A Fast Algorithm for Level Set Based Optimization*

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Abstract

In this article, we develop a fast method to solve a class of optimization problem with level set representation. When applying this algorithm to the Chan-Vese image segmentation model (2001, IEEE Trans. on Image Processing. **10**, 2), it improves the computational speed dramatically. This approach differs from previous methods in that we do not need to solve the Euler-Lagrange equation of the underlying variational problem. Instead, we calculate the energy directly and check if the energy is decreased when we change a point inside the level set to outside or vice versa. We analyze the algorithm and prove that under most initial conditions, we only need one sweep over the pixels to converge to the correct solution for 2-phase images. Another advantage of this method is that the gradient of the functional is not needed. This enables it to be applied to broader range of optimization problems. The complexity of our algorithm to do tasks such as image segmentation is $O(N)$, where N is the number of pixels in the image.

1 Introduction

Variational methods have been extensively used and studied in image processing in the past decade because of their flexibility in modeling and various advantages in the numerical implementation. Examples of this include image segmentation [3, 4, 12], object tracking [13], texture synthesis and vector field visualization [1]. The basic idea of variational methods is to minimize a

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cost or energy functional. This functional generally will depend on the features of the image. The classical way to solve the minimization problem is to solve the corresponding Euler-Lagrange equation. This PDE based method sometimes is not very efficient because of numerical stability constraints.

In this paper, we present a new algorithm for solving a kind of optimization problems that can be formulated by level sets. That is, our algorithm is defined for:

$$\min_{\phi} F(H(\phi)), \tag{1}$$

where H is the Heaviside function, F is any functional dependent on $H(\phi)$ and ϕ is the level set function. In particular, the Chan-Vese model [3] is in this form. This minimization problem is usually solved by solving the Euler-Lagrange equation:

$$\phi_t = -\nabla_\phi F(H(\phi)). \tag{2}$$

The prerequisite for this method is that F must be differentiable with respect to ϕ. But for many problems, this may not be true. For example, if we want to segment cluttered images, F can be defined as the distance between the histogram of $\phi > 0$ and that of the training set [17]. To overcome this difficulty, in this paper, we use, instead of solving the corresponding Euler-Lagrange equation, a direct method to solve the variational problem. The main advantage is that we do not need to solve a PDE, thus have no numerical stability constraints. We simply test each point to check if the energy decreases or not when we change a point inside the level set to outside or vice versa. When we apply this method to the Chan-Vese image segmentation model [4], it improves the computational speed dramatically (at least 10 times). For 2-phase images, it will converge in one sweep under a suitable initial condition which usually will be satisfied. For multi-phase images, this method also converges very fast compared to PDE methods. It is easy to extend this method to higher dimensional problems such as 3-D segmentation and clustering. Another virtue of this method is that the gradient of functional F is no longer required. Thus we can apply this method to a broader range of optimization problems.

The possibility of obtaining a fast algorithm for the Chan-Vese model by solving ordinary differential equation was pointed out to the second author by Fedkiw in [5]. This has led to the recent work of Gibou and Fedkiw in [6], which is similar, but not identical, to our current work.[1] The methods

[1] Apparently, we were working independently and were not aware of each other's investigation.

are similar in the sense that they both are designed to speed up the computation of the Chan-Vese model, and they both exploit the fact that we only need the sign of ϕ but not its value. They are different in their specific approaches. The method in [6] is motivated by considering large time steps for a simplified Euler-Lagrange equation of the Chan-Vese model by ignoring the length regularization term. The regularization is then put back in by a subsequent anisotropic diffusion process. Our approach is a general level set based optimization framework, with the Chan-Vese model as a particular application. We use the values of the objective function directly, without the need for its gradient or the corresponding Euler-Lagrange equation, to determine the sign of ϕ. Our model also allows the full Chan-Vese model with the length term included. We will have a more detailed comparison later in the paper.

The paper is organized as follows. In section 2, we briefly review the level set methods and give our new algorithm to solve the optimization problem (1). In section 3, we review the Chan-Vese model and analyze its properties in section 4. These properties are for the Chan-Vese model only. In section 5 we apply the algorithm to image segmentation by piecewise linear functions. Numerical implementation and experimental results are given in section 6, and we end the paper with a brief concluding section.

2 The Level Set Based Optimization

We begin by reviewing the standard level set method, then we give a new method to solve the optimization problem (1) and set the conventions that will be used throughout the paper.

The level set method, as initiated in [11], has been widely used in many areas including computational physics, image processing and computer graphics. The idea of the level set formulation is that it represents the front as the zero level set of a function defined in a higher dimensional space.

