Value of Loss in n-Person Games[1]

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The paper extends the Shapley value by incorporation of a stochastic cooperative structure into the n-person game. We value the individual's marginal contribution in forming the cooperation. At the same time, we also recognize the potential loss in the absence of the individual. The generalized value is then defined by the expected marginal contribution and the expected potential loss. In this generalization, the Shapley value assumes that the cooperation formation depends on its size alone. JEL Classification: C71.

KEYWORDS: Shapley value, cooperation formation, n-person game

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1. THE VALUE

We consider the n-person game where $N = \overline{12\cdots n}$ being the set of players $1, 2, \cdots, n$ with $n \geq 3$. For any coalition $T \subseteq N$, it alone can achieve the maximal utility or payoff $v(T)$, called *characteristic function*. Without loss of generality, the empty coalition $\emptyset$ has the payoff 0, i.e. $v(\emptyset) = 0$. However, the coalition T could not secure the payoff $v(T)$ unless it involves in certain value-generating activity or task. In many situations, no player knows the characteristic function unless all possible coalitions are formed and their respective payoffs are observed. We assume that the nature (or God) knows the characteristic function and propose a solution concept for the nature.

Without knowing the perspective $v(\cdot)$, some or all of the players may still try to form a potential cooperation structure (i.e. a firm) or a few structures to maximize their payoffs or to achieve some goals which are beyond personal ability. In this paper, we only consider the simplest case of one potential cooperation structure. Let $\mathbf{S}$ be the set of players in the potential cooperation to take the value-generating task and $\mathcal{P}$ be the probability distribution of $\mathbf{S}$. For an example, consider the formation of a firm where there are n potential employees. In the following, we shall use lower-case italic letters or numerals for players and italic capitals for coalitions. Set subtraction will be indicated by " $\setminus$ ". We shall employ the vinculum in naming the players of a coalition, thus "$\overline{abc}$" for "$\{a, b, c\}$". The number of elements of a finite set X is denoted $|X|$.

There are many solution concepts for the n-person game. The Shapley value assumes the super-additivity (i.e. $v(Z \cup T) \geq v(Z) + v(T)$ for any $Z \cap T = \emptyset$) and counts the weighted marginal contribution in forming the grand coalition N,

$$\phi_i[v] \stackrel{\text{def}}{=} \sum_{T \subseteq N} \frac{(|T|-1)!(n-|T|)!}{n!}[v(T) - v(T \setminus \bar{i})]. \tag{1}$$

The formation of the grand coalition is necessary to achieve the Shapley value. When we treat the n-person game as a social group of n people, however, the super-additivity assumption may be false. For example, consider the formation of a firm. The admission of a new employee does not necessarily bring additional value. For another example, consider the unsuccessful M&A cases in business. Therefore, the players may not generally form the grand coalition, especially when the payoff function v is unknown to them before a coalition is formed.

For simplicity, we let $P_T \stackrel{\text{def}}{=} \text{Prob}(\mathbf{S} = T)$ for any $T \subseteq N$. For the potential cooperation $\mathbf{S}$, player i's value to $\mathbf{S}$ comes from two aspects. If $i \in \mathbf{S}$, he contributes in bringing the added payoff $v(\mathbf{S}) - v(\mathbf{S} \setminus \bar{i})$ (could be negative). On the other hand, if $i \notin \mathbf{S}$, then $\mathbf{S}$ faces a potential loss $v(\mathbf{S} \cup \bar{i}) - v(\mathbf{S})$ without the participation of player i. In this case, the nature knows the potential "value" $v(\mathbf{S} \cup \bar{i}) - v(\mathbf{S})$ of player i though the "value" is not recognized by the players of $\mathbf{S}$. Player i's value to the task is defined by player i's expected marginal contribution and potential loss,

$$\begin{aligned}\phi_i[v,\mathcal{P}] &\stackrel{\text{def}}{=} \textstyle\sum_{T\subseteq N} P_T[v(T)-v(T\setminus\bar{i})]+\sum_{T\subseteq N} P_T[v(T\cup\bar{i})-v(T)] \\ &= \textstyle\sum_{T\subseteq N} P_T[v(T\cup\bar{i})-v(T\setminus\bar{i})]=\mathrm{E}[v(\mathbf{S}\cup\bar{i})-v(\mathbf{S}\setminus\bar{i})].\end{aligned} \tag{2}$$

To relate the value with $\phi_i[v]$, we list two uniformity conditions:

U-I: $|\mathbf{S}|$ is uniformly distributed on $\{0,1,2,\cdots,n\}$;

U-II: P_T is a function of the size $|T|$.

