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Martin Burger · Vincenzo Capasso ·
Daniela Morale

On an Aggregation Model with Long and Short Range Interactions

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Abstract In recent papers the authors had proposed a stochastic model for swarm aggregation, based on individuals subject to long range attraction and short range repulsion, in addition to a classical Brownian random dispersal. Under suitable laws of large numbers they showed that, for a large number of individuals, the evolution of the empirical distribution of the population can be expressed in terms of an approximating nonlinear degenerate and nonlocal parabolic equation, which describes the limit.

In this paper the well-posedness of such evolution equations is investigated, which invokes a notion of *entropy solutions* extended to the nonlocal case. We motivate entropy solutions from the discrete particle system and use them to prove uniqueness. Moreover, we provide existence results and discuss some basic properties of solutions. Finally, we apply a Lagrangian numerical scheme to perform numerical simulations in spatial dimension one.

1 Introduction

In recent papers [25, 26] the authors had proposed a stochastic model for swarm aggregation, based on a number of individuals subject to long range attraction, and short range repulsion, in addition to a classical Brownian random dispersal.

The biological motivation of this series of papers arises from some field experiments [2] showing an example of animal swarming due to the interaction among the individuals of a population of ants, *Polyergus* rufuscens,

Martin Burger
Industrial Mathematics Institute, Johannes Kepler Universität Linz,
E-mail: martin.burger@jku.it

V. Capasso, D. Morale
ADAMSS & Department of Mathematics, University of Milan, Italy
E-mail: {capasso,morale}@mat.unimi.it

and their relation with the environment. The colonies of these species are characterized by the absence of polyethism in the worker cast which is composed only by soldiers, unable to attend any task (e.g. brood tending or nest maintenance) other than raiding activity [12]. These indispensable tasks are performed by individuals belonging to few specific species which have been kidnapped by *Polyergus* soldiers when they were newborn or pupae, and grew up in the *Polyergus* nest. To keep constant the slave's population in their nest, *Polyergus* ants periodically raid ant nests of the slave species. In these circumstances *Polyergus* soldiers aggregate in an army of 300-1000 individuals, 10-40 cm wide and some meters long.

In our analysis, we neglect the main direction of movement of the army since in this phase we are interested only in the aggregation mechanism. The ants clearly aggregate in a transversally organized army whose width, by the way, seems to depend on the type of terrain they move on.

We mention that problems of this kind and quite similar models as the one derived below appear in several biological applications, e.g., in the modelling of other animal swarms (cf. [24,31]) and in chemotaxis (cf. [19,8,15,16] and the references therein).

Let $N \in \mathbb{N} - \{0\}$ be the constant size of the population. In the Lagrangian stochastic model proposed by the authors in [26], the k-th individual, out of N, is located at the random position $X_N^k(t) \in \mathbb{R}^d, t \geq 0$, so that $\{X_N^k(t), t \in \mathbb{R}_+\}$ is a stochastic process in the state space $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$, on a common probability space $(\Omega, \mathcal{F}, P)$.

The dynamics underlying the system of stochastic processes is given by a system of Itô type stochastic differential equations (EDS's)

$$dX_N^k(t) = F_N[X_N(t)](X_N^k(t))dt + \sigma_N dW^k(t), \quad k = 1, \dots, N, \tag{1.1}$$

where the randomness is modelled by additive independent standard Wiener processes $\{W^k, k = 1, \dots, N\}$. Furthermore the common variance σ_N^2 might depend on the total number of individuals and

$$\lim_{N \to \infty} \sigma_N^2 = \sigma_\infty^2 \geq 0. \tag{1.2}$$

The drift term F_N describes the mutual interaction among individuals; it depends on the relative location of the specific individual $X_N^k(t)$ with respect to all other individuals, via the empirical measure of the whole system of individuals

$$X_N(t) = \frac{1}{N} \sum_{k=1}^{N} \epsilon_{X_N^k(t)} \in \mathcal{M}_P(\mathbb{R}^d); \tag{1.3}$$

$\mathcal{M}_P(\mathbb{R}^d)$ is the space of all probability measures on $\mathbb{R}^d$. This measure provides the spatial distribution of the system of N individuals at time t, so that its evolution provides a (discrete) Eulerian description of the army.

In order to find an expression for the drift operator F_N, the authors have made the following assumptions [26]

(i) individuals tend to aggregate subject to their interaction within a range of size $R > 0$ (finite or not). This corresponds to the assumption that each individual has a limited knowledge of the spatial distribution of its neighbors and interacts within a bounded region; this kind of interaction is modelled via a reference kernel $G : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ with compact support of radius R. This kind of interaction is known as *McKean-Vlasov* type. The aggregation drift can be expressed mathematically in terms of the so called "generalized gradient" of the empirical measure $X_N(t)$ convoluted with the kernel G, i.e. by

$$\nabla G * X_N(t)(x) = \sum_{j=1}^{N} \nabla G(x - X_N^j(t)), \qquad x \in \mathbb{R}^d. \tag{1.4}$$

Note that

$$[(\nabla G) * X_N(t)](x) = \nabla(G * X_N(t))(x), \qquad x \in \mathbb{R}^d. \tag{1.5}$$

Given a measure μ on $\mathbb{R}^d$, we recall that its convolution with a kernel $K : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$K * \mu(x) = \int_{\mathbb{R}^d} K(x - y)\ \mu(dy), \qquad x \in \mathbb{R}^d. \tag{1.6}$$

If G is an even function, (1.4) expresses a force of attraction on an individual in the direction of increasing concentration of the others;

(ii) individuals are subject to repulsion when they come "too close" to each other. Any accumulation in a single point in space is avoided [26]. A repulsion kernel $V_N : \mathbb{R}^d \longrightarrow \mathbb{R}_+$, rescaled by the total number N of interacting individuals is considered

$$V_N(x) = N^{\beta} V_1(N^{\beta/d} x), \qquad \beta \in (0,1), \qquad \forall\ x \in \mathbb{R}^d, \tag{1.7}$$

where V_1 is a continuous and symmetric probability density on $\mathbb{R}^d$. It is clear that

$$\lim_{N \to +\infty} V_N = \delta_0, \tag{1.8}$$

where δ_0 is Dirac's delta function. So repulsion is modelled by the negative generalized gradient of the empirical measure $X_N(t)$ convoluted with the kernel V_N,

$$-\nabla V_N * X_N(t)(x) = -\sum_{j=1}^{N} \nabla V_N(x - X_N^j(t)), \qquad x \in \mathbb{R}^d. \tag{1.9}$$

The type of interaction induced by (1.7) is called *moderate* interaction; a "mesoscale" has been introduced, since the range of the interaction is much smaller than the size of the whole space but much larger than the typical distance between two individuals. At this "mesoscale", for large N, we may have enough individuals to apply suitable laws of large numbers [28]. Hence, all interactions are modelled via the drift term F_N

$$F_N[X_N(t)](X_N^k(t)) \;=\; \frac{1}{N}[\nabla(G - V_N) * X_N(t)](X_N^k(t)). \tag{1.10}$$

The drift term can be considered as the negative derivative of the energy functional

$$E_N[X_N] = \frac{1}{N}\sum_{j,k=1}^{N}\left[V_N(X_N^j - X_N^k) - G(X_N^j - X_N^k)\right], \tag{1.11}$$

with respect to the particle positions X_k. The deterministic trajectories (i.e. the case $\sigma_N = 0$) correspond to a gradient flow for this energy in the form

$$\frac{dX_N^k}{dt}(t) = -\frac{\partial E_N}{\partial X_N^k}[X_N(t)]. \tag{1.12}$$

Under sufficient regularity on the kernels, the stochastic process of empirical measures $\{X_N(t), t \in \mathbb{R}_+\}$ has been shown to converge for $N \to \infty$ (in a suitable sense) to a deterministic process $\{X_\infty(t), t \in \mathbb{R}_+\}$ (cf. [26]), whenever $X_N(0)$ converges (in a suitable sense) to $X_\infty(0)$. Furthermore, if $X_\infty(0)$ admits a density ρ_0 with respect to the usual Lebesgue measure on $\mathbb{R}^d$, then, for any $t \in \mathbb{R}_+$, $X_\infty(t)$ admits a density $\rho(\cdot, t)$, i.e.