Consider a closed moving interface $\Gamma(t)$ in R^n. Let $\Omega(t)$ be the region (possibly multi-connected) that $\Gamma(t)$ encloses. We associate with $\Omega(t)$ an auxiliary function $\phi(x,t)$, called the level set function, which is Lipschitz continuous and satisfies:

$$\begin{cases} \phi(x,t) > 0 & \text{for } x \in \Omega \\ \phi(x,t) = 0 & \text{for } x \in \partial\Gamma \\ \phi(x,t) < 0 & \text{for } x \in \Omega^c, \end{cases} \tag{3}$$

where $x \in R^n, t \in R^+$. Then we have the following simple facts:

$$\hat{n} = -\frac{\nabla\phi}{|\nabla\phi|}, \tag{4}$$

$$|\Gamma(t)| = \int |\nabla H(\phi)| dx, \tag{5}$$

$$|\Omega(t)| = \int H(\phi) dx, \tag{6}$$

where $\hat{n}$ is the outward normal of the interface $\Gamma(t)$, $H(\phi)$ is the 1D Heaviside function which takes 0 for $\phi < 0$ and 1 otherwise. In 2D, $|\Gamma(t)|$ is simply the arc length of $\Gamma(t)$ and $|\Omega(t)|$ the area of $\Omega(t)$, while in 3D, $|\Gamma(t)|$ is the surface area of $\Gamma(t)$ and $|\Omega(t)|$ the volumn of $\Omega(t)$.

If we know ϕ, we can locate the interface by finding the zero level set of ϕ. That is, $\Gamma(t) = \{x : \phi(x,t) = 0\}$. So moving the interface is equivalent to updating ϕ, which can be done by solving a Hamilton-Jacobi type equation such as (2). This is usually slow because of the CFL condition. When solving (1), if we really need the value of level set ϕ, we have to solve (2). But for some special cases, for example, the Chan-Vese segmentation model, we do not need the value of ϕ, but only its sign. From the optimization point of view, this opens to us the possibility to use other direct methods to solve the minimization problem like (1). Based on this observation, we give a new direct algorithm to solve (1).

The outline of the algorithm is as follows:

Step 1. Initialize. Construct an initial partition, one part for $\phi > 0$, one part for $\phi < 0$ and compute the value of F according to ϕ.

Step 2. Advance. For each point x in image, if the energy F decreases when we change $\phi(x)$ to $-\phi(x)$, then update this point by $\phi(x) = -\phi(x)$, otherwise, $\phi(x)$ remains unchanged. We sweep the pixels in some prescribed order. For example, in image segmentation, we can sweep the pixels row by row. We can use either Gauss-Seidel or Jacobi iteration in each sweep.

Step 3. Repeat the step 2 until the energy F remains unchanged.

In step 1, there are several ways to set the initial value of ϕ. For example, we can use signed distance to the zeros level set as initial value or just let $\phi = 1$ inside the level set and $\phi = -1$ outside. Which one will be used depends on the expression of F. Note that after initialization, we only

change the sign of ϕ in each sweep. There is no need to compute derivatives and F need not be differentiable. On the other hand, in the above heuristic approach, we can not guarantee F has a unique solution. We also do not know whether the algorithm can converge even to a local minimum.

To illustrate this algorithm, we apply it to the Chan-Vese image segmentation model [4] in the next section and see how it improves the convergence speed. We also prove some properties of this method.

3 Application to the Chan-Vese Model

The Chan-Vese image segmentation model is a variational model for 2-phase image segmentation. The basic idea is to look for a particular partition of a given image into two regions, one representing the objects to be detected and one representing the background. Assuming that the image u_0 is a 2-phase image with piecewise constant values u_0^i and u_0^o and that the object to be detected is represented by the value u_0^i. Let C_0 denote the boundary of the object. The "fitting energy" is defined as :

$$F_1(C) + F_2(C) = \int_{inside(C)} |u_0 - c_1|^2 + \int_{outside(C)} |u_0 - c_2|^2,$$

where C is any other variable curve, and the constants c_1, c_2 are the averages of u_0 inside and outside of C respectively. The fitting energy will be minimized if $C = C_0$. In the Chan-Vese model, they also have a regularizing term, such as the length of C and the area inside C to control the smoothness of the boundary. Therefore, the energy $F(C, c_1, c_2)$ is define by:

$$F(C, c_1, c_2) = \mu \cdot (length(C)) + \lambda_1 \int_{inside(C)} |u_0 - c_1|^2 + \lambda_2 \int_{outside(C)} |u_0 - c_2|^2 \tag{7}$$

