SUPRA-CONVERGENCE OF LINEAR EQUATIONS ON IRREGULAR CARTESIAN GRIDS

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ABSTRACT. We prove the second order convergence of a class of finite difference schemes for linear heat equations, and wave equations on irregular grids. Numerical examples and convergence studies are provided to demonstrate our theoretical results.

1. INTRODUCTION

In numerical computations for applications such as electromagnetics, the boundaryof the domain of interest usually cannot be accurately represented by grids with uniform mesh sizes. One way to represent the boundary accurately is to introduce grid points that lie exactly on it and henceforth introduce irregular mesh sizes. A simple analysis using Taylor expansion on the consistent "central" difference operators for $\partial/\partial x$ and $\partial^2/\partial x^2$ shows that the approximation errors are locally first or second order accurate. However, it is shown in [5] that a second order convergence can be achieved for a class of ordinary differential equations. This is called supra-convergence, a phenomenon that has been observed in practice. There has been only a limited number of people working on problems related to supra-convergence of finite difference schemes. See [1] and [7][8, 9]. We will comment on their approaches when we reach the related topics in the following sections.

In [7][9], the authors look at the supra-convergence of a class of fully discretized schemes for two dimensional transport equations. Instead of looking at the error cancellation properties of the schemes directly, they sought a predictor-corrector scheme to find the intermediate grid functions, of $O(||h||_\infty^2)$, that cancel the first order error terms. Also in [7], a Lax-Wendroff type scheme with an artificial viscosity regularization whose coefficient vanishes with Δt is introduced.

We extend the result of [5] to the heat, and the wave equations on compact domains with Dirichlet boundary conditions. Our approach is similar to the method of lines; namely we focus on the discretization of the spatial derivatives, and prove the stability for the resulting systems of ordinary

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differential equations. We first prove that each of the corresponding difference operators admits supra-convergence for the corresponding steady state equations. We then study the resolvent equations of each of the two equations. The resolvent equations are the Laplace transforms of the original equations. We prove second order bounds on the error terms and establish the second order bounds for the error for $\mathrm{Re}(s) \geq \eta_0$, where s is the variable introduced by the Laplace transform. By Parseval's relations, we can then conclude that the ODE systems are stable, in the sense that the second order error committed by the spatial discretizations on irregular grids can be bounded. In the last section of this paper, we will discretize the time derivative by some standard second or higher order ODE solvers and present some numerical convergence studies to illustrate our theoretical results. For linear transport equations, central differencing is equivalent to the well understood finite volume method which is second order accurate already. We will therefore skip our discussion over this type of equations and refer the readers to [4].

We will begin by introducing some notations that we will use throughout the paper, followed by a series of lemmas and propositions that describe our results for each of the three model equations.

1.1. **Notations.** We consider the differential operator $P(\partial^2/\partial x^2) = \partial^2/\partial x^2$ on the space $C^\infty([0,1]$ subject to Dirichlet boundary condition. Due to linearity, we can further assume zero boundary conditions at $x = 0$ and 1.

Given an irregular partition $\{x_0, x_1, \cdots, x_n\}$ of the interval $[0,1]$, we define

$$h = \{h_i = x_{i+1} - x_i > 0 : i = 0, \cdots n-1\},$$
$$\tilde{h}_i = \frac{h_i + h_{i-1}}{2}.$$

We assume that

$$\frac{h_{\max}}{h_{\min}} \leq \text{Constant},$$

and consequently, we have $h_j/h_k \leq C < \infty, \forall j,k \in \{0, \cdots n-1\}$. By $h \longrightarrow 0$ we mean $h_{\max} = ||h||_\infty \longrightarrow 0$. The difference operators are defined on the Banach space $\mathcal{X}^h$ ={complex valued grid functions v on h} equipped with the l^∞-norm or suitable discrete L^2 norm; in particular, we use $\mathcal{X}_0^h = \{v \in \mathcal{X}^h : v_0 = v_n = 0\}$.

We use the following notation for the differencing:

$$D_+ v_i = \frac{v_{i+1} - v_i}{h_i}, \; D_- v_i = \frac{v_i - v_{i-1}}{h_{i-1}}, \; D_0 v_i = \frac{v_{i+1} - v_{i-1}}{2\tilde{h}_i},$$
$$D_+ D_- v_i = \frac{D_- v_{i+1} - D_- v_i}{\tilde{h}_i} = D_- D_+ v_i = \frac{D_+ v_i - D_+ v_{i-1}}{\tilde{h}_i},$$

and

$$D_+^j v_i = \underbrace{D_+ \cdots D_+}_{j\,\text{times}} v_i := j! v[x_i, \cdots, x_{i+j}],$$

where $v[x_i, \cdots, x_{i+j}]$ is the divided difference of v.