If U-I is satisfied, then $\mathrm{Prob}(|\mathbf{S}|=t)=\sum_{|T|=t} P_T=\frac{1}{n+1}$ for any $t=0,1,\cdots,n$. U-II is equivalent to that $\mathbf{S}$ is uniformly distributed on $\{T\subseteq N||T|=t\}$ given $|\mathbf{S}|=t$, or $\mathrm{Prob}(\mathbf{S}=T||\mathbf{S}|=|T|=t)=\frac{t!(n-t)!}{n!}$.

PROPOSITION 1.1. *If $\mathcal{P}$ satisfies U-I and U-II, then $\phi_i[v,\mathcal{P}]=\phi_i[v]$ for any $i\in N$.*

Proof: See Appendix A1.

The following proposition shows the difference between the value and the probabilistic value.

PROPOSITION 1.2. *There exists no cooperative structure $\mathcal{P}$ such that $E[v(\mathbf{S})-v(\mathbf{S}\setminus\bar{i})]=\phi_i[v]$ for all $i\in N$; there exists no cooperative structure $\mathcal{P}$ such that $E[v(\mathbf{S}\cup\bar{i})-v(\mathbf{S})]=\phi_i[v]$ for all $i\in N$.*

Proof: See Appendix A2.

A large class of the n-person games assumes the super-additivity. For a game in this class, one could assume the following normality axiom.

N-A: $\sum_{i \in N} \phi_i[v, \mathcal{P}] = v(N)$.

PROPOSITION 1.3. *N-A implies U-I.*

Proof: See Appendix A3.

The inverse is not true in general. As shown in Proposition 1.1, U-I, together with U-II, implies N-A.

2. SOME REMARKS

From the above analysis, we have the alternative axioms for the Shapley value:

A-1: $\phi_i[v, \mathcal{P}] = \mathrm{E}[v(\mathbf{S} \cup \bar{i}) - v(\mathbf{S} \setminus \bar{i})]$;

A-2: $\mathcal{P}$ satisfies N-A;

A-3: $\mathcal{P}$ satisfies U-II.

One could relax either A-2 or A-3 to investigate the value in some specific classes of cooperation. One could also believe that the marginal contribution should be more weighted than the potential loss. This defines a weighted value. For example,

$$\mathrm{E}[\alpha[v(\mathbf{S}) - v(\mathbf{S} \setminus \bar{i})] + (2-\alpha)[v(\mathbf{S} \cup \bar{i}) - v(\mathbf{S})]]$$

for some $0 \leq \alpha \leq 2$. An expanding firm, however, may put more weight on the potential loss.

We note the deterministic cooperation is a special stochastic cooperation. Say, T is a deterministic cooperation which could not be broken up. Then $\text{Prob}(T \subseteq \mathbf{S}) + \text{Prob}(T \subseteq N \setminus \mathbf{S}) = 1$. Another way to generalize the value model is to allow several (random) number of value-generating tasks. This is the case in the real world. If each player can undertake multiple tasks at the same time without worsening the quality of the tasks, then we still have the same index. When the time factor is incorporated in the model, one may value the individual using the real option theory in which either the layoff of the player or the admission of the player is an option in the future.