If we use the level set to represent C, that is, C is the zero level set of a Lipschitz function $\phi : R^2 \to R$, then we can replace the unknown variable C by the unknown variable ϕ, and the energy functional $F(C, c_1, c_2)$ can be written as:

$$\begin{aligned} F(H(\phi), c_1, c_2) = \mu\Big(\int_\Omega |\nabla H(\phi)|\Big) + \lambda_1 \int_\Omega |u_0 - c_1|^2 H(\phi) dx \\ + \lambda_2 \int_\Omega |u_0 - c_2|^2 (1 - H(\phi)) dx. \end{aligned} \tag{8}$$

where c_1, c_2 are also functions of $H(\phi)$. Note that F is in the form of (1). In equation (8), the two fitting terms are easy to compute directly. We can

approximate $\int |\nabla H(\phi)|dx$ by:

$$\sum_{i,j} \sqrt{(H(\phi_{i+1,j}) - H(\phi_{i,j}))^2 + (H(\phi_{i,j+1}) - H(\phi_{i,j}))^2},$$

where $\phi_{i,j}$ is the value of ϕ at the i,jth pixel. The summand can only take the values 0, 1 or $\sqrt{2}$, depending on whether the 3 distinct pair of points from the set $\{\ \phi_{i,j}, \phi_{i+1,j}, \phi_{i,j+1}\}$ belong to the same or different regions. Thus the length term can be easily computed knowing only $H(\phi)$, and there is no need to know ϕ. This computed value can be interpreted as the discretized length of zero level set. Note that to apply our algorithm, we do not need F differentiable in (8), which would have necessitied $\delta(\phi)$ in Euler-Lagrange equation of (8).

To solve the minimization problem (8), the usual approach is to derive its Euler-Lagrange equation, then use explicit time marching or implicit iteration. Because of the CFL condition, the time step should be very small, thus it needs a lot of iterations to converge. We can also use implicit iterative methods, such as those used in the original paper [3]. In this case, the time step is not as restricted as in explicit time marching.

Our algorithm for the Chan-Vese model is:

1. Give any initial partition of the image, set $\phi = 1$ for one part and $\phi = -1$ for another part.

2. Assume that the value of current pixel is x, c_1 and c_2 are averages for $\phi = 1$ and $\phi = -1$ respectively, m and n are number of pixels for $\phi = 1$ and $\phi = -1$. If $\phi(x) = 1$, then compute the difference between the new and the old energy:

$$\Delta F_{12} = (x - c_2)^2 \frac{n}{n+1} - (x - c_1)^2 \frac{m}{m-1}.$$

If $\Delta F_{12} < 0$, then change $\phi(x)$ from 1 to -1.
And similarly for the phi(x)=-1 case. If we consider the length term, then the change of the length is easy to compute since only four neighbor points will be affected when we change the value of a point.

3. Repeat the step 2 until the total energy F remains unchanged.

4 Analysis of the Algorithm for Chan-Vese Model

When we apply our new algorithm to the Chan-Vese model, it always converges in a finite number of sweeps (usually less than 10). If we do not

consider the length term ($\mu = 0$), it even converges in less than 5 sweeps. This leads us to analyze a simplified form of the algorithm that leaves out the regularization in the Chan-Vese model.

Considering a two phase image, the object is represented by A (maybe multi-connected), the background is B, the corresponding value for A and B is a and b, Given an initial partition $\phi > 0$ and $\phi < 0$, denoted by ϕ_1 and ϕ_2. Assume there are m points in ϕ_1 and n points in ϕ_2. Let c_i, F_i be the average and energy for ϕ_i, $i = 1, 2$. Assume a point $P \in \phi_1$ with value x. If we change P to from ϕ_1 to ϕ_2, let $\tilde{c_1}$, $\tilde{c_2}$ be the new average for ϕ_1 and ϕ_2, respectively, and $\tilde{F_1}, \tilde{F_2}$ be the new energy for ϕ_1 and ϕ_2. Then we can easily calculate:

$$\begin{aligned}
\tilde{c_1} &= c_1 + \frac{c_1 - x}{m - 1} \\
\tilde{c_2} &= c_2 - \frac{c_2 - x}{n + 1} \\
\tilde{F_1} &= F_1 - (x - c_1)^2 \frac{m}{m - 1} \\
\tilde{F_2} &= F_2 + (x - c_2)^2 \frac{n}{n + 1}.
\end{aligned}$$

The difference between the new energy and old energy is:

$$\Delta F_{12} = (x - c_2)^2 \frac{n}{n + 1} - (x - c_1)^2 \frac{m}{m - 1}.$$

Similarly, if P change from ϕ_2 to ϕ_1, the change of energy is:

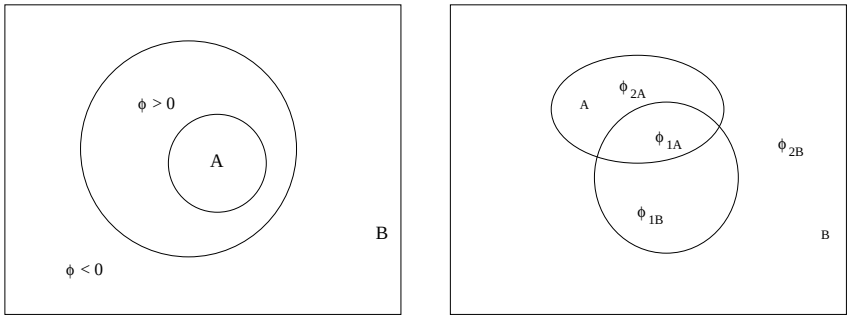
$$\Delta F_{21} = (x - c_1)^2 \frac{m}{m + 1} - (x - c_2)^2 \frac{n}{n - 1}.$$