$$\lim_{N\to\infty} X_N(t) = \rho(\cdot, t)dx. \tag{1.13}$$

Under the above assumptions, the density satisfies the equation

$$\frac{\partial\rho}{\partial t} = \frac{\sigma_\infty^2}{2}\triangle\rho + \text{div}\;(\rho\nabla(\rho - G * \rho)) \qquad \text{on } \mathbb{R}^d \times \mathbb{R}_+. \tag{1.14}$$

with initial condition

$$\rho(x, 0) = \rho_0(x), \qquad x \in \mathbb{R}^d. \tag{1.15}$$

The limit dynamics is different depending upon the limit of the diffusion coefficient; indeed, if $\sigma_\infty > 0$ the dynamics of the density is smoothed by the diffusive term. This is due to the memory of the fluctuations existing when the number of individuals N is finite. In this case the limit dynamics of the k-th individual is still stochastic and described by the hybrid model

$$dY(t) = (\nabla G * \rho(\cdot, t))(Y(t)) - (\nabla\rho(Y(t), t))dt + \sigma_\infty dW^k(t). \tag{1.16}$$

For $\sigma_\infty = 0$ all stochasticity disappears and (1.14) becomes a degenerate equation

$$\frac{\partial\rho}{\partial t} = \text{div}\;(\rho\nabla(\rho - G * \rho)) \qquad \text{on } \mathbb{R}^d \times \mathbb{R}_+. \tag{1.17}$$

Equation (1.14), or (1.17), can be interpreted as describing the time variation of the density of a large population subject to long-range aggregation and "infinitesimally local" repulsion. In the literature such equations are known as the equations for the "mean field" ρ. As in the case of deterministic particles, one can also find the associated energy functional for the mean field, given by

$$E[\rho] = \frac{1}{2}\int_{\mathbb{R}^d} \rho(\rho - G * \rho)\ dx + \frac{\sigma_\infty^2}{2}\int_{\mathbb{R}^d} \rho \log \rho\ dx, \qquad (1.18)$$

and (1.14) can be rewritten as

$$\frac{\partial \rho}{\partial t} = \text{div}\ (\rho \nabla (E'[\rho])).$$

The mathematical analysis of (1.14) and in particular (1.17) is a challenging problem since the model combines a nonlinear degenerate parabolic differential operator with a nonlocal flux. From an applied point of view it is of particular interest to understand the equations as an Eulerian description of the dynamics of a large population of individuals subject to specific rules of interaction and random movement.

Non-linear degenerate diffusion models have been used to describe dispersal of biological populations [14,29] or aggregation of animal populations [24,26,27]. The interest in models with nonlocal effects has led to the investigation of integro-differential equation models; they describe interactions at distance. More recently the influence of the aggregation and repulsion kernels has been investigated by Mogilner and Keshet in [24]. They found that if the density dependence in the repulsion term is of a higher order than in the attraction term, then the swarm profile is realistic; i.e. the swarm has a constant interior density, with sharp edges, as observed in biological examples. The diffusion they consider is linear, which is essential for the proof of existence and uniqueness of the solutions. On the other hand the linear diffusion usually dominates the aggregation case and leads to solutions with infinite support. The main advantages of nonlinear diffusion are a finite speed of propagation, the existence of non-trivial stationary states, and non-decaying time dependent solutions. For a related chemotaxis model, a detailed study of linear and nonlinear diffusion models has been carried out recently (cf. [3]), with the result that linear diffusion always leads to decaying solutions for large time (contradicting the motivation of an aggregation model) and nonlinear diffusion allows for non-trivial stationary solutions with compact support.

A major issue is to find the right notion of solutions for equations like (1.17). It is well-known that classical solutions do not exist in general for degenerate equations, in particular for equations like (1.17) one has to expect that the solution is not differentiable at the boundary of the (compact) support. A usual way to overcome such difficulties for parabolic equations is to use *weak solutions*, which are obtained by multiplying with smooth test functions, integrating and using Gauss' Theorem in order to obtain derivatives of the test functions instead of derivatives of u in the weak formulation. However, for degenerate equations like the general form

$$\frac{\partial v}{\partial t} + \text{div}\ f(x,t,v) - \Delta a(v) = 0, \qquad (1.19)$$

the weak solution is not unique, and a different concept of solutions, so-called *entropy solutions*, has to be used in order to obtain uniqueness (cf. [4]). Another example of nonuniqueness has been found for transport equations with nonlocal nonlinearity (cf. [7]) corresponding to (1.17) without the diffusion term. To our knowledge the only uniqueness result for an equation like (1.17) is due to [27], but it holds only in 1D for a very special convex long-range interaction kernel.

The above issues concerning uniqueness motivate the study of weak and entropy solutions for (1.17). In conservation laws, entropy solutions are usually obtained as vanishing viscosity limits and well motivated from a physical point of view. In the case of biological models, entropy solutions are hardly used and not well motivated so far. We provide a formal analysis in Section 2, which indicates that entropy solutions can be obtained in a natural way as limits of interacting particle systems. In Section 3 we use a fixed point argument based on continuous dependence estimates for an equation with local flux to construct a local-in-time solution and to prove uniqueness. In Section 4 we discuss some further properties of the model such as scaling, mass conservation, and energy dissipation. Section 5 finally presents some numerical simulations in spatial dimension one, highlighting some properties of the aggregation model.

Throughout the paper we shall use the following notation: in general, we will denote the time variable by $t \in [0,T]$ (if necessary also by s, τ) and the spatial variable by $x \in \Omega \subset \mathbb{R}^d$ (if necessary also by y, z). Unless further noticed, the domain Ω is assumed to be either a bounded domain with Lipschitz boundary or $\Omega = \mathbb{R}^d$. Furthermore, we shall use the notation $\Omega_T := \Omega \times [0,T] \subset \mathbb{R}^{d+1}$ for the space-time domain. Functions on Ω_T will be denoted by u, v, in some cases also by the original variable ρ for the population density. The function spaces used as well as detailed assumptions on the aggregation kernel G are given in the Appendix.

2 Weak and Entropy Solutions

In this section we shall introduce the notions of weak and entropy solutions, respectively, and motivate them from the discrete particle system. For the sake of simplicity we shall only consider deterministic particles, i.e., $\sigma_N = 0$ in (1.1). In the following Ω is either a sufficiently regular domain, or $\Omega = \mathbb{R}^d$.

2.1 Definition of Solutions

We start with the definition of a weak solution without assuming any sign of the density ρ (whereas in the end we are only interested in nonnegative functions due to their interpretation as a population density). In order to obtain solutions in such a generality, we rewrite the model as

$$\frac{\partial u}{\partial t} + \text{div }\,(u\nabla(G * u)) - \Delta a(u) = 0 \qquad \text{in } \mathbb{R}^d \times (0,T] \tag{2.1}$$

$$u = u_0 \qquad \text{in } \mathbb{R}^d \times \{0\}, \tag{2.2}$$

where a is the function $a(u) := \frac{1}{2}u|u|$. It is easy to see that (2.1) is equivalent to (1.17) if the solution u is nonnegative.

Definition 1 A weak solution u of the system (2.1), (2.2) is a function $u \in C([0,T]; L^3(\Omega))$ with $u^2 \in L^2([0,T]; H_0^1(\Omega))$, $u_t \in L^2([0,T]; H^{-1}(\Omega))$ such that the initial condition (2.2) is satisfied and the identity

$$\int_0^T \int_\Omega \left(u\frac{\partial \phi}{\partial t} - \frac{1}{2}\nabla u^2 \nabla \phi + u(\nabla G * u)\nabla \phi \right) \, dx \, dt = 0 \tag{2.3}$$

holds for all $\phi \in C_0^\infty(\Omega)$.

In general, we cannot expect the uniqueness of a weak solution, so that we have to pick the right weak solution subject to some criteria. From the motivation of the continuum equation as the limit of a particle system, it seems clear that these criteria should also be satisfied by the solution of the particle model, too, at least in a weaker sense. As we shall see below, this property is true for so-called *entropy solutions*, which are usually introduced in conservation laws and degenerate parabolic equations in order to obtain physically correct entropy dissipation (cf. e.g. [22] for a detailed discussion). Moreover, we shall prove the existence and uniqueness of the entropy solution.

