp \right| + \frac{1 - c_0}{2} \psi'(0^+) c_1 \|u_\perp\| && \text{[by (57)]} \\
\leq \ & c_0 \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u_\perp| + \frac{1 - c_0}{2} \psi'(0^+) c_1 \|u_\perp\| && \text{[by (27)]} && (58) \\
\leq \ & c_0 \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u_\perp| + \frac{1 - c_0}{2} \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u_\perp| && \text{[by (33)]} \\
= \ & \frac{c_0 + 1}{2} \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u_\perp|. && (59)
\end{aligned}
$$

On the other hand, (56) shows that for every $(x', y') \in B_{\hat{h}}((\hat{x}, y), \mu)$ and for all $u_\perp \in T_{\hat{h}}^\perp \cap S$ we have

$$
\begin{aligned}
\delta_1 \mathcal{F}(x', y')(u_\perp) \ & \geq \ \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u_\perp| - \left| D_1 \mathcal{F}_{\hat{h}}(x', y').u_\perp \right| \\
& \geq \ \left(1 - \frac{c_0 + 1}{2} \right) \psi'(0^+) \sum_{i \in \hat{h}} |a_i^T u_\perp| > 0, \qquad \text{[by (59)].}
\end{aligned}
$$

12

The last inequality is strict since for every $u_\perp \in T_{\hat h}^\perp \cap S$, there is at least one index $i \in \hat h$ for which $a_i^T u_\perp \neq 0$. $\diamond$

We now complete the proof of Theorem 1. Consider $\nu > 0$ and $\mu > 0$ the radii found in Lemmas 2 and 3, and $\mathcal{X}$ the function exhibited in Lemma 2. By the continuity of $\mathcal{X}$, there exists $\xi \in (0, \min\{\mu, \nu\}]$ such that $\mathcal{X}(y') \in B(\hat x, \mu)$ for every $y' \in B(y, \xi)$. For any $y' \in B(y, \xi)$, consider the point $\hat x' := \mathcal{X}(y')$. From Lemma 2, $\hat x' \in \Theta_{\hat h}(y')$ and $\hat x'$ is a strict local minimizer of $\mathcal{F}|_{\overline{\Theta_{\hat h}(y')}}(., y')$. From Lemma 3, $\delta_1 \mathcal{F}(\hat x', y')(u_\perp) > 0$, $\forall u_\perp \in T_{\hat h}^\perp \cap S$. All the conditions of Proposition 1 being satisfied, $\mathcal{F}(., y')$ reaches a strict local minimum at $\hat x'$. It follows that $\mathcal{X} : B(y, \xi) \to R^p$ is the sought $\mathcal{C}^{m-1}$ minimizer function. $\diamond$

We now focus on the assumptions involved in this theorem. Assumption 2 is nothing else but the very classical sufficient condition for strict local minimum of a smooth function over an affine subspace. Assumption 3 was used in Proposition 1 and was discussed therein.

Remark 6 (on assumption 1.) The subset $\{a_i : i \in \hat h\}$ in assumption 1 is determined by (6). With the notations introduced in the beginning of Lemma 2, $y_{\hat h} := I_{\hat h} y \in R^{\# \hat h}$ belongs to the range of $A_{\hat h}$, denoted by $\mathcal{R}(A_{\hat h})$. Since $\dim \mathcal{R}(A_{\hat h}) = \operatorname{rank} A_{\hat h}$, it follows that if $\operatorname{rank} A_{\hat h} < \# \hat h$, then all $y'_{\hat h}$ belonging to $\mathcal{R}(A_{\hat h})$ belong to a subspace of dimension strictly smaller than $\# \hat h$. Thus, assumption 1 fails to hold only if y is included in a subspace of dimension smaller than q. But the chance that noisy data y belong to such a subspace is null. Reciprocally, assumption 1 is satisfied for all $y \in R^q$ except those belonging to a closed, negligible subset of R^q.

It is worth emphasizing that the independence of the whole set $\{a_i : i \in \{1, \ldots, q\}\}$ is not required. Thus, Theorem 1 addresses any matrix A whether it be ill-conditioned, or singular, or invertible. $\diamond$

Theorem 1 entails some important consequences which are discussed next.

Remark 7 (Stability of minimizers.) The fact that there is a $\mathcal{C}^{m-1}$ local minimizer function shows that, in spite of the non-smoothness of $\mathcal{F}$, for any y, all the strict local minimizers of $\mathcal{F}(., y)$ which satisfy the conditions of the theorem, are *stable under weak perturbations of data y*. This result extends Lemma 1 to non-smooth functions of the form (11)-(12). Moreover, if for every $y \in R^q$, the function $\mathcal{F}(., y)$ is strictly convex, then the unique minimizer function $\mathcal{X} : R^q \to R^p$, relevant to $\mathcal{F}(., R^q)$, is $\mathcal{C}^{m-1}$ on R^q. $\diamond$

Remark 8 (Stability of $\hat h$.) The result formulated in (40) means that *the set-valued function $y' \to \mathcal{H}(\mathcal{X}(y'), y')$ is constant on N, i.e. that $\mathcal{H}$ is constant under small perturbations of y. Equivalently, all residuals $(a_i^T \mathcal{X}(y') - y_i')$ for $i \in \hat h$ are null on N.* $\diamond$

Remark 9 (Data domain.) Theorem 1 reveals that the data domain R^q contains *volumes of positive measure* composed of data that lead to local minimizers which fit exactly the data entries belonging to the same set. (*E.g.*, for A invertible, $\alpha = 0$ yields $\hat h = \{1, \ldots, q\}$ and the data volume relevant to this $\hat h$ is R^q.) For a meaningful choice of ψ, Φ and α, there are volumes corresponding to various $\hat h$, and they are large enough so that noisy data come across them. That is why in practice, non-smooth data-fidelity terms yield minimizers fitting exactly a certain number of the data entries. The resultant numerical effect is observed in § 7. $\diamond$

Next we present a simple example which illustrates Theorem 1.

13

Example 1 (Non-smooth data-fidelity term.) Consider the function

$$\mathcal{F}(x,y) = \sum_{i=1}^{q} |x_i - y_i| + \alpha \sum_{i=1}^{q} \frac{x_i^2}{2},$$

where $\alpha > 0$. For every $y \in \mathrm{R}^q$, the function $\mathcal{F}(.,y)$ is strictly convex, so it has a unique minimizer and the latter is strict. Moreover

$$\min_x \mathcal{F}(x,y) \;=\; \sum_{i=1}^{q} \min_{x_i} f(x_i, y_i)$$

$$\text{where} \quad f(x_i, y_i) \;=\; |x_i - y_i| + \frac{\alpha x_i^2}{2} \quad \text{for} \quad i = 1, \ldots, q.$$

For $y \in \mathrm{R}^q$, let $\hat{x}$ be the minimizer of $\mathcal{F}(.,y)$. Now $\hat{h} = \{i : \hat{x}_i = y_i\}$. For every i, the fact that $f(.,y_i)$ has a minimum at $\hat{x}_i$ means that $\delta_1 f(\hat{x}_i, y_i)(u) \geq 0$, for every $u \in \mathrm{R}$. Then for every $u \in \mathrm{R}$ we have

$$\text{if} \;\; i \in \hat{h}^c \;\Leftrightarrow\; \hat{x}_i \neq y_i, \quad \text{then} \quad \delta_1 f(x_i, y_i)(u) \;=\; Df(x_i, y_i).u = (\mathrm{sign}(x_i - y_i) + \alpha x_i).u \geq 0;$$

$$\text{if} \;\; i \in \hat{h} \;\Leftrightarrow\; \hat{x}_i = y_i, \quad \text{then} \quad \delta_1 f(\hat{x}_i, y_i)(u) \;=\; |u| + (\alpha y_i).u \geq 0.$$

From Proposition 1, the entries of the minimizer function $\mathcal{X}$ are

$$\text{if} \;\; i \in \hat{h}^c \;\Leftrightarrow\; |y_i| > \frac{1}{\alpha}, \quad \text{then} \quad \mathcal{X}_i(y) \;=\; \frac{1}{\alpha} \, \mathrm{sign}(y_i);$$

$$\text{if} \;\; i \in \hat{h} \;\Leftrightarrow\; |y_i| \leq \frac{1}{\alpha}, \quad \text{then} \quad \mathcal{X}_i(y) \;=\; y_i.$$

Theorem 1 applies provided that $|y_i| \neq 1/\alpha$ for every $i \in \hat{h}$, which corresponds to assumption 3. In such a case, we can take for the neighborhood exhibited in Theorem 1

$$N = B(y, \xi) \;\; \text{with} \;\; \xi = \min_{i=1}^{q} \left| \, |y_i| - \frac{1}{\alpha} \, \right|.$$

We see that $y' \to \mathcal{H}(\mathcal{X}(y'), y')$ reads

$$\mathcal{H}(\mathcal{X}(y'), y') = \left\{ i \in \{1, \ldots, q\} : |y_i'| \leq \frac{1}{\alpha} \right\}$$

and is constant on N. The above expression shows also that the cardinality of $\hat{h}$ increases when α decreases.