Now, we have:

Lemma 1. *If $A \subseteq \phi_1$ and $\frac{n}{n+1} > \frac{(m-A)^2}{m^2} \frac{m}{m-1}$, where A represent the totality of points in object A, then, after one sweep, the algorithm will converge. The object is either $\phi > 0$ or $\phi < 0$.*

Proof. Assume $a < b$, then $c_2 = b, a < c_1 < b$. We only need to consider every possible case. See Fig. 1(a). Assume $P \in A$. If we change P to ϕ_2, the change of energy will be:

$$\begin{aligned}
\Delta F_{12} &= \frac{n}{n + 1}(a - b)^2 - (a - c_1)^2 \frac{m}{m - 1} \\
&= (\frac{n}{n + 1} - (\frac{m - A}{m})^2 \frac{m}{m - 1})(a - b)^2.
\end{aligned}$$



(a) (b)

Figure 1: different initial condition, (a), the object is completely contained in $\phi > 0$, (b), the object A has nonempty intersection with $\phi > 0$ and $\phi < 0$

By assumption, this is positive, so P will not change.
If $P \in B \cap \phi_1$, then

$$\Delta F_{12} = -(b - c_1)^2 \frac{m}{m-1} < 0,$$

so P will change to ϕ_2.
If $P \in B \cap \phi_2$, then if change P from ϕ_2 to ϕ_1,

$$\Delta F_{21} = (b - c_1)^2 \frac{m}{m+1} > 0,$$

therefore, P will not change.
So, after one sweep, all points are computed correctly. The object is represented by $\phi > 0$. This complete our proof. □

Remark 1. For $A \subseteq \phi_2$, the proof is the same. In practice, $\frac{n}{n+1} \approx 1, \frac{m}{m-1} \approx 1$, so if ϕ_1 is close enough to A, then $m - A << m$, and the condition in lemma 1 will be satisfied.

Remark 2. In the above proof, we use Jacobi iteration. The sweeping order is not important since we do not use the improved values of F until after a complete sweep. We can use any sweeping order in step 2. If we use the Gauss-Seidel iteration, that is, for each point, we use the most recent value, we still have the same result. To prove this, we only need to show that the condition in lemma 1 will be satisfied for each point. Given a point P, if $P \in A$ or $P \in B \cap \phi_2$, then from the proof of lemma 1, $\phi(P)$ will not change. If $P \in B \cap \phi_1$, then P will change from ϕ_1 to ϕ_2. Note that after

this calculation, $m = m - 1$, $n = n + 1$, so the condition in lemma 1 is still satisfied. This completes our proof.

Remark 3. Strictly speaking, our method does not use level set evolution, We just use level set to formulate our problem.

We now consider the other case, i.e. A and B can not be completely contained in $\phi > 0$ or $\phi < 0$, we have:

Lemma 2. *Let $\phi_{1A} = A \cap \phi_1$, $\phi_{2A} = A \cap \phi_2$, $\phi_{1B} = B \cap \phi_1$, $\phi_{2B} = B \cap \phi_2$, we assume none of them are empty, i.e., A and B can not be completely contained in ϕ_1 or ϕ_2. See Fig.1(b). If ϕ_{1A} does not change from ϕ_1 to ϕ_2, then ϕ_{2A} must change sign. Similar for ϕ_{1B} and ϕ_{2B}.*

Proof. Since ϕ_{1A} does not change sign, then

$$\Delta F_{12} = \frac{n}{n+1}(a - c_2)^2 - (a - c_1)^2 \frac{m}{m-1} \geq 0,$$

so we have

$$\frac{(a - c_2)^2}{(a - c_1)^2} \geq \frac{m}{m-1}\frac{n+1}{n} > 1.$$

Now, if ϕ_{2A} does not change from ϕ_2 to ϕ_1, then

$$\Delta F_{21} = \frac{m}{m+1}(a - c_1)^2 - (a - c_2)^2 \frac{m}{m-1} \geq 0,$$

so we have

$$\frac{(a - c_2)^2}{(a - c_1)^2} \leq \frac{m}{m+1}\frac{n-1}{n} < 1.$$

This is a contradiction. □

This means if one part of A does not change, the other part of A must change to the same sign.

Remark 4. It is possible that ϕ_{1A} and ϕ_{1B} change sign at the same time. In this case, it must satisfy the following condition:

$$\frac{n-1}{n}\frac{m}{m+1} \leq \frac{(a - c_2)^2}{(a - c_1)^2} \leq \frac{m}{m-1}\frac{n+1}{n}. \tag{9}$$

For example, if $c_1 = c_2$, then this condition will be satisfied. In the next section, we give an example satisfying this condition, and the algorithm ceases

to work. This condition means that c_1 and c_2 are very close. In general, if the object is not similar to the background and the initial partition is close to object, then condition (9) will not be satisfied and the algorithm will converge very quickly.

