On Winning Structures

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The traditional power indices in a voting body measure weighted marginal contributions in bringing losing coalitions to winning. However a single result may mean winning to some voters to some extent, but losing to some others. The dual approach defines the asymmetric Banzhaf and asymmetric Shapley-Shubik power indices. Personal winning structure is defined by his preference to vote for the bill. In addition, the bill itself can be put in a dual way: one statement, or its negative, or others. We incorporate a stochastic cooperative structure to the voting body and discuss its relation with the personal preference, the voting rules, the power indices, and the various statements of the bill. *Journal of Economic Literature* Classification: C71, D71, D72, D74

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1. PRELIMINARIES

When we talk about the result after an election, a football game, a war or an agreement between multi-party bargaining, we have a scenario of diverse responses. The same result may mean winning to some players to certain degrees; it also means losing to some others. Notice, however, that the degrees to which the players win the result are not the same even within the same winning side. Many may even be indifferent to the result. Actually the players have different levels of support for their sides before the election is voted, the game is played, and so on.

Let us consider the voting problem only. For a generic bill in the voting body, it should relate to the problems confronted with the body and its members. We assume each player has to vote for (YES to) the bill or vote against (NO to) it. However we do not assume that the players have the symmetric roles (e.g. the simple majority rule) nor the body is fair to all of them. The committee of bill proposers or initiators may propose bills in favor of some players. They may also select voting rules in favor of some players. Any player may choose the vote of either YES or NO in a specific bill. For the generic bill, say, his choice follows a distribution function which takes binary values (either 1 or 0). We expect that one player's distribution function may be different from that of another player. We also expect that the votes of any two players may not be independent. For example, consider the United Nations Security Council from 1950 to 1990. In the council, USA's votes have higher correlation with UK's than that with those of USSR. However USA's votes were not always identical to those of UK. On the other hand, the bills in the council may be in favor of some members by and large. As a consequence, these members then supported the bills more often than others. It might be improper to address the uncertainty problem or the potential cooperation among the players by some deterministic methods. In this paper, we shall incorporate a stochastic structure $\mathcal{P}$ into the voting body. As each player has to choose either YES or NO, it is sufficient for the cooperative structure $\mathcal{P}$ to specify the probability in forming each potential coalition of players who vote YES. In other words, if we partition N into any two disjoint coalitions, then $\mathcal{P}$ is a probability measure on the set of all partitions $\{(T, N \setminus T) | T \subseteq N\}$.

Let us consider the finite simple game $\Gamma(N, \mathcal{W})$ where $N = \overline{12\cdots n}$ being the set of players (or voters) $1, 2, \cdots, n$. The $\mathcal{W}$ denotes the set of winning coalitions such that: (a) the empty coalition $\emptyset$ never wins; (b) N always wins; (c) any superset of a winning coalition also wins. Those not in $\mathcal{W}$ are called *losing*. A subset of a losing coalition is then losing. Simple games have been used as voting rules in various voting bodies. In this paper, we shall use lower-case italic letters or numerals for players, italic capitals for coalitions, script capitals (except $\mathcal{P}$) for sets of coalitions. For the set of coalitions $\mathcal{S}$, we denote $\mathcal{S}^+$ the set of all supersets of elements in $\mathcal{S}$. Set subtraction will be indicated by "$\setminus$". In naming the players of a coalition, we shall employ the vinculum, thus "$\overline{137}$" for "$\{1, 3, 7\}$". The number of elements of a finite set X is denoted by $|X|$. Let the characteristic function

$v : 2^N \to \{0,1\}$ defined by

$$v(T) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } T \in \mathcal{W}; \\ 0, & \text{otherwise.} \end{cases}$$

Thus T gets 1 credit if the bill is passed by T, or 0 otherwise. We call i a *dictator* (*dummy*, *veto player*, or *master*) if $v(T \cup \bar{i}) - v(T \setminus \bar{i}) = 1$ ($v(T \cup \bar{i}) - v(T \setminus \bar{i}) = 0$, $v(T \setminus \bar{i}) = 0$, or $v(T \cup \bar{i}) = 1$ respectively) for any $T \subseteq N$. Thus a dictator is necessarily a veto player and a master. We notice that a simple game has at most one dictator and if it has one, then all other players are necessarily dummies.

The traditional notions of an individual's power index or relative strength in the voting body $\Gamma(N, \mathcal{W})$ come down ultimately to the question of his being essential in turning losing coalitions to winning. For any $T \subseteq N$, we say i is a *swinger* of T if $T \in \mathcal{W}$ but $T \setminus \bar{i} \notin \mathcal{W}$, i.e. $v(T) - v(T \setminus \bar{i}) = 1$. Here the swinger i is necessarily in T. The Banzhaf index $b_i[\Gamma]$ simply counts, divided by 2^{n-1}, the coalitions in which player i swings.

$$b_i[\Gamma] \stackrel{\text{def}}{=} \frac{1}{2^{n-1}} \sum_{T \subseteq N} [v(T) - v(T \setminus \bar{i})].$$

The coalition T may have multiple swingers. It may also have no swinger. The Shapley-Shubik(S-S) index $\phi_i[\Gamma]$, on the other hand, applies the "value" concept to the simple game,

$$\phi_i[\Gamma] \stackrel{\text{def}}{=} \sum_{T \subseteq N} \frac{(|T|-1)!(n-|T|)!}{n!} [v(T) - v(T \setminus \bar{i})]. \tag{1}$$

Therefore both $b_i[\Gamma]$ and $\phi_i[\Gamma]$ are weighted marginal contribution in forming winning coalitions: $v(T) - v(T \setminus \bar{i})$. In other words, both of them are weighted numbers of swingings in $\Gamma(N, \mathcal{W})$. More generally, when the probability distribution or the stochastic cooperative structure $\mathcal{P}$ is regarded, player i's expected number of swingings in $\Gamma(N, \mathcal{W})$ is then

$$\mathrm{E}[v(\mathbf{S}) - v(\mathbf{S} \setminus \bar{i})|\Gamma, \mathcal{P}] = \sum_{T \subseteq N} \mathrm{Prob}(\mathbf{S} = T|\Gamma, \mathcal{P})[v(T) - v(T \setminus \bar{i})] \quad (2)$$

where $\mathbf{S}$ denotes the stochastic coalition of players who vote for the bill. If i is a veto player and j $(\neq i)$ is a master, however, neither $b_i[\Gamma]$ nor $\phi_i[\Gamma]$ is well-defined. If this happens, we let the binary model satisfy the condition $\mathrm{Prob}(\overline{ij} \subseteq \mathbf{S}|\Gamma, \mathcal{P}) + \mathrm{Prob}(\overline{ij} \subseteq N \setminus \mathbf{S}|\Gamma, \mathcal{P}) = 1$. In this case, $\overline{ij}$ makes a dictatorship.

PROPOSITION 1.1. *These exists no cooperative structure $\mathcal{P}$, independent of Γ, such that $E[v(\mathbf{S}) - v(\mathbf{S} \setminus \bar{i})|\Gamma, \mathcal{P}] = b_i[\Gamma]$ for all $i \in N$ or $E[v(\mathbf{S}) - v(\mathbf{S} \setminus \bar{i})|\Gamma, \mathcal{P}] = \phi_i[\Gamma]$ for all $i \in N$.*

Proof. See Appendix A1. ▮

We shall show the existence of some fixed $\mathcal{P}$'s such that the asymmetric indices (defined later) lead to the Banzhaf or Shapley-Shubik index. These are stated in Proposition 3.2 and 3.6. The above indices are defined from the view point of passing the bill. From the view point of a blocker, however, the blocker does not have to vote for the bill and turn losing coalitions to winning. Instead, there are situations such that his vote of NO turns potential winning coalitions (if he votes YES) to losing. Our asymmetric power indices will recognize these situations.

As designed on the ballot of vote, the bill could be stated in one of many statements, such as one or its negative. However, the cooperative structure $\mathcal{P}$ may not response the variant forms in a proper way. Under certain conditions, we show that the power indices are invariant with these forms.

We next treat a special case of $\mathcal{P}$ in which all players are assumed to vote independently. As a consequence, the players' personal interest shall be fully valued by himself. In this case, $\mathcal{P}$ has an explicit expression. For a player in the voting body, the paper will address some aspects of the following issues:

- before the vote, how much he supports the bill and what his ideal statement of the bill should be;
- in the voting process, how much chance his vote is influential in forming the result;
- after the vote, how much he loses or wins and how much credit or responsibility he should take for the winning or losing result.

In §2, we address the dual logics of the voting rule and the bill. §3 defines the asymmetric Banzhaf index as the expected number of the swinging positions in the voting rule and its dual. It also defines the asymmetric Shapley-Shubik index as the expected contribution in the formed winning or blocked coalition. We analyze the personal winning structure in §4. §5 concludes the paper with several advanced issues. All proofs are in §6.

2. THE DUAL LOGIC OF VOTING

To address a bill to vote, we have several elements in mind: the winning rule, the statement of the bill, the players' response to the statement, the players' interactive behavior among each other, the voting procedure, and so on. However, there are equivalent or variant forms to the elements and these forms may cause fallacies to the players. In words, we may have a few related voting problems. If no confusion or fallacy occurs, we expect that the power indices should be the same in one voting problem and its equivalent problem. For variant problems, additional conditions may be required to make the indices equal.

The voting rule $\Gamma(N, \mathcal{W})$ is set up from the point of "passing the bill". From the view point of blockers, we can also define "winning" by that "the bill is not passed" or "the bill is blocked" and re-define its "winning" rule Γ^*. A coalition T in Γ^* then has the players who vote against the bill. A "winning" T in Γ^* then makes "passing the bill" impossible, or *blocks* forming any winning coalition in Γ. Therefore it is "winning" in Γ^* if and only if it has nonempty intersection with all winning coalitions in Γ. Actually, Γ^* itself is also a simple game, denoted $\Gamma^*(N, \mathcal{W}^*)$ where

$$\mathcal{W}^* \stackrel{\text{def}}{=} \{Z | Z \cap T \neq \emptyset, \forall T \in \mathcal{W}\} = \{N \setminus T | T \notin \mathcal{W}\}.$$

The game $\Gamma^*(N, \mathcal{W}^*)$ is called the *dual* of $\Gamma(N, \mathcal{W})$. Clearly the dual's dual is itself: $\Gamma^{**} = \Gamma$. The characteristic function of Γ^* is $v^*(T) \stackrel{\text{def}}{=} 1 - v(N \setminus T)$. It is 1 if $T \in \mathcal{W}^*$ or 0 otherwise. Let $\mathbf{S}^* = N \setminus \mathbf{S}$, the stochastic coalition of players who vote against the bill.

PROPOSITION 2.1. *1. A dictator (or dummy) in Γ is also a dictator (or dummy respectively) in Γ^*; 2. a veto player (or master) in Γ is a master (or veto player respectively) in Γ^*, and vice verse.*

The proof is simply from their definitions. For a single bill, we have defined two notions of "winning" from two "controversial" points of view. As each voter has to choose either YES or NO, the formation of the YES coalition $\mathbf{S} = T$ in $\Gamma(N, \mathcal{W})$ is then equivalent to the formation of the YES coalition $\mathbf{S} = N \setminus T$ in $\Gamma^*(N, \mathcal{W}^*)$. Therefore we can use the same $\mathcal{P}$ for both Γ and Γ^*. Actually, any coalition formation should be based on the voters' own interest, the voters' collective structure with others and randomness as well, not based on the notion of "winning" from which side. As $\mathcal{P}$ specifies the distribution of $\mathbf{S}$ and $\mathbf{S}^*$ in Γ, it also specifies the distribution of $\mathbf{S}$

and $\mathbf{S}^*$ in Γ^*. The duality of the two games are mathematically abstracted in the simple games $\Gamma(N, \mathcal{W})$ and $\Gamma^*(N, \mathcal{W}^*)$ as well as their cooperative structure,

$$\begin{aligned} &\text{Prob}(\mathbf{S} = T|\Gamma, \mathcal{P}) = \text{Prob}(\mathbf{S}^* = N \setminus T|\Gamma, \mathcal{P}) \\ = \; &\text{Prob}(\mathbf{S} = N \setminus T|\Gamma^*, \mathcal{P}) = \text{Prob}(\mathbf{S}^* = T|\Gamma^*, \mathcal{P}). \end{aligned}$$

In the ballot of the bill, if Γ is stated as "$A \Rightarrow B$" (read "If A, then B"), the Γ^* is then $\sim (A \Rightarrow B)$ where $\sim$ means NOT or AGAINST. Now, player i's expected number of swingings in Γ^* is

$$\begin{aligned} &\text{E}[v^*(\mathbf{S}) - v^*(\mathbf{S} \setminus \bar{i})|\Gamma^*, \mathcal{P}] = \text{E}[v(N \setminus \mathbf{S} \cup \bar{i}) - v(N \setminus \mathbf{S})|\Gamma^*, \mathcal{P}] \\ = \; &\text{E}[v(\mathbf{S}^* \cup \bar{i}) - v(\mathbf{S}^*)|\Gamma^*, \mathcal{P}] = \text{E}[v(\mathbf{S} \cup \bar{i}) - v(\mathbf{S})|\Gamma, \mathcal{P}] \\ = \; &\textstyle\sum_{T \subseteq N} [v(T \cup \bar{i}) - v(T))\text{Prob}(\mathbf{S} = T|\Gamma, \mathcal{P}] \end{aligned}$$

Proposition 2.2 (invariant). *For any $i \in N$, $b_i[\Gamma^*] = b_i[\Gamma]$ and $\phi_i[\Gamma^*] = \phi_i[\Gamma]$.*

Proof. See Appendix A2. ▌

Let us explain the idea from the statement of the bill. The bill can be stated in two ways: one statement or its negative. For example, the statement "no smoking is allowed in the public" (i.e. "If a man is in public, then he is not allowed smoking.") is the negative of that "smoking is allowed in the public". As usual, any statement should be clear enough to avoid potential misunderstanding and confusion to the voters. Formally, the statement $A \Rightarrow B$ has the negative $A \Rightarrow\sim B$ (read "If A then NOT B"). In a ballot of vote, it can be expressed as "Are You for the bill $(A \Rightarrow B)$?" with two boxes "YES" and "NO" leaving blank for a voter to choose. We let the negative statement $A \Rightarrow\sim B$ has the winning rule $\tilde{\Gamma}$. To vote the negative statement $A \Rightarrow\sim B$, one could just apply the voting rule Γ or Γ^*, instead of creating a new one.