We now illustrate Remark 9. For $h \subset \{1, \ldots, q\}$, put

$$V_h := \left\{ y \in \mathrm{R}^q : \; |y_i| \leq \frac{1}{\alpha}, \forall i \in h \;\; \text{and} \;\; |y_i| > \frac{1}{\alpha}, \forall i \in h^c \right\}.$$

Obviously, every $y' \in V_h$ gives rise to a minimizer $\hat{x}'$ of $\mathcal{F}(.,y')$ satisfying $\mathcal{H}(\hat{x}', y') = h$. That is, the function $y' \to \mathcal{H}(\mathcal{X}(y'), y')$ is constant on V_h. Note that $V_\emptyset = \{y \in \mathrm{R}^q : |y_i| > 1/\alpha, \forall i\}$ and that $V_\emptyset = \emptyset$ if $\alpha = 0$. Moreover, *for every $h \subset \{1, \ldots, q\}$, the set V_h has a positive volume in R^q*, whereas the family of all V_h, when h ranges over the family of all the subsets of $\{1, \ldots, q\}$ (including the empty set), is a *partition* of R^q. ◇

5 Smooth data-fidelity terms

In this section we focus on smooth cost-functions with the goal to check whether we can get minimizers which fit exactly a certain number of data entries. We start with an illuminating example.

Example 2 (Smooth cost-function.) For $A \in \mathbb{R}^{q \times p}$ and $G \in \mathbb{R}^{r \times p}$ with $r \in \mathbb{N}^*$, consider the cost-function $\mathcal{F} : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$,

$$\mathcal{F}(x, y) = \|Ax - y\|^2 + \alpha\|Gx\|^2. \tag{60}$$

Recall that since [37], cost-functions of this form are among the most widely used tools in signal and in image estimation [25, 22, 35, 13]. Under the classical assumption $\ker A^T A \cap \ker G^T G = \emptyset$, it is seen that for every $y \in \mathbb{R}^q$, $\mathcal{F}(., y)$ is strictly convex and its unique minimizer $\hat{x}$ is determined by solving the equation

$$D\mathcal{F}(\hat{x}, y) = 0 \quad \text{where} \quad D\mathcal{F}(\hat{x}, y) = 2(A\hat{x} - y)^T A + 2\alpha \hat{x}^T G^T G.$$

The relevant minimizer function $\mathcal{X} : \mathbb{R}^q \to \mathbb{R}^p$ reads

$$\mathcal{X}(y) = (A^T A + \alpha\, G^T G)^{-1} A^T . y. \tag{61}$$

We now determine the set of *all* data points $y \in \mathbb{R}^q$ for which $\hat{x} := \mathcal{X}(y)$ fits exactly the ith data entry y_i. To this end, we have to solve with respect to y the equation

$$a_i^T \mathcal{X}(y) = y_i. \tag{62}$$

Using (61), this is equivalent to solving the equation

$$
\begin{aligned}
p_i(\alpha).y &= 0, \\
\text{where} \quad p_i(\alpha) &= a_i^T (A^T A + \alpha\, G^T G)^{-1} A^T - e_i^T.
\end{aligned}
\tag{63}
$$

We can have $p_i(\alpha) = 0$ only if α belongs to the discrete set of several values which satisfy a data-independent system of q polynomials of degree p. However, α will almost never belong to such a set, so in general, $p_i(\alpha) \neq 0$. Then (63) implies $y \in \{p_i(\alpha)\}^\perp$. More generally, we have the implication

$$\exists i \in \{1, \ldots, q\} \quad \text{such that} \quad \mathcal{X}_i(y) = y_i \quad \Rightarrow \quad y \in \bigcup_{j=1}^{q} \{p_j(\alpha)\}^\perp.$$

Since every $\{p_i(\alpha)\}^\perp$ is a subspace of $\mathbb{R}^q$ of dimension $q - 1$, the union on the right side above is a *closed, negligible subset of* $\mathbb{R}^q$. The chance that noisy data come across this union is null. Hence, the chance that noisy data y yield a minimizer $\mathcal{X}(y)$ which fits even one data entry, *i.e.* that there is at least one index i such that (62) holds, is null. $\diamond$

The theorem stated below generalizes this example.

Theorem 2 *Consider a C^m-function $\mathcal{F} : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$, with $m \geq 2$, of the form (1)-(2), and let $h \subset \{1 \ldots, q\}$ be nonempty. Assume the following:*

1. *$\forall i = 1, \ldots, q$, the functions $\psi_i : \mathbb{R} \to \mathbb{R}$ satisfy $\psi_i''(t) > 0$, $\forall t \in \mathbb{R}$;*

2. *A is invertible (recall that for every $i = 1, \ldots, q$, the ith row of A is a_i^T);*

3. *there is an open domain $N_0 \subset \mathbb{R}^q$, so that $\mathcal{F}(., N_0)$ admits a C^{m-1} local minimizer function $\mathcal{X} : N_0 \to \mathbb{R}^p$, such that $D_1^2 \mathcal{F}(\mathcal{X}(y), y)$ is positive definite, $\forall y \in N_0$;*

15

4. for every $x \in \mathcal{X}(N_0) \subset \mathbb{R}^p$ and for every $i \in h$ we have $D^2\Phi(x).[A^{-1}]_i \neq 0$ where $[A^{-1}]_i$ denotes the ith column of A^{-1}, for $i = 1, \ldots, q$.

For a given set of constants $\{\theta_i, i \in h\}$, and for any $N \subset N_0$ a closed subset of $\mathbb{R}^q$, put

$$\Upsilon_h := \left\{ y \in N : a_i^T \mathcal{X}(y) = y_i + \theta_i, \ \forall i \in h \right\}. \tag{64}$$

Then Υ_h is a closed subset of $\mathbb{R}^q$ which is negligible with respect to the Lebesgue measure on $\mathbb{R}^q$.

Proof. For every h nonempty we have

$$\Upsilon_h = \bigcap_{i \in h} \Upsilon_{\{i\}}.$$

It is hence sufficient to show that $\Upsilon_{\{i\}}$ is closed and negligible in $\mathbb{R}^q$ for some $i \in h$. For simplicity, in the following we write Υ_i for $\Upsilon_{\{i\}}$. Since $\mathcal{X}$ is continuous on N, every Υ_i is closed in N, and hence in $\mathbb{R}^q$. Our reasoning below is developed *ad absurdum*. So suppose that Υ_i is of positive measure. Then Υ_i contains an open, connected subset of $\mathbb{R}^q$, say $\tilde{N} \subset \Upsilon_i \subset N$. We can hence write down

$$a_i^T \mathcal{X}(y) = y_i + \theta_i, \quad \forall y \in \tilde{N}. \tag{65}$$

Differentiating both sides of this identity with respect to y yields

$$a_i^T D\mathcal{X}(y) = e_i^T, \quad \forall y \in \tilde{N}. \tag{66}$$

We next determine the form of $D\mathcal{X}$. Since for every $y \in \tilde{N}$ the point $\mathcal{X}(y)$ is a local minimizer of $\mathcal{F}(., y)$, it satisfies $D_1\mathcal{F}(\mathcal{X}(y), y) = 0$. Differentiating both sides of the latter identity leads to

$$D_1^2\mathcal{F}\left(\mathcal{X}(y), y\right) D\mathcal{X}(y) + D_{1,2}\mathcal{F}\left(\mathcal{X}(y), y\right) = 0, \quad \forall y \in \tilde{N}. \tag{67}$$

The Hessian of $x \rightarrow \mathcal{F}(x, y)$, say $H(x, y) := D_1^2\mathcal{F}(x, y)$, reads

$$\begin{aligned} H(x, y) &= D_1^2\Psi(x, y) + \alpha D^2\Phi(x) \\ &= A^T \operatorname{Diag}\left(\ddot{\psi}(x, y)\right) A + \alpha D^2\Phi(x), \end{aligned} \tag{68}$$

where for every x and y, $\ddot{\psi}(x, y) \in \mathbb{R}^q$ is the vector whose entries read

$$[\ddot{\psi}(x, y)]_i = \psi_i''(a_i^T x - y_i), \quad \text{for} \ \ i = 1, \ldots, q.$$

By assumption 3, $H(\mathcal{X}(y), y)$ is an invertible matrix, $\forall y \in \tilde{N}$. Furthermore,

$$D_{1,2}\mathcal{F}(x, y) = -A^T \operatorname{Diag}\left(\ddot{\psi}(x, y)\right).$$

Inserting the last expression and (68) into (67) shows that

$$D\mathcal{X}(y) = \left(H(\mathcal{X}(y), y)\right)^{-1} A^T \operatorname{Diag}\left(\ddot{\psi}(\mathcal{X}(y), y)\right), \quad \forall y \in \tilde{N}. \tag{69}$$

Introducing now (69) into (66) yields

$$a_i^T \left(H(\mathcal{X}(y), y)\right)^{-1} A^T \operatorname{Diag}\left(\ddot{\psi}(\mathcal{X}(y), y)\right) = e_i^T, \quad \forall y \in \tilde{N}. \tag{70}$$