If w is a smooth function on $[0,1]$, Taylor expansion shows that

$$D_\pm w_i = w_x(x_i) + O(||h||_\infty),$$
$$D_+ D_- w_i = w_{xx}(x_i) + O(||h||_\infty).$$

Definition 1.1. (discrete L^2 norm) Given two complex valued grid functions v and w, we have the following notations:

$$(v,w)_{r,s} = \sum_{i=r}^{s} v_i \bar{w}_i \tilde{h}_i;$$
$$||v||_{r,s}^2 = (v,v)_{r,s}.$$

It is clear from the definition that $||\cdot||_{1:n-1}$ defines a norm in the space $\mathcal{X}^h$, and is equivalent to the sup norm $||\cdot||_\infty$.

1.2. **Some Properties of The Difference Operators on Irregular Grids.** The finite difference operators discussed in this article share their resemblance to the standard finite difference operators. They appear to be in similar forms and, together with the discrete L_2 norm defined above, possess similar properties with respect to operations such as summation by parts (see e.g. [4]). We will list the properties that we need for our proofs.

Lemma 1.2. *(Summation by parts) Let* $\{v_i\}, \{w_i\} \in \mathcal{X}_0^h$,

$$(D_+ D_- v, w)_{1,n-1} = -\sum_{i=1}^{n} D_- v_i D_- \bar{w}_i h_{i-1}$$

Proof. We prove this by direct manipulation of the summation:

$$\begin{aligned}
\sum_{i=1}^{n-1} D_+ D_- v_i \cdot \bar{w}_i \tilde{h}_i &= \sum_{i=1}^{n-1} \frac{D_- v_{i+1} - D_- v_i}{\tilde{h}_i} \cdot \bar{w}_i \tilde{h}_i \\
&= (D_- v_n \cdot \bar{w}_{n-1} - D_- v_{n-1} \cdot \bar{w}_{n-1}) \\
&\quad + (D_- v_{n-1} \cdot \bar{w}_{n-2} - D_- v_{n-2} \cdot \bar{w}_{n-2}) + \cdots \\
&= D_- v_n \cdot \bar{w}_{n-1} - \sum_{i=2}^{n-1} D_- v_i D_- \bar{w}_i h_{i-1} - D_- v_1 \cdot \bar{w}_1.
\end{aligned}$$

Using the boundary conditions $w_n = w_0 = 0$,

$$\begin{aligned}
D_- v_n \cdot \bar{w}_{n-1} &= -D_- v_n \cdot \frac{\bar{w}_n - \bar{w}_{n-1}}{h_{n-1}} h_{n-1} \\
D_- v_1 \cdot \bar{w}_1 &= D_- v_1 \cdot \frac{\bar{w}_1 - \bar{w}_0}{h_0} h_0.
\end{aligned}$$

Therefore, we have

$$(D_+D_-v,w)_{1,n-1} = -\sum_{i=1}^{n} D_-v_i D_-\bar{w_i}\, h_{i-1}$$

□

Replacing w by v, we have:

Corollary 1.3. *Let* $\{v_i\} \in \mathcal{X}_0^h$,

$$(D_+D_-v,v)_{1,n-1} = -\sum_{i=1}^{n} |D_-v_i|^2\, h_{i-1}$$

The following is a direct consequence of the above lemma.

Corollary 1.4. D_+D_-*, as a linear operator on* $\mathcal{X}_0^h$*, has only negative eigenvalues.*

Proof. Suppose the unit vector $w \in \mathcal{X}_0^h$ is an eigenvector corresponding to an eigenvalue $\lambda > 0$; i.e.

$$D_+D_-w = \lambda w.$$

$$(D_+D_-w,w)_{1,n-1} = \sum_{i=1}^{n-1} D_+D_-w_i \cdot \bar{w_i}\, \tilde{h}_i = \sum_{i=1}^{n-1} \lambda w_i \cdot \bar{w_i}\, \tilde{h}_i = \lambda ||w||^2_{1,n-1}.$$

By Lemma 1.3, λ can only be non-positive. Now, w is in the null space of D_+D_-, by Lemma 1.3, we have

$$D_-w_i = 0,\ \ i = 1,\cdots,n$$

From the boundary conditions, $w_0 = 0, w_n = 0$, we see that $D_+D_-w = 0$ if and only if $w = 0$. Hence, we conclude that D_+D_-on $\mathcal{X}_0^h$ has only negative eigenvalues. □

Proposition 1.5. *Let* $\{v_i\} \in \mathcal{X}_0^h$*, there exists a constant C independent of* h_i *such that*

$$(D_+D_-v,v)_{1,n-1} = -\sum_{i=1}^{n} |D_-v_i|^2\, h_{i-1} \le -C||v||^2_{1,n}.$$