However, that i cooperates with j in voting for $A \Rightarrow B$ does not necessarily imply that i cooperates with j in voting for $A \Rightarrow\sim B$. So the bill $A \Rightarrow\sim B$ should have a different cooperative structure from that of $A \Rightarrow B$. Given $A \Rightarrow B$'s cooperative structure $\mathcal{P}$, we let $\tilde{\mathcal{P}}$ be $\tilde{\Gamma}$'s cooperative structure. We do, however, observe several forms of "symmetry" in reality. When the voting rule Γ is applied to $A \Rightarrow\sim B$ (or $\Gamma = \tilde{\Gamma}$), the YES coalition to the bill $A \Rightarrow B$ would turn to the NO coalition to the bill $A \Rightarrow\sim B$. This is the condition S-I as stated in the follows. Secondly, when the rule Γ^* is applied to $A \Rightarrow\sim B$ (or $\Gamma^* = \tilde{\Gamma}$), the YES coalition to the bill $A \Rightarrow B$ under the rule Γ would turn to the NO coalition in the bill $A \Rightarrow\sim B$ under the dual rule Γ^*. This is the condition S-II.

S-I: $\tilde{\Gamma} = \Gamma$ and $\text{Prob}(\mathbf{S} = T|\tilde{\Gamma}, \tilde{\mathcal{P}}) = \text{Prob}(\mathbf{S}^* = T|\Gamma, \mathcal{P})$ for $\forall T \subseteq N$.

S-II: $\tilde{\Gamma} = \Gamma^*$ and $\text{Prob}(\mathbf{S} = T|\tilde{\Gamma}, \tilde{\mathcal{P}}) = \text{Prob}(\mathbf{S}^* = T|\Gamma, \mathcal{P})$ for $\forall T \subseteq N$.

In either S-I or S-II, the cooperative structure is preserved, though the voting rule $\tilde{\Gamma}$ takes the form of Γ or Γ^* respectively. In this paper, the cooperative structure is independent of the voting rule, though this may not true in reality. For their own sakes, S-I and S-II provide a special connection between the voting rule and the cooperative structure.

PROPOSITION 2.3. *If the vote obeys either S-I or S-II, then* $b_i[\tilde{\Gamma}] = b_i[\Gamma]$ *and* $\phi_i[\tilde{\Gamma}] = \phi_i[\Gamma]$ *for any* $i \in N$.

The proof is simply from Proposition 2.2.

The game $\Gamma(N, \mathcal{W})$ is *proper* if $T \cap Z \neq \emptyset$ for $\forall T, Z \in \mathcal{W}$. Therefore two disjoint coalitions can not both win the game, or $v(T)+v(N\setminus T) \leq 1$ for all $T \subseteq N$. If Γ is proper, its dual game Γ^* may be improper. For example, in the game $\Gamma(\overline{123}, \{\overline{123}\})$, Γ is proper but the dual game $\Gamma^*(N, \{\overline{1}, \overline{2}, \overline{3}\}^+)$ is not proper. The game $\Gamma(N, \mathcal{W})$ is called *decisive* if $v(T)+v(N\setminus T) = 1$ for all $T \subseteq N$. Therefore in a decisive game, a winning coalition has a losing complement, and vice verse.

PROPOSITION 2.4. *1.* $\Gamma(N, \mathcal{W})$ *is decisive if and only if both* Γ *and its dual* Γ^* *are proper; 2.* $\Gamma(N, \mathcal{W})$ *is decisive if and only if* $\Gamma^*(N, \mathcal{W}^*)$ *is decisive; 3.* $\Gamma(N, \mathcal{W})$ *is decisive if and only if* $v(T) = v^*(T)$ *for any* $T \subseteq N$*; 4.* $\Gamma(N, \mathcal{W})$ *is decisive if and only if* $\mathcal{W} = \mathcal{W}^*$.

The first two parts are from $v^*(T)+v^*(N\setminus T) = 2-[v(T)+v(N\setminus T)]$ for all $T \subseteq N$. The last two parts are from $v^*(T) = 1-v(N\setminus T)$ for any $T \subseteq N$. The simple majority rule with n odd is generally decisive. But the two-third majority rule is not decisive. Say, two complemental coalitions, each having almost half of the players, are both losing. The General Assembly of the United Nations has different voting rules according to different proposals. For example, it adopts the two-third majority voting rule regarding an issue specified as important, such as the admittance of the People's Republic of China. This further implies that the negative bill of an important issue should not be specified as "important" in the voting system. As we see, the same preference or votes by the players can lead to different result under different voting procedure or voting rule.

3. ASYMMETRIC POWER INDICES

As the bill could be either passed or blocked, player i's total contribution comes from two aspects. In the first aspect, $i \in \mathbf{S} \in \mathcal{W}$. His vote contributes in successfully passing the bill. In the second aspect, $i \notin \mathbf{S} \notin \mathcal{W}$ or $i \in \mathbf{S}^* \in \mathcal{W}^*$. He contributes in successfully blocking the bill. Of course there are two more cases he gets not credit: 1. $i \in \mathbf{S} \notin \mathcal{W}$ (he votes YES but the bill is blocked); 2. $i \notin \mathbf{S} \in \mathcal{W}$ (he votes NO but the bill is passed). In these two cases, he is not on the side of the successful voters. From the voter's point of view, he is winning if he votes YES and $\mathbf{S} \in \mathcal{W}$. He is also winning if he votes NO and $\mathbf{S}^* \in \mathcal{W}^*$. In this sense, we credit him if and only if he is winning. The asymmetric Banzhaf index $b_i[\Gamma, \mathcal{P}]$ is defined by player i's total expected number of swingings in both Γ and Γ^*,

$$\begin{aligned}
b_i[\Gamma,\mathcal{P}] &\stackrel{\text{def}}{=} \mathrm{E}[v(\mathbf{S}) - v(\mathbf{S}\setminus\bar{i})|\Gamma,\mathcal{P}] + \mathrm{E}[v^*(\mathbf{S}) - v^*(\mathbf{S}\setminus\bar{i})|\Gamma^*,\mathcal{P}] \\
&= \textstyle\sum_{T\subseteq N} \mathrm{Prob}(\mathbf{S}=T|\Gamma,\mathcal{P})[v(T) - v(T\setminus\bar{i})] \\
&\quad + \textstyle\sum_{T\subseteq N} \mathrm{Prob}(\mathbf{S}=T|\Gamma,\mathcal{P})[v(T\cup\bar{i}) - v(T)] \qquad (3) \\
&= \textstyle\sum_{T\subseteq N} \mathrm{Prob}(\mathbf{S}=T|\Gamma,\mathcal{P})[v(T\cup\bar{i}) - v(T\setminus\bar{i})] \\
&= \mathrm{E}[v(\mathbf{S}\cup\bar{i}) - v(\mathbf{S}\setminus\bar{i})|\Gamma,\mathcal{P}].
\end{aligned}$$

PROPOSITION 3.1 (invariant). *$b_i[\Gamma,\mathcal{P}] = b_i[\Gamma^*,\mathcal{P}]$ for all $i \in N$.*

Proof. See Appendix A3. ▮

PROPOSITION 3.2. *1. If Prob$(\mathbf{S} = T|\Gamma,\mathcal{P}) = \frac{1}{2^n}$ for $\forall T \subseteq N$, then $b_i[\Gamma,\mathcal{P}] = b_i[\Gamma]$. 2. For $\forall k = 0,1,\cdots,n$, if Prob$(|\mathbf{S}| = k|\Gamma,\mathcal{P}) = \frac{1}{n+1}$ and given $|\mathbf{S}| = k$, $\mathbf{S}$ is uniformly distributed on $\{T \subseteq N||T| = k\}$, then $b_i[\Gamma,\mathcal{P}] = \phi_i[\Gamma]$.*

Proof. See Appendix A4. ▮

By the proposition, the Banzhaf index assumes that the YES coalition $\mathbf{S}$ is uniformly distributed on 2^N. For the S-S index, $|\mathbf{S}|$ is uniformly distributed on $\{0,1,\cdots,n\}$.

Proposition 3.3 (S-I-II). *1. If the vote obeys S-I and the game is decisive, then* $b_i[\tilde{\Gamma}, \tilde{\mathcal{P}}] = b_i[\Gamma, \mathcal{P}]$ *for any* $i \in N$. *2. If the vote obeys S-II, then* $b_i[\tilde{\Gamma}, \tilde{\mathcal{P}}] = b_i[\Gamma, \mathcal{P}]$ *for all* $i \in N$.

Proof. See Appendix A5. ▮

The index is invariant with the notion of "winning" and the choice of the statements the bill uses, as we have expected.

Example 1:

To see the decisiveness is necessary in Proposition 3.3, let $N = \overline{123}$, $\mathcal{W} = \{\overline{12}, \overline{13}\}^+$ and

$$\begin{aligned} & \text{Prob}(\mathbf{S} = \overline{12}|\Gamma, \mathcal{P}) = \text{Prob}(\mathbf{S} = \overline{13}|\Gamma, \mathcal{P}) \\ = \ & \text{Prob}(\mathbf{S} = \overline{2}|\tilde{\Gamma}, \tilde{\mathcal{P}}) = \text{Prob}(\mathbf{S} = \overline{3}|\tilde{\Gamma}, \tilde{\mathcal{P}}) = .5. \end{aligned}$$

This vote obeys S-I; but it is not decisive since $v(\overline{1}) + v(\overline{23}) \neq 1$. Now

$$\begin{aligned} b_2[\Gamma, \mathcal{P}] &= \text{Prob}(\mathbf{S} = \overline{12}|\Gamma, \mathcal{P})[v(\overline{12}) - v(\overline{1})] \\ &\quad + \text{Prob}(\mathbf{S} = \overline{13}|\Gamma, \mathcal{P})[v(\overline{123}) - v(\overline{13})] = .5; \\ b_2[\tilde{\Gamma}, \tilde{\mathcal{P}}] &= \text{Prob}(\mathbf{S} = \overline{2}|\tilde{\Gamma}, \tilde{\mathcal{P}})[v(\overline{2}) - v(\emptyset)] \\ &\quad + \text{Prob}(\mathbf{S} = \overline{3}|\tilde{\Gamma}, \tilde{\mathcal{P}})[v(\overline{23}) - v(\overline{3})] = 0 \neq b_2[\Gamma, \mathcal{P}]. \end{aligned}$$

Neither $b_i[\Gamma]$'s nor $b_i[\Gamma, \mathcal{P}]$'s, however, sum to 1 or any other constant. To overcome the problem, we shall define the asymmetric Shapley-Shubik power index. Every bill or proposal must be either passed or blocked: it can not be partly passed or partly blocked. We may, however, say how much the bill is passed or blocked. Given $\mathbf{S} \in \mathcal{W}$, it could win by a margin of just one vote in that $\mathbf{S}$ has swingers. In this case, the swingers should be more important or critical than other players in $\mathbf{S}$. We also notice that the coalition of $\mathbf{S}$'s swingers is not necessarily winning in Γ. $\mathbf{S}$ could win by a significant leading of many votes. In this case, $\mathbf{S}$ contains no swinger. Given $\mathbf{S} \in \mathcal{W}$, we may also say how much one should be credited for passing the bill. As $\mathbf{S}$'s gain is $\mathbf{S}^*$'s loss, one could also say how much responsibility or blame one (in $\mathbf{S}^*$) should take for losing the game, and so on.