By assumption 1, $\text{Diag}\left(\ddot{\psi}(\mathcal{X}(y),y)\right)$ is invertible for every $y \in \tilde{N}$. Its inverse is a diagonal matrix whose diagonal terms are $\left(\psi_i''(a_i^T\mathcal{X}(y)-y_i)\right)^{-1}$, for $i=1\ldots,q$. Noticing that

$$e_i^T\left(\text{Diag}\left(\ddot{\psi}(\mathcal{X}(y),y)\right)\right)^{-1} = \frac{e_i^T}{\psi_i''\left(a_i^T\mathcal{X}(y)-y_i\right)},$$

we find that (70) equivalently reads

$$\psi_i''(a_i^T\mathcal{X}(y)-y_i).a_i^T\left(H(\mathcal{X}(y),y)\right)^{-1} = e_i^T A^{-T}, \quad \forall y \in \tilde{N},$$

where $A^{-T} := \left(A^T\right)^{-1}$. Then, taking into account (68),

$$\psi_i''(a_i^T\mathcal{X}(y)-y_i).a_i^T = e_i^T A^{-T}\left(A^T\,\text{Diag}\left(\ddot{\psi}(\mathcal{X}(y),y)\right)A + \alpha D^2\Phi(\mathcal{X}(y))\right), \quad \forall y \in \tilde{N}.$$

By the invertibility of A (assumption 2), and noticing that $e_i^T A = a_i^T$, the latter expression is simplified to

$$\psi_i''\left(a_i^T\mathcal{X}(y)-y_i\right).a_i^T = \psi_i''\left(a_i^T\mathcal{X}(y)-y_i\right).a_i^T + \alpha e_i^T A^{-T}D^2\Phi(\mathcal{X}(y)), \quad \forall y \in \tilde{N},$$

and finally to

$$D^2\Phi(\mathcal{X}(y)).A^{-1}e_i = 0, \quad \forall y \in \tilde{N}.$$

However, the obtained identity contradicts assumption 4. We conclude that Υ_h is negligible. $\quad\diamond$

Let us comment the assumptions taken in this theorem. Recall first that assumption 3 was discussed in Lemma 1 and in Remark 1. In the typical case when Ψ is a data-fidelity measure, every ψ_i is a strictly convex function satisfying $\psi_i(0) = 0$ and $\psi_i(t) = \psi_i(-t)$.

Remark 10 (on assumption 2.) This proposition also addresses the case when

$$\mathcal{F}(x,y) = \|Ax - y\|^2 + \alpha\Phi(x) \quad\text{with}\quad \text{rank}A = p \leq q.$$

Indeed, for $p < q$, $\mathcal{F}$ can equivalently be expressed in terms of a $p \times p$-matrix $\tilde{A}$ with $\text{rank}\tilde{A} = p$, in place of A. Every minimizer $\hat{x}$ of $\mathcal{F}(.,y)$ satisfies $2A^TA\hat{x} + \alpha D\Phi(\hat{x}) = 2A^Ty$. Then equivalently, $\mathcal{F}(x,y) = \|\tilde{A}x - \tilde{A}^{-T}A^Ty\| + \alpha\Phi(x)$ where we can put $\tilde{y} := \tilde{A}^{-T}A^Ty$ to be an equivalent data vector of length p. Clearly, $\hat{x}$ satisfies the same conditions for minimum.

Remark 11 (on assumption 4.) By the invertibility of A (assumption 2), we see that $[A^{-1}]_i = A^{-1}e_i \neq 0$, for every $i = 1,\ldots,q$. It would be a pathological situation to have some of the columns of A^{-1} in $\ker D^2\Phi(x)$, for some x. For instance, focus on the classical case given in (4) with $G_i^T : \mathrm{R}^p \to \mathrm{R}$. Let G denote the $r \times p$ matrix whose rows are G_i^T, for $i = 1,\ldots,r$. Then $D^2\Phi(x) = G^T\text{Diag}\left(\ddot{\varphi}(Gx)\right)G$ where $\ddot{\varphi}(Gx) \in \mathrm{R}^r$ is the vector with entries $[\ddot{\varphi}(Gx)]_i = \varphi''(G_i^Tx)$ for $i = 1,\ldots,r$. Focus on the case when $\varphi''(t) > 0$, $\forall t \in \mathrm{R}$ (*e.g.* φ is strictly convex) and G yields first-order differences between neighboring samples. Then $\text{Ker}D^2\Phi(x)$ is composed of the constant vectors, $\kappa[1,\ldots,1]^T$, $\kappa \in \mathrm{R}$. Then assumption 4 is satisfied provided that A^{-1} does not involve constant columns.

Remark 12 (Meaning of the theorem.) If for some $y \in \mathrm{R}^q$ a minimizer $\hat{x}$ of $\mathcal{F}(.,y)$ satisfies an affine equation of the form $a_i^T\hat{x} = y_i + \theta_i$, then Theorem 2 asserts that y belongs to a closed, negligible subset of R^q. There is no chance that noisy data y yield local minimizers of a smooth cost-function $\mathcal{F}(.,y)$ satisfying such an equation.

The next proposition states the same conclusions but under different assumptions.

Proposition 2 *Consider a C^m-function $\mathcal{F} : \mathrm{R}^p \times \mathrm{R}^q \to \mathrm{R}$, with $m \geq 2$, of the form (1)-(2) and let $h \subset \{1 \ldots, q\}$ be nonempty. Assume the following:*

1. *there is a domain $N_0 \subset \mathrm{R}^q$ so that $\mathcal{F}(., N_0)$ admits a C^{m-1} local minimizer function $\mathcal{X} : N_0 \to \mathrm{R}^p$, such that $D_1^2 \mathcal{F}(\mathcal{X}(y), y)$ is positive definite, $\forall y \in N_0$;*

2. *for every $y \in N_0$ and for every $i \in h$ there exists $j \in \{1, \ldots, q\}$ such that the following function $\mathcal{K}_{i,j}$*

$$\mathcal{K}_{i,j}(y') := \psi_i'' \left(a_j^T \mathcal{X}(y') - e_j^T y' \right) . a_i^T \left(H(\mathcal{X}(y'), y') \right)^{-1} . a_j,$$

where H was given in (68), is nonconstant on any neighborhood of y.

For $\{\theta_i \in \mathrm{R} : i \in h\}$ given, and for every $N \subset N_0$ a closed subset of R^q, put

$$\Upsilon_h := \left\{ y \in N : a_i^T \mathcal{X}(y) = y_i + \theta_i, \ \forall i \in h \right\}. \tag{71}$$

Then Υ_h is a closed negligible subset of R^q.

Proof. As in the proof of Theorem 2, we focus on Υ_i for $i \in h$ and develop our reasoning by contradiction. So suppose that Υ_i has a positive measure in R^q. Using the same arguments as in the proof of Theorem 2, we deduce that Υ_i must contain an open ball $\tilde{N}$. Then (65) and (66) are true. In particular, comparing (66) for $y' \neq y$ with the same equality for y yields

$$a_i^T D\mathcal{X}(y') = a_i^T D\mathcal{X}(y), \quad \forall y' \in \tilde{N}. \tag{72}$$

Notice that $A^T \mathrm{Diag} \left(\ddot{\psi}(x, y') \right)$ is a matrix whose jth column reads $\psi''(a_j^T x - y_j') . a_j$. Introducing (69) into (72) show that the latter is equivalent to the system

$$\mathcal{K}_{i,j}(y') = \mathcal{K}_{i,j}(y), \quad \forall j \in \{1, \ldots, q\}, \quad \forall y' \in \tilde{N}.$$

The obtained result contradicts assumption 2. It remains that Υ_i is negligible. ◇

Remark 13 (on assumption 2.) Although a general proof of the validity of this assumption appears to be more intricate than important, we conjecture that it is usually satisfied, except possibly in some pathological cases. The intuitive arguments are the following. Let us focus on the classical case when Φ is as in (4). The entries of $H(x', y')$ read

$$[H(x', y')]_{m,n} = \sum_{j=1}^q \eta_{j,m}^2 \psi''(a_j x' - y_j') + \sum_{j=1}^r \kappa_{j,n}^2 \varphi''(G_j x'), \quad \text{for } (m, n) \in \{1, \ldots, p\}^2, \tag{73}$$

where $\eta_{j,m}, j = 1, \ldots, q$ and $\kappa_{j,n}, j = 1, \ldots, r$ are constants that are calculated from G and A. From Cramer's rule for matrix inversion, for every j, the term $a_i^T \left(H(x', y') \right)^{-1} a_j$ is the fraction of two polynomials. The entries of the numerator read $\beta_{s,m,n}([H(x', y')]_{m,n})^s, \forall (m, n) \in \{1, \ldots, p\}^2$ with $\beta_{s,m,n} \in \mathrm{R}$, for $s = 0, \ldots p - 1$. In the denumerator we have $\gamma_{s,m,n}([H(x', y')]_{m,n})^s, \forall (m, n) \in \{1, \ldots, p\}^2$ with $\gamma_{s,m,n} \in \mathrm{R}$, for $s = 0, \ldots p$. For $\mathcal{X}$ a minimizer function and j and i given, $\mathcal{K}_{i,j}$ has the form

$$\mathcal{K}_{i,j}(y') = \psi'' \left(a_j^T \mathcal{X}(y') - y_j' \right) . \frac{\sum_{s=1}^{p-1} \sum_{(m,n)} \beta_{s,m,n}([H(\mathcal{X}(y'), y')]_{m,n})^s}{\sum_{s=1}^{p} \sum_{(m,n)} \gamma_{s,m,n}([H(\mathcal{X}(y'), y')]_{m,n})^s}. \tag{74}$$

18

Assumption 2 requires that for $i \in h$, there is at least one index $j \in \{1, \ldots, q\}$ for which the relevant function $\mathcal{K}_{i,j}$ does not remain constant on any neighborhood of y.