Proof. By definition and Cauchy-Schwarz inequality,

$$|v_j|^2 = \left|\sum_{i=1}^{j} D_-v_i h_{i-1}\right|^2 \le \sum_{i=1}^{j} |D_-v_i|^2 h_{i-1} \sum_{i=1}^{j} h_{i-1} \le \sum_{i=1}^{n} |D_-v_i|^2 h_{i-1};$$

therefore,

$$||v||^2_{1,n} = \sum_{j=1}^{n} |v_j|^2 \tilde{h_j} \le \sum_{i=1}^{n} |D_-v_i|^2 h_{i-1} \sum_{j=1}^{n} \frac{h_j + h_{j-1}}{2} \le \sum_{i=1}^{n} |D_-v_i|^2 h_{i-1}.$$

This proves our statement. □

If we define a new norm

$$|||v|||_{1,n}^2 = \sum_{j=1}^{n} |v_j|^2 h_{j-1},$$

it is not hard to show that $|||\cdot|||_{1:n}$ is equivalent to $||\cdot||_{1:n}$ throught the bound on $h_{\max}/h_{\min}$. Hence, we have the discrete Poincaré inequality:

$$||v||_{1,n}^2 \leq C||D_-v||_{1,n}^2,$$

and consequently,

$$(D_+D_-v, v)_{1,n-1} \leq -\tilde{C}||v||_{1,n}^2.$$

We remark that D_+D_- on $\mathcal{X}_0^h$ can be written as a tridiagonal matrix, whose entries in the two off diagonals are of one sign. The Corollary above implies that the eigenvalues of this matrix are are all negative. The following theorem shows that this type of matrices is diagonalizable.

Theorem 1.6. *Let A be an invertible, tridiagonal matrix with $A_{i,i+1}$ and $A_{i,i-1}$ are real of one sign and nonzero, then A is diagonalizable.*

Proof. Since A is a tridiagonal matrix with $A_{i,i+1}$ and $A_{i,i-1}$ are of one sign and nonzero, we can symmetrize A by diagonal scaling such that $\tilde{A} = DAD^{-1}$ is hermitian, and $\tilde{A}_{ii} = A_{ii}$. Here D is a diagonal matrix with

$$D_{ii} = d_i.$$

It can be easily shown by simple calculation that d_i satisfies

$$\frac{d_i A_{i,i+1}}{d_{i+1}} = \frac{d_{i+1} A_{i+1,i}}{d_i} \Longrightarrow \frac{d_i^2}{d_{i+1}^2} = \frac{A_{i+1,i}}{A_{i,i+1}}, i = 1, \cdots, n.$$

We can then diagonalize the Hermitian matrix $\tilde{A}$ by unitary matrix $U : \tilde{A} = U\Lambda U^*$, $UU^* = I$. Therefore, Λ is a diagonal matrix with strictly negative entries. $\tilde{A} = DAD^{-1}$, $A = (D^{-1}U\Lambda U^*D)$. $(U^*D)^{-1} = D^{-1}U$. □

The following corollary is straight forward from the above proof.

Corollary 1.7. *If $A_{i,i}$ are purely imaginary and $A_{i,i+1} > 0$ and $A_{i,i-1} < 0$ or . $A_{i,i+1} < 0$ and $A_{i,i-1} > 0$, then there exists D such that DAD^{-1} is anti-Hermitian.*

The finite difference operators presented here on irregular grids $\mathcal{X}^h$ can be viewed as Newton divided differences. In addition, they can be related in $||\cdot||_\infty$ by

$$||D_+^j D_-^k v||_\infty = ||D_+^{j+k} v||_\infty.$$

To justify the convergence of our numerical approximation to smooth solutions, we need to study the convergence of the interpolants constructed from the grid functions we computed. We can find smooth Hermite interpolant Int v of the grid function v with

$$\frac{d^j}{dx^j}\text{Int } v(x_i) = D_+^j v_i,\ \ j = 0,1,2,\ \ i = 0,\cdots N-1.$$

In addition, the derivatives of the interpolant can be bounded by the finite difference operators of the same order:

Lemma 1.8. *Let Int $w(x)$ be a C^2 interpolant of the grid function v. The derivatives of Int w can be bounded by the divided differences of v:*

$$||D_+^j v||_\infty \le ||\frac{d^j}{dx^j} Int\ w||_\infty \le \delta ||v||_\infty + C(\delta)||D_+^j v||_\infty.$$

Please see [6] and [3, 2] for more detail. Using this result, we can bound the finite differences of v by a higher difference of v and v itself:

Lemma 1.9. *For every $\delta > 0$, there exists constants $C_j(\delta)$ independent of h and v such that*

$$||D_+^j v||_h \le \delta ||v||_h + C(\delta)||D_+^2 v||_h,\ \ j = 0,1. \tag{1.1}$$