Given $\mathbf{S} = T \in \mathcal{W}$, all players in T vote YES to the bill and all players in $N \setminus T$ vote NO. To distribute the contribution of $v(T) = 1$, we note that a swinger in T should receive more credit than a non-swinger. We also note that a dummy should receive no credit at all. These further imply that the contribution should not be evenly credited to all players of T. In addition, the players of $N \setminus T$ should receive no credit for passing the bill since they vote against it. Now we define the *local game* $\Gamma(T, \mathcal{W}_T)$ by $\mathcal{W}_T = \{Z \in \mathcal{W} | Z \subseteq T\}$. It is a well-defined simple game for any $T \in \mathcal{W}$. Its characteristic function is $v(Z)$ for $\forall Z \subseteq T$. As all players in T vote YES, a fair allocation of the contribution is of course the S-S index in the local game $\Gamma(T, \mathcal{W}_T)$. So the players of $N \setminus T$ get no credit. As the swingers of T make up the veto players in the local game, any swinger gets the same amount of credit, and gets more credit than a non-swinger. The total credit of passing the bill by T is 1. In the example with $n = \overline{12345}$ and $\mathcal{W} = \{\overline{123}, \overline{124}, \overline{134}, \overline{145}, \overline{2345}\}^+$, if $\mathbf{S} = \overline{1234}$, then $\mathcal{W}_{\overline{1234}} = \{\overline{123}, \overline{124}, \overline{134}\}^+$. So the fair contribution to form the winning coalition $\overline{1234}$ is the S-S indices $(\frac{3}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ in the local game $\Gamma(\overline{1234}, \mathcal{W}_{\overline{1234}})$, instead of all credited to player 1. Player 1 is a veto player in the local game; player 5 receives no credit as he votes against the bill.

Similarly, given $\mathbf{S}|\Gamma^* = T \in \mathcal{W}^*$, we let $\mathcal{W}_T^* \stackrel{\text{def}}{=} \{Z \in \mathcal{W}^* | Z \subseteq T\}$. Then the local game $\Gamma^*(T, \mathcal{W}_T^*)$ of $\Gamma^*(N, \mathcal{W}^*)$ is also a simple game for any $T \in \mathcal{W}^*$. As before, the fair allocation of the contribution for blocking the bill by T should be the S-S index in the local game $\Gamma^*(T, \mathcal{W}_T^*)$. In the example with $n = \overline{12345}$ and $\mathcal{W} = \{\overline{123}, \overline{124}, \overline{134}, \overline{145}, \overline{2345}\}^+$, if $\mathbf{S}|\Gamma = \overline{15}$, then it loses and $\mathbf{S}|\Gamma^* = \overline{234}$. As $\mathcal{W} = \{\overline{12}, \overline{14}, \overline{15}, \overline{24}, \overline{34}, \overline{235}\}^+$, $\mathcal{W}^*_{\overline{234}} = \{\overline{24}, \overline{34}\}^+$ and the fair contribution for blocking the bill by $\overline{234}$ is $(\frac{1}{6}, \frac{1}{6}, \frac{2}{3})$. Players 1 and 5 votes YES and receive no credit for blocking the bill successfully; player 4 is a veto player in the local game and he receives more credit than player 2 or 3.

Let $\phi_i[\Gamma, \mathcal{W}_T]$ denote player i's S-S index in the game $\Gamma(T, \mathcal{W}_T)$ if $i \in T \in \mathcal{W}$ or 0 otherwise; let $\phi_i[\Gamma^*, \mathcal{W}_T^*]$ denote i's S-S index in the game $\Gamma^*(T, \mathcal{W}_T^*)$ if $i \in T \in \mathcal{W}^*$ or 0 otherwise. Extending (3), we now define the asymmetric Shapley-Shubik index $\phi_i[\Gamma, \mathcal{P}]$ by player i's expected contribution in both

Γ and Γ^*,

$$\begin{aligned}\phi_i[\Gamma,\mathcal{P}] &\stackrel{\text{def}}{=} \mathrm{E}\phi_i[\Gamma,\mathcal{W}_{\mathbf{S}}] + \mathrm{E}\phi_i[\Gamma^*,\mathcal{W}^*_{\mathbf{S}}] \\ &= \textstyle\sum_{i\in T\in\mathcal{W}} \mathrm{Prob}(\mathbf{S}=T|\Gamma,\mathcal{P})\phi_i[\Gamma,\mathcal{W}_T] \\ &\quad + \textstyle\sum_{i\in T\in\mathcal{W}^*} \mathrm{Prob}(\mathbf{S}=T|\Gamma^*,\mathcal{P})\phi_i[\Gamma^*,\mathcal{W}^*_T].\end{aligned} \tag{4}$$

LEMMA 3.1. *$\phi_i[\Gamma,\mathcal{P}]$ has a simple formula,*

$$\textstyle\sum_{T\subseteq N} Prob(\mathbf{S}=T|\Gamma,\mathcal{P}) \Big[\sum_{Z\subseteq T} \frac{(|Z|-1)!(|T|-|Z|)!}{(|T|)!}(v(Z)-v(Z\setminus\bar{i})) + \sum_{Z\supseteq T} \frac{(n-|Z|-1)!(|Z|-|T|)!}{(n-|T|)!}(v(Z\cup\bar{i})-v(Z))\Big].$$

Proof. See Appendix A6. ▮

PROPOSITION 3.4 (invariant). *$\phi_i[\Gamma^*,\mathcal{P}] = \phi_i[\Gamma,\mathcal{P}]$ for any $i\in N$.*

Proof. See Appendix A7. ▮

PROPOSITION 3.5. *1. $\phi_i[\Gamma,\mathcal{P}]\geq 0$ and $\sum_{i=1}^n \phi_i[\Gamma,\mathcal{P}]=1$; 2. if i is a dummy, then $\phi_i[\Gamma,\mathcal{P}]=0$; 3. if i is a dictator, then $\phi_i[\Gamma,\mathcal{P}]=1$; 4. if $Prob(\mathbf{S}=N|\Gamma,\mathcal{P})=1$ or $Prob(\mathbf{S}^*=N|\Gamma,\mathcal{P})=1$, then $\phi_i[\Gamma,\mathcal{P}]=\phi_i[\Gamma]$.*

Proof. See Appendix A8. ▮

A simple observation is that $\phi_i[\Gamma,\mathcal{P}]=0$ implies $b_i[\Gamma,\mathcal{P}]=0$. But $\phi_i[\Gamma,\mathcal{P}]=0$ does not imply that i is a dummy. As $\phi_i[\Gamma,\mathcal{W}_T]$ and $\phi_i[\Gamma^*,\mathcal{W}^*_T]$

are weighted marginal contribution, the index $\phi_i[\Gamma, \mathcal{P}]$ is actually a composite of weighted marginal contribution. In particular if we apply the conditions in Proposition 3.2, our composite method leads to the same index (Proposition 3.6). As a remark, we shall see that there exist other bill-based and cooperation-based fairly acceptable conditions, instead of these special probability distributions. In a multi-party voting, for example, partial cooperations are more likely formed within each party; and $\mathcal{P}$ would relate to their collective interests to the bill.

However if i is a veto player, then $\phi_i[\Gamma, \mathcal{P}] \geq \phi_j[\Gamma, \mathcal{P}]$ is not generally true for $\forall j \in N$. This may come as a surprise. For example, consider the USA and China in the United Nations Security Council from 1970 to 2000. Both were veto players. As USA had closer relation with UK and France, however, we believe that USA had more power than China in the council.

PROPOSITION 3.6. *1. For* $\forall k = 0, 1, \cdots, n$, *if* $Prob(|\mathbf{S}| = k|\Gamma, \mathcal{P}) = \frac{1}{n+1}$ *and given* $|\mathbf{S}| = k$, $\mathbf{S}$ *is uniformly distributed on* $\{T \subseteq N||T| = k\}$, *then* $\phi_i[\Gamma, \mathcal{P}] = \phi_i[\Gamma]$. *2. If* $Prob(\mathbf{S} = T|\Gamma, \mathcal{P}) = \frac{1}{2^n}$ *for* $\forall T \subseteq N$, *then* $\phi_i[\Gamma, \mathcal{P}] = \phi_i[\Gamma]$.

Proof. See Appendix A9. ▮

PROPOSITION 3.7 (S-I-II). *1. If the vote obeys the S-I and Γ is decisive, then $\phi_i[\tilde{\Gamma}, \tilde{\mathcal{P}}] = \phi_i[\Gamma, \mathcal{P}]$ 2. If the vote obeys the S-II, then $\phi_i[\tilde{\Gamma}, \tilde{\mathcal{P}}] = \phi_i[\Gamma, \mathcal{P}]$*

Proof. See Appendix A10. ▮

To see the decisiveness is necessary in Proposition 3.7, let us revisit Example 1. In this example, $\mathcal{W}^* = \{\overline{1}, \overline{23}\}^+$; player 2 has S-S index .5 in the local game $\Gamma(\overline{12}, \mathcal{W}_{\overline{12}})$. Therefore

$$\begin{aligned}
\phi_2[\Gamma, \mathcal{P}] &= \text{Prob}(\mathbf{S} = \overline{12}|\Gamma, \mathcal{P})\phi_2[\Gamma, \mathcal{W}_{\overline{12}}] \\
&\quad + \text{Prob}(\mathbf{S} = \overline{13}|\Gamma, \mathcal{P})\phi_2[\Gamma, \mathcal{W}_{\overline{13}}] = .25; \\
\phi_2[\Gamma, \mathcal{P}^*] &= \text{Prob}(\mathbf{S}^* = \overline{2}|\Gamma, \mathcal{P})\phi_2[\Gamma, \mathcal{W}_{\overline{2}}] \\
&\quad + \text{Prob}(\mathbf{S}^* = \overline{3}|\Gamma, \mathcal{P})\phi_2[\Gamma, \mathcal{W}_{\overline{3}}] = 0 \neq \phi_2[\Gamma, \mathcal{P}].
\end{aligned}$$

Example 2:

Let $N = \overline{123}$, $\mathcal{W} = \{\overline{12}, \overline{13}\}^+$ and $\mathcal{W}^* = \{\overline{1}, \overline{23}\}^+$. In Table I, we assign or estimate the probability distribution of $\mathbf{S}|\Gamma$ and calculate the S-S indices for all potential $\Gamma(T, \mathcal{W}_T)$ or $\Gamma^*(T, \mathcal{W}_T^*)$. By the table, we have $b_1[\Gamma, \mathcal{P}] = .15 + .1 + .15 + .2 + .2 + .05 = .85$, $b_2[\Gamma, \mathcal{P}] = .1 + .15 = .25$, and $b_3[\Gamma, \mathcal{P}] = .1 + .2 = .3$. The $\phi_i[\Gamma, \mathcal{P}]$'s are

$$\begin{aligned}
&.05(\tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{6}) + .1(0, \tfrac{1}{2}, \tfrac{1}{2}) + .15(1, 0, 0) + .1(1, 0, 0) \\
&+ .15(\tfrac{1}{2}, \tfrac{1}{2}, 0) + .2(\tfrac{1}{2}, 0, \tfrac{1}{2}) + .2(1, 0, 0) + .05(\tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{6}) = (\tfrac{83}{120}, \tfrac{17}{120}, \tfrac{20}{120}).
\end{aligned}$$

We compare these indices as follows:

$$b_i[\Gamma]: \quad (\tfrac{90}{120}, \tfrac{30}{120}, \tfrac{30}{120}); \qquad \phi_i[\Gamma]: \quad (\tfrac{80}{120}, \tfrac{20}{120}, \tfrac{20}{120});$$

$$b_i[\Gamma, \mathcal{P}]: \ (\tfrac{102}{120}, \tfrac{30}{120}, \tfrac{36}{120}); \qquad \phi_i[\Gamma, \mathcal{P}]: \ (\tfrac{83}{120}, \tfrac{17}{120}, \tfrac{20}{120}).$$

TABLE I.