6 Non-smooth regularization versus non-smooth data-fidelity

In this section we compare cost-functions involving non-smooth data-fidelity terms to cost-functions involving non-smooth regularization terms. The visual effects produced by these classes of cost-functions can be seen in § 7.

Cost-functions with *non-smooth regularization* typically have the form (1) where Ψ is a C^m-function, $m \geq 2$, whereas Φ is as in (4) with φ non-smooth at zero. The most often, $\Psi(x, y) = \|Ax - y\|^2$. Non-smooth functions φ are for instance the L^1 and the concave functions in (5). Since [33, 18], such cost-functions are customarily used in signal and image restoration [18, 1, 14, 11, 12, 38]. Visually, the obtained minimizers exhibit a *stair-casing effect* since they typically involve many constant regions—see for instance Figs. 6 and 10 in § 7. This effect is discussed by many authors [18, 15, 14, 12]. In particular, the ability of the L^1-function to recover non-correlated "nearly-black" images in the simplest case when $G_i = e_i$, $\forall i$, was interpreted in [15] using mini-max decision theory. Total-variation methods, corresponding to $\varphi(t) = |t|$ also, were observed to yield "blocky images" [14, 12]. The concave function was shown to transform a ramp-shaped signal into a step-shaped minimizer [19].

A theoretical explanation of stair-casing was given in [26, 27, 28]. It was shown there that regularization of the form (4) with φ non-smooth at zero yield local minimizers $\hat{x}$ which satisfy $G_i\hat{x} = 0$ *exactly*, for many indexes i. For instance, if G_i^T, $i = 1, \ldots, r$, yield first-order differences between neighboring samples (if x is a signal of $\mathbb{R}^p$, $G_i x = x_i - x_{i+1}$ for $i = 1, \ldots, p-1$), the relevant minimizers $\hat{x}$ are constant over many zones. If G_i^T, $i = 1, \ldots, r$, yield second-order differences, then $\hat{x}$ involves many zones over which it is affine, *etc.* More generally, the sets of indexes i for which $G_i\hat{x} = 0$ determine zones which can be said to be *strongly homogeneous* [27]. Stair-casing is due to a special form of stability property which is explained next. Let a data point y give rise to a local minimizer $\hat{x}$ which satisfies $G_i^T\hat{x} = 0$, $\forall i \in \hat{h}$ where $\hat{h} \neq \emptyset$. It is shown in [26, 27, 28] that y is in fact contained in a neighborhood $N \in \mathbb{R}^q$ whose elements $y' \in N$ (noisy data) give rise to local minimizers $\hat{x}'$ of $\mathcal{F}(., y')$, placed near to $\hat{x}$, which satisfy $G_i^T\hat{x}' = 0$ for all $i \in \hat{h}$. Since every such N is a volume of positive measure, noisy data come across these volumes and yield minimizers satisfying $G_i^T\hat{x}' = 0$ for many indexes i. Notice that this behavior is due to the non-smoothness of φ at zero since it cannot occur with differentiable cost-functions [27].

The behavior of the minimizers of cost-functions with *non-smooth data-fidelity*, as considered in Theorem 1, is opposite. If y leads to a minimizer $\hat{x}$ which fits exactly a set $\hat{h}$ of entries of y, Theorem 1 shows that y is contained in a neighborhood N such that the relevant minimizer function $\mathcal{X}$ follows closely every small variation all data entries y_i' for $i \in \hat{h}$ when y' ranges over N. Thus $a_i^T\mathcal{X}(y')$ is never constant in the vicinity of y for $i \in \hat{h}$.

7 Non-smooth data-fidelity to detect and to smooth outliers

Our objective now is to process data in order to detect, and possibly to smooth, outliers and impulsive noise. To this end, take $a_i = 1$ for every $i \in \{1, \ldots, q\}$ in (2). Focus on

$$\mathcal{F}(x, y) = \sum_{i=1}^{q} \psi(x_i - y_i) + \alpha \sum_{i=1}^{r} \varphi(G_i^T x) \tag{75}$$

where $G_i^T : \mathrm{R}^p \to \mathrm{R}$ for $i = 1, \ldots, r$, yield differences between neighboring samples (*e.g.* $G_i^T x = x_i - x_{i+1}$ if x is a signal), ψ and φ are even and strictly increasing on $[0, \infty)$, with $\psi'(0^+) > 0$ and φ smooth on R. Suppose that $\hat{x}$ is a strict minimizer of $\mathcal{F}(., y)$ and put $\hat{h} = \mathcal{H}(\hat{x}, y)$. Based on the results in § 4, we naturally come to the following method for the detection of outliers. Since every y_i corresponding to $i \in \hat{h}$ is kept intact in the minimizer $\hat{x}$, that is $\hat{x}_i = y_i$, every such y_i can be considered as a *faithful data entry*. In contrast, every y_i with $i \in \hat{h}^c$ corresponds to $\hat{x}_i \neq y_i$ which can indicate that this y_i is aberrant. In other words, *given $y \in \mathrm{R}^q$, we posit that $\hat{h}^c$, the complementary of $\hat{h} = \mathcal{H}(\mathcal{X}(y), y)$, provides an estimate of the locations of the outliers in y.* The possibility to keep intact all faithful data entries is both spectacular and precious from a practical point of view, *e.g.* to pre-process data.

Remark 14 (Stability of the detection of outliers.) If a minimizer $\hat{x}$ of $\mathcal{F}(., y)$, for $y \in \mathrm{R}^q$, gives rise to $\hat{h} = \mathcal{H}(\hat{x}, y)$, Theorem 1 ensures that all data y' placed near to y yield minimizers $\hat{x}'$ which recover exactly the same set of outlier positions $\hat{h}^c$. Hence, the suggested method for detection of outliers is stable under small data variations.

We can also envisage to *smooth* outliers since the value of every $\hat{x}_i$, for $i \in \hat{h}^c$, is obtained from the values of neighboring data samples through the terms $\alpha \varphi(G_j^T \hat{x})$ for j neighbor of i. Small values of α make the weight of Ψ more important, so the relevant minimizers $\hat{x}$ fit larger sets of data entries, *i.e.* $\hat{h}$ is larger. At the same time, all samples $\hat{x}_i$ for $i \in \hat{h}^c$ incur an only weak smoothing and may remain close to y_i. In contrast, large values of α improve smoothing since they increase the weight of Φ. To resume, small values of α are more adapted for the detection of outliers while large values of α are better suited for smoothing of outliers. We are hence faced with a compromise between efficiency of detection and quality of smoothing. The next example, as well as the experiments presented below, corroborate this conjecture.

Example 3 Consider the following cost-function:

$$\mathcal{F}(x, y) = \sum_{i=1}^{q} |x_i - y_i| + \alpha \sum_{i=1}^{p-1} (x_i - x_{i+1})^2.$$

Let $\hat{x}$ be a minimizer of $\mathcal{F}(., y)$ for which $\hat{h} := \mathcal{H}(\hat{x}, y)$ is nonempty. Focus on $i \in \hat{h}^c$. Since $\hat{x}_i \neq y_i$, then

$$0 = \frac{\partial \mathcal{F}(\hat{x}, y)}{\partial \hat{x}_i} = \mathrm{sign}(\hat{x}_i - y_i) + 2\alpha \left((\hat{x}_i - \hat{x}_{i+1}) - (\hat{x}_{i-1} - \hat{x}_i) \right),$$

which yields

$$\hat{x}_i = \frac{\hat{x}_{i-1} + \hat{x}_{i+1}}{2} - \frac{\mathrm{sign}(\hat{x}_i - y_i)}{4\alpha}. \tag{76}$$

Hence, $\hat{x}_i$ takes the form (76) only if we have

$$\text{either} \quad y_i > \frac{\hat{x}_{i-1} + \hat{x}_{i+1}}{2} + \frac{1}{4\alpha}, \quad \text{or} \quad y_i < \frac{\hat{x}_{i-1} + \hat{x}_{i+1}}{2} - \frac{1}{4\alpha}.$$

Remark that (76) does not involve y_i but only the sign of $(\hat{x}_i - y_i)$. Thus, if y_i is an outlier, the value of $\hat{x}_i$ relies only on faithful data entries y_j for $j \in \hat{h}$, by means of $\hat{x}_{i-1}$ and $\hat{x}_{i+1}$. Moreover, the smoothing incurred by $\hat{x}_i$ is stronger for large values of α, since then $\hat{x}_i$ is closer to the mean of $\hat{x}_{i-1}$ and $\hat{x}_{i+1}$. Otherwise, if $i \in \hat{h}$, we have $\delta\mathcal{F}(\hat{x}, y)(e_i) \geq 0$, which yields

$$\hat{x}_i = y_i \quad \Leftrightarrow \quad \frac{\hat{x}_{i-1} + \hat{x}_{i+1}}{2} - \frac{1}{4\alpha} \leq y_i \leq \frac{\hat{x}_{i-1} + \hat{x}_{i+1}}{2} + \frac{1}{4\alpha}.$$

This inequality is easier to satisfy if α is small in which case numerous data samples are fitted exactly, whereas only a few samples are detected as outliers. ◇

Concrete results depend on the shape of ψ, φ, $\{G_i^T\}$ and of α. We leave this crucial question for future work. In order to recover and to smooth outliers, we take the following cost-function:

$$\mathcal{F}(x, y) = \sum_{i=1}^{q} |x_i - y_i| + \alpha \sum_{i=1}^{p} \sum_{j \in \mathcal{N}(i)} |x_i - x_j|^\nu \quad \text{for} \quad \nu \in (1, 2], \tag{77}$$

where for every $i = 1, \ldots, p$, the set $\mathcal{N}(i)$ contains the indexes of all samples j which are neighbors to i. In all the restorations presented below, $\mathcal{N}(i)$ is composed of the 8 nearest neighbors. Since [9], we can expect that $\nu > 1$ but close to 1 allow edges to be better preserved when outliers are smoothed. Based on this, all the experiments with (77) in the following correspond to $\nu = 1.1$.