Proof. For any $C^2([0,1])$ function w, $0 \le j \le k$, we have the Sobolev inequality

$$||\frac{d^j}{dx^j} w||_2^2 \le \delta^2 ||\frac{d^2}{dx^2} w||_2^2 + C_{j,k}^2(\delta)||w||_2^2,$$

which implies

$$||\frac{d^j}{dx^j} w||_2^2 \le (\delta ||\frac{d^2}{dx^2} w||_2 + C_{j,k}(\delta)||w||_2)^2;$$

i.e.

$$||\frac{d^j}{dx^j} w||_2 \le (\delta ||\frac{d^2}{dx^2} w||_2 + C_{j,k}(\delta)||w||_2).$$

Since the domain $[0,1]$ is bounded, we have $||w||_2 \le \text{Const} \cdot ||w||_\infty$. Inequality (1.1) is then a direct application of lemma 1.8. □

2. Main Results

In this section, we begin by presenting the supra-convergence of compact central schemes for a class of first and second order ODEs. They are the building block of our analysis for time dependent PDEs. We prove the supra-convergence by direct manipulation of the result presented in [5]. We then analyze the supra-convergence property of the the two model equations. Our results can be generalized to equations with variable coefficients rather easily, after suitable modification on the boundary conditions and assumptions. In principle, the generalization follows the standard localization principle.

2.1. $u_{xx} = f$. We consider the second order boundary value problem

$$\begin{cases} u_{xx} = f \\ u(0) = u(1) = 0 \end{cases} \tag{2.1}$$

and the discretization

$$\begin{cases} D_+D_-U_i = f_i, \text{ for } i = 1, \cdots, n-1 \\ U_0 = U_n = 0 \end{cases} . \tag{2.2}$$

Let u be the smooth solution to the problem, we have

$$D_+D_-u_i = f_i + \frac{h_i - h_{i-1}}{3} f_i' + \frac{h_i^2 - h_i h_{i-1} + h_{i-1}^2}{12} f_i'' + O(||h||_\infty^3).$$

Define the error function

$$e_i = u_i - U_i,$$

we have that the error satisfies

$$D_+D_-e_i = \frac{h_i - h_{i-1}}{3} f_i' + \frac{h_i^2 - h_i h_{i-1} + h_{i-1}^2}{12} f_i'' + O(||h||_\infty^3),$$

for $i = 1, \cdots, n-1$, and $e_0 = e_1 = 0$. Multiplying the above equation by $(h_i + h_{i-1})/2$ to get rid of the denominator on the left hand side, we have

$$\begin{aligned} \frac{e_{i+1} - e_i}{h_i} - \frac{e_i - e_{i-1}}{h_{i-1}} &= \frac{h_i^2 - h_{i-1}^2}{6} f_i' + O(||h||_\infty^3) \\ &= \frac{1}{6} h_i^2 f_i' - \frac{1}{6} h_{i-1}^2 f_{i-1}' + O(||h||_\infty^3). \end{aligned}$$

We then sum this recursion relation from $i = 1, \cdots, k < n$, and obtain

$$\begin{aligned} \sum_{i=1}^{k} \frac{e_{i+1} - e_i}{h_i} - \frac{e_i - e_{i-1}}{h_{i-1}} &= \frac{e_{k+1} - e_k}{h_k} - \frac{e_1 - e_0}{h_0} \\ &= \frac{1}{6} h_k^2 f_k' - \frac{1}{6} h_0^2 f_0' + \sum_{i=1}^{k} O(||h||_\infty^3). \end{aligned}$$

Again, we multiply the above equation by h_k, and obtain

$$e_{k+1} - e_k - \frac{h_k}{h_0}(e_1 - e_0) = \frac{1}{6} h_k^3 f_k' - \frac{1}{6} h_k h_0^2 f_0' + O(||h||_\infty^3). \tag{2.3}$$

In [5], the authors studied the half plane problem with boundary conditions on x_0 and x_1, which lead to at most $O(||h||_\infty^3)$ errors in e_0 and e_1, and

$$e_{k+1} - e_k = O(||h||_\infty^3).$$

Then summing up from $k=0$ to k', they concluded that the error $e_{k'} = O(||h||_\infty^2)$.

We have a slightly different problem, since we are considering two point boundary value problem. Nevertheless, we will show that the error for our problem is also second order. We will prove this claim by Gaussian elimination. First off, with the left boundary condition, we have $e_0 = 0$, and telescoping (2.3), we have

$$e_{k'} - (k'-1)\frac{h_1}{h_0}e_1 = O(k'||h||_\infty^3).$$

We will prove that $e_1 = O(||h||_\infty^3)$ so that $e_{k'} = O(||h||_\infty^2)$ for $0 \le k' \le n$.