Now we have our main theorem:

Theorem 1. *If the condition in inequality (9) is not satisfied, and $|c_1-c_2| > c$, where c is some constant, then the algorithm (with $\mu = 0$) with either Jacobi or Gauss-Seidel converges in one sweep for a 2-phase image.*

Proof. First we consider Jacobi type iteration. Since inequality (9) is not satisfied, without loss of generality, we assume $\frac{(a-c_2)^2}{(a-c_1)^2} \geq \frac{m}{m-1}\frac{n+1}{n}$. We further assume $c_2 > c_1$, $a < b$. Then $a < c_1 < c_2 < b$. If A or B is completely contained in ϕ_1 or ϕ_2, then by lemma 1, it will converge in one sweep. Otherwise, both A, B have intersection with ϕ_1 and ϕ_2. Since $\frac{(a-c_2)^2}{(a-c_1)^2} \geq \frac{m}{m-1}\frac{n+1}{n}$, so

$$\Delta E_{12} = \frac{n}{n+1}(a-c_2)^2 - (a-c_1)^2\frac{m}{m-1} \geq 0,$$

which means ϕ_{1A} does not change sign. By lemma 2, ϕ_{2A} will change sign from ϕ_2 to ϕ_1. So $A \in \phi_1$. Now consider a point P in ϕ_{2B}. If it changes to ϕ_1, then

$$\Delta E_{21} = \frac{n}{n+1}(b-c_1)^2 - (b-c_2)^2\frac{m}{m-1} \geq 0.$$

This means ϕ_{2B} will not change sign. By lemma 2, ϕ_{1B} will change sign from ϕ_1 to ϕ_2 and we have $B \in \phi_2$. If we use Jacobi iteration, then, after one sweep, $A \in \phi_1$ and $B \in \phi_2$. If we use Gauss-Seidel iteration, as the argument in remark 2 shows, each time when we change a point, the condition in the theorem is still satisfied. This completes our proof. □

Remark 5. In practice, we do not expect one sweep convergence by just considering local change since the Chan-Vese model is a global model. In this model, the only global sharing information is the average. In our algorithm, when we change a point, we know how it affects the average. That's why it is possible to converge in one sweep.

Generally, it is easy to have initial conditions that satisfies the hypothesis for theorem 1. Our experiments also show that it is correct for 2-phase images. For multi-phase images, we no longer have finite convergence result of theorem 1, but we still can expect fast convergence.

In practice, when we apply our algorithm to the Chan-Vese model, we have several choices. The first one is to directly apply the algorithm to the model with the length term included. Or we can first consider $\mu = 0$, then followed by $\mu > 0$ to have the full effect of regularization. Another choice is to consider $\mu = 0$, followed by a PDE-based algorithm. We will give an example in the section 6.

5 Application to Other Optimization Problems

A simple extension to the Chan-Vese model is to use linear approximation instead of constant [17, 18]. This model is more appropriate when the image has region of linear shading instead of piecewise constant intensities. We consider a 2-phase image segmentation, each phase can be approximated by a linear function. Thus, using the level set representation, the functional can be represented as:

$$\begin{aligned} F(H(\phi)) = \mu\big(\int_\Omega |\nabla H(\phi)|\big) + \lambda_1 \int_\Omega |u_0 - a_0 - a_1 x - a_2 y|^2 H(\phi) dxdy \\ + \lambda_2 \int_\Omega |u_0 - b_0 - b_1 x - b_2 y|^2 (1 - H(\phi)) dxdy, \end{aligned} \tag{10}$$

where a_i, b_i , $i = 0, 1, 2$, are coefficients of a linear function which depend on $H(\phi)$. The a_i can be computed by linear system:

$$\frac{\partial}{\partial a_i} \int_\Omega |u_0 - a_0 - a_1 x - a_2 y|^2 H(\phi) dxdy = 0, i = 0, 1, 2.$$

Similarly for b_i. It is as easy to evaluate F and apply our algorithm as in the Chan-Vese model. But in this model, updating a_i and b_i is not so easy when we change ϕ of a point. So it is better to use Jacobi iteration than Gauss-Seidel iteration. The experiment result will be given in next section.