The Swingers, Probability and Contribution in Example 2

$\mathbf{S}\|\Gamma$	Probability	Swinger	T	$\mathcal{W}_T$	$\mathcal{W}^*_T$	Contribution
$\emptyset$	.05		$\overline{123}$		$\{\overline{1}, \overline{23}\}^+$	$(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$
$\overline{1}$	.1	2,3	$\overline{23}$		$\{\overline{23}\}$	$(0, \frac{1}{2}, \frac{1}{2})$
$\overline{2}$	.15	1	$\overline{13}$		$\{\overline{1}, \overline{13}\}^+$	$(1, 0, 0)$
$\overline{3}$	.1	1	$\overline{12}$		$\{\overline{1}, \overline{12}\}^+$	$(1, 0, 0)$
$\overline{12}$	.15	1,2	$\overline{12}$	$\{\overline{12}\}$		$(\frac{1}{2}, \frac{1}{2}, 0)$
$\overline{13}$	.2	1,3	$\overline{13}$	$\{\overline{13}\}$		$(\frac{1}{2}, 0, \frac{1}{2})$
$\overline{23}$	.2	1	$\overline{1}$		$\{\overline{1}\}$	$(1, 0, 0)$
$\overline{123}$	.05	1	$\overline{123}$	$\{\overline{12}, \overline{13}\}^+$		$(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$

Example 3:

There have been many manipulative congresses in the histroy. According to the Constitution of a country, for example, the major political party (say, CCP) should have at least 50% quota of the members in the Congress. For most bills, the voting rule is the simple majority rule. CCP can determine the votes of their party members. By the S-S index or the Banzhaf index, any congressman has the same power. By the asymmetric S-S power index, however, non-CCP members never gain power whenever they have conflicting votes with these CCP members. Their powers are from the situations when they comply with the party. The asymmetric Banzhaf indices of the non-CCP members are 0 for any distribution function $\mathcal{P}$. Therefore in the congress, the non-CCP members are actually "dummies" and the CCP is actually a "dictator". The non-CCP members are used by the dictator as pumpets to cheat the people of the country.

4. PERSONAL WINNING STRUCTURES

In this section, we shall assume that all players vote the bill independently and analyze their personal winning structures. For a bill to vote, player i has certain level of preference to vote either YES or NO, or certain level of intensity to support the bill. We use the preference or intensity to denote his personal winning structure. It is natural to consider a whole spectrum of preference or intensity from "Absolutely vote NO", "Strongly Preferred NO", "Slightly Preferred NO", "Indifference between YES or NO", "Slightly Preferred YES", "Strongly Preferred YES", to "Absolutely Vote YES", among other levels. A normal way to quantify the whole spectrum is to map, by one-to-one, the preference levels onto $[0,1]$. We denote i's preference to voting YES or the intensity to support the bill by p_i. So a

p_i of 1, .5 or 0 indicates the preference level of "Absolutely vote YES", "Indifference between YES and NO", or "Absolutely vote NO", respectively.

Let us explain this by the stochastic preference in which p_i denotes player i's probability to vote for the bill. Denote the vote YES (or NO) by the numerical 1 (or 0 respectively). Given that no one else will affect player i's choice, his vote or vote function $\mathbf{U}_i : [0,1] \to \{0,1\}$ is then determined by his preference p_i in a stochastic way,

$$\mathbf{U}_i = \begin{cases} 1, & \text{with probability } p_i \\ 0, & \text{with probability } 1 - p_i, \end{cases}$$

i.e., $\mathbf{U}_i$ is a Bernoulli random variable with parameter p_i. We use the stochastic preference rather than a static preference since there exists no continuous vote function from the continuum of the preference space $[0,1]$ onto the discrete space $\{0,1\}$. The intensity of support is much diversely distributed though each voter has to choose either YES or NO. That "player i has more preference to vote YES than player j has" is indicated by $p_i > p_j$, or by $\text{Prob}(\mathbf{U}_i = 1) > \text{Prob}(\mathbf{U}_j = 1)$, or by "$i$ is more likely to vote YES than j", but not by "$\mathbf{U}_i \geq \mathbf{U}_j$". This, however, does not exclude the case that "i votes NO and j votes YES". This case, however, has the probability $(1-p_i)p_j < \min\{(1-p_i)p_i, (1-p_j)p_j\} \leq .25$. The more likely cases are that $\mathbf{U}_i \geq \mathbf{U}_j$.

More generally, one may explain the stochastic preference by a process. As each bill generally takes some period to be proposed and to be discussed in public before it is formally voted by the players. In the process, p_i is player i's preference at the time when the bill is proposed and all players learn the potential good and bad benefits if the bill is passed or blocked. Finally they have to decide $\mathbf{U}_i$'s for themselves. We can imagine that larger p_i generally leads to larger $\mathbf{U}_i$, but not always. The potential cooperative structure also depends on the p_i's. To determine the preference p_i, we could assume that i expects the (random) payoff $\mathbf{c}_{i,0}$ if the bill is blocked or (random) $\mathbf{c}_{i,1}$ if the bill is passed. The payoffs are not assumed the same distribution functions. Given the lack of the information about other players' preference, his sincere strategy p_i would maximize his expected utility function

$$\max_{p_i \in [0,1]} \mathrm{E}[u_i(p_i \mathbf{c}_{i,1} + (1-p_1)\mathbf{c}_{i,0})]$$

where $u_i(\cdot)$ is i's utility function.

We note that player i's preference to Γ^* is necessarily $1 - p_i$. In this section, we will use the vector $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ denote the preference to the bill Γ, or equivalently $\mathbf{1} - \mathbf{p}$ for the preference to the bill Γ^* where $\mathbf{1} = (1, 1, \cdots, 1) \in R^n$. The coalition $\mathbf{S}$ then has the probability distribution

$$\begin{aligned} \mathrm{Prob}(\mathbf{S} = T|\Gamma, \mathbf{p}) &= \mathrm{Prob}(\mathbf{U}_i = 1, \forall i \in T; \mathbf{U}_j = 0, \forall j \notin T|\Gamma, \mathbf{p}) \\ &= \prod_{j \in T} p_j \prod_{j \notin T} (1 - p_j). \end{aligned}$$

and

$$\mathrm{Prob}(\mathbf{S}=T|\Gamma^*,\mathbf{1}-\mathbf{p})=\prod_{j\in T}(1-p_j)\prod_{j\notin T}p_j=\mathrm{Prob}(\mathbf{S}=N\setminus T|\Gamma,\mathbf{p}).$$

Therefore, with the independence assumptation, the cooperative structure $\mathcal{P}$ determines both $\mathrm{Prob}(\mathbf{S}=T|\Gamma,\mathbf{p})$ and $\mathrm{Prob}(\mathbf{S}=T|\Gamma^*,\mathbf{1}-\mathbf{p})$, and vice verse. As already shown (Owen, 1972), the probability that $\mathbf{S}$ passes the bill is given by its multilinear extension $h(\Gamma,\mathbf{p})$:

$$\begin{aligned}\mathrm{Prob}(\mathbf{S}\in\mathcal{W}|\Gamma,\mathbf{p}) &= \textstyle\sum_{T\in\mathcal{W}}\mathrm{Prob}(\mathbf{S}=T|\Gamma,\mathbf{p})\\ &= \textstyle\sum_{T\subseteq N}\mathrm{Prob}(\mathbf{S}=T|\Gamma,\mathbf{p})v(T)=\mathrm{E}[v(\mathbf{S}|\Gamma,\mathbf{p})]\\ &= \textstyle\sum_{T\subseteq N}v(T)\prod_{j\in T}p_j\prod_{j\notin T}(1-p_j)\\ &\stackrel{\mathrm{def}}{=} h(\Gamma,\mathbf{p}).\end{aligned}\tag{5}$$

The $1-h(\Gamma,\mathbf{p})$ measures the probability that the bill is blocked.

PROPOSITION 4.1. *1.*$h(\Gamma^*,\mathbf{1}-\mathbf{p})=1-h(\Gamma,\mathbf{p})$*; 2.*$h(\Gamma^*,\mathbf{p})=h(\Gamma,\mathbf{p})$ *if* Γ *is decisive.*

Proof. See Appendix A11. ∎

If $p_i=0$, then his perfect objective is $\mathrm{E}[v(\mathbf{S})|\Gamma,\mathbf{p}]=0$ such that no chance of passing the bill exists. If $p_i=1$, on the other hand, his objective is $\mathrm{E}[v(\mathbf{S})|\Gamma,\mathbf{p}]=1$ such that the bill is passed for sure. For the generic player i with $0<p_i<1$, he is actually unsatisfactory with both $v(\mathbf{S})=0$ or $v(\mathbf{S})=1$ with different degrees. His objective is that $\mathrm{E}[v(\mathbf{S})|\Gamma,\mathbf{p}]=p_i$.

If this is true, then he can represent the group on this issue. On the other point, i may propose a new and modified bill to substitute the original one. For example, the bill "Smoking is NOT allowed in public" could be modified as "Smoking is not allowed in a corporate building and its adjacent area within 50-meter; Smoking can be allowed in restricted area (smoking area) in restaurants". For another bill, "If the city has an earthquake of 6.7 degree, then the state will allocate the city 100 million dollars of emergency relief" can have such modified bill as "If the city has an earthquake of 7.0 degree, then the state will allocate the city 100 million dollars of emergency relief", or "If the city has an earthquake of 6.7 degree, then the state will allocate the city 80 million dollars of emergency relief", or "If the city has an earthquake of 6.7 degree and the earthquake causes 1 billion of loss, then the state will allocate the city 100 million dollars of emergency relief", etc. Therefore for any given preference $p \in [0,1]$, we can come up many statements which represent p.

PROPOSITION 4.2. *$b_i[\Gamma, \mathbf{p}] = \sum_{T \subseteq N \setminus \bar{i}} Prob(\mathbf{S} \setminus \bar{i} = T | \Gamma, \mathbf{p})[v(T \cup \bar{i}) - v(T)] = \frac{\partial h}{\partial p_i}(\Gamma, \mathbf{p})$ which is independent of p_i.*

Proof. See Appendix A12. ▮

COROLLARY 4.1. *1. $b_i[\Gamma] = b_i[\Gamma, \frac{1}{2}\mathbf{1}]$; 2. $\phi_i[\Gamma] = \int_0^1 b_i[\Gamma, x\mathbf{1}]\,dx$;*

Proof. See Appendix A13. ∎

The corollary can also be found in Owen(1972) in a different setting. Besides the independence assumption, the S-S index implies that all voters have the same preference which is uniformly distributed on $[0,1]$. So the probability to form the YES coalition T is

$$\text{Prob}(\mathbf{S}=T|\Gamma,\mathbf{p}) = \int_0^1 x^{|T|}(1-x)^{n-|T|}dx = \frac{(|T|)!(n-|T|)!}{(n+1)!}.$$

For any given size $0 \leq z \leq n$, the probability $\text{Prob}(|\mathbf{S}|=z|\Gamma,\mathbf{p})$ to form the YES coalition with the size z is then

$$\sum_{|T|=z} \text{Prob}(\mathbf{S}=T|\Gamma,\mathbf{p}) = \frac{(z)!(n-z)!}{(n+1)!}\begin{pmatrix} n \\ z \end{pmatrix} = \frac{1}{n+1}.$$

That is, the size $|\mathbf{S}|$ is uniformly distributed on the set $\{0,1,\cdots,n\}$. So $\emptyset$ and N are more likely to be formed than a coalition whose size is near $\frac{n}{2}$. On the other hand, the Banzhaf index assumes that all voters have indifference preference on the bill. Or they do not care about the result. The probability to form any coalition is then $\frac{1}{2^n}$. That is, $\mathbf{S}$ is uniformly distributed on 2^N. Finally the probability of passing the vote without the participation of i is (see Appendix A12 for the second equality)

$$\begin{aligned} g_i(\Gamma,\mathbf{p}) &\stackrel{\text{def}}{=} \text{Prob}(\mathbf{S}\setminus\bar{i} \in \mathcal{W}|\Gamma,\mathbf{p}) = \text{E}[v(\mathbf{S}\setminus\bar{i})|\Gamma,\mathbf{p}] \\ &= \textstyle\sum_{i\notin T\subset N} v(T\setminus\bar{i})\text{Prob}(\mathbf{S}\setminus\bar{i}=T|\Gamma,\mathbf{p}) \\ &= \textstyle\sum_{i\notin T\in\mathcal{W}} \prod_{j\in T} p_j \prod_{j\notin T, j\neq i}(1-p_j). \end{aligned}$$

This is also the probability that i does not swing in $\mathbf{S}$. Notice that the probability is independent of p_i.

PROPOSITION 4.3. $h(\Gamma, \mathbf{p}) = p_i b_i[\Gamma, \mathbf{p}] + g_i(\Gamma, \mathbf{p})$.

Proof. See Appendix A14. ▮

By the proposition, if player i knows $g_i(\Gamma, \mathbf{p})$ and $b_i[\Gamma, \mathcal{P}]$, say by the historical observation, his sincere strategy p_i then maximizes

$$\max_{p_i \in [0,1]} \mathrm{E}[\mathbf{c}_{i,0}(1 - p_i b_i[\Gamma, \mathbf{p}] - g_i(\Gamma, \mathbf{p})) + \mathbf{c}_{i,1}(p_i b_i[\Gamma, \mathbf{p}] + g_i(\Gamma, \mathbf{p}))].$$

The strategic choice of p_i is, however, ultimately related to the formation of cooperation with other players.