In the next definition we introduce the concept of *entropy solutions* to our system. Entropy solutions have been introduced and analyzed recently for equations of the form (1.19) (cf. e.g. [4,17]). We adapt this notion to our system, closely following this approach with a simple modification enforced by the nonlocal convolution operator, which leads us to the following definition:

Definition 2 An entropy solution of (2.1) is a measurable function u on Ω_T satisfying the following conditions:

(i) $u \in L^\infty(\Omega_T) \cap C(0,T; L^1(\Omega))$
(ii) $u^2 \in L^2(0,T; H^1(\Omega))$
(iii) For all $c \in \mathbb{R}$ and all nonnegative test functions $\phi \in C_0^\infty \Omega_T)$ the following entropy inequality holds:

$$\int_0^T \int_\Omega \Big(|u-c|(\frac{\partial \phi}{\partial t} + (\nabla G * u)\nabla \phi) - c \, \mathrm{sign}(u-c)(\Delta G * u)\phi + \frac{1}{2}|u^2 - c^2|\Delta \phi \Big) \, dx \, dt \geq 0 \tag{2.4}$$

(iv) Essentially, as $t \downarrow 0$: $\int_\Omega |u(x,t) - u_0(x)| \, dx \to 0$.

This definition of an entropy solution is an extension of the corresponding one for an equation of the form (1.19) as investigated in [17], with $f(x,t,u) = F(x,t)u$ in our case. The flux F is given by $F = \nabla G * u$ in our case, i.e., it depends on the solution, but in a nonlocal and smooth way. The rationale behind this approach is that by construction a fixed point map as the concatenation of $u \mapsto F$ and $F \mapsto v$, where v is the unique entropy solution of (1.19), which will be used below to analyze entropy solutions of (2.1), (2.2).

In order to be coherent with the definition of weak solutions given above, we verify that each entropy solution is also a weak solution:

Proposition 1 *Let u be an entropy solution of* (2.1), (2.2)*. Then u is also a weak solution.*

Proof First, we choose $c > \|u\|_{L^\infty(\Omega_T)}$, then the entropy inequality becomes

$$\int_0^T \int_{\mathbb{R}^d} \Big((c-u)\frac{\partial \phi}{\partial t} - (u(\nabla G * u))\, \nabla \phi - \frac{1}{2}(u^2 - c^2)\Delta \phi \Big)\; dx\; dt \geq 0,$$

for $\phi \in C_0^\infty(\Omega_T)$, where we have used Gauss' Theorem on Ω to eliminate the terms containing $c(\nabla G * u)$. Because of

$$\int_0^T \int_{\mathbb{R}^d} c\frac{\partial \phi}{\partial t}\; dx\; dt = \int_{\mathbb{R}^d} c(\phi(x,T) - \phi(x,0))\; dx = 0$$
$$\int_0^T \int_{\mathbb{R}^d} (u^2 - c^2)\Delta \phi\; dx\; dt = \int_0^T \int_{\mathbb{R}^d} \nabla (u^2 - c^2) \nabla \phi\; dx\; dt,$$

where we have used $u^2 \in L^2(0,T;H^1(\Omega))$ and the compact support of ϕ, we obtain that for all nonnegative $\phi \in C_0^\infty(\Omega_T)$ the inequality

$$\int_0^T \int_{\mathbb{R}^d} \Big(-u\frac{\partial \phi}{\partial t} - (u(\nabla G * u))\, \nabla \phi + \frac{1}{2}\nabla(u^2)\nabla \phi \Big)\; dx\; dt \geq 0.$$

Similary, by choosing $c < -\|u\|_{L^\infty(\Omega_T)}$, we may deduce the reverse inequality and thus,

$$\int_0^T \int_{\mathbb{R}^d} \Big(-u\frac{\partial \phi}{\partial t} - (u(\nabla G * u))\, \nabla \phi - \frac{1}{2}\nabla(u^2)\nabla \phi \Big)\; dx\; dt = 0$$

for all nonnegative test functions ϕ. For general ϕ we can construct two sequences of nonnegative test functions $\phi_k^+, \phi_k^- \in C_0^\infty(\Omega)$ such that $\phi_k^+ \to \max\{\phi, 0\}$, $\phi_k^- \to -\min\{\phi, 0\}$ in $C(\Omega_T)$ and

$$\begin{aligned} 0 = & \int_0^T \int_{\mathbb{R}^d} \Big(-u\frac{\partial \phi_k^+}{\partial t} - (u(\nabla G * u))\, \nabla \phi_k^+ - \frac{1}{2}\nabla(u^2)\nabla \phi_k^+ \Big)\; dx\; dt \\ & - \int_0^T \int_{\mathbb{R}^d} \Big(-u\frac{\partial \phi_k^-}{\partial t} - (u(\nabla G * u))\, \nabla \phi_k^- - \frac{1}{2}\nabla(u^2)\nabla \phi_k^- \Big)\; dx\; dt \\ \to & \int_0^T \int_{\mathbb{R}^d} \Big(-u\frac{\partial \phi}{\partial t} - (u(\nabla G * u))\, \nabla \phi - \frac{1}{2}\nabla(u^2)\nabla \phi \Big)\; dx\; dt, \end{aligned}$$

which completes the proof.

We finally mention that in several situations, e.g. for certain parabolic equations without spatial dependence of the coefficients, the equivalence of weak and entropy solutions can be shown (cf. [20]), which is however not to be expected in general. Since the concept of entropy solutions seems to be more appropriate in the case we consider, we shall base our study of existence and uniqueness on the concept of entropy solutions.

2.2 Solutions of the Particle System

In the following we consider the deterministic version of the particle system, and to avoid technicalities, we also ignore the diffusion term (noticing that the nonuniqueness of weak solutions is mainly caused by the nonlocal transport term, cf. [7]). Hence, the model under consideration is (1.1) with $\sigma_N = 0$ and $V_N \equiv 0$, which can be rewritten as

$$\frac{dX_N^k}{dt} = \frac{1}{N}\sum_{j=1}^{N} \nabla G(X_N^k - X_N^j). \tag{2.5}$$

Since we assume that $G \in W^{2,\infty}(\mathbb{R}^d)$, i.e., ∇G is Lipschitz continuous, the existence and uniqueness of the trajectories $X_N^k(t)$ is guaranteed by the Picard-Lindelöf Theorem for ordinary differential equations.

The continuum limit ($N \to \infty$) corresponding to (2.5) is given by the transport equation

$$\frac{\partial \rho}{\partial t} = -\text{div }(\rho \nabla (G * \rho)) \qquad \text{on } \mathbb{R}^d \times \mathbb{R}_+, \tag{2.6}$$

with the corresponding entropy condition

$$\int_0^T \int_{\mathbb{R}^d} \left[|\rho - c|(\frac{\partial \phi}{\partial t} + (\nabla G * \rho)\nabla\phi) - \text{sign}(\rho - c)\text{div}(c(\nabla G * \rho))\phi \right] dx\ dt \geq 0$$

for all test functions $\phi \in C_0^\infty(\mathbb{R}^d \times (0,T))$.