Another extension to the Chan-Vese model is to consider multiphase constant approximation to the image [3]. The idea is using $\log n$ level set functions to represent n phases or segments. Therefore, the energy to be minimized is given by:

$$F_n(C, \Phi) = \sum_{1 \le I \le = 2^m} \int_\Omega |u_0 - c_I| \chi_I dxdy + \sum_{1 \le i \le m} \nu \int_\Omega |\nabla H(\phi_i)|. \tag{11}$$

Here, the set of curves C is represented by the union of the zero level sets of the functions ϕ_i. For example, we can use two level set functions ϕ_1 and

ϕ_2 to represent a 4 phase image which consists of four regions: $\{\phi_1 > 0,$ $\phi_2 > 0\}$, $\{\phi_1 > 0,$ $\phi_2 < 0\}$, $\{\phi_1 < 0,$ $\phi_2 > 0\}$, $\{\phi_1 < 0,$ $\phi_2 < 0\}$. If we use PDE evolution to solve (11), we need to solve two PDE's for ϕ_1 and ϕ_2. Applying our algorithm to this minimization, we only need to test how the energy will change when we change a point from one region to other three regions, then choose the region that has minimum energy.

6 Experimental Results

In this section, we present numerical results using our new algorithm on the Chan-Vese segmentation model on various synthetic and real images. All of the calculation use Jacobi iterations. The numerical implementation used is quite simple and only requires a few lines of C++ code. In our experiment, we set $\phi = 1$ inside the level set and -1 outside, $\mu = 0.045$, $\lambda_1 = 1$ and $\lambda_2 = 1$.

First, we show the segmentation results on 2-phase image in Fig. 2. The length term is omitted since there is no noise. The image size is 100×100. We use four different initial conditions, all of them converge to correct solution in one sweep. In fact, it is hard to find an initial condition that it does not work. This also shows that the algorithm is quite robust.

In Fig. 3, we show an example that this algorithm does not always work. The initial condition is $\phi = 1$ on the left side and $\phi = -1$ on the right side. Then the averages c_1 and c_2 will be equal and not satisfy the condition in theorem 1. After one sweep, the part in which $\phi = 1$ will become $\phi = -1$ and vice versa. In the second sweep, it will change back to the initial condition. It will just change back and forth so it does not converge. If we use a Gauss-Seidel iteration instead of Jacobi iteration, then it will converge in one sweep.

In Fig. 4, we apply our algorithm to an image with noise. In this example, the length term in the energy expression is considered. Convergence is achieved in less than 5 sweeps, as compared to more than 400 steps in the original method. We show the result at sweep 1, 2, 3 and 4. We note that there is no special handling needed to detect the interior contours.

In Fig. 5, we show different initial conditions (a), (b), (c) and plot the energy versus iterations. The length term is considered. Note how fast our algorithm is. All of them converge in less then 8 sweeps. They also converge to the same result (d).

Fig. 6 shows the results of the Chan-Vese model without the length term. It converges in 4 sweeps. Note that it does not converge in exactly

one sweep because it is not a 2-phase piecewise constant image. This result is very close to the result we want and we can either do a few more steps of denoising or use a PDE-based algorithm to reach the final result.

In Fig. 7, we show the result of our new algorithm followed by level set evolution. We first use 4 sweeps of new algorithm and then use 5 iteration of original Chan-Vese method. We can also use other methods such as denoising or diffusion to remove the noise.

Fig. 8 shows the segmentation result of a noise image. In this example, we do not consider the length term. Although noise is presented, we still get correct result after 6 sweeps. The image size is 240×240.

Fig. 9 shows the segmentation result for a 3-phases image. We use two level sets to represent the image. (a) is the initial contours overlay on original image, (b) is the segmentation results. The image is 64x64 and converges in 1 sweep. Since the energy which is minimized is not convex, and also there is no uniqueness for the minimizers, the algorithm may not converge. We also have no theoretical result like theorem (1) for multiphase images.

Fig. 10 shows an example of segmenting a shading image. (a) is the original image which consists of two linear parts. (b) is the segmentation result of piecewise constant approximation. (c) is the segmentation result of piecewise linear approximation, it needs only 6 sweeps to converge. We can see that it converges to correct segmentation.

7 Concluding Remarks

In this paper, we proposed a new algorithm to solve a special kind of optimization problems which can be formulated by level set. Instead of solving the corresponding Euler-Lagrange equation, we compute the energy directly and see how the energy will change when we change the sign of a point. Another main advantage is that the gradient of the functional is not needed. Thus, it can be applied to more broad range of optimization problems. This algorithm is very successful when we apply it to the Chan-Vese model. We proved that for a 2-phase image, it converges in one sweep with most initial condition. The method can be easily extended to higher dimensional problems such as 3-D segmentation and clustering. In the future, we will also consider extending this method to multi-phase piecewise constant case.