The minimizer $\hat{x}$ of $\mathcal{F}(., y)$, for $y \in \mathbb{R}^q$, is calculated by continuation. Using that the Huber function (5),

$$\psi_\nu(t) = \begin{cases} t^2 & \text{if} \quad |t| \leq \nu, \\ \nu(\nu + 2|t - \nu|) & \text{if} \quad |t| > \nu, \end{cases} \quad \text{where} \quad \nu > 0,$$

satisfies $\psi_\nu(t) \to |t|$ when $\nu \downarrow 0$, we construct a family of functions $\mathcal{F}_\nu(., y)$ indexed by $\nu > 0$:

$$\mathcal{F}_\nu(x, y) := \sum_{i=1}^{q} \psi_\nu(a^T x - y_i) + \Phi(x).$$

Being strictly convex and differentiable, every $\mathcal{F}_\nu(., y)$ has a unique minimizer, denoted by $\hat{x}_\nu$, which is calculated by gradient descent. Since by construction having $\nu > \nu'$ entails $\mathcal{F}_\nu(x, y) \geq \mathcal{F}_{\nu'}(x, y)$, $\forall x \in \mathbb{R}^p$, we see that $\mathcal{F}_\nu(\hat{x}_\nu, y)$ decreases monotonically when ν decreases to 0. It is easy to check that moreover, as $\nu \downarrow 0$, we have $\mathcal{F}_\nu(\hat{x}_\nu, y) \to \mathcal{F}(\hat{x}, y)$, and hence $\hat{x}_\nu \to \hat{x}$, since every $\mathcal{F}_\nu(., y)$ has a unique minimizer and the latter is strict. Total variation methods are similar from numerical point of view since they involve $\phi(t) = |t|$. Many authors used smooth approximations [33, 38], e.g. $\phi_\nu = \sqrt{t^2 + \nu}$. However, approximation using Huber function has the numerical advantage to involve only quadratic and affine segments. At the same time, the fact that ψ'_ν is discontinuous at $\pm\nu$ is of no practical importance since the chance to get a minimizer $\hat{x}_\nu$ involving a difference whose modulus is exactly ν, is null [27].

First experiment

The original image x in Fig. 1(a) can be supposed to be a noisy version of an ideal piecewise constant image. Data y in Fig. 1(b) are obtained by adding aberrant impulsions to x whose locations are seen in Fig. 4-left.

Recall that our goal is to detect, and possibly to smooth, the outliers in y, while preserving all the remaining entries of y.

The image in Fig. 2(a) is the minimizer of the cost-function $\mathcal{F}(.,y)$ proposed in (77), with $\nu = 1.1$ and $\alpha = 0.14$. The outliers are well visible although their amplitudes are considerably reduced. The image of the residuals $y - \hat{x}$, shown in Fig. 2(b), is null everywhere except at the positions of the outliers in y. The pixels corresponding to non-zero residuals (*i.e.* the elements of $\hat{h}^c$) provide a faithful estimate of the locations of the outliers in y, as seen in Fig. 4-middle. Next, in Fig. 3(a) we show a minimizer $\hat{x}$ of the same $\mathcal{F}(.,y)$ obtained for $\alpha = 0.25$. This minimizer does not contain visible outliers and is very close to the original image x. The image of the residuals $y - \hat{x}$ in Fig. 3(b) is null only on restricted areas, but has a very small magnitude everywhere beyond the outliers. However, applying the above detection rule now leads to numerous false detections, as seen in Fig. 4-right. These experiments confirm our conjecture about the role of α.

The issue of the minimization of a smooth cost-function, namely $\mathcal{F}$ in (75) with $\psi(t) = \varphi(t) = t^2$ and $\alpha = 0.2$, is shown in Fig. 5(a). As expected, edges are blurred whereas outliers are well seen. The residuals in Fig. 5(b) are large everywhere, which shows that $\hat{x}$ does not fit any data entry. The minimizer in Fig. 6(a) is obtained using non-smooth regularization, where $\mathcal{F}$ is of the form (75) with $\psi(t) = t^2$, $\varphi(t) = |t|$ and $\alpha = 0.2$. In accordance with our discussion in § 6, $\hat{x}$ is constant on very large regions.

(a) Original x.

(b) Data $y = x$+outliers.
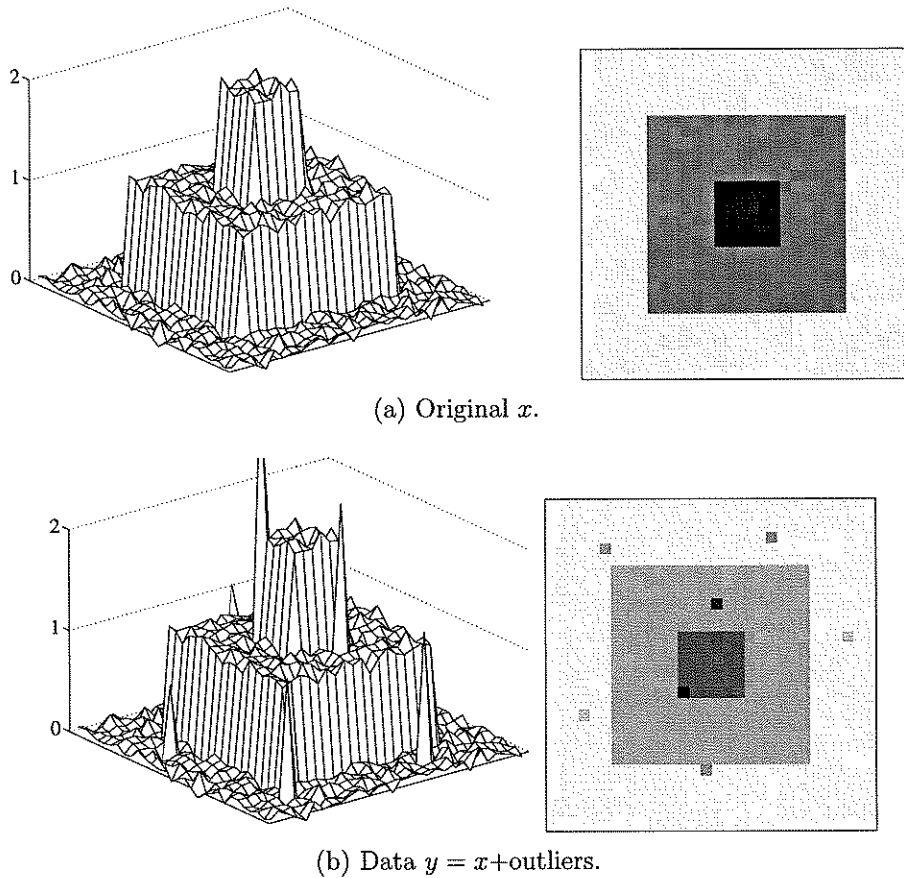
Figure 1: Original x and data y degraded by outliers.

(a) Restoration $\hat{x}$ for $\alpha = 0.14$.
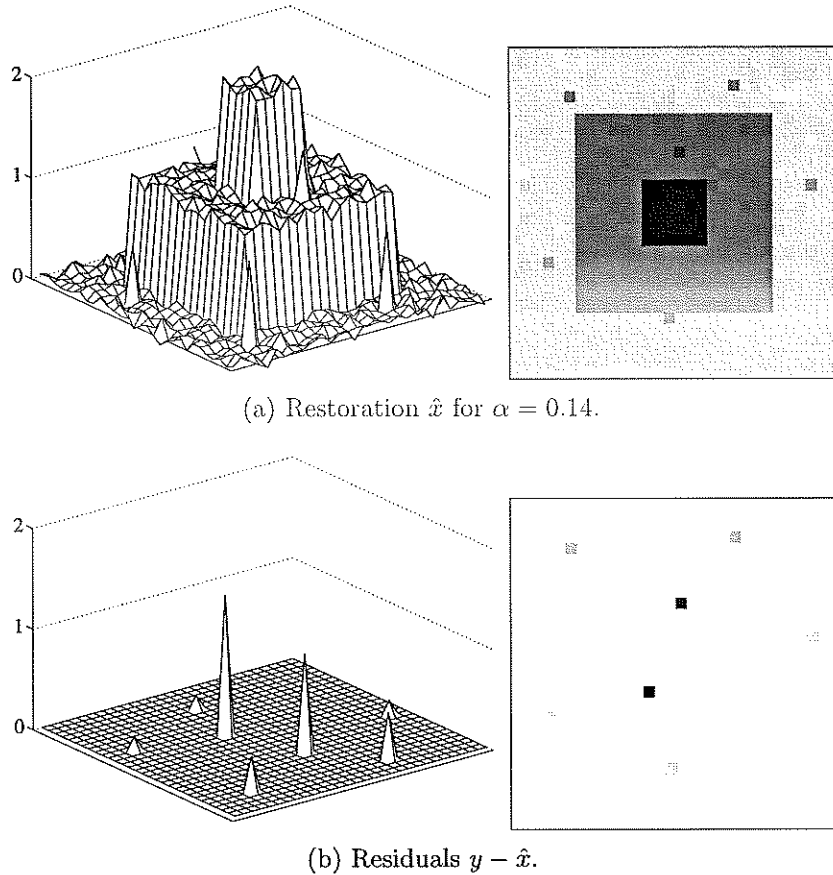
(b) Residuals $y - \hat{x}$.