Example 4:

Consider a public firm with the shareholders $N = \overline{xy_1y_2 \cdots y_{9000}}$ where one person x holding 10% of the voting share and 9000 people $y_1 - y_{9000}$ holding .01% each. The quota or threshold in the weighted majority game is .5+. For a policy to vote, we assume $p_x = .4$ and $p_{y_i} = .45$ for all i. The major holder x swings $\mathbf{S}$ in Γ or Γ^* if and only if $\mathbf{S} \setminus \overline{x} \in \{S \subseteq N \setminus \overline{x} | 4000 < |S| \leq 5000\}$. By the normal approximation, $b_x[\Gamma, \mathbf{p}] = \sum_{s=4001}^{5000} \sum_{|S|=s, S \subseteq N \setminus \overline{x}} \mathrm{Prob}(\mathbf{S} \setminus \overline{x} = S | \Gamma, \mathbf{p}) \approx .850416178$. For the minor shareholder y_i, he swings in two situations: (1) the major holder votes NO and exactly 5000 other minor holders vote YES; or (2) the major holder votes YES and exactly 4000 other minor holders vote YES. Therefore by Proposition 4.2,

$$b_{y_i}[\Gamma, \mathbf{p}] = (1 - .4)\begin{pmatrix} 8999 \\ 5000 \end{pmatrix}(.45)^{5000}(1 - .45)^{3999} + .4\begin{pmatrix} 8999 \\ 4000 \end{pmatrix}(.45)^{4000}(1 - .45)^{4999} \approx .00195003.$$

In this example, the major shareholder controls 8.02% of power by the asymmetric Banzhaf index. To contrast, the S-S index $\phi_x[\Gamma] \approx \frac{1}{9}$. By the Banzhaf index, he holds almost 100% of power.

One could extend the authority distribution, as defined in Hu and Shapley(2002), with the asymmetric S-S index to define the asymmetric authority distribution. For an abstract organization with members $N = \overline{12 \cdots n}$, we associate each member $i \in N$ with a simple game $\Gamma_i(N, \mathcal{W}_i)$, called *command game*, to designate his local authority structure. If $T \setminus \bar{i} \in \mathcal{W}_i$, then we say $T \setminus \bar{i}$ can *boss* i; if $T \in \mathcal{W}$ but $T \setminus \bar{i} \notin \mathcal{W}_i$, then $T \setminus \bar{i}$ has the right to *approve* or veto i's proposal. For any $T \in \mathcal{W}_i$, i has to obey any command by T regardless of his own desire and judgment. Let $P = [P_{ij}]$, called *power transition matrix*, be the $n \times n$ matrix with $P_{ij} = \phi_j[\Gamma_i]$, j's S-S index in i's command game. The authority distribution $\pi \in [0, 1]^n$ satisfies the normalization condition $\sum_{i=1}^{n} \pi_i = 1$ and the *counterbalance equilibrium equation* $\pi = \pi P$. As shown in Hu and Shapley(2002), π measures the general administrative power in the organization and the long-term influence between the members, and the members to the organization as a whole. As asymmetric power indices are concerned, we should extend the matrix P with $P_{ij} = \phi_j[\Gamma_i, \mathcal{P}]$ and define the *asymmetric authority distribution.*

For a voter $i \in N$ with $p_i = 0$, he is winning if and only if the bill is blocked. But if the bill is passed, then he has to "obey" the stipulations of the bill. We may associate the game $\Gamma(N, \mathcal{W})$ as his "command" game Γ_i. If $p_i = 1$, then he is winning if and only if the bill is passed. Otherwise he is unsatisfactory with the result and his interest may be affected. We associate the game $\Gamma^*(N, \mathcal{W}^*)$ as his "command" game Γ_i. Now for any voter i with $0 < p_i < 1$, we take his "command" game Γ_i by some stochastic mechanism of Γ and Γ^*, say, $p_i\Gamma^* + (1-p_i)\Gamma$. In this case, $P_{ij} = \phi_j[\Gamma_i, \mathcal{P}]$, by Proposition 3.1, for any $i \in N$. Thus $\{\pi_i = \phi_i[\Gamma, \mathcal{P}]\}_{i=1}^n$ is a solution to the counterbalance equilibrium equation $\pi = \pi P$. That is, the asymmetric S-S index is a special authority distribution.

Hu(2000) proposed the linear approximation $\beta[\Gamma, \mathbf{p}] \stackrel{\text{def}}{=} \sum_{i \in N} p_i\phi_i[\Gamma]$ of $\mathrm{E}[v(\mathbf{S}|\Gamma, \mathbf{p})] = h(\Gamma, \mathbf{p})$ as the collective preference for the game $\Gamma(N, \mathcal{W})$. Let us consider a multi-person bargaining problem as a sequence of voting problems where the statements are constantly modified to satisfy the players until a compromised solution reaches the grand agreement. Denote $p_i(t)$ be player i's preference to the original voting issue at period $t = 0, 1, \cdots$. Say, at period t, the statement is modified such that it responds to the collective preference $\beta[\Gamma, \mathbf{p}(t)]$. Let us consider a simple compromising scheme by

$$p_i(t+1) = (1-\alpha_i)p_i(t) + \alpha_i\beta[\Gamma, \mathbf{p}(t)] \tag{6}$$

where $0 < \alpha_i < 1$. The α_i's are the adoption rates.

PROPOSITION 4.4. *Given $p_i(t)$ defined by (6), there exists $p \in [0,1]$ such that $p_i(t) \to p$ for all $i \in N$. And p solves $p = \beta[\Gamma, p\mathbf{1}]$.*

Proof. See Appendix A15. ▌

As a remark, any $x \in [0,1]$ solves the equation $x = \beta[\Gamma, x\mathbf{1}]$. If another compromising scheme applies, we expect the solution p should also satisfy $p = \beta[\Gamma, p\mathbf{1}]$.

5. CONCLUSION

The asymmetry of power distribution in a voting body comes not only from the asymmetry of the players' roles in the voting rule, but also from the asymmetry of the internal cooperative structure. This paper incorporates a stochastic cooperative structure to the voting rule and generalizes the Banzhaf and Shapley-Shubik power indices. We have analyzed their properties and compared them with those of the traditional power indices.

The duality of the voting rule provides an equivalent form to the rule. When we value the contribution of turning losing coalitions to winning, we believe that the dual contribution of turning winning coalitions to losing is also worth mentioning. The expected number of swinging positions in both games defines the asymmetric Banzhaf index; the expected contribution (the Shapley value) in the formed coalition, either winning or losing, defines the asymmetric Shapley-Shubik power index. When we limit the cooperative structure to two special cases, we then have the traditional indices.

It is also worth mentioning the variant forms of the cooperative structure $\mathcal{P}$. On one hand, it has a variant when the rule takes its dual; the variant is equivalent to the original structure. On the other hand, it has a different but very closed form when the statement of the bill is changed to its negative. We provide some conditions such that the power indices are invariant with the variant structures. Especially if we modify the original statement, the structure of support intensity may be changed.

A simple cooperative structure assumes the independence among the players. This form only takes the players' personal interest into account and assumes that the players cast sincere votes, not strategic votes, by some randomized mechanism. The probabilities to vote for the bill are interpreted as their stochastic preference to the bill. We provide a simple solution to the collective preference for the simple game and a compromising schem for a multi-person bargaining problem.

From a theoretical point of views, there are several related issues worth mentioning. First, In many real votings (e.g. the US presidential election and the UN Security Council), each voter has the option to waive his right to vote and therefore he has three alternatives of vote: YES, NO or NEUTRAL (or WAIVE). Hu(2001) has extended the binary model to the trinary cases in which each voter has three choices and the result may also have three outcomes. In this generalization, $\mathcal{P}$ is a probability measure on the set of N's partitions of three coalitions. If we limit $\mathcal{P}$ to have zero probability for the choice of the third alternative, then we have the binary model.

Secondly, to keep the analysis focused, we have not explained the common cooperative structures in details. A typical one is the bi-party voting system (such as the politics in the United States) or multi-party system (with two or three major parties and a few small parties which are sometimes critical between the major parties). Another one is the representative system in which a representative has some common interest with his neighboring representatives and is almost independent of all other representatives. We believe the in-depth analysis of the common cooperative structures may guide us in such issues in optimal campaign strategies, structural analysis of policy, etc.

Thirdly, in our analysis, the voting games are assumed simple games. Hence, any outsider could not interfere with the internal affairs. However any concrete voting body has to be subject to the changes of the outside, such as the environment, business, investment and political policies, etc. The incorporation of the external influence with the simple game seems challenging, but interesting.

Notice, however, that the set-up has not taken the players' private information into account. For example, the senior players in the voting body may have much more private information about the cooperative structure than the junior players may hold. They could make strategic votes or form implicit cooperation which is invisible to the junior players. Of course, to gather the information about the cooperative structure for the junior players is not costless. One could incorporate the cost features in our model and investigate their contexts and interpretations.

As a remark, we may concern about the uncertainty in mapping preference to $[0, 1]$. For example, "YES strongly preferred to NO" could means a preference .9, but an estimated number between .85 to .95 may also sound acceptable. When the interactions are regarded, such as in a highly organizated system, even if i knows his own preference, he may not precisely perceive those of other players. One could allow some level of trust to address the uncertainty problem. In general, it is customary to apply the probability theory for the belief-related distributions.

Finally, an voting body is dynamic instead of just a static structure. The promotion or replacement of a player is then a restructure of the body. To keep the analysis valid for long term, it is useful to relate the results of the paper with some dynamic features, such as the terms of the members, or geographic distribution of the players, and so on.

6. APPENDIX: PROOFS

A1: PROOF OF PROPOSITION 1.1

On the contrary, if there exists such a $\mathcal{P}$ such that $\mathrm{E}[v(\mathbf{S})-v(\mathbf{S}\setminus\bar{i})|\Gamma,\mathcal{P}]=b_i[\Gamma]$ for any $i\in N$, then

$$\sum_{i\in T}\mathrm{Prob}(\mathbf{S}=T|\Gamma,\mathcal{P})[v(T)-v(T\setminus\bar{i})]=\sum_{i\in T}\frac{1}{2^{n-1}}[v(T)-v(T\setminus\bar{i})].$$

As $\mathbf{S}$ is independent of the choice of $v(\cdot)$, $\mathrm{Prob}(\mathbf{S}=T|\Gamma,\mathcal{P})=\frac{1}{2^{n-1}}$ for any T such that $i\in T$. As the probabilities over these T's sum to 1, $i\in\mathbf{S}$ with probability 1. Now we let i vary in N to obtain that $\mathbf{S}=N$ with probability 1. Pick a Γ such that N has no swinger, then $\mathrm{E}[v(\mathbf{S})-v(\mathbf{S}\setminus\bar{i})|\Gamma,\mathcal{P}]=0$ for all $i\in N$. But Γ's Banzhaf indices are not all 0's. Therefore, there exists at least one i such that $\mathrm{E}[v(\mathbf{S})-v(\mathbf{S}\setminus\bar{i})|\Gamma,\mathcal{P}]\neq b_i[\Gamma]$. The contradiction shows the nonexistence of $\mathcal{P}$.

By the similar argument, we can show that there exists no $\mathcal{P}$, independent of Γ, such that $\mathrm{E}[v(\mathbf{S})-v(\mathbf{S}\setminus\bar{i})|\Gamma,\mathcal{P}]=\phi_i[\Gamma]$ for all $i\in N$. *Q.E.D.*

A2: PROOF OF PROPOSITION 2.2

$$\begin{aligned}
b_i[\Gamma^*] &= \tfrac{1}{2^{n-1}} \textstyle\sum_{i\in Z} [v^*(Z) - v^*(Z \setminus \bar{i})] \\
&= \tfrac{1}{2^{n-1}} \textstyle\sum_{i\in Z} [v(N \setminus Z \cup \bar{i}) - v(N \setminus Z)] \\
&\overset{T=N\setminus Z\cup\bar{i}}{=} \tfrac{1}{2^{n-1}} \textstyle\sum_{i\in T} [v(T) - v(T \setminus \bar{i})] = b_i[\Gamma]. \\
\phi_i[\Gamma^*] &= \textstyle\sum_{i\in Z} \tfrac{(|Z|-1)!(n-|Z|)!}{n!} [v^*(Z) - v^*(Z \setminus \bar{i})] \\
&= \textstyle\sum_{i\in Z} \tfrac{(|Z|-1)!(n-|Z|)!}{n!} [v(N \setminus Z \cup \bar{i}) - v(N \setminus Z)] \\
&\overset{T=N\setminus Z\cup\bar{i}}{=} \textstyle\sum_{i\in T} \tfrac{(n-|T|)!(|T|-1)!}{n!} [v(T) - v(T \setminus \bar{i})] = \phi_i[\Gamma].
\end{aligned}$$

Q.E.D.