For test functions ϕ as above, we now take a look at the change of ϕ along the trajectories, more precisely

$$\begin{aligned} 0 = \frac{1}{N}\sum_{k=1}^{N} \phi(X_N^k(t),t)|_0^T &= \frac{1}{N}\sum_{k=1}^{N}\int_0^T \frac{d}{dt}\phi(X_N^k(t),t)dt \\ &= \frac{1}{N}\sum_{k=1}^{N}\int_0^T \left(\frac{\partial \phi}{\partial t}(X_N^k(t),t) + \nabla\phi(X_N^k(t),t)\cdot \frac{dX_N^k}{dt}(t)\right) dt \end{aligned}$$

Inserting (2.5) we obtain

$$0 = \int_0^T \int_{\mathbb{R}^d} \left(\frac{\partial \phi}{\partial t} + \nabla\phi \cdot \nabla G * X_N \right) dX_N dt.$$

Moreover, due to the fact that ϕ has compact support in the space domain and $\phi(.,T) = 0$ we obtain

$$|c| \int_0^T \int_{\mathbb{R}^d} \frac{\partial \phi}{\partial t}\ dx\ dt = |c| \int_{\mathbb{R}^d} \phi\ dx \Big|_0^T = 0,$$

and, together with Gauss' Theorem,

$$|c| \int_0^T \int_{\mathbb{R}^d} \nabla\phi \cdot \nabla G * X_N\ dx\ dt = -|c| \int_0^T \int_{\mathbb{R}^d} \phi\ \text{div}(\nabla G * X_N) dx\ dt.$$

Adding the last three equalities we end up with

$$0 = \int_0^T \int_{\mathbb{R}^d} \left(\frac{\partial \phi}{\partial t} + \nabla \phi \cdot \nabla G * X_N \right) (dX_N + |c| dX) dt$$
$$- c \ \mathrm{sign}(-c) \int_0^T \int_{\mathbb{R}^d} \phi \ \mathrm{div}(\nabla G * X_N) \ dx \ dt. \qquad (2.7)$$

If we formally write the density corresponding to X_N as

$$\rho_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{X_N^j},$$

then due to the properties of Dirac delta distributions we have sign $(\rho_N - c) =$ sign$(-c)$ and $|\rho_N - c| = \rho_N + |c|$ (which can be made rigorous in the sense of total variation of measures). Hence, we can formally rewrite (2.7) as

$$\int_0^T \int_{\mathbb{R}^d} \Big[|\rho_N - c| \big(\frac{\partial \phi}{\partial t} + (\nabla G * \rho_N) \nabla \phi \big) - c \ \mathrm{sign}(\rho_N - c)(\Delta G * \rho_N) \phi \Big] dx \ dt \geq 0,$$

i.e., the entropy condition is automatically satisfied by the particle system and should be conserved by the limit, so that the entropy solution should be correct for the continuum equation.

We finally mention that there is even stronger motivation for the entropy solution as the limit of the stochastic particle system, since the Brownian motion acts like viscosity, and the entropy solution is usually obtained as the vanishing viscosity limit. In the case of non-vanishing linear diffusion, it is known that a weak solution is also an entropy solution (cf. [9]).

3 Existence and Uniqueness of Entropy Solutions

In the following we investigate the existence and uniqueness of entropy solutions for (2.1), (2.2).

3.1 An Auxiliary Local Problem

We start our investigation with the local problem

$$\frac{\partial u}{\partial t} + \mathrm{div} \ (u \nabla F) = \Delta a(u) \qquad \text{on } \Omega \times (0, T) \qquad (3.1)$$

with initial value $u = u_0$ at $t = 0$. Equation (3.1) is a *Fokker-Planck* equation with space-time dependent flux F. We assume in the following that

$$F \in L^\infty(0, T; W^{3,1}(\Omega)) \cap L^\infty(0, T; W^{2,\infty}(\Omega)) \cap L^\infty(0, T; C^1(\Omega)), \qquad (3.2)$$

The motivation for investigating (3.1) is a fixed point argument used later for proving existence and uniqueness of solutions, we shall consider the map $\rho \mapsto F = G * \rho$, concatenated with $F \mapsto u$. In the degenerate case the function

a is given by $a(u) = \frac{1}{2}u|u|$, but we can equally treat the viscous case with the choice $a(u) = \frac{1}{2}u|u| + \sigma_\infty u$, and even the non-diffusive case with $a \equiv 0$. All the arguments in the following do not depend on the specific choice of a, but only on the fact that a is a monotone C^1-function.

The definition of entropy solutions for (3.1) is analogous as in the nonlocal case above, the entropy inequality becomes

$$\int_0^T \int_\Omega \Big(|u-c|(\frac{\partial \phi}{\partial t} + \nabla F \cdot \nabla \phi) - c \,\mathrm{sign}(u-c)(\Delta F)\phi \\ + |A(u) - A(u)|\Delta\phi \Big) \, dx\, dt \geq 0 \qquad (3.3)$$

By analogous reasoning to Theorem 12 and 13 in [4] we can deduce existence and uniqueness for the local problem:

Proposition 2 *For $u_0 \in L^1(\Omega)$, there exists a unique entropy solution of* (3.1).

Since we have a fixed point-argument for the nonlocal equation in mind, we need a continuous dependence estimate for the solution of (3.1) in dependence of the flux F. Such continuous dependence estimates for nonlinear degenerate equations have been derived recently in a very general setup (cf. [11,17]) using additional regularity assumptions on the initial value and the solution, namely $u_0 \in BV(\Omega) \cap L^\infty(\Omega_T)$ and $u \in L^\infty(0,T;BV(\Omega)) \cap L^\infty(\Omega_T)$. The existence of solutions in this class can be shown as in [18]. Moreover, from a standard L^1-contraction result (cf. [17, Theorem 1.2]) one can deduce that

$$\|u(.,t)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} \qquad (3.4)$$

for almost every $t \in (0,T)$. We now state the continuous dependence estimate for (3.1), by specializing a result from [17]. For this sake, we consider the equations

$$\frac{\partial u^i}{\partial t} = \mathrm{div}\ (u^i \nabla(u^i - F^i)) \qquad \text{on } \Omega \times (0,T) \qquad (3.5)$$

with initial values $u^i = u_0^i$ for $i = 1,2$.

Lemma 1 *Let F^i satisfy* (3.2) *and let u^1, u^2 be entropy solutions of* (3.5) *in $L^\infty(0,T;BV(\Omega)) \cap L^\infty(\Omega_T)$. Then the estimate*

$$\|u^1(.,t) - u^2(.,t)\|_{L^1(\Omega)} \leq \|u_0^1 - u_0^1\|_{L^1(\Omega)} + tC_0 \|\nabla F^1 - \nabla F^2\|_{L^\infty(0,t;L^\infty(\Omega))} \\ + tC_\infty |\nabla F^1 - \nabla F^2|_{L^\infty(0,t;BV(\Omega))} \qquad (3.6)$$

holds for almost every $t \in (0,T)$, with constants

$$C_0 = \min\{|u^1|_{L^\infty(0,t;BV(\Omega))}, |u^2|_{L^\infty(0,t;BV(\Omega))}\}$$
$$C_\infty = \min\{\|u^1\|_{L^\infty(0,t;L^\infty(\Omega))}, \|u^2\|_{L^\infty(0,t;L^\infty(\Omega))}\}$$

Proof The proof can be carried out in an analogous way as the proof of Theorem 1.3 in [17] with the special choice $f(u) = g(u) = u$, taking into account the time dependence of the flux function F and (3.4).

Lemma 1 is a good basis for applying a fixed-point argument, since it allows to estimate $u^1 - u^2$ in terms of $F^1 - F^2$ with a constant that decreases with T. However, the constants C_0 and C_∞ in (3.6) still depends on the bounded variation and supremum norm of the solutions, which we have not yet estimated yet. In order to obtain an estimate for the supremum norm, one can use comparison principles for entropy solutions, which also yield nonnegativity:

Lemma 2 *Let u be an entropy solution of* (3.1) *with F satisfying* (3.2)*. Moreover, let $u_0 \in L^\infty(\Omega)$ be nonnegative. Then,*

$$0 \leq u(.,t) \leq e^{C_F t} \|u_0\|_{L^\infty(\Omega)} \tag{3.7}$$

with $C_F = \|(\Delta F)^-\|_{L^\infty(\Omega_T)}$, where $a^- := \min\{a, 0\}$.