Figure 2: Restoration using the proposed cost-function $\mathcal{F}$ with non-smooth data-fidelity in (77) for $\nu = 1.1$ and $\alpha = 0.14$. The residuals provide a faithful estimator for the locations of outliers.

Second experiment

The original, clean image x is shown in Fig. 7(a). The data y, shown in Fig. 7(b), are obtained by adding to x 770 impulsions with random locations and random amplitude in the interval $(0, 1.2)$.

In Fig. 8(a) we show a zoom of the histograms of x (up) and of y (down). Fig. 8(b) shows the result from applying to y two iterations of median filtering. The obtained image contains only a few outliers with weak amplitude but the entire image is degraded and especially the edges are blurred. The ℓ_1-norm of the error $\|\hat{x} - x\|_1 = \sum_i |\hat{x}_i - x_i|$ is 523. The next two restorations in Fig. 9 are obtained by minimizing the cost-function $\mathcal{F}$ with non-smooth data-fidelity proposed in (77) where $\nu = 1.1$. The minimizer in (a) corresponds to $\alpha = 0.2$ and it fits exactly the data everywhere except for several hundred pixels where it detects outliers. This detection gives rise to 50 erroneous non-detections and to 15 false alarms, the remaining detections being correct. The next image (b) is obtained for $\alpha = 0.55$. The minimizer $\hat{x}$ does not contain outliers any longer but it fits exactly only a restricted number of the data entries. Nevertheless, it remains very close to all data entries which are not outliers, since the ℓ_1-norm of the error is 126. This minimizer provides a very clean restoration where both edges and smoothly varying areas are nicely preserved. The restoration in Fig. 10(a) results from a smooth cost-function $\mathcal{F}$ as in (75) with $\psi(t) = \varphi(t) = t^2$ and $\alpha = 0.2$. This image fits no data entry while edges are

23
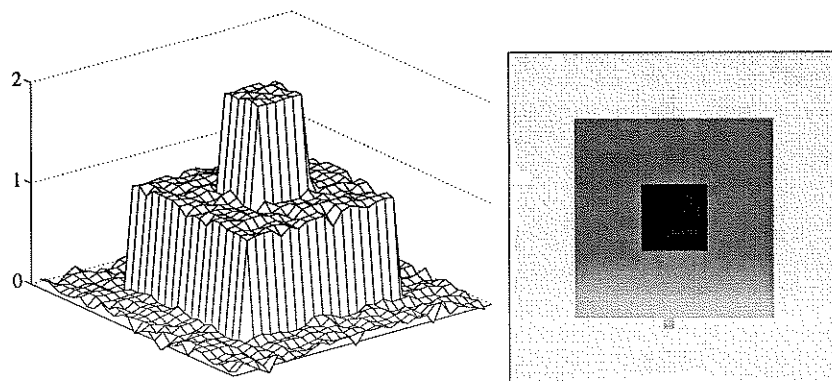
(a) Restoration $\hat{x}$ for $\alpha = 0.25$.
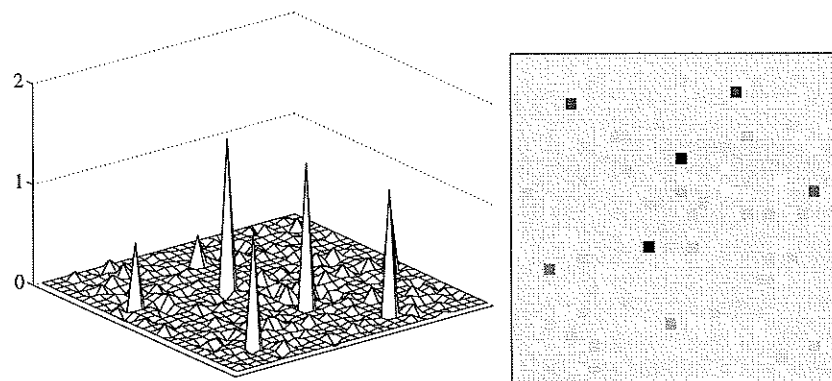
(b) Residuals $y - \hat{x}$.

Figure 3: Restoration using the proposed cost-function $\mathcal{F}$ in (77) for $\nu = 1.1$ and $\alpha = 0.25$. The outliers are well smoothed in $\hat{x}$ whereas the residuals remain small everywhere beyond the outliers locations.

smooth. Fig. 10(b) illustrates the stair-casing effect induced by non-smooth regularization. This minimizer corresponds to $\mathcal{F}$ as in (75) with $\psi(t) = t^2$ and $\varphi(t) = |t|$, for $\alpha = 0.4$ and it still contains several outliers.

8   Conclusion

We showed that taking non-smooth data-fidelity terms in a regularized cost-function yields minimizers which fit exactly a certain number of the data entries. In contrast, this cannot occur for a smooth cost-function. These are strong properties which can be used in different ways. We proposed a cost-function with a non-smooth data-fidelity term in order to process outliers. The obtained results advocate the use of non-smooth data-fidelity terms in image processing.

References

[1] R. ACAR AND C. VOGEL, *Analysis of bounded variation penalty methods for ill-posed problems*, IEEE Transactions on Image Processing, 10 (1994), pp. 1217–1229.

[2] S. ALLINEY, *Digital filters as absolute norm regularizers*, IEEE Transactions on Medical Imaging, MI-12 (1993), pp. 173–181.
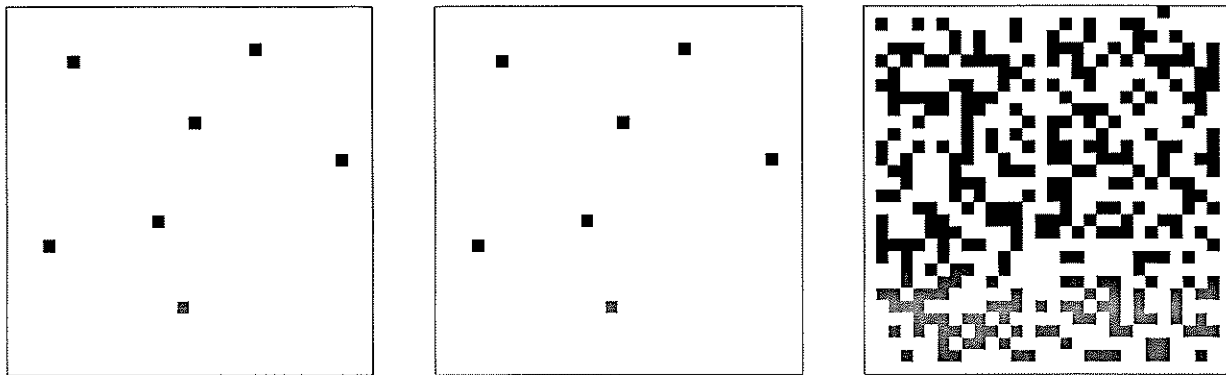
Figure 4: Left: the locations of the outliers in y. Next—the locations of the pixels of a minimizer $\hat{x}$ at which $\hat{x}_i \neq y_i$. Middle: these locations for the minimizer obtained for $\alpha = 0.14$, Fig. 2. Right: the same locations for the minimizer relevant to $\alpha = 0.25$, see Fig. 3.

[3] ———, *A property of the minimum vectors of a regularizing functional defined by means of absolute norm*, IEEE Transactions on Signal Processing, 45 (1997), pp. 913–917.

[4] S. ALLINEY AND S. A. RUZINSKY, *An algorithm for the minimization of mixed l_1 and l_2 norms with application to Bayesian estimation*, IEEE Transactions on Signal Processing, SP-42 (1994), pp. 618–627.

[5] A. AVEZ, *Calcul différentiel*, Masson, 1991.

[6] J. E. BESAG, *On the statistical analysis of dirty pictures (with discussion)*, Journal of the Royal Statistical Society B, 48 (1986), pp. 259–302.

[7] M. BLACK AND A. RANGARAJAN, *On the unification of line processes, outlier rejection, and robust statistics with applications to early vision*, International Journal of Computer Vision, 19 (1996), pp. 57–91.

[8] B. BLOOMFIELD AND W. L. STEIGER, *Least Absolute Deviations: Theory, applications and algorithms*, Birkhäuser, Boston, 1983.

[9] C. BOUMAN AND K. SAUER, *A generalized Gaussian image model for edge-preserving MAP estimation*, IEEE Transactions on Image Processing, IP-2 (1993), pp. 296–310.