A3: PROOF OF PROPOSITION 3.1

$$\begin{aligned}
b_i[\Gamma^*, \mathcal{P}] &= \mathrm{E}[v^*(\mathbf{S} \cup \bar{i}) - v^*(\mathbf{S} \setminus \bar{i}) | \Gamma^*, \mathcal{P}] \\
&= \mathrm{E}[(1 - v(\mathbf{S}^* \setminus \bar{i})) - (1 - v(\mathbf{S}^* \cup \bar{i})) | \Gamma^*, \mathcal{P}] \\
&= \mathrm{E}[v(\mathbf{S} \cup \bar{i}) - v(\mathbf{S} \setminus \bar{i}) | \Gamma, \mathcal{P}] = b_i[\Gamma, \mathcal{P}].
\end{aligned}$$

Q.E.D.

A4: PROOF OF PROPOSITION 3.2

1. Given that $\mathrm{Prob}(\mathbf{S} = T | \Gamma, \mathcal{P}) = \frac{1}{2^n}$ for $\forall T \subseteq N$, by (3),

$$\begin{aligned}
b_i[\Gamma, \mathcal{P}] &= \textstyle\sum_{i\in T} \tfrac{1}{2^n} [v(T) - v(T \setminus \bar{i})] + \textstyle\sum_{i\notin T} \tfrac{1}{2^n} [v(T \cup \bar{i}) - v(T)] \\
&\overset{Z=T\cup\bar{i}}{\equiv} \tfrac{1}{2^n} \textstyle\sum_{i\in T} [v(T) - v(T \setminus \bar{i})] + \tfrac{1}{2^n} \textstyle\sum_{i\in Z} [v(Z) - v(Z \setminus \bar{i})] \\
&\overset{Z\equiv T}{\equiv} \tfrac{1}{2^{n-1}} \textstyle\sum_{i\in T} [v(T) - v(T \setminus \bar{i})] = b_i[\Gamma].
\end{aligned}$$

2. First, $\text{Prob}(\mathbf{S} = T|\Gamma, \mathcal{P}) = \frac{1}{n+1} \div \begin{pmatrix} n \\ |T| \end{pmatrix} = \frac{(|T|)!(n-|T|)!}{(n+1)!}$. Now by (3),

$$\begin{aligned} b_i[\Gamma, \mathcal{P}] &= \sum_{i \in T} \frac{(|T|)!(n-|T|)!}{(n+1)!}[v(T) - v(T \setminus \bar{i})] \\ &\quad + \sum_{i \notin T} \frac{(|T|)!(n-|T|)!}{(n+1)!}[v(T \cup \bar{i}) - v(T)] \\ &\overset{Z=T\cup\bar{i}}{=} \sum_{i \in T} \frac{(|T|)!(n-|T|)!}{(n+1)!}[v(T) - v(T \setminus \bar{i})] \\ &\quad + \sum_{i \in Z} \frac{(|Z|-1)!(n-|Z|+1)!}{(n+1)!}[v(Z) - v(Z \setminus \bar{i})] \\ &\overset{T=Z}{=} \sum_{i \in Z} \frac{(|Z|)!(n-|Z|)! + (|Z|-1)!(n-|Z|+1)!}{(n+1)!}[v(Z) - v(Z \setminus \bar{i})] \\ &= \sum_{Z \subseteq N} \frac{(|Z|-1)!(n-|Z|)!}{n!}[v(Z) - v(Z \setminus \bar{i})] = \phi_i[\Gamma]. \end{aligned}$$

Q.E.D.

A5: PROOF OF PROPOSITION 3.3

1. As S-I is satisfied, $\tilde{\Gamma} = \Gamma$ and so $\tilde{\Gamma}$ has the characteristic function $v(\cdot)$,

$$\begin{aligned} b_i[\tilde{\Gamma}, \tilde{\mathcal{P}}] &= \sum_{T \subseteq N} \text{Prob}(\mathbf{S} = T|\tilde{\Gamma}, \tilde{\mathcal{P}})[v(T \cup \bar{i}) - v(T \setminus \bar{i})] \\ &\overset{\text{S-I}}{=} \sum_{T \subseteq N} \text{Prob}(\mathbf{S}^* = T|\Gamma, \mathcal{P})[v(T \cup \bar{i}) - v(T \setminus \bar{i})] \\ &\overset{Z=N\setminus T}{=} \sum_{Z \subseteq N} \text{Prob}(\mathbf{S} = Z|\Gamma, \mathcal{P})[v(N \setminus Z \cup \bar{i}) - v(N \setminus Z \setminus \bar{i})] \\ &\overset{\text{decisive}}{=} \sum_{Z \subseteq N} \text{Prob}(\mathbf{S} = Z|\Gamma, \mathcal{P})[v(Z \cup \bar{i}) - v(Z)] = b_i[\Gamma, \mathcal{P}]. \end{aligned}$$

2. As S-I is satisfied, $\tilde{\Gamma} = \Gamma^*$ and so $\tilde{\Gamma}$ has the characteristic function $v^*(\cdot)$,

$$\begin{aligned} b_i[\tilde{\Gamma}, \tilde{\mathcal{P}}] &\overset{\text{S-II}}{=} \sum_{T \subseteq N} \text{Prob}(\mathbf{S} = T|\tilde{\Gamma}, \tilde{\mathcal{P}})[v^*(T \cup \bar{i}) - v^*(T \setminus \bar{i})] \\ &= \sum_{T \subseteq N} \text{Prob}(\mathbf{S} = T|\tilde{\Gamma}, \tilde{\mathcal{P}})[v(N \setminus T \cup \bar{i}) - v(N \setminus T \setminus \bar{i})] \\ &\overset{\text{S-II}}{=} \sum_{T \subseteq N} \text{Prob}(\mathbf{S}^* = T|\Gamma, \mathcal{P})[v(N \setminus T \cup \bar{i}) - v(N \setminus T \setminus \bar{i})] \\ &\overset{Z=N\setminus T}{=} \sum_{Z \subseteq N} \text{Prob}(\mathbf{S} = Z|\Gamma, \mathcal{P})[v(Z \cup \bar{i}) - v(Z \setminus \bar{i})] = b_i[\Gamma, \mathcal{P}]. \end{aligned}$$

Q.E.D.

A6: PROOF OF LEMMA 3.1

Denote $\delta(t,z) = \frac{(z-1)!(t-z)!}{t!}$ for all $0 \le z \le t$. First,

$$\begin{aligned}
&\textstyle\sum_{i\in T\in\mathcal{W}} \text{Prob}(\mathbf{S}=T|\Gamma,\mathcal{P})\phi_i[\Gamma,\mathcal{W}_T]\\
&= \textstyle\sum_{i\in T\in\mathcal{W}} \text{Prob}(\mathbf{S}=T|\Gamma,\mathcal{P})\sum_{i\in Z\subseteq T}\delta(|T|,|Z|)(v(Z)-v(Z\setminus\bar{i}))\\
&= \textstyle\sum_{i\in T\in\mathcal{W}} \text{Prob}(\mathbf{S}=T|\Gamma,\mathcal{P})\sum_{Z\subseteq T}\delta(|T|,|Z|)(v(Z)-v(Z\setminus\bar{i}))\\
&= \textstyle\sum_{T\in\mathcal{W}} \text{Prob}(\mathbf{S}=T|\Gamma,\mathcal{P})\sum_{Z\subseteq T}\delta(|T|,|Z|)(v(Z)-v(Z\setminus\bar{i}))\\
&= \textstyle\sum_{T\subseteq N} \text{Prob}(\mathbf{S}=T|\Gamma,\mathcal{P})\sum_{Z\subseteq T}\delta(|T|,|Z|)(v(Z)-v(Z\setminus\bar{i}))
\end{aligned}$$

Secondly,

$$\begin{aligned}
&\textstyle\sum_{i\in T\in\mathcal{W}^*} \text{Prob}(\mathbf{S}=T|\Gamma^*,\mathcal{P})\phi_i[\Gamma^*,\mathcal{W}^*_T]\\
&= \textstyle\sum_{i\in T\in\mathcal{W}^*} \text{Prob}(\mathbf{S}^*=T|\Gamma,\mathcal{P})\sum_{i\in Z\subseteq T}\delta(|T|,|Z|)(v^*(Z)-v^*(Z\setminus\bar{i}))\\
&= \textstyle\sum_{i\in T\in\mathcal{W}^*} \text{Prob}(\mathbf{S}^*=T|\Gamma,\mathcal{P})\sum_{Z\subseteq T}\delta(|T|,|Z|)(v^*(Z)-v^*(Z\setminus\bar{i}))\\
&= \textstyle\sum_{T\in\mathcal{W}^*} \text{Prob}(\mathbf{S}^*=T|\Gamma,\mathcal{P})\sum_{Z\subseteq T}\delta(|T|,|Z|)(v^*(Z)-v^*(Z\setminus\bar{i}))\\
&\overset{\overline{T}=N\setminus T}{=} \textstyle\sum_{\overline{T}\notin\mathcal{W}} \text{Prob}(\mathbf{S}=\overline{T}|\Gamma,\mathcal{P})\sum_{Z\subseteq N\setminus\overline{T}}\delta(n-|\overline{T}|,|Z|)(v^*(Z)-v^*(Z\setminus\bar{i}))\\
&\overset{\overline{Z}=N\setminus Z}{=} \textstyle\sum_{\overline{T}\notin\mathcal{W}} \text{Prob}(\mathbf{S}=\overline{T}|\Gamma,\mathcal{P})\sum_{\overline{Z}\supseteq\overline{T}}\delta(n-|\overline{T}|,n-|\overline{Z}|)(v(\overline{Z}\cup\bar{i})-v(\overline{Z}))\\
&= \textstyle\sum_{\overline{T}\subseteq N} \text{Prob}(\mathbf{S}=\overline{T}|\Gamma,\mathcal{P})\sum_{\overline{Z}\supseteq\overline{T}}\delta(n-|\overline{T}|,n-|\overline{Z}|)(v(\overline{Z}\cup\bar{i})-v(\overline{Z}))\\
&= \textstyle\sum_{T\subseteq N} \text{Prob}(\mathbf{S}=T|\Gamma,\mathcal{P})\sum_{Z\supseteq T}\delta(n-|T|,n-|Z|)(v(Z\cup\bar{i})-v(Z)).
\end{aligned}$$

The proof is complete by the above results and (4). *Q.E.D.*

A7: PROOF OF PROPOSITION 3.4

By Lemma 3.1, $\phi_i[\Gamma^*, \mathcal{P}]$ has a formula,

$$\sum_{T\subseteq N} \text{Prob}(\mathbf{S} = T|\Gamma^*, \mathcal{P}) \ \Big[\sum_{Z\subseteq T} \delta(|T|, |Z|)(v^*(Z) - v^*(Z \setminus \bar{i}))+ \sum_{Z\supseteq T} \delta(n - |T|, n - |Z|)(v^*(Z \cup \bar{i}) - v^*(Z))\Big].$$

If we replace T with $N\setminus T$ and replace Z with $N\setminus Z$ in the above expression, then $\phi_i[\Gamma^*, \mathcal{P}]$ reduces to

$$\sum_{T\subseteq N} \text{Prob}(\mathbf{S} = T|\Gamma, \mathcal{P}) \ \Big[\sum_{Z\supseteq T} \delta(n - |T|, n - |Z|)(v(Z \cup \bar{i}) - v(Z))+ \sum_{Z\subseteq T} \delta(|T|, |Z|)(v(Z) - v(Z \setminus \bar{i}))\Big]$$

which is just $\phi_i[\Gamma, \mathcal{P}]$. *Q.E.D.*

A8: PROOF OF PROPOSITION 3.5

1. $\phi_i[\Gamma, \mathcal{P}] \geq 0$ since $\phi_i[\Gamma, \mathcal{W}_T] \geq 0$ and $\phi_i[\Gamma^*, \mathcal{W}_T^*] \geq 0$. Next,

$$\begin{aligned}
& \sum_{i=1}^n \phi_i[\Gamma, \mathcal{P}] \\
= \ & \sum_{i=1}^n \sum_{i\in T\in\mathcal{W}} \text{Prob}(\mathbf{S} = T|\Gamma, \mathcal{P})\phi_i[\Gamma, \mathcal{W}_T] \\
& + \sum_{i=1}^n \sum_{i\in T\in\mathcal{W}^*} \text{Prob}(\mathbf{S} = T|\Gamma^*, \mathcal{P})\phi_i[\Gamma^*, \mathcal{W}_T^*] \\
= \ & \sum_{T\in\mathcal{W}} \text{Prob}(\mathbf{S} = T|\Gamma, \mathcal{P}) \sum_{i\in T} \phi_i[\Gamma, \mathcal{W}_T] \\
& + \sum_{T\in\mathcal{W}^*} \text{Prob}(\mathbf{S}^* = T|\Gamma, \mathcal{P}) \sum_{i\in T} \phi_i[\Gamma^*, \mathcal{W}_T^*] \\
= \ & \sum_{T\in\mathcal{W}} \text{Prob}(\mathbf{S} = T|\Gamma, \mathcal{P}) + \sum_{T\in\{N\setminus Z|Z\notin\mathcal{W}\}} \text{Prob}(\mathbf{S} = N \setminus T|\Gamma, \mathcal{P}) \\
= \ & \sum_{T\in\mathcal{W}} \text{Prob}(\mathbf{S} = T|\Gamma, \mathcal{P}) + \sum_{Z\notin\mathcal{W}} \text{Prob}(\mathbf{S} = Z|\Gamma, \mathcal{P}) = 1.
\end{aligned}$$

2. If i is a dummy, then he is a dummy in $\Gamma(T, \mathcal{W}_T)$ if $i \in T \in \mathcal{W}$ and $\Gamma^*(T, \mathcal{W}_T^*)$ if $i \in T \in \mathcal{W}^*$. Therefore $\phi_i[\Gamma, \mathcal{W}_T] = \phi_i[\Gamma^*, \mathcal{W}_T^*] = 0$ for any $T \in \mathcal{W}$ or $T \in \mathcal{W}^*$. Thus $\phi_i[\Gamma, \mathcal{P}] = 0$.