Let $e = (e_1, \cdots, e_{n-1})$ and A be the matrix version of the left hand side of (2.3); i.e.

$$\begin{pmatrix} -1+\frac{h_1}{h_0} & 1 & & & 0 \\ \frac{h_2}{h_0} & -1 & 1 & & \\ \vdots & & \ddots & \ddots & \\ \frac{h_{n-2}}{h_0} & & & -1 & 1 \\ \frac{h_{n-1}}{h_0} & & & & -1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{n-2} \\ e_{n-1} \end{pmatrix} = O(||h||_\infty^3).$$

Now getting rid of the upper diagonal using Gaussian elimination just involves adding the rows, starting the last row all the way up to the second row. Thus the right hand side is $O(||h||_\infty^2)$ due to the summing. The left hand side becomes $\tilde{A}e$, where

$$\tilde{A} = \begin{pmatrix} -1+\sum_{p=1}^{n-1}\frac{h_p}{h_0} & & & & 0 \\ \sum_{p=2}^{n-1}\frac{h_p}{h_0} & -1 & & & \\ \vdots & & \ddots & & \\ \frac{h_{n-2}}{h_0}+\frac{h_{n-1}}{h_0} & & & -1 & \\ \frac{h_{n-1}}{h_0} & & & & -1 \end{pmatrix}.$$

We are only interested in

$$\tilde{A}_{1,1} = -1 + \frac{1}{h_0}\sum_{p=1}^{n-1} h_p = -1 + \frac{1}{h_0}(1-h_0).$$

Hence, we have

$$\tilde{A}_{1,1}e_1 = O(||h||_\infty^2) \Longrightarrow e_1 = O(||h||_\infty^3).$$

In summary, we have proved:

Proposition 2.1. *The solution to (2.2) gives a second order approximation to the solution of (2.1), that is,* $u_i - U_i = e_i = O(||h||_\infty^2)$ *for* $i = 0, \cdots, n$.

2.2. **Heat Equation.** Consider the linear heat equation with the two point boundary conditions

$$u_t - Lu = g,\ u(0,t) = u(1,t) = 0,\ u(x,0) = u_0(x)$$

with L representing $\partial^2/\partial x^2$. Its Laplace transform is

$$s\hat{u} - L\hat{u} = f,\ \ \hat{u}(0,s) =, \hat{u}(1,s) = 0, \tag{2.4}$$

where $f(x) = \hat{g}(x) + u_0(x)$. Let v be the solution to

$$sv - Qv = f\ \ v(0,s) = 0, v(1,s) = 0, \tag{2.5}$$

where Q denotes the operator D_+D_-. Writing

$$L\hat{u} = Q\hat{u} + (L\hat{u} - Q\hat{u}) = Q\hat{u} - \frac{h_i - h_{i-1}}{3}\hat{u}^{(3)}(x_i) - R(x_i),$$

subtracting (2.5) from (2.4), we have

$$(sI - Q)(\hat{u} - v) = \frac{h_i - h_{i-1}}{3}\hat{u}^{(3)}(x_i) - R(x_i) =: \tilde{R}_i, \tag{2.6}$$

with boundary conditions $(\hat{u} - v)(0,s) = 0, (\hat{u} - v)(1,s) = 0$. Let $\tilde{e}$ solves the steady state two-point boundary problem

$$Q\tilde{e}_i = -\tilde{R}_i, i = 1, \cdots n-1,\ \ \tilde{e}_0 = 0, \tilde{e}_1 = 0.$$

Now assume that $\hat{u} - v = e + \tilde{e}$,

$$(sI - Q)e_i = -s\tilde{e}_i, i = 0, \cdots, n,\ \ e_0 = 0, e_1 = 0.$$

We now look at the inner products

$$((sI - Q)e, e)_{1,n-1} = s(\tilde{e}, e)_{1,n-1}.$$

We want to bound $||e||_{1:n-1}$ by $||\tilde{e}||_{1:n-1}$. By Corollary 1.3, and $s = a + ib$, we have the resolvent equation

$$(a+ib)||e||^2_{1,n-1} + \underbrace{\sum_{i=1}^{n} |D_- e_i|^2 h_{i-1}}_{A \geq 0} = (a+ib)(\tilde{e}, e)_{1,n-1} \tag{2.7}$$

Taking the norm on both sides of the equation, we have for $a \geq 0$:

$$\sqrt{a^2+b^2}||e||^2 \leq \sqrt{(a||e||^2 + A)^2 + b^2||e||^4} \leq \sqrt{a^2+b^2}||\tilde{e}||\,||e||,$$

where the norm $||\cdot||$ is understood as the one defined by $(\cdot,\cdot)^{1/2}_{1:n-1}$. Therefore, together with Proposition 2.1, which implies that $||\tilde{e}|| = O(||h||^2_\infty)$, we conclude

$$||e|| \leq ||\tilde{e}|| = O(||h||^2_\infty) \text{ for } \mathrm{Re}(s) \geq 0.$$

Thus, we proved:

Proposition 2.2. *For* $Re(s) \geq 0$, *the operator* $(sI - Q)^{-1}$ *is bounded in the* $||\cdot||$ *operator norm, and the error e of the approximation (2.5) is second order; i.e.* $||e|| = O(||h||_\infty^2)$.