Lastly, let us consider another class of n-person games. In a simple game, $v(T)$ is either 1 or 0. Moreover, $v(\cdot)$ is weakly increasing in that $v(T) \leq v(Z)$ whenever $T \subseteq Z$. When the simple game is the winning rule for voting a bill, $\mathbf{S}$ is the set of voters who support the bill while $N \setminus \mathbf{S}$ the set of players who are against the bill. And $v(\mathbf{S}) = 1$ if and only if the bill is passed. There are two major power indices to quantify a player's relative strength in the game. The Shapley-Shubik index is just the Shapley value with the special characteristic function v (the super-additivity is not assumed). The Banzhaf index, on the other hand, is defined by

$$b_i[v] \stackrel{\text{def}}{=} \frac{1}{2^{n-1}} \sum_{T \subseteq N} [v(T) - v(T \setminus \bar{i})].$$

We have the proposition.

PROPOSITION 2.1. *If* $\mathbf{S}$ *is uniformly distributed on* 2^N*, i.e.* $P_T = \frac{1}{2^n}$ *for any* $T \subseteq N$*, then* $\phi_i[v, \mathcal{P}] = b_i[v]$.

Proof: See Appendix A4.

3. APPENDIX: PROOFS

A1: PROOF OF PROPOSITION 1.1

Given U-I and U-II, $P_T = \frac{(|T|)!(n-|T|)!}{(n+1)!}$ for any $T \subseteq N$. By (2),

$$\begin{aligned}
\phi_i[v, \mathcal{P}] &= \sum_{i \in T} \frac{(|T|)!(n-|T|)!}{(n+1)!}[v(T) - v(T \setminus \bar{i})] \\
&\quad + \sum_{i \notin T} \frac{(|T|)!(n-|T|)!}{(n+1)!}[v(T \cup \bar{i}) - v(T)] \\
&\stackrel{Z=T\cup\bar{i}}{=} \sum_{i \in T} \frac{(|T|)!(n-|T|)!}{(n+1)!}[v(T) - v(T \setminus \bar{i})] \\
&\quad + \sum_{i \in Z} \frac{(|Z|-1)!(n-|Z|+1)!}{(n+1)!}[v(Z) - v(Z \setminus \bar{i})] \\
&\stackrel{T=Z}{=} \sum_{Z \subseteq N} \frac{(|Z|-1)!(n-|Z|)!}{n!}[v(Z) - v(Z \setminus \bar{i})] = \phi_i[v].
\end{aligned}$$

Q.E.D.

A2: PROOF OF PROPOSITION 1.2

On the contrary, if there exists such a $\mathcal{P}$ such that $\mathrm{E}[v(\mathbf{S})-v(\mathbf{S}\setminus\bar{i})] = \phi_i[\Gamma]$ for any $i \in N$, then

$$\sum_{i\in T} \mathrm{Prob}(\mathbf{S}=T)[v(T)-v(T\setminus\bar{i})] = \sum_{i\in T} \frac{(|T|-1)!(n-|T|)!}{n!}[v(T)-v(T\setminus\bar{i})].$$

As $\mathbf{S}$ is independent of the choice of $v(\cdot)$, $\mathrm{Prob}(\mathbf{S}=T) = \frac{(|T|-1)!(n-|T|)!}{n!}$ for any T such that $i \in T$. As the probabilities over these T's sum to 1, $i \in \mathbf{S}$ with probability 1. Now we let i vary in N to obtain that $\mathbf{S} = N$ with probability 1. Pick a special v in which

$$v(T) = \begin{cases} 0, & \text{if } |T| = 0, 1; \\ \frac{|T|-1}{n-2}, & \text{if } |T| = 2, 3, \cdots, n-2; \\ 1, & \text{if } |T| = n-1, n. \end{cases}$$

In this game, $\phi_i[v] = \frac{1}{n}$ but $\sum_{i\in T} \mathrm{Prob}(\mathbf{S}=T)[v(T)-v(T\setminus\bar{i})] = 0$. We get a contradiction.