Proof Let u_0^δ, F^δ and a^δ be appropriate C^∞-approximations of F, u_0, and $a(u)$, respectively (converging as $\delta \to 0$), and let $a^\delta \geq \delta > 0$, $u_0^\delta \geq 0$. Due to standard results for parabolic problems (cf. [21]) there exists a unique classical solution v^δ of

$$\frac{\partial v^\delta}{\partial t} = \mathrm{div}(a^\delta \nabla v^\delta - v^\delta \nabla F^\delta).$$

Now assume that there exists $x \in \Omega$, $t > 0$ such that $v^\delta(x,t) = 0$ and $v^\delta \geq 0$ in $\Omega \times [0,t)$. Then $v^\delta(.,t)$ attains a minimum at x and hence, $\nabla v^\delta(x,t) = 0$ and the Hessian of v^δ at (x,t) is positive semidefinite. Thus,

$$\frac{\partial v^\delta}{\partial t} = a^\delta \Delta v^\delta \geq 0.$$

Hence, $v^\delta(x,.)$ remains positive in $(t, t+\epsilon)$ for some $\epsilon > 0$. As a consequence, $v^\delta \geq 0$ in Ω_T.

Now let $w^\delta(x,t) = e^{-C_{F^\delta} t} v^\delta t$, then w^δ is a classical solution of

$$\frac{\partial w^\delta}{\partial t} + C_{F^\delta} w^\delta = \mathrm{div}(a^\delta \nabla w^\delta - w^\delta \nabla F^\delta).$$

Since the zero-order coefficient $(C_{F^\delta} + \Delta F^\delta) w^\delta$ is nonnegative and $w^\delta(x) \to 0$ for $|x| \to 0$, a standard maximum principle implies that w^δ attains its maximum at $t = 0$. Thus,

$$v^\delta(x,t) \leq e^{C_{F^\delta} t} \|u_0^\delta\|_{L^\infty(\Omega)}.$$

The estimate (3.7) is obtained in a standard way in the limit $\delta \to 0$.

A quantitative estimate for the bounded variation seminorm can be obtained again from the continuous dependence estimate. We sketch the derivation in the case of $\Omega = \mathbb{R}^d$, but the same result can be obtained (with additional technicalities) on bounded domains. For this sake let $h > 0$ and take $u^1 = u$ and $u^2 = u(.+h,.)$. Then u^2 solves the equation with flux $F^2 = F(.+h,.)$ and hence, (3.6) implies

$$\begin{aligned} h^{-1}\|u(.,t) - u(.+h,t)\|_{L^1(\Omega)} \leq\ & h^{-1}\|u_0 - u_0(.+h)\|_{L^1(\Omega)} + \\ & th^{-1}C_0(u)\|\nabla F - \nabla F(.+h)\|_{L^\infty(0,t;L^\infty(\Omega))} + \\ & th^{-1}C_\infty(u)|\nabla F - \nabla F(.+h)|_{L^\infty(0,t;BV(\Omega))} \end{aligned}$$

As $h \downarrow 0$, we obtain

$$\begin{aligned} |u|_{L^\infty(0,t;BV(\Omega))} \leq\ & |u_0|_{BV(\Omega)} + t|u|_{L^\infty(0,t;BV(\Omega))}\|F\|_{L^\infty(0,t;W^{2,\infty}(\Omega))} \\ & + t\|u\|_{L^\infty(0,t;L^\infty(\Omega))}\|F\|_{L^\infty(0,t;W^{3,1}(\Omega))}. \end{aligned}$$

Hence, for T such that

$$2T\|F\|_{L^\infty(0,t;W^{2,\infty}(\Omega))} < 1 \tag{3.8}$$

we obtain

$$|u(.,t)|_{L^\infty(0,t;BV(\Omega))} \leq 2|u_0|_{BV(\Omega)} + 2te^{C_F T}\|u_0\|_{L^\infty(\Omega)}\|F\|_{L^\infty(0,t;W^{3,1}(\Omega))}. \tag{3.9}$$

3.2 Uniqueness

We now start with the analysis of the nonlocal problem and derive one of the major results of this paper, namely global uniqueness of entropy solutions. Note again that the possibility of obtaining uniqueness is a major motivation for considering entropy instead of weak solutions. The main tool for the proof is the correspondence of solutions of (2.1) and (3.1) with flux $F = G * u$, combined with the above continuous dependence estimate for (3.1).

Theorem 1 (Uniqueness) *Let $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$ be nonnegative and let G satisfy* (6.1)*. Then there is at most one entropy solution of* (2.1), (2.2) *in the class $L^\infty(0,T;BV(\Omega)) \cap L^\infty(\Omega_T)$.*

Proof Assume that u^1 and u^2 are two different entropy solutions and let $S \in [0,T)$ be the maximal time such that $u^1 = u^2$ in $\Omega \times [0,S)$. Now use $u_0^i = u^i(.,S)$ as the new initial values and shift the time variable to $t - S$. Then the u^i, $i = 1,2$ are entropy solutions of (3.5) in the smaller time interval $(0, T-S)$, with fluxes $F^i(.,t-S) := G * u^i(.,t)$.

$$\begin{aligned} &\|u^1(.,t) - u^2(.,t)\|_{L^1(\Omega)} \\ &\quad \leq (t-S)C_0\|\nabla F^1 - \nabla F^2\|_{L^\infty(0,t-S;L^\infty(\Omega))} + \\ &\qquad (t-S)C_\infty|\nabla F^1 - \nabla F^2|_{L^\infty(0,t-S;BV(\Omega))} \\ &\quad \leq (t-S)\left[C_0\|G\|_{W^{1,\infty}(\Omega)} + C_\infty\|G\|_{H^2(\Omega)}\right]\|u^1 - u^2\|_{L^\infty(S,t;L^1(\Omega))}, \end{aligned}$$

where we have used standard Fourier theorems for convolutions (see the Appendix) to obtain the second estimate. Hence, we conclude an estimate of the form

$$\|u^1 - u^2\|_{L^\infty(S,\tau;L^1(\Omega))} \leq (\tau - S)C\|u^1 - u^2\|_{L^\infty(S,\tau;L^1(\Omega))}$$

and choosing $\tau - S$ sufficiently small implies $u^1 = u^2$ in $\Omega \times (0, \tau)$, which contradicts the choice of S as being maximal in $(0, T)$. Hence, $S = T$ and therefore the entropy solution is unique.

3.3 Existence

In order to obtain local existence of solutions to (2.1), (2.2) we apply a fixed point-argument, assuming that T is so small that (3.8) is satisfied. The fixed point problem is given by

$$u = \mathcal{F}(u), \tag{3.10}$$

with $\mathcal{F} : L^\infty(0, T; L^1(\Omega)) \to L^\infty(0, T; L^1(\Omega))$ being the concatenation $\mathcal{F} = \mathcal{H} \circ \mathcal{G}$ of the convolution operator

$$\begin{aligned} \mathcal{G} : L^\infty(0, T; L^1(\Omega)) &\to \mathcal{V} \\ u &\mapsto F = G * u \end{aligned} \tag{3.11}$$

and the parameter-to-solution map

$$\begin{aligned} \mathcal{H} : \mathcal{V} &\to L^\infty(0, T; L^1(\Omega)) \\ F &\mapsto u \text{ being entropy solution of (3.1).} \end{aligned} \tag{3.12}$$

The function space $\mathcal{V}$ used for the flux function is given by

$$\mathcal{V} := \{\ F \in L^\infty(\Omega_T) \mid F \text{ satisfies (3.2) }\},$$

with the corresponding norm being the maximum of those used for the function spaces in (3.2).