3. If i is a dictator, then all other players are dummies. That $\phi_i[\Gamma, \mathcal{P}] = 1$ is from the first part and $\phi_j[\Gamma, \mathcal{P}] = 0$ for $\forall j \neq i$.

4. If $\text{Prob}(\mathbf{S} = N|\Gamma, \mathcal{P}) = 1$, then $\mathbf{S} = N$ a.s. The second summation in (4) is just 0 and the first one reduces to only one item $\phi_i[\Gamma, \mathcal{W}_N]$ which is just $\phi_i[\Gamma]$. If $\text{Prob}(\mathbf{S} = N|\Gamma^*, \mathcal{P}) = 1$, then (4) reduces to $\phi_i[\Gamma^*, \mathcal{W}_N^*] = \phi_i[\Gamma^*]$ which is $\phi_i[\Gamma]$ by Proposition 2.2. *Q.E.D.*

A9: PROOF OF PROPOSITION 3.6

1. For $\forall i \in N$, by the proof of Proposition 3.2 (Appendix A4), the first summation in (4) is

$$\begin{aligned}
& \sum_{i \in T \in \mathcal{W}} \text{Prob}(\mathbf{S} = T|\Gamma, \mathcal{P}) \phi_i[\Gamma, \mathcal{W}_T] \\
= & \sum_{i \in T \in \mathcal{W}} \tfrac{(|T|)!(n-|T|)!}{(n+1)!} \sum_{Z \subseteq T} \tfrac{(|Z|-1)!(|T|-|Z|)!}{(|T|)!} [v(Z) - v(Z \setminus \bar{i})] \\
= & \sum_{i \in Z} \tfrac{(|Z|-1)!}{(n+1)!} [v(Z) - v(Z \setminus \bar{i})] \sum_{T \in \mathcal{W}, T \supseteq Z} (n - |T|)!(|T| - |Z|)! \\
= & \sum_{i \in Z} \tfrac{(|Z|-1)!}{(n+1)!} [v(Z) - v(Z \setminus \bar{i})] \sum_{T \supseteq Z} (n - |T|)!(|T| - |Z|)! \\
\stackrel{k=|T|}{=} & \sum_{i \in Z} \tfrac{(|Z|-1)!}{(n+1)!} [v(Z) - v(Z \setminus \bar{i})] \sum_{k=|Z|}^{n} (n - k)!(k - |Z|)! \begin{pmatrix} n - |Z| \\ k - |Z| \end{pmatrix} \\
= & \sum_{i \in Z} \tfrac{(|Z|-1)!}{(n+1)!} [v(Z) - v(Z \setminus \bar{i})] \sum_{k=|Z|}^{n} (n - |Z|)! \\
= & \sum_{i \in Z} \tfrac{(|Z|-1)!(n-|Z|+1)!}{(n+1)!} [v(Z) - v(Z \setminus \bar{i})].
\end{aligned}$$

Similarly, together with $\text{Prob}(\mathbf{S} = T|\Gamma^*, \mathcal{P}) = \text{Prob}(\mathbf{S} = N \setminus T|\Gamma, \mathcal{P}) = \frac{(|T|)!(n-|T|)!}{(n+1)!}$, the second summation in (4) is

$$\begin{aligned} & \textstyle\sum_{i \in T \in \mathcal{W}^*} \text{Prob}(\mathbf{S} = T|\Gamma^*, \mathcal{P}) \phi_i[\Gamma^*, \mathcal{W}^*_T] \\ = \; & \textstyle\sum_{i \in Z} \frac{(|Z|-1)!(n-|Z|+1)!}{(n+1)!}[v^*(Z) - v^*(Z \setminus \bar{i})] \\ = \; & \textstyle\sum_{i \in Z} \frac{(|Z|-1)!(n-|Z|+1)!}{(n+1)!}[v(N \setminus Z \cup \bar{i}) - v(N \setminus Z)] \\ \overset{T=N\setminus Z}{=} \; & \textstyle\sum_{i \notin T} \frac{(n-|T|-1)!(|T|+1)!}{(n+1)!}[v(T \cup \bar{i}) - v(T)] \\ \overset{Z=T\cup\bar{i}}{=} \; & \textstyle\sum_{i \in Z} \frac{(n-|Z|)!(|Z|)!}{(n+1)!}[v(Z) - v(Z \setminus \bar{i})]. \end{aligned}$$

We combine the two results,

$$\begin{aligned} \phi_i[\Gamma, \mathcal{P}] &= \textstyle\sum_{i \in Z} \frac{(|Z|-1)!(n-|Z|+1)! + (n-|Z|)!(|Z|)!}{(n+1)!}[v(Z) - v(Z \setminus \bar{i})] \\ &= \textstyle\sum_{i \in Z} \frac{(|Z|-1)!(n-|Z|)!}{n!}[v(Z) - v(Z \setminus \bar{i})] = \phi_i[\Gamma]. \end{aligned}$$

2. For $\forall i \in N$, the first summation in (4) is

$$\begin{aligned} & \textstyle\sum_{i \in T \in \mathcal{W}} \text{Prob}(\mathbf{S} = T|\Gamma, \mathcal{P}) \phi_i[\Gamma, \mathcal{W}_T] = \frac{1}{2^n} \sum_{i \in T \in \mathcal{W}} \phi_i[\Gamma, \mathcal{W}_T] \\ = \; & \textstyle\frac{1}{2^n} \sum_{i \in T \in \mathcal{W}} \sum_{Z \subseteq T} \frac{(|Z|-1)!(|T|-|Z|)!}{(|T|)!}[v(Z) - v(Z \setminus \bar{i})] \\ = \; & \textstyle\frac{1}{2^n} \sum_{i \in Z} (|Z|-1)![v(Z) - v(Z \setminus \bar{i})] \sum_{T \supseteq Z} \frac{(|T|-|Z|)!}{(|T|)!} \\ \overset{k=|T|}{=} \; & \textstyle\frac{1}{2^n} \sum_{i \in Z} (|Z|-1)![v(Z) - v(Z \setminus \bar{i})] \sum_{k=|Z|}^{n} \frac{(k-|Z|)!}{k!} \begin{pmatrix} n-|Z| \\ k-|Z| \end{pmatrix} \\ = \; & \textstyle\frac{1}{2^n} \sum_{i \in Z} (|Z|-1)!(n-|Z|)![v(Z) - v(Z \setminus \bar{i})] \sum_{k=|Z|}^{n} \frac{1}{k!(n-k)!} \\ = \; & \textstyle\frac{1}{2^n} \sum_{i \in Z} (|Z|-1)!(n-|Z|)! B(n, |Z|)[v(Z) - v(Z \setminus \bar{i})]. \end{aligned}$$

where $B(n,k) \stackrel{\text{def}}{=} \sum_{t=k}^{n} \frac{1}{t!(n-t)!}$. The second summation in (4) is

$$\begin{aligned}
& \sum_{i\in T\in\mathcal{W}^*} \text{Prob}(\mathbf{S}=T|\Gamma^*,\mathcal{P})\phi_i[\Gamma^*,\mathcal{W}^*_T] \\
= \quad & \frac{1}{2^n}\sum_{i\in Z}(|Z|-1)!(n-|Z|)!B(n,|Z|)[v^*(Z)-v^*(Z\setminus\bar{i})] \\
= \quad & \frac{1}{2^n}\sum_{i\in Z}(|Z|-1)!(n-|Z|)!B(n,|Z|)[v(N\setminus Z\cup\bar{i})-v(N\setminus Z)] \\
\stackrel{T=N\setminus Z}{=} \quad & \frac{1}{2^n}\sum_{i\notin T}(n-|T|-1)!(|T|)!B(n,n-|T|)[v(T\cup\bar{i})-v(T)] \\
\stackrel{Z=T\cup\bar{i}}{=} \quad & \frac{1}{2^n}\sum_{i\in Z}(n-|Z|)!(|Z|-1)!B(n,n-|Z|+1)[v(Z)-v(Z\setminus\bar{i})].
\end{aligned}$$

We observe that

$$\begin{aligned}
& B(n,|Z|)+B(n,n-|Z|+1) \\
= \quad & \sum_{t=|Z|}^{n}\frac{1}{t!(n-t)!}+\sum_{j=n-|Z|+1}^{n}\frac{1}{j!(n-j)!} \\
\stackrel{t=n-j}{=} \quad & \sum_{t=|Z|}^{n}\frac{1}{t!(n-t)!}+\sum_{t=0}^{|Z|-1}\frac{1}{t!(n-t)!}=\sum_{t=0}^{n}\frac{1}{t!(n-t)!}=\frac{2^n}{n!}.
\end{aligned}$$

Finally we apply the above results to (4),

$$\phi_i[\Gamma,\mathcal{P}]=\sum_{i\in Z}\frac{(n-|Z|)!(|Z|-1)!}{n!}[v(Z)-v(Z\setminus\bar{i})]=\phi_i[\Gamma].$$

Q.E.D.

A10: PROOF OF PROPOSITION 3.7

1. By S-I and Lemma 3.1, $\phi_i[\tilde{\Gamma},\tilde{\mathcal{P}}]$ has a formula,

$$\begin{aligned}
\sum_{T\subseteq N}\text{Prob}(\mathbf{S}=T|\tilde{\Gamma},\tilde{\mathcal{P}})\ \Big[&\sum_{Z\subseteq T}\delta(|T|,|Z|)(v(Z)-v(Z\setminus\bar{i})) \\
&+\sum_{Z\supseteq T}\delta(n-|T|,n-|Z|)(v(Z\cup\bar{i})-v(Z))\Big].
\end{aligned}$$

By S-I, the above expression is equal to

$$\begin{aligned}
\sum_{T\subseteq N}\text{Prob}(\mathbf{S}^*=T|\Gamma,\mathcal{P})\ \Big[&\sum_{Z\subseteq T}\delta(|T|,|Z|)(v(Z)-v(Z\setminus\bar{i})) \\
&+\sum_{Z\supseteq T}\delta(n-|T|,n-|Z|)(v(Z\cup\bar{i})-v(Z))\Big].
\end{aligned}$$

As Γ is decisive, $v(N \setminus Z) - v(N \setminus Z \setminus \bar{i}) = v(Z \cup \bar{i}) - v(Z)$ and $v(N \setminus Z \cup \bar{i}) - v(N \setminus Z) = v(Z) - v(Z \setminus \bar{i})$. If we replace T with $N \setminus T$ and replace Z with $N \setminus Z$ in the above expression, then $\phi_i[\tilde{\Gamma}, \tilde{\mathcal{P}}]$ reduces to

$$\sum_{T \subseteq N} \text{Prob}(\mathbf{S} = T | \Gamma, \mathcal{P}) \left[\sum_{Z \supseteq T} \delta(n - |T|, n - |Z|)(v(Z \cup \bar{i}) - v(Z)) + \sum_{Z \subseteq T} \delta(|T|, |Z|)(v(Z) - v(Z \setminus \bar{i})) \right]$$

which is just $\phi_i[\Gamma, \mathcal{P}]$.