Taking $s = i\xi$, *we can invert e back to the time domain:*

$$\check{e} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi t} e(\xi) d\xi.$$

Invoking the Parseval's relations, we conclude:

Theorem 2.3. *The error* $\check{e} = O(||h||_\infty^2)$.

2.3. **Wave Equation.** Consider the wave equation with periodic boundary condition

$$u_{tt} - u_{xx} = 0,\ u(0) = u(1) = 0, u(x,0) = f(x), u_t(x,0) = g(x).$$

The corresponding resolvent equation is

$$(s^2 I - \frac{\partial^2}{\partial x^2})\hat{u}(x,s) = \hat{F}(x,s),\ \text{where } \tilde{F}(x,s) = sf(x) + g(x).$$

The resolvent equation for the error is very similar to (2.7) of the heat equation. With the same notation that we used for the heat equation, we have

$$s^2||e||_{1,n-1}^2 + \underbrace{\sum_{i=1}^{n} |D_- v_i|^2 h_{i-1}}_{A\geq 0} = s^2(\tilde{e}, e)_{1,n-1}. \tag{2.8}$$

Let $s = i\xi + \eta$, and $s^2 = (\eta^2 - \xi^2) + i(2\eta\xi)$. Let us discuss the case in which $\eta \geq 0$, and $\eta = \alpha\xi$. (2.8) becomes

$$((\alpha^2 - 1)\xi^2 + i2\alpha\xi^2)||e||^2 + A = ((\alpha^2 - 1)\xi^2 + i2\alpha\xi^2)(\tilde{e}, e)_{1,n-1}.$$

Here we drop the subscript of $||\cdot||_{1:n-1}$ for convenience. Taking the absolute value of the imaginary part of the above equation, we have

$$\begin{aligned} 2|\alpha|\xi^2||e||^2 &= |\mathrm{Im}\,((\alpha^2-1)\xi^2 + i2\alpha\xi^2)(\tilde{e}, e)_{1:n-1}| \\ &\leq |((\alpha^2-1)\xi^2 + i2\alpha\xi^2)||(\tilde{e}, e)_{1:n-1}| \\ &\leq ((\alpha^2-1)^2\xi^4 + 4\alpha^2\xi^4)^{1/2}||\tilde{e}||\,||e|| \end{aligned}$$

and thus

$$||e|| \leq \frac{((\alpha^2-1)^2\xi^4 + 4\alpha^2\xi^4)^{1/2}}{2|\alpha|\xi^2}||\tilde{e}|| = \frac{((\alpha^2-1)^2 + 4\alpha^2)^{1/2}}{2|\alpha|}||\tilde{e}||.$$

Therefore, for $||h||_\infty \ll \alpha \leq 1$,

$$||e|| = O(||h||_\infty^2).$$

On the other hand, for $1 < \alpha$, we have $\mathrm{Re}(s^2) = \eta^2 - \xi^2 = (\alpha^2 - 1)\xi^2 > 0$, and the analysis is the same as for the heat equation. Hence, we proved:

Proposition 2.4. *For s with $Re(s) > \eta_0$, for any $\eta_0 > 0$, $||e||_{1,n-1}$ defined in (2.8) is $O(||h||_\infty^2)$.*

The above analysis shows that one can invert the error for any $s = \eta + i\xi$, if $\eta > 0$. This implies that the error can be bound by $C||h||_\infty^2 e^{\eta t}$, with an exponential growth factor in time, after inverting the Laplace transform. We are going to show that for $0 \le t \le T$, the error can only grow linearly in time. To do so, we go back to the physical space and look at our equation

$$e_{tt} = Qe + C||h||_\infty^2 q,$$

where e is the vector $(e_1, \cdots, , e_{n-1})$, q is assumed to be nice smooth function, and Q is a tridiagonal matrix that corresponds to D_+D_- and the homogeneous zero boundary condition. In fact, the diagonal of Q is strictly negative

$$Q_{ii} = -\frac{2}{h_i h_{i-1}} < 0, \ i = 1, \cdots, n-1,$$

while the two off diagonals are strictly positive

$$Q_{i,i+1} = \frac{2}{h_i(h_i + h_{i-1)}} > 0, \ i = 1, \cdots, n-2,$$

and

$$Q_{i+1,i} = \frac{2}{h_{i+1}(h_{i+1} + h_i)} > 0 \ i = 1, \cdots, n-2.$$

By Corollary 1.4 and Theorem 1.6, Q has only negative eigenvalues and can be diagonalized by U^*D, i.e. $Q = (D^{-1}U\Lambda U^*D)$, where Λ is a diagonal matrix with strictly negative entries, and U, D can be found following the proof of Theorem 1.6. Let $\tilde{e} = U^*De$. First, we notice that

$$\frac{d_i^2}{d_{i+1}^2} = \frac{Q_{i+1,i}}{Q_{i,i+1}} = O(1),$$

so we can choose the diagonal entries $d_i = O(1)$, and that $\tilde{e}/e = O(1)$.