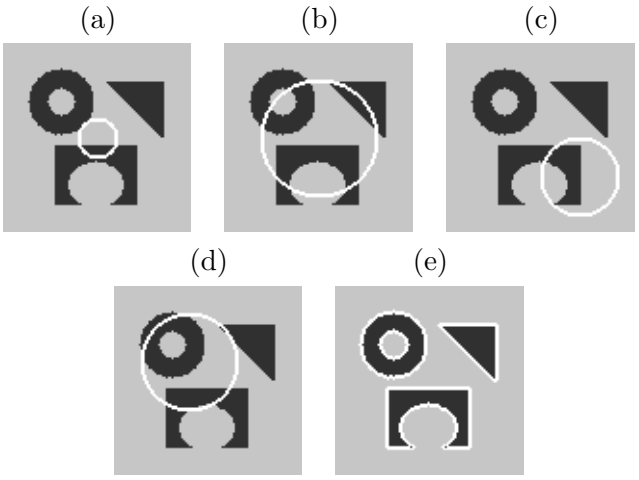


Figure 2: Segmentation of 2-phase image. (a),(b), (c), (d) are four different initial conditions, all of them have the same result after 1 sweep. The result is in (e). The image size is 100×100. Note that interior contour of the circle is automatically detected.

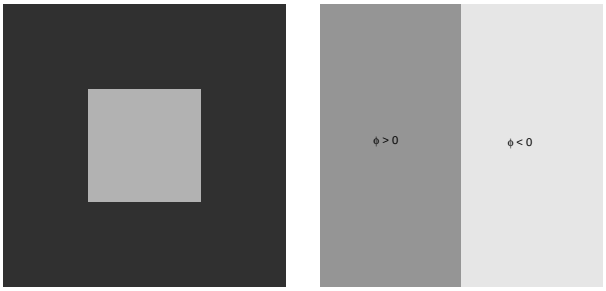


Figure 3: An example to show that the algorithm does not always work. Left: image we want to segment. Right: the initial ϕ. $\phi = 1$ on left side and -1 right side. The average on left side and right side are equal, so the necessary condition of theorem 1 does not apply.

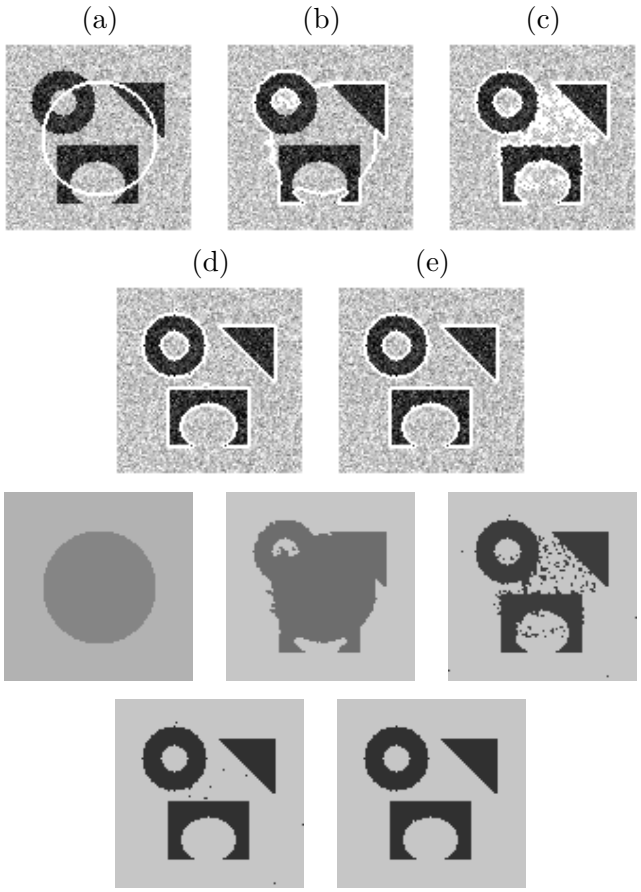


Figure 4: Detection of different objects from a very noise image. Top: u_0 and the contour. (a) initial ϕ, (b) after 1 sweep. (c) after 2 sweeps, (d) after 3 sweeps, (e) after 4 sweeps. Bottom: the piecewise constant approximation of u_0. Even though theorem 1 does not strictly apply, the algorithm still converges very fast.