2. By S-II and Lemma 3.1, $\phi_i[\tilde{\Gamma}, \tilde{\mathcal{P}}]$ has a formula,

$$\sum_{T \subseteq N} \text{Prob}(\mathbf{S} = T | \tilde{\Gamma}, \tilde{\mathcal{P}}) \left[\sum_{Z \subseteq T} \delta(|T|, |Z|)(v^*(Z) - v^*(Z \setminus \bar{i})) + \sum_{Z \supseteq T} \delta(n - |T|, n - |Z|)(v^*(Z \cup \bar{i}) - v^*(Z)) \right].$$

By S-II, the above expression is equal to

$$\sum_{T \subseteq N} \text{Prob}(\mathbf{S}^* = T | \Gamma, \mathcal{P}) \left[\sum_{Z \subseteq T} \delta(|T|, |Z|)(v(N \setminus Z \cup \bar{i}) - v(N \setminus Z)) + \sum_{Z \supseteq T} \delta(n - |T|, n - |Z|)(v(N \setminus Z) - v(N \setminus Z \setminus \bar{i})) \right].$$

If we replace T with $N \setminus T$ and replace Z with $N \setminus Z$ in the above expression, then $\phi_i[\tilde{\Gamma}, \tilde{\mathcal{P}}]$ reduces to

$$\sum_{T \subseteq N} \text{Prob}(\mathbf{S} = T | \Gamma, \mathcal{P}) \left[\sum_{Z \supseteq T} \delta(n - |T|, n - |Z|)(v(Z \cup \bar{i}) - v(Z)) + \sum_{Z \subseteq T} \delta(|T|, |Z|)(v(Z) - v(Z \setminus \bar{i})) \right]$$

which is just $\phi_i[\Gamma, \mathcal{P}]$. *Q.E.D.*

A11: PROOF OF PROPOSITION 4.1

By the independence assumption and $v^*(T) = 1 - v(N \setminus T)$,

$$\begin{aligned}
h(\Gamma^*, \mathbf{1} - \mathbf{p}) &= \textstyle\sum_{T \subseteq N} \mathrm{Prob}(\mathbf{S} = T | \Gamma^*, 1 - \mathbf{p}) v^*(T) \\
&= \textstyle\sum_{T \subseteq N} \mathrm{Prob}(\mathbf{S} = T | \Gamma^*, 1 - \mathbf{p}) 1 \\
&\quad - \textstyle\sum_{T \subseteq N} \mathrm{Prob}(\mathbf{S} = T | \Gamma^*, 1 - \mathbf{p}) v(N \setminus T) \\
&= 1 - \textstyle\sum_{T \subseteq N} v(N \setminus T) \prod_{i \in T} (1 - p_i) \prod_{i \notin T} p_i \\
&\overset{Z = N \setminus T}{=} 1 - \textstyle\sum_{Z \subseteq N} v(Z) \prod_{i \notin Z} (1 - p_i) \prod_{i \in Z} p_i \\
&= 1 - h(\Gamma, \mathbf{p}).
\end{aligned}$$

If Γ is decisive,

$$\begin{aligned}
h(\Gamma^*, \mathbf{p}) &= \textstyle\sum_{T \subseteq N} \mathrm{Prob}(\mathbf{S} = T | \Gamma^*, \mathbf{p}) v^*(T) \\
&\overset{\text{decisive}}{=} \textstyle\sum_{T \subseteq N} \mathrm{Prob}(\mathbf{S} = T | \Gamma^*, \mathbf{p}) v(T) \\
&= \textstyle\sum_{T \subseteq N} v(T) \prod_{i \in T} p_i \prod_{i \notin T} (1 - p_i) \\
&= h(\Gamma, \mathbf{p}).
\end{aligned}$$

Q.E.D.

A12: PROOF OF PROPOSITION 4.2

For any $Z \in N \setminus \bar{i}$,

$$\begin{aligned}
\mathrm{Prob}(\mathbf{S} \setminus \bar{i} = Z | \Gamma, \mathbf{p}) &= \mathrm{Prob}(\mathbf{S} = Z | \Gamma, \mathbf{p}) + \mathrm{Prob}(\mathbf{S} = Z \cup \bar{i} | \Gamma, \mathbf{p}) \\
&= (1 - p_i) \Pi_{j \in Z} p_j \Pi_{j \notin Z, j \neq i} (1 - p_j) \\
&\quad + p_i \Pi_{j \in Z} p_j \Pi_{j \notin Z, j \neq i} (1 - p_j) \\
&= \Pi_{j \in Z} p_j \Pi_{j \notin Z, j \neq i} (1 - p_j).
\end{aligned}$$

Now,

$$\begin{aligned} b_i[\Gamma,\mathbf{p}] &= \textstyle\sum_{i\in T}\Pi_{j\in T}p_j\Pi_{j\notin T}(1-p_j)[v(T)-v(T\setminus\bar{i})] \\ &\quad +\textstyle\sum_{i\notin T}\Pi_{j\in T}p_j\Pi_{j\notin T}(1-p_j)[v(T\cup\bar{i})-v(T)] \\ &\overset{Z=T\setminus\bar{i}}{=} p_i\textstyle\sum_{Z\in N\setminus\bar{i}}\text{Prob}(\mathbf{S}\setminus\bar{i}=Z|\Gamma,\mathbf{p})[v(Z\cup\bar{i})-v(Z)] \\ &\quad +(1-p_i)\textstyle\sum_{Z\in N\setminus\bar{i}}\text{Prob}(\mathbf{S}\setminus\bar{i}=Z|\Gamma,\mathbf{p})[v(Z\cup\bar{i})-v(Z)] \\ &= \textstyle\sum_{Z\in N\setminus\bar{i}}\text{Prob}(\mathbf{S}\setminus\bar{i}=Z|\Gamma,\mathbf{p})[v(Z\cup\bar{i})-v(Z)]. \end{aligned}$$

and

$$\begin{aligned} \tfrac{\partial h}{\partial p_i}(\Gamma,\mathbf{p}) &= \textstyle\sum_{i\in T}\Pi_{j\in T,j\neq i}p_j\Pi_{j\notin T}(1-p_j)v(T) \\ &\quad -\textstyle\sum_{i\notin T}\Pi_{j\in T}p_j\Pi_{j\notin T,j\neq i}(1-p_j)v(T) \\ &\overset{Z=T\setminus\bar{i}}{=} \textstyle\sum_{Z\in N\setminus\bar{i}}\text{Prob}(\mathbf{S}\setminus\bar{i}=Z|\Gamma,\mathbf{p})[v(Z\cup\bar{i})-v(Z)]. \end{aligned}$$

Q.E.D.

A13: PROOF OF COROLLARY 4.1

By the proof of Proposition 4.2 (see Appendix A12), $\text{Prob}(\mathbf{S}\setminus\bar{i}=Z|\Gamma,x\mathbf{1}) = x^{|Z|}(1-x)^{n-|Z|-1}$ for any $Z\in N\setminus\bar{i}$,

$$\begin{aligned} b_i[\Gamma,\tfrac{1}{2}\mathbf{1}] &= \textstyle\sum_{Z\in N\setminus\bar{i}}\text{Prob}(\mathbf{S}\setminus\bar{i}=Z|\Gamma,\tfrac{1}{2}\mathbf{1})[v(Z\cup\bar{i})-v(Z)] \\ &\overset{T=Z\cup\bar{i}}{=} \textstyle\sum_{i\in T}\frac{1}{2^{n-1}}[v(T)-v(T\setminus\bar{i})]=b_i[\Gamma]; \\ \textstyle\int_0^1 b_i[\Gamma,x\mathbf{1}]dx &= \textstyle\int_0^1\sum_{Z\in N\setminus\bar{i}}x^{|Z|}(1-x)^{n-1-|Z|}[v(Z\cup\bar{i})-v(Z)]dx \\ &\overset{T=Z\cup\bar{i}}{=} \textstyle\sum_{i\in T}[v(T)-v(T\setminus\bar{i})]\int_0^1 x^{|T|-1}(1-x)^{n-|T|}dx \\ &= \textstyle\sum_{i\in T}\frac{(|T|-1)!(n-|T|)!}{n!}[v(T)-v(T\setminus\bar{i})]=\phi_i[\Gamma]. \end{aligned}$$

Q.E.D.

A14: PROOF OF PROPOSITION 4.3

By Proposition 4.2, we have that $h(\Gamma,\mathbf{p}) = p_i b_i[\Gamma,\mathbf{p}] + c$ for some c independent of p_i. Let $p_i = 0$ in the above expression, then c is the probability of passing the bill given $p_i = 0$. Therefore

$$c = \sum_{i \notin T \subseteq N} (1-p_i)\Pi_{j\in T} p_j \Pi_{j\notin T, j\neq i}(1-p_j) = g(\Gamma,\mathbf{p}).$$

This proves the proposition. *Q.E.D.*

A15: PROOF OF PROPOSITION 4.4

Let $\chi = \{i \in N | \phi_i[\Gamma] > 0\}$, $\delta_t = \min_{i\in\chi} p_i(t)$ and $\Delta_t = \max_{i\in\chi} p_i(t)$. It is easy to see, by the bargaining scheme (6), that δ_t is weakly increasing and Δ_t is weakly decreasing. Therefore the limits $\delta = \lim_{t\to\infty}\delta_t$ and $\Delta = \lim_{t\to\infty}\Delta_t \geq \delta$ exist.

Say, $\limsup_{t\to\infty} p_i(t) = \Delta$ for some $i \in \chi$. Then by (6),

$$\begin{aligned}\Delta &= \limsup_{t\to\infty} p_i(t+1)\\ &\leq (1-\alpha_i)\limsup_{t\to\infty} p_i(t) + \alpha_i \textstyle\sum_{j\in\chi}\phi_j[\Gamma]\limsup_{t\to\infty} p_j(t)\\ &\leq (1-\alpha_i)\Delta + \alpha_i\Delta\textstyle\sum_{j\in\chi}\phi_j[\Gamma] = \Delta.\end{aligned}$$

This implies that $\limsup_{t\to\infty} p_i(t) = \Delta$ for any $i \in \chi$. Similarly, we have $\liminf_{t\to\infty} p_i(t) = \delta$ for any $i \in \chi$. Now for any $\epsilon > 0$, there exists K such that for all $t > K$, we have 1. $\delta - \epsilon \leq p_i(t) \leq \Delta + \epsilon$ for any $i \in \chi$; 2. $\min_{j\in\chi} p_j(t) \in [\delta-\epsilon, \delta+\epsilon]$; 3. $\max_{j\in\chi} p_j(t) \in [\delta-\epsilon, \delta+\epsilon]$. Now for any $t > K+1$, there exists $i, j \in \chi$ such that $p_i(t+1) = \max_{s\in\chi} p_s(t+1)$ and

$p_j(t) = \min_{s\in\chi} p_s(t)$. Now

$$\begin{aligned} \Delta - \epsilon &\leq p_i(t+1) = (1-\alpha_i)p_i(t) + \alpha_i \textstyle\sum_{s\in\chi} p_s(t)\phi_s[\Gamma] \\ &\leq (1-\alpha_i)(\Delta+\epsilon) + \alpha_i(\Delta+\epsilon) + \alpha_i\phi_j[\Gamma](\delta-\Delta) \\ &= \Delta + \epsilon - \alpha_i\phi_j[\Gamma](\Delta-\delta). \end{aligned}$$

Since ϵ is arbitrary, $\Delta < \Delta$ if $\delta < \Delta$. Therefore, $\delta = \Delta$. We let $p = \delta$.

Next, for any $i \notin \chi$, the bargaining scheme (6) reduces to

$$p_i(t+1) = (1-\alpha_i)p_i(t) + \alpha_i \sum_{j\in\chi} p_j(t)\phi[\Gamma].$$

For any $\epsilon > 0$, there exists $k(\epsilon)$ such that $|\sum_{j\in\chi} p_j(t)\phi_j[\Gamma] - p| \leq \epsilon$ for all $t \geq k(\epsilon)$. For any $t \geq k(\epsilon)$,

$$\begin{aligned} |p_i(t) - p| &= |(1-\alpha_i)(p_i(t-1)-p) + \alpha_i[\textstyle\sum_{j\in\chi} p_j\phi_j[\Gamma] - p]| \\ &\leq (1-\alpha_i)|p_i(t-1)-p| + \alpha_i\epsilon \\ &\leq (1-\alpha_i)^2|p_i(t-2)-p| + (1-\alpha_i)\alpha_i\epsilon + \alpha_i\epsilon \\ &\leq \cdots\cdots\cdots \\ &\leq (1-\alpha_i)^{k(\epsilon)}|p_i(t-k(\epsilon))-p| + (1-\alpha_i)^{k(\epsilon)-1}\alpha_i\epsilon + \cdots + \alpha_i\epsilon \\ &= (1-\alpha_i)^{k(\epsilon)}|p_i(t-k(\epsilon))-p| + [1-(1-\alpha_i)^{k(\epsilon)}]\epsilon. \end{aligned}$$

As ϵ could be arbitrarily small and thus $k(\epsilon)$ arbitrarily large, the above could be arbitrarily small. This completes the proof. *Q.E.D.*

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