We then have

$$\tilde{e}_{tt} = \Lambda\tilde{e} + C||h||_\infty^2 \tilde{q},$$

$\tilde{q} = U^*Dq$. We look at each component of the diagonalized system: (the index indicating the component is dropped for clarity)

$$\tilde{e}_{tt} = -\lambda\tilde{e} + C|h||_\infty^2 \tilde{q}, \lambda > 0.$$

We rewrite this into a system of ODEs:

$$\begin{pmatrix} \tilde{e} \\ \tilde{e}_t \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix} \begin{pmatrix} \tilde{e} \\ \tilde{e}_t \end{pmatrix} + \begin{pmatrix} 0 \\ Ch^2\tilde{q} \end{pmatrix}.$$

It is clear at this stage that we can diagonalize this system and consider the resulting two scalar equations of the form

$$\tilde{\tilde{e}}_t = \pm i\sqrt{\lambda}\tilde{\tilde{e}} + \tilde{C}||h||_\infty^2 \tilde{\tilde{q}}.$$

By Duhamel's principle and the initial conditions $e(0) = e_t(0) = 0$, we have

$$\tilde{\tilde{e}} = \int_0^T e^{\pm i\sqrt{\lambda}(T-s)}\tilde{C}||h||_\infty^2 \tilde{\tilde{q}} ds,$$

which implies that

$$|\tilde{\tilde{e}}| = O(T||h||_\infty^2).$$
$$e_i(t) = O(T||h||_\infty^2).$$

We can therefore conclude with the following theorem.

Theorem 2.5. *For $0 \le t \le T$, the error e grow linearly in time and $||e||_\infty = O(T||h||_\infty^2)$.*

3. NUMERICS AND EXAMPLES

We follow the methods of line by first discretizing the spatial derivatives using the operators discussed above, and then solve the resulting systems of ODEs by standard ODE solvers such as Forward/Backward Euler, Runge-Kutta, Leap Frog, or Crank-Nicholson.

3.1. **Generation of mesh.** We first generate a uniform mesh $\{x_i\}$ on $[0,1]$ such that $x_0 = 0$, and $x_n = 1$. We then perturb the grid nodes by a uniformly distributed random number σ scaled to lie between $[-\sigma,\sigma] \subset [-0.5, 0.5]$ such that the the nodes remain ordered; i.e. $x_i > x_{i-1}$ for $i = 1,\cdots,n$. Hence,

$$h_{\max} = (1+2\sigma)h \longrightarrow 0 \text{ as } h \longrightarrow 0,$$
$$h_{\min} = (1-2\sigma)h \longrightarrow 0 \text{ as } h \longrightarrow 0,$$

and

$$\frac{h_{\max}}{h_{\min}} = \frac{1+2\sigma}{1-2\sigma}.$$

3.2. **Heat Equation.**

$$u_t = u_{xx} + f(x), \; u(0) = u(1) = 0.$$

In our first example, we have

$$f(x) = 0,$$

and the initial data

$$u(x,0) = \sin(3\pi x).$$

Therefore, we have the analytical solution

$$u(x,t) = e^{-9\pi^2 t}\sin(3\pi x).$$

We use the Matlab's ode45 to evolve the system of ODE resulted from our discretization. We show a numerical convergence result in table 1.

3.3. **Wave Equation.** The wave equation

$$u_{tt} = c^2 u_{xx}$$

is always posed as an initial (boundary) value problem, with the initial conditions

$$u(x,0) = f(x) \text{ and } u_t(x,0) = g(x).$$

One can pose the homogeneous boundary conditions

$$u(x_0,t) = 0 \,\text{and}\, u(x_1,t) = 0.$$

This condition may be viewed as a string with fixed ends.

We discretize the wave equation directly using central differencing both in time and in space.

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = c^2 D_+ D_- u_i^n.$$

This leads to

$$u_i^{n+1} = (\Delta t^2 c^2 D_+ D_- + 2)u_i^n - u_i^{n-1}.$$

The initial conditions are approximated with second order accuracy

$$u_i^0 = u(x_i,0) \text{ and } \frac{u_i^1 - u_i^{-1}}{2\Delta t} = u_t(x_i,0).$$

In the test program, we simply put in true solution u_i^1, and start crunching.