Theorem 2 (Local Existence) *Let $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$ be nonnegative and let G satisfy* (6.1)*. Then, there exist $T > 0$ and a nonnegative function $u \in L^\infty(0, T; BV(\Omega)) \cap L^\infty(\Omega_T)$ being an entropy solution of* (2.1)*,* (2.2)*.*

Proof From standard results for convolution operators (see appendix) and Lemma 1 one observes that $\mathcal{G}$ and $\mathcal{H}$ are continuous operators. Now let $F^i = \mathcal{G}(v^i)$ and $u^i = \mathcal{F}(v^i)$. Then, as in the proof of Theorem 1, we can use (3.6) to deduce

$$\begin{aligned} &\|u^1(.,t) - u^2(.,t)\|_{L^1(\Omega)} \leq \\ &\qquad t \left[C_0(T)\|G\|_{W^{1,\infty}(\Omega)} + C_\infty(T)\|G\|_{H^2(\Omega)}\right] \|v^1 - v^2\|_{L^\infty(0,t;L^1(\Omega))}, \end{aligned}$$

and thus,

$$\begin{aligned} &\|\mathcal{F}(v^1) - \mathcal{F}(v^2)\|_{L^\infty(0,T;L^1(\Omega))} \leq \\ &\qquad T \left[C_0(T)\|G\|_{W^{1,\infty}(\Omega)} + C_\infty(T)\|G\|_{H^2(\Omega)}\right] \|v^1 - v^2\|_{L^\infty(0,T;L^1(\Omega))}. \end{aligned}$$

Due to Lemma 2, we can further choose $C_\infty(T) = e^{C_F(T)T}\|u_0\|_{L^\infty(\Omega)}$,

$$C_F(T) = \|\Delta G\|_{L^\infty(\Omega)} \min\{\|v^i\|_{L^\infty(0,T;L^1(\Omega))}\}.$$

Moreover, from (3.4) we obtain

$$\|v^i\|_{L^\infty(0,T;L^1(\Omega))} \leq \|u_0\|_{L^1(\Omega)}$$

and hence,

$$C_F(T) \leq \|\Delta G\|_{L^\infty(\Omega)} \|u_0\|_{L^1(\Omega)}.$$

Moreover, we have

$$\begin{aligned} C_0(T) &= |u|_{L^\infty(0,T;BV(\Omega))} \\ &\leq 2|u_0|_{BV(\Omega)} + 2Te^{C_F T}\|u_0\|_{L^\infty(\Omega)}\|G * v^i\|_{L^\infty(0,T;W^{3,1}(\Omega))} \\ &\leq 2|u_0|_{BV(\Omega)} + 2Te^{C_F T}\|u_0\|_{L^\infty(\Omega)}\|G\|_{W^{3,2}(\Omega)}\|u_0\|_{L^1(\Omega)} \end{aligned}$$

Hence, there exists $T > 0$ sufficiently small (and dependent only on G, $\|u_0\|_{L^\infty(\Omega)}$, and $\|u_0\|_{BV(\Omega)}$) such that

$$\gamma(T) := T\left[C_0(T)\|G\|_{W^{1,\infty}(\Omega)} + C_\infty(T)\|G\|_{H^2(\Omega)}\right] < 1.$$

Since $\gamma(T)$ is the Lipschitz-constant of the operator $\mathcal{F}$, the Banach fixed point theorem (cf. [6]) implies the existence of a fixed point or, equivalently, of an entropy solution of (2.1), (2.2).

We finally consider the behaviour globally in time.

Theorem 3 *Let $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$ be nonnegative and let G satisfy* (6.1). *Then, there exist $T > 0$ and a nonnegative function $u \in L^\infty(\Omega_T) \cap L^\infty(0,T;BV(\Omega))$ being an entropy solution of* (2.1), (2.2), *such that*

1. $T = +\infty$, *or*
2. $|u(.,t)|_{BV(\Omega)} \to \infty$ *as* $t \to T$.

Proof Suppose that $(0,T)$ is the maximal existence interval of an entropy solution. If T is finite, and both $\|u\|_{L^\infty(0,T;L^\infty(\Omega))}$ and $\|u\|_{L^\infty(\Omega)}$ are bounded, then we can find δ arbitrarily small such that $u(.,t-\delta) \in L^\infty(\Omega) \times BV(\Omega)$. For $\delta \geq 0$ and $\tau > 0$ sufficiently small we can apply the same fixed point procedure as in Theorem 2 to construct an entropy solution in $(T-\delta, T+\tau)$, which contradicts the maximality of T. Thus, T is infinite or the solution blows up in $BV(\Omega)$ or $L^\infty(\Omega)$. The supremum norm grows at most exponentially as an estimate analogous to the estimate of $C_\infty(T)$ in the proof of Theorem 2 shows, and hence, there is no blow-up in $L^\infty(\Omega)$. Since the L^1-norm is uniformly bounded, blow-up can only happen in the seminorm of $BV(\Omega)$.

We finally mention again that our proofs are valid for models of pure aggregation ($a \equiv 0$), too. In this case one expects that the density aggregates to one or multiple Dirac δ distributions (as confirmed by computations below), so that blow-up happens in L^∞, but not in finite time.

4 Further Properties of Solutions

In the previous section we have discussed the existence and uniqueness of solutions, as a side product we have obtained their nonnegativity. Besides nonnegativity of the density, there are various other properties that a physically relevant solution should satisfy such as conservation laws and energy dissipation relations. We shall investigate such properties in the following, together with the scaling properties of the model.

4.1 Conserved Quantities

If the model is considered on the whole space, then a physically relevant solution should conserve the total mass of particles (since none of them can be lost), i.e.,

$$\int_{\mathbb{R}^d} u(.,t)dx = \int_{\mathbb{R}^d} u_0 dx, \qquad \forall\, t \in [0,T]. \tag{4.1}$$

This conservation is quite natural for diffusion equations and can be derived from the definition of the entropy solutions with appropriate test functions. For the sake of simplicity we only provide a formal computation in the following. Using (2.1) we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} u(.,t)dx &= \int_{\mathbb{R}^d} \frac{\partial u}{\partial t}(.,t)dx \\ &= \int_{\mathbb{R}^d} \operatorname{div}(\nabla a(u) - u(\nabla G * u))dx \end{aligned}$$

and since both u and $\nabla a(u)$ tend to zero as $|x| \to \infty$, the last integral evaluates to zero. Hence we obtain (4.1) by integrating with respect to time.

Another interesting property of particle models is conservation of the center of mass. In the case of a continuum model, this property is reflected by conservation of the first moment, i.e.,

$$\int_{\mathbb{R}^d} u(.,t)x dx = \int_{\mathbb{R}^d} u_0 x dx, \qquad \forall\, t \in [0,T], \tag{4.2}$$

if $\int_{\mathbb{R}^d} |u_0||x|dx$ is finite, In general, such a property is not true for nonzero fluxes, but it holds for the aggregation model if we make the reasonable assumption that the kernel G is symmetric, i.e., $G(x) = G(-x)$ for all $x \in \mathbb{R}^d$. An analogous computation as above yields

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} u(.,t)x dx &= \int_{\mathbb{R}^d} \frac{\partial u}{\partial t}(.,t)x dx \\ &= \int_{\mathbb{R}^d} \operatorname{div}(\nabla a(u) - u(\nabla G * u))x dx \\ &= -\int_{\mathbb{R}^d} [\operatorname{div}(\mathbf{I}a(u)) - u(\nabla G * u)]dx \end{aligned}$$

with $\mathbf{I}$ being the $d \times d$ identity matrix, and we obtain $\int_{\mathbb{R}^d} \mathrm{div}(\mathbf{I}a(u))dx = 0$ since $a(u) \to 0$ as $|x| \to \infty$. For the second term we use the symmetry of G (and the resulting anti-symmetry of ∇G) together with Fubini's Theorem to deduce

$$\begin{aligned}\int_{\mathbb{R}^d} u(\nabla G * u)dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(x)u(y)\nabla G(x-y)\ dy\ dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(x)u(y)\nabla G(x-y)\ dy\ dx \\ &\quad -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(y)u(x)\nabla G(y-x)\ dx\ dy \\ &= 0.\end{aligned}$$

4.2 Energy Dissipation

As mentioned in the introduction, the model (2.1) has a gradient flow structure for an energy functional given by

$$E[u] = \int_{\mathbb{R}^d} [A(u) - \frac{1}{2}u(G * u)]\ dx, \tag{4.3}$$

where A is determined from $A''(p) = a'(p)/p$, i.e.,

$$A(p) = \frac{1}{2}p^2 + \sigma_\infty p \log p$$

in the diffusion case. Equation (2.1) in terms of A reads

$$\frac{\partial u}{\partial t} = \mathrm{div}(u\nabla(A'(u) - G * u)).$$

Again we assume that G is a symmetric kernel in the following.