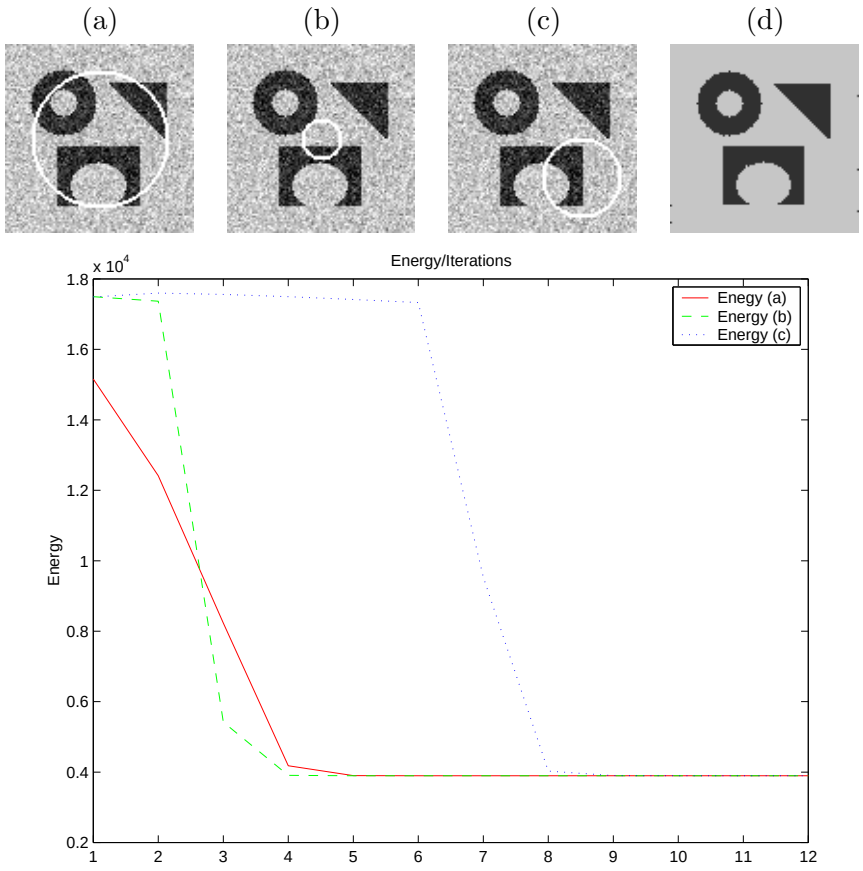


Figure 5: Three different initial conditions and the corresponding energy versus iterations. In this example, the length term is included in the objective functional. Note that all 3 initial conditions converge to the same result.

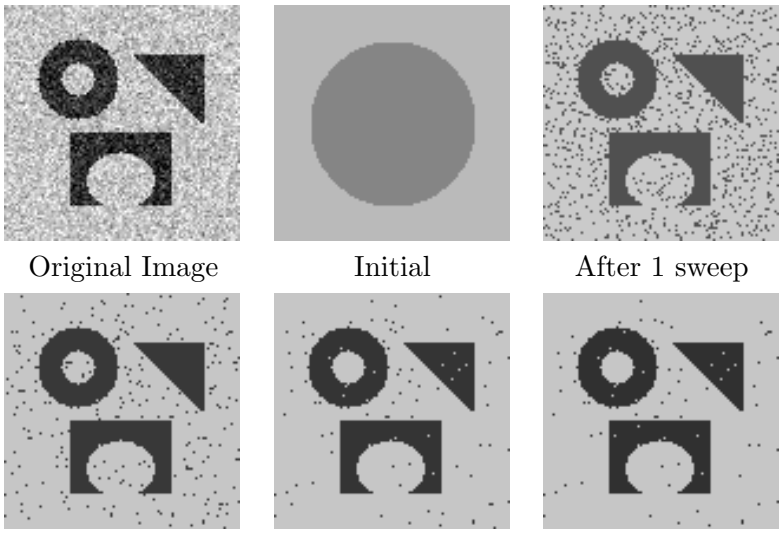


Figure 6: Chan-Vese model without length term. Note that even though theorem 1 does not strictly apply, the algorithm still converges in only 4 iterations.

Figure 7: Combining optimization and PDE evolution. (a) Initialization, (b) Result of fast algorithm using 4 sweeps, note that there are still some pixels which have not converge and show as "noise". (c) Using 5 sweeps PDE evolution starting from (b). Note that all the "noise" have converged.

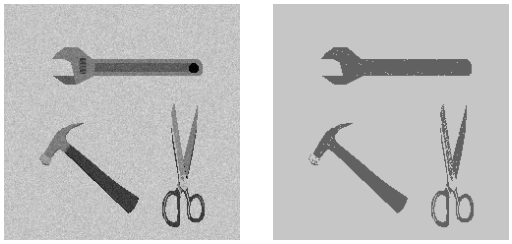


Figure 8: Segmentation of a noisy image. (a) Original image, (b) Result of piecewise constant approximation. The size of image is 240 x 240 and it converges in 6 sweeps. In this example, we did not consider the length term.

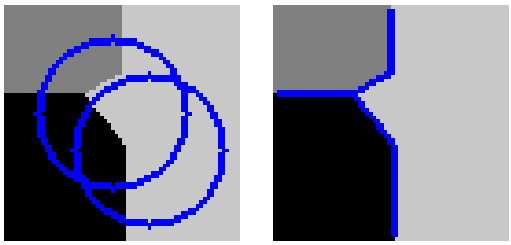


Figure 9: Segmentation of 3-phase image. (a) is the initial contours overlay on original image, (b) is the segmentation results. The image is 64x64 and converges in 1 sweep.

(a) (b) (c)

Figure 10: Piecewise linear approximation to image. (a) Original image, (b) Result of piecewise constant approximation, converges in 4 sweeps. (c) Result of piecewise linear approximation, converges in 6 sweeps.

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