[10] ———, *A unified approach to statistical tomography using coordinate descent optimization*, IEEE Transactions on Image Processing, 5 (1996), pp. 480–492.

[11] A. CHAMBOLLE AND P.-L. LIONS, *Image recovery via total variation minimization and related problems*, Numerische Mathematik, 76 (1997), pp. 167–188.

[12] T. F. CHAN AND C. K. WONG, *Total variation blind deconvolution*, IEEE Transactions on Image Processing, 7 (1998), pp. 370–375.

[13] G. DEMOMENT, *Image reconstruction and restoration : Overview of common estimation structure and problems*, IEEE Transactions on Acoustics Speech and Signal Processing, ASSP-37 (1989), pp. 2024–2036.

[14] D. DOBSON AND F. SANTOSA, *Recovery of blocky images from noisy and blurred data*, SIAM Journal on Applied Mathematics, 56 (1996), pp. 1181–1199.

[15] D. DONOHO, I. JOHNSTONE, J. HOCH, AND A. STERN, *Maximum entropy and the nearly black object*, Journal of the Royal Statistical Society B, 54 (1992), pp. 41–81.

[16] S. DURAND AND M. NIKOLOVA, *Stability of image restoration by minimizing regularized objective functions (to appear)*, in IEEE Int. Conf. on Computer Vision / Workshop on Variational and Level-Set Methods, Vancouver, Canada, 2001.

[17] D. GEMAN, *Random fields and inverse problems in imaging*, vol. 1427, École d'Été de Probabilités de Saint-Flour XVIII - 1988, Springer-Verlag, lecture notes in mathematics ed., 1990, pp. 117–193.

[18] D. GEMAN AND G. REYNOLDS, *Constrained restoration and recovery of discontinuities*, IEEE Transactions on Pattern Analysis and Machine Intelligence, PAMI-14 (1992), pp. 367–383.

[19] D. GEMAN AND C. YANG, *Nonlinear image recovery with half-quadratic regularization*, IEEE Transactions on Image Processing, IP-4 (1995), pp. 932–946.
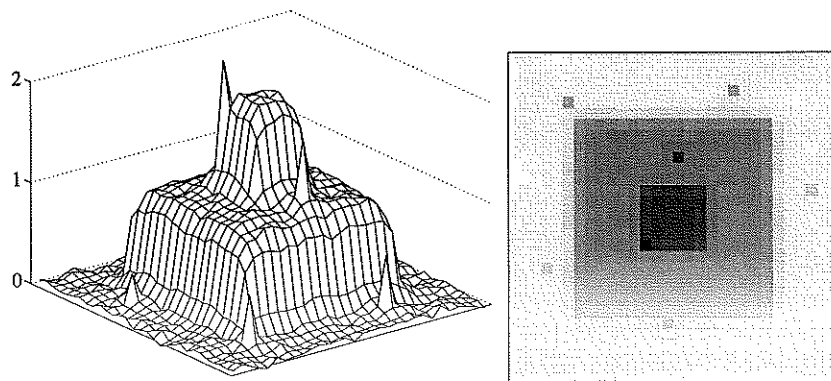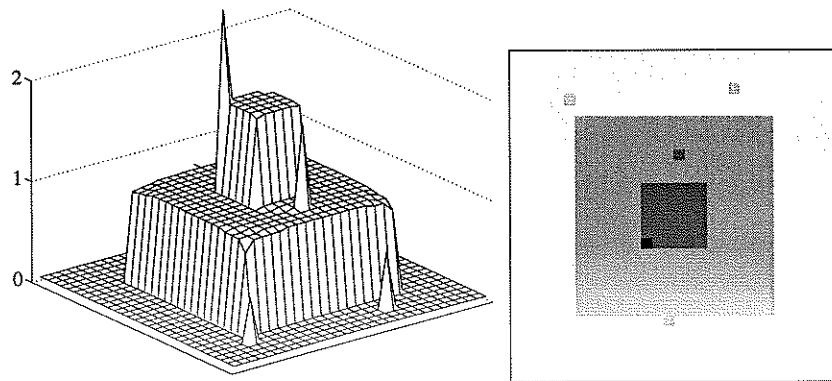
(a) Restoration from yo $\hat{x}$ for $\alpha = 0.2$.
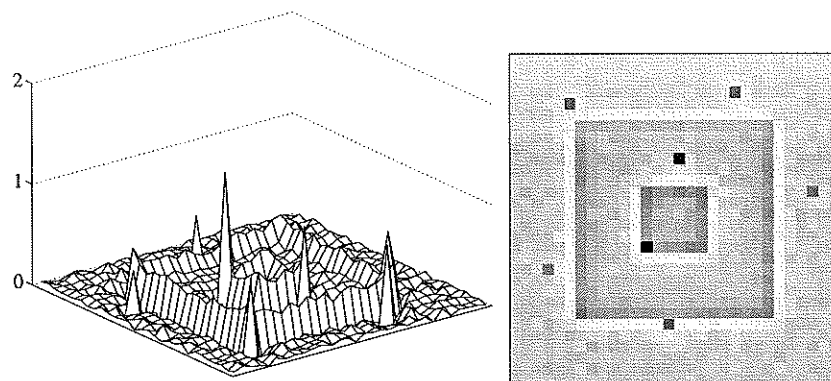
(b) Residuals $y - \hat{x}$.

Figure 5: Restoration using a smooth cost-function, namely $\mathcal{F}$ in (75) with $\psi(t) = \varphi(t) = t^2$ and $\alpha = 0.2$.

[20] S. GEMAN AND D. McCLURE, *Statistical methods for tomographic image reconstruction*, in Proc. of the 46-th Session of the ISI, Bulletin of the ISI, vol. 52, 1987, pp. 22–26.

[21] P. J. GREEN, *Bayesian reconstructions from emission tomography data using a modified EM algorithm*, IEEE Transactions on Medical Imaging, MI-9 (1990), pp. 84–93.

[22] T. KAILATH, *A view of three decades of linear filtering theory*, IEEE Transactions on Information Theory, IT-20 (1974), pp. 146–181.

[23] A. KAK AND M. SLANEY, *Principles of Computerized Tomographic Imaging*, IEEE Press, New York, NY, 1987.

[24] S. LI, *Markov Random Field Modeling in Computer Vision*, Springer-Verlag, New York, 1 ed., 1995.

[25] K. S. MILLER AND M. M. ROCHWARGER, *Least-squares methods for ill-posed problems with a prescribed bound*, SIAM Journal on Mathematical Analysis, 1 (1970), pp. 52–74.

[26] M. NIKOLOVA, *Estimées localement fortement homogènes*, Comptes-Rendus de l'Académie des Sciences, t. 325, série 1 (1997), pp. 665–670.

[27] ———, *Local strong homogeneity of a regularized estimator*, SIAM Journal on Applied Mathematics, 61 (2000), pp. 633–658.

[28] ———, *Weakly constrained minimization. application to the estimation of images and signals involving constant regions*, tech. report, CMLA—ENS de Cachan, France, 2001.

[29] P. PERONA AND J. MALIK, *Scale-space and edge detection using anisotropic diffusion*, IEEE Transactions on Pattern Analysis and Machine Intelligence, PAMI-12 (1990), pp. 629–639.

[30] T. T. PHAM AND R. J. P. DE FIGUEIREDO, *Maximum likelihood estimation of a class of non-Gaussian densities with application to l_p deconvolution*, IEEE Transactions on Signal Processing, 37 (1989), pp. 73–82.

(a) Restoration $\hat{x}$ for $\alpha = 0.2$.
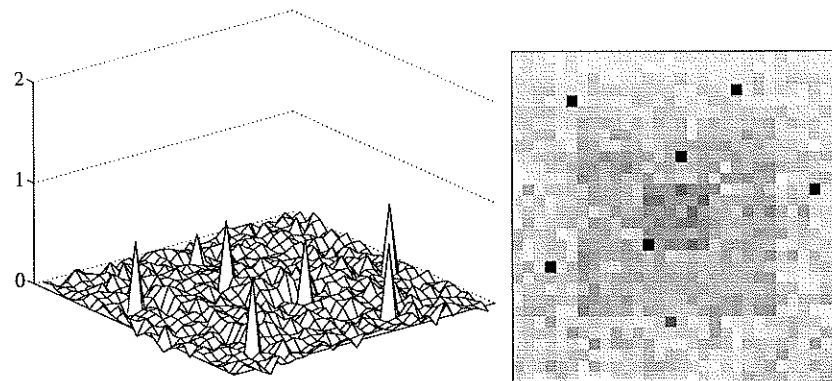
(b) Residuals $y - \hat{x}$.

Figure 6: Restoration involving non-smooth regularization: $\mathcal{F}$ is as in (75) with $\psi(t) = t^2$ and $\varphi(t) = |t|$, for $\alpha = 0.2$. The minimizer $\hat{x}$ is constant over large regions.

[31] J. R. RICE AND J. S. WHITE, *Norms for smoothing and estimation*, SIAM Rev., 6 (1964), pp. 243–256.

[32] R. T. ROCKAFELLAR AND J. B. WETS, *Variational analysis*, Springer-Verlag, New York, 1997.

[33] L. RUDIN, S. OSHER, AND C. FATEMI, *Nonlinear total variation based noise removal algorithm*, Physica, 60 D (1992), pp. 259–268.