In our first test case, we consider the vibrating string problem with two fixed end points $u(0,t) = u(1,t) = 0$, $u_t(x,0) = 2\pi$, and $c = 1$.

$$u_0(x) = \sin(2\pi x).$$

Some numerical error results are shown in table 2.

4. SUMMARY

In this paper, we study the supra-convergence of a class of finite difference schemes for linear heat and wave equations. The method of lines approach is adopted. We discretize the spatial derivatives and investigate the stability of the resulting ODE systems by studying the corresponding resolvent equations. Our results state that supra-convergence is achieved in each model equation when the spatial derivatives are discretized appropriately. In the end, we demonstrate our theoretical results by providing some numerical convergence studies.

We look forward to extending further our studies in supra-convergence to nonlinear equations in higher dimensions.

$n-1$	2σ	T	$\|\|e\|\|_\infty$	rate	$\|\|e\|\|$	rate	$h_{\max}/h_{\min}$
25	0.4	0.01	0.004135		0.002285		1.74
50	0.4	0.01	0.001020	2.02	0.000662	1.79	2.14
100	0.4	0.01	0.000237	2.10	0.000148	2.16	2.16
200	0.4	0.01	0.000060	1.98	0.000037	1.99	2.24
400	0.4	0.01	0.000014	2.08	0.000009	2.02	2.14

$n-1$	2σ	T	$\|\|e\|\|_\infty$	rate	$\|\|e\|\|$	rate	$h_{\max}/h_{\min}$
20	0.9	0.01	0.012072		0.006631		3.78
40	0.9	0.01	0.002236	2.43	0.001101	2.59	4.62
80	0.9	0.01	0.000522	2.10	0.000309	1.83	10.99
160	0.9	0.01	0.000127	2.04	0.000081	1.94	12.69
320	0.9	0.01	0.000030	2.11	0.000018	2.18	11.31

$n-1$	σ	T	$\|\|e\|\|_\infty$	rate	$\|\|e\|\|$	rate	$h_{\max}/h_{\min}$
20	0.99	0.002	0.013215		0.004915		24.64
40	0.99	0.002	0.002460	2.43	0.000879	2.48	10.79
80	0.99	0.002	0.000512	2.27	0.000204	2.11	14.68
160	0.99	0.002	9.85663e-05	2.38	4.9228e-05	2.05	65.33
320	0.99	0.002	1.8102e-05	2.44	9.73304e-06	2.34	11.09

TABLE 1. Heat equation, example one.

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$n-1$	2σ	T	$\|\|e\|\|_\infty$	rate	$\|\|e\|\|$	rate	$h_{\max}/h_{\min}$
25	0.80	0.597	0.00812472	NaN	0.00496178	NaN	4.43
50	0.80	0.596	0.0024858	1.71	0.00125931	1.98	7.04
100	0.80	0.596	0.000467766	2.41	0.000309024	2.03	7.95
200	0.80	0.595	0.000127734	1.87	7.65891e-05	2.01	7.35
400	0.80	0.595	3.04849e-05	2.07	1.85865e-05	2.04	7.12

$n-1$	2σ	T	$\|\|e\|\|_\infty$	rate	$\|\|e\|\|$	rate	$h_{\max}/h_{\min}$
25	0.90	2.099	0.0292956	NaN	0.0186708	NaN	5.34
50	0.90	2.098	0.00729352	2.01	0.0047248	1.98	9.05
100	0.90	2.098	0.0017639	2.05	0.00109397	2.11	7.63
200	0.90	2.097	0.000415597	2.09	0.000261007	2.07	7.14
400	0.90	2.097	0.00011673	1.83	7.44967e-05	1.81	12.78

$n-1$	2σ	T	$\|\|e\|\|_\infty$	rate	$\|\|e\|\|$	rate	$h_{\max}/h_{\min}$
30	0.70	2.097	0.0198392	NaN	0.0123258	NaN	4.60
60	0.70	2.097	0.00424768	2.22	0.00260563	2.24	3.76
120	0.70	2.096	0.000986974	2.11	0.000626667	2.06	3.22
240	0.70	2.096	0.000262193	1.91	0.000164727	1.93	4.88
480	0.70	2.096	6.74956e-05	1.96	4.30043e-05	1.94	4.95

$n-1$	2σ	T	$\|\|e\|\|_\infty$	rate	$\|\|e\|\|$	rate	$h_{\max}/h_{\min}$
25	0.20	2.000	0.000807662	NaN	0.00037136	NaN	1.33
50	0.20	2.000	8.1055e-05	3.32	3.57583e-05	3.38	1.33
100	0.20	2.000	9.56991e-06	3.08	3.80512e-06	3.23	1.47
200	0.20	2.000	1.43395e-06	2.74	3.46698e-07	3.46	1.46
400	0.20	2.000	2.28414e-07	2.65	4.62027e-08	2.91	1.47

TABLE 2. Wave equation, example one.

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