The time variation of the energy functional is determined by

$$\begin{aligned}\frac{d}{dt}E[u(.,t)] &= \int_{\mathbb{R}^d} [A'(u) - (G * u)]\frac{\partial u}{\partial t}\ dx \\ &= \int_{\mathbb{R}^d} [A'(u) - (G * u)]\ \mathrm{div}(u\nabla(A'(u) - G * u))\ dx \\ &= -\int_{\mathbb{R}^d} u|\nabla[A'(u) - (G * u)]|^2\ dx,\end{aligned}$$

where we have inserted the rewritten equation and applied Gauss Theorem. Since the term on the right hand-side is negative, $\frac{d}{dt}E[u(.,t)] \le 0$, i.e.,

$$E[u(.,t)] \le E[u_0] \qquad \forall\ t \in [0,T] \tag{4.4}$$

if $E[u_0]$ is finite.

4.3 Scaling Properties

In order to obtain further insight into the properties of the model, we perform a nondimensional analysis, for the nonlinear diffusion $a(p) = \frac{\lambda p^2}{2} + \sigma_\infty p$. We introduce new variables

$$\tilde{\rho} = \frac{\rho}{\rho_0}, \quad \tilde{x} = \frac{x}{\ell}, \qquad \tilde{t} = \frac{t}{\tau}, \qquad \tilde{G} = \frac{G}{g_0},$$

with a typical density ρ_0, a typical length ℓ, a typical time τ, and a typical scale g_0 for the kernel G. It is reasonable to choose g_0 such that $\int \tilde{G}^2(x)d\tilde{x} = 1$, and hence the rescaled convolution operator has norm one on $L^1(\mathbb{R}^d)$. The rescaled problem in non-dimensional variables reads

$$\frac{\partial\tilde{\rho}}{\partial\tilde{t}} = \frac{\rho_0\tau\lambda}{\ell^2}\mathrm{div}(\tilde{\rho}\nabla(\tilde{\rho} - \frac{g_0\ell^d}{\lambda}\tilde{G} * \tilde{\rho}) + \frac{\sigma_\infty\tau}{\ell^2}\Delta\tilde{\rho} \tag{4.5}$$

For the sake of simplicity we drop the superscript tilde in the following.

An appropriate scaling for the spatial aggregation is $\ell = (\lambda/g_0)^{1/d}$, and the diffusion time scale is given by $\tau = \frac{\ell^2}{\rho_0\lambda}$. With these settings, the rescaled model becomes

$$\frac{\partial\rho}{\partial t} = \mathrm{div}(\rho\nabla(\rho - G * \rho)) + \epsilon\Delta\rho, \tag{4.6}$$

with $\epsilon = \frac{\sigma_\infty}{\rho_0\lambda}$. For $\sigma_\infty = 0$, we even obtain the parameter-free model

$$\frac{\partial\rho}{\partial t} = \mathrm{div}(\rho\nabla(\rho - G * \rho)). \tag{4.7}$$

Since we still have some freedom to choose ρ_0, it suffices to consider (4.6) with unit mass, i.e., $\int_{\mathbb{R}^d}\rho dx = 1$. Under this scaling, it becomes rather obvious that the effect of the linear diffusion is stronger for smaller initial densities, since ϵ scales with $1/\rho_0$. Moreover, one observes that the effect of the aggregation kernel, since ℓ scales with $(\lambda/g_0)^{1/d}$ we obtain smaller spatial scales (i.e., stronger aggregation) for larger kernels (compared to the diffusion size), and since $\tau \sim (\lambda/g_0)^{2/d}$ the aggregation happens at a fast time scale.

5 Numerical Simulation

In the following we present some numerical simulations of the model (1.17), in spatial dimension one, more precisely $\Omega = \mathbb{R}^1$. In this case we can use a Lagrangian scheme avoiding the approximation by a finite domain. Let m be the mass $m = \int_{\mathbb{R}}\rho dx$ and let $F : \mathbb{R} \times [0, T] \to [0, m]$ be the distribution function satisfying

$$F_x(x,t) = \rho(x,t) \quad \text{a.e,} \qquad \lim_{x\to-\infty} F(x,t) = 0 \quad \forall\ t.$$

The pseudo-inverse $w : [0, m] \times [0, T] \to \mathbb{R}$ is defined via

$$w(\xi, t) = \sup\{\ x \in \mathbb{R} \mid F(x,t) \le \xi\ \}.$$

By analogous reasoning as in [13, 23] we can derive the equation

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial \xi} a \left(\left(\frac{\partial w}{\partial \xi} \right)^{-1} \right) + \int_0^m G'(w - w(\eta, .)) d\eta \tag{5.1}$$

to be satisfied by the pseudo-inverse u, with the nonlinearity $a(p) = \frac{p^2}{2}$. By linear rescaling time and the kernel G we can actually obtain (5.1) with the nonlinearity Note that this equation corresponds to the Eulerian description of the system, roughly speaking the pseudo-inverse describes the location of particles (which would be exact for piecewise constant u). In the construction of a finite-difference method we follow the approach [13] for a granular medium equation, which can be carried over in a one-to-one fashion by just changing the kernel and the nonlinear diffusion function A. If we use a grid $0 = \xi_0 < \xi_1 < \ldots < \xi_N = m$ and denote $w_k(t) = w(x_k, t)$, then a step of an explicit upwind finite difference method can be written as

$$w_k(t + \tau) = w(t) - \tau D_+ a \left(\frac{1}{D_- w_k(t)} \right) + \tau \sum_j w_j G'(w_k(t) - w_j(t)),$$

where D_+ and D_- denote the standard forward and backward difference quotients.

We mention that a particular advantage of this Lagrangian scheme is the possibility to represent steep densities and even Dirac delta distributions, in a numerically stable way. The pseudo inverse function corresponding to a Dirac delta distribution is just a constant function whose value equals the location of the Dirac delta. As we shall see in the following this allows even to treat the case of pure aggregation and zero diffusion in a robust way.

5.1 Pure Aggregation

We start with an example of pure aggregation, i.e., only the aggregation kernel G is included in the model, not the nonlinear diffusion term (respectively $a \equiv 0$ in (5.1)). The initial density is given by

$$\rho_0(x) = \begin{cases} 0.25 & \text{if } |x| \leq 2 \\ 0 & \text{else.} \end{cases}$$

For the numerical computation we use a regular grid with $N = 201$ and a time step $\tau = 0.01$

We show the results of two simulations with different aggregation kernels, in the first case with the Gaussian kernel

$$G(p) = \frac{1}{2} \exp(-2(p.^2)),$$

and in the second with the kernel

$$G(p) = \frac{1}{2\pi} \cos(\pi p), \qquad p \in [-0.5, 0.5]$$

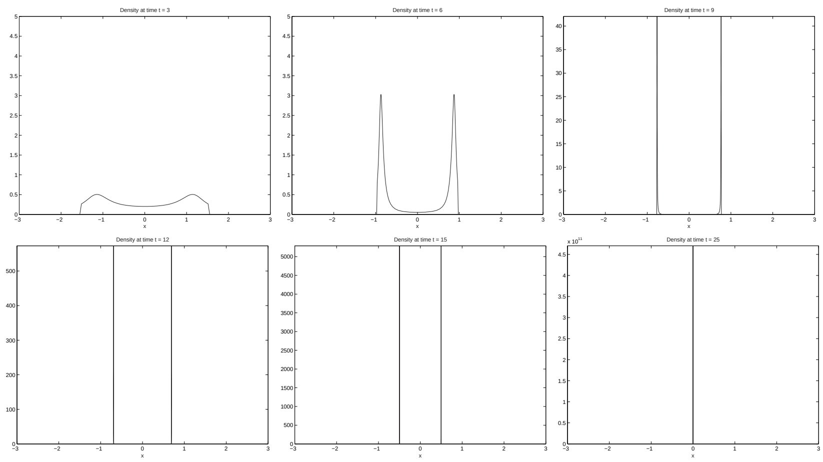

Fig. 1 Evolution of the density in the absence of diffusion and with a Gaussian aggregation kernel, at time steps $t = 3, 6, 9, 12, 15, 25$.

continued by zero for $|p| < 0.5$. For the first kernel (plots of the density at different times in Figure 1) one observes that two peaks form initially, but due to the global interaction with the Gaussian aggregation kernel there is still attraction and the peaks merge for larger time. In the large time limit one observes convergence of the pseudo inverse w to zero, i.e., convergence of the density ρ towards a Dirac delta distribution. For the cosine kernel with local support (plots of the density at different times in Figure 1) there are also two peaks forming after some time, but since there distance is larger than the size of the kernel support there is no further interaction. In this case there is also a smaller peak in the middle, which further aggregates in time. In the large time limit the density tends to the sum of three Dirac delta distributions located at points with distance greater than the size of the support of G, and hence they do not interact further.