By a similar argument, we can show the second part. *Q.E.D.*

A3: PROOF OF PROPOSITION 1.3

For any fixed $i \in N$, (2) can be re-written as $\phi_i[v,\mathcal{P}]$ as

$$\phi_i[v,\mathcal{P}] = \sum_{i\in T} v(T)[P_T + P_{T\setminus\bar{i}}] - \sum_{i\notin T} v(T)[P_T + P_{T\cup\bar{i}}]. \tag{3}$$

We compare the coefficient of $v(N)$ in the equation $v(N) = \sum_{i\in N} \phi_i[v,\mathcal{P}]$,

$$nP_N + \sum_{i\in N} P_{N\setminus\bar{i}} = 1. \tag{4}$$

In general, for any fixed T such that $T \neq N$ and $T \neq \emptyset$, N-A and (3) imply

$$|T|P_T + \sum_{i \in T} P_{T \setminus \bar{i}} - \sum_{i \notin T} [P_T + P_{T \cup \bar{i}}] = 0.$$

Therefore

$$(2|T| - n)P_T = \sum_{i \notin T} P_{T \cup \bar{i}} - \sum_{i \in T} P_{T \setminus \bar{i}}.$$

For any fixed $t = 1, \cdots, n-1$, we take the summation over the coalitions with the size t,

$$\begin{aligned}
(2t-n)\sum_{|T|=t} P_T &= \sum_{|T|=t}\sum_{i \notin T} P_{T \cup \bar{i}} - \sum_{|T|=t}\sum_{i \in T} P_{T \setminus \bar{i}} \\
&\overset{Z=T\cup\bar{i}}{=} \sum_{|Z|=t+1}\sum_{i \in Z} P_Z - \sum_{|T|=t}\sum_{i \in T} P_{T \setminus \bar{i}} \\
&\overset{Y=T\setminus\bar{i}}{=} (t+1)\sum_{|Z|=t+1} P_Z - \sum_{|Y|=t-1}\sum_{i \notin Y} P_Y \\
&= (t+1)\sum_{|T|=t+1} P_T - (n-t+1)\sum_{|T|=t-1} P_T.
\end{aligned}$$

Let $x_t = \sum_{|T|=t} P_T$ for any $t = 0, 1, \cdots, n$. The above equation reduces to

$$(n-t+1)x_{t-1} + (2t-n)x_t - (t+1)x_{t+1} = 0, \ \forall t = 1, 2, \cdots, n-1.$$

Together with (4) and $\sum_{t=0}^{n} x_t = 1$, we have the linear equation

$$\begin{bmatrix} n & 2-n & -2 & & & & & & \\ & n-1 & 4-n & -3 & & & & & \\ & & n-2 & 6-n & -4 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & & 3 & n-4 & 1-n & \\ & & & & & & 2 & n-2 & -n \\ & & & & & & & 1 & n \\ 1 & 1 & 1 & \cdots & \cdots & \cdots & 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

We expand the determinant of the coefficient matrix by the last row (from left to right),

$$n![1 - n + \frac{1}{2} - \frac{1}{3} + \cdots + (-1)^n \frac{1}{n}] \neq 0.$$

Therefore $\{x_i = \frac{1}{n+1}\}_{i=0}^{n}$ is the unique solution to the linear equation, i.e. $\sum_{|T|=t} P_T = \frac{1}{n+1}$ for any $t = 0, 1, \cdots, n$. This completes the proof.

As a final remark, when $n = 2$, the linear equation

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

has multiple solutions. This is the reason why we have assumed $n \geq 3$.

Q.E.D.

A4: PROOF OF PROPOSITION 2.1

Given $P_T = \frac{1}{2^n}$ for any $T \subseteq N$,

$$\begin{aligned}
\phi_i[v, \mathcal{P}] &= \tfrac{1}{2^n} \textstyle\sum_{i \in T} [v(T) - v(T \setminus \bar{i})] + \tfrac{1}{2^n} \sum_{i \notin T} [v(T \cup \bar{i}) - v(T)] \\
&\overset{Z=T\cup\bar{i}}{=} \tfrac{1}{2^n} \textstyle\sum_{i \in T} [v(T) - v(T \setminus \bar{i})] + \tfrac{1}{2^n} \sum_{i \in Z} [v(Z) - v(Z \setminus \bar{i})] \\
&\overset{Z=T}{=} \tfrac{1}{2^{n-1}} \textstyle\sum_{T} [v(T) - v(T \setminus \bar{i})] = b_i[v].
\end{aligned}$$

Q.E.D.

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