[34] K. SAUER AND C. BOUMAN, *A local update strategy for iterative reconstruction from projections*, IEEE Transactions on Signal Processing, SP-41 (1993), pp. 534–548.

[35] A. TARANTOLA, *Inverse problem theory : Methods for data fitting and model parameter estimation*, Elsevier Science Publishers, Amsterdam, 1987.

[36] S. TEBOUL, L. BLANC-FÉRAUD, G. AUBERT, AND M. BARLAUD, *Variational approach for edge-preserving regularization using coupled PDE's*, IEEE Transactions on Image Processing, 7 (1998), pp. 387–397.

[37] A. TIKHONOV AND V. ARSENIN, *Solutions of Ill-Posed Problems*, Winston, Washington DC, 1977.

[38] C. R. VOGEL AND M. E. OMAN, *Fast, robust total variation-based reconstruction of noisy, blurred images*, IEEE Transactions on Image Processing, 7 (1998), pp. 813–824.

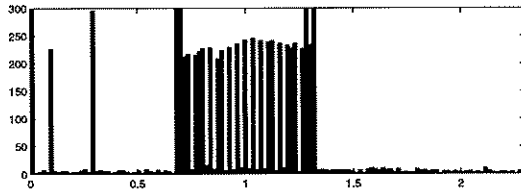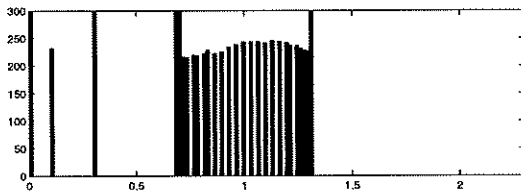[39] J. WEICKERT, *Anisotropic Diffusion in Image Processing*, B.G. Teubner, Stuttgart, 1998.
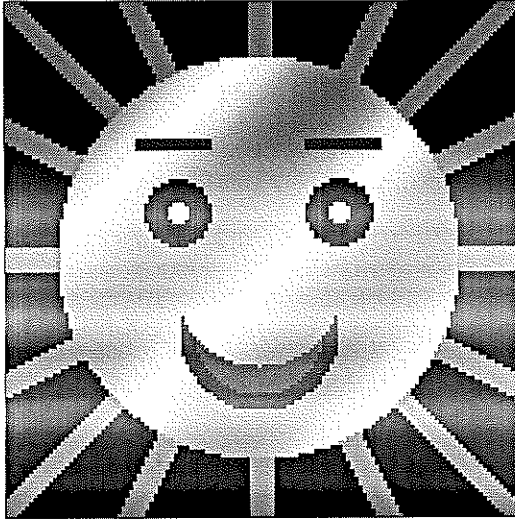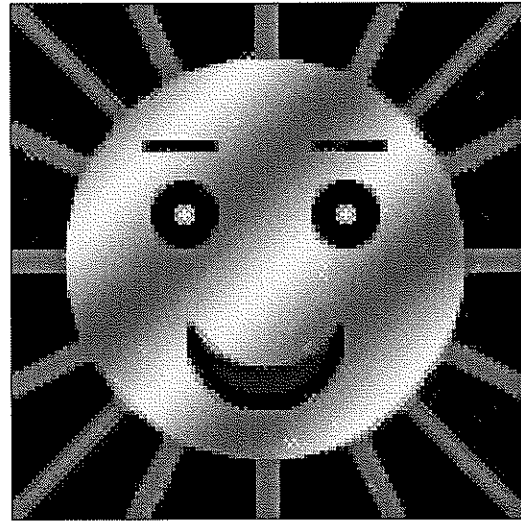
(a) Original image x
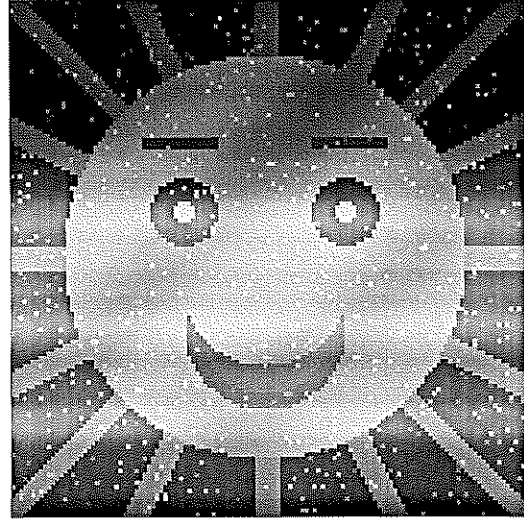
(b) Data $y = x + 770$ outliers.

Figure 7: Original image x and data y obtained by adding to x 770 outliers with random location and random amplitude.
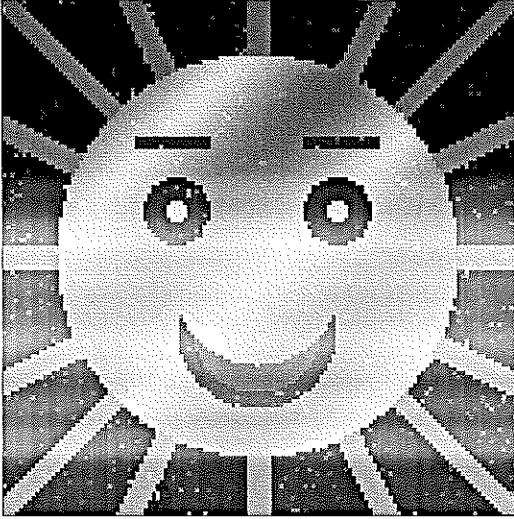
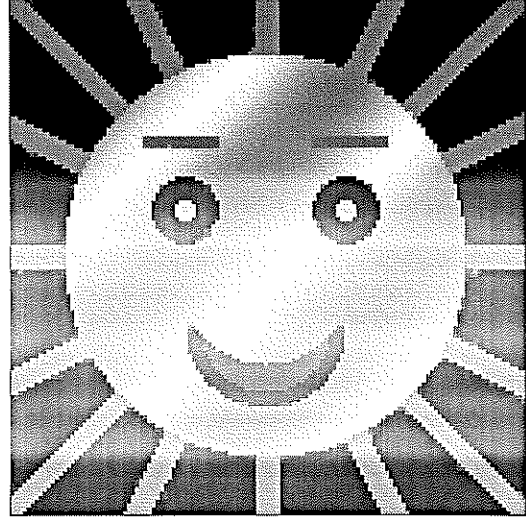

(a) Histograms: x (up), y (down).

(b) Restoration by median filtering.

Figure 8: (a) Zoom of the histograms of the original x (up) and of the data y. (b) Restoration using 2 iterations of median filtering.
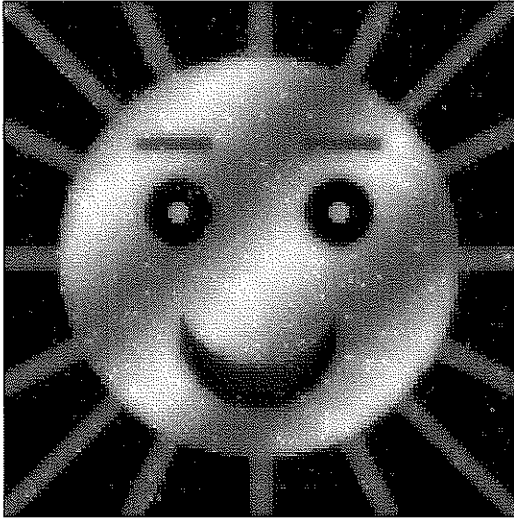
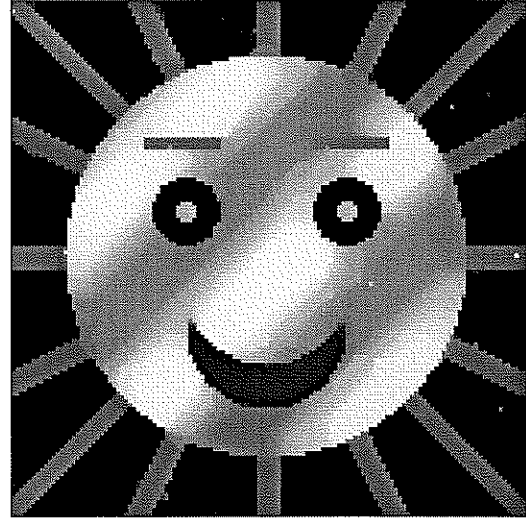(a) Minimizer obtained for $\alpha = 0.2$.

(b) Minimizer calculated for $\alpha = 0.55$

Figure 9: Minimizers obtained using the proposed cost-function $\mathcal{F}$ in (77), involving non-smooth data-fidelity. (a) For $\alpha = 0.2$ there are 720 correct and to 65 erroneous detections of outliers. Outliers are only weakly smoothed. (b) For $\alpha = 0.55$ outliers are well smoothed and the error is weak.

(a) Smooth cost-function.

(b) Non-smooth regularization.

Figure 10: Minimizers obtained by minimizing $\mathcal{F}$ of the form (75). (a) For $\psi(t) = t^2 = \varphi(t)$ and $\alpha = 0.2$. Outliers are well seen whereas edges are degraded. (b) For $\psi(t) = t^2$ and $\varphi(t) = |t|$, and $\alpha = 0.4$. Only several outliers remain visible. Stair-casing is well present.

29