5.2 Small Nonlinear Diffusion

In the following we consider the same setup as in the previous section, but now with $\lambda = 0.01$ in the nonlinear diffusion function a. In order to obtain stability of the explicit scheme in the diffusion case we choose the time step $\tau = \frac{1}{(N-1)^2}$.

In this case the dynamics is rather similar to case $\lambda = 0$ considered above, but the width of the peaks is larger due to the additional diffusion. The corresponding density for the Gaussian and the cosine kernel are plotted in Figure 3 and Figure 4 respectively. For the Gaussian kernel again two peaks form first, but they still interact and later merge to a single one, which becomes stationary. For the cosine kernel one observes aggregation of two peaks, and since the distance of their boundaries is larger than the size of the kernel support, there is no interaction for large time and the two peaks

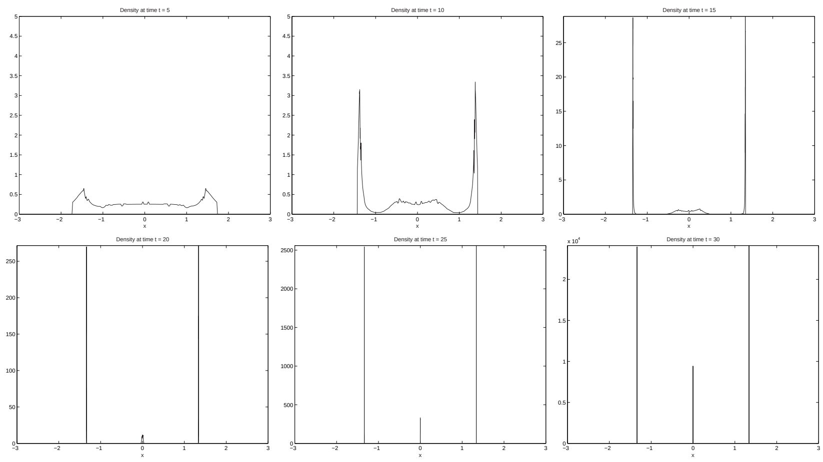

Fig. 2 Evolution of the density in the absence of diffusion and with a locally supported aggregation kernel, at time steps $t = 5, 10, 15, 20, 25, 30$.

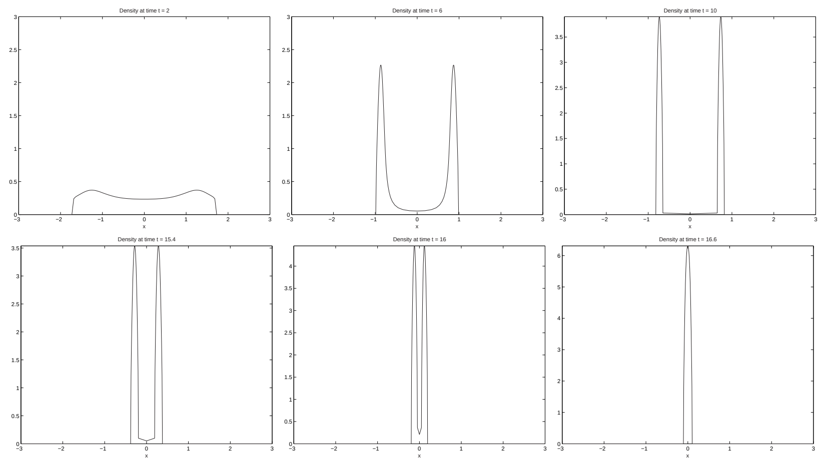

Fig. 3 Evolution of the density with small diffusion and Gaussian aggregation kernel, at time steps $t = 10, 30, 50, 77, 80, 83$.

become stationary. A third peak does not occur in this situation, note that due to the finite size of the peaks the third one would interact with the others in this case.

5.3 Large Nonlinear Diffusion

As a last example we consider a larger value for the diffusion, namely $\lambda = 1$, again with the same initial value and kernels as above. The time step is $\tau = \frac{5}{(N-1)^2}$.

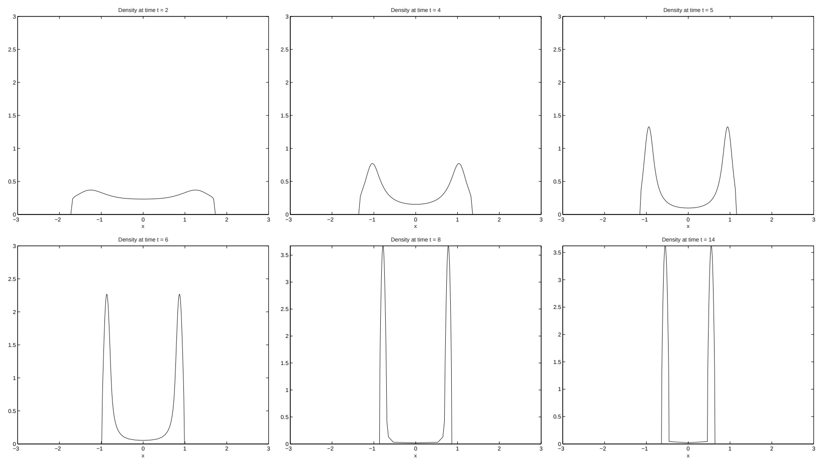

Fig. 4 Evolution of the density with small diffusion and a locally supported aggregation kernel, at time steps $t = 10, 20, 25, 30, 40, 70$.

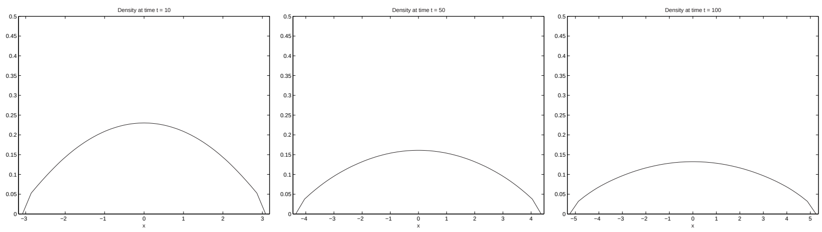

Fig. 5 Evolution of the density with small diffusion and Gaussian aggregation kernel, at time steps $t = 10, 50, 100$.

The evolution of the density is illustrated by plots at three different time steps in Figures 5 and 6 respectively. For both choices of the kernel, the effect of diffusion is stronger in this case, The density starts to spread out and decay very similar to the porous medium equation, which corresponds to the dominating part in the equation in this case. However, the (small) impact of the aggregation part is still observable from a comparison of the respective evolutions. The decay is faster in the case of the cosine kernel due to the small support and the lower magnitude compared to the Gaussian kernel.

6 Appendix: Notations and Assumptions

In the following we recall some basic notations and definitions of function spaces to be used in the subsequent analysis. Moreover, we give some basic assumptions on the aggregation kernel G and the initial value u_0 in (2.2), which will be used in the subsequent analysis without further notice.

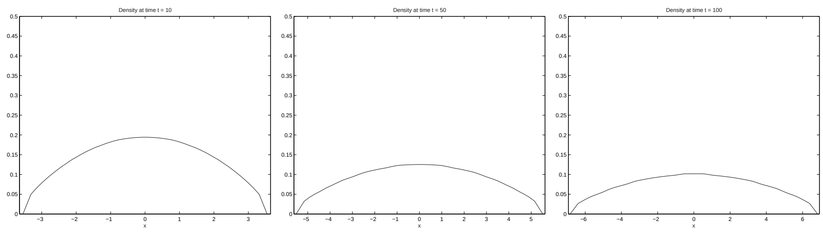

Fig. 6 Evolution of the density with small diffusion and a locally supported aggregation kernel, at time steps $t = 10, 50, 100$.

The Convolution Kernel G

In the above model for the aggregation kernel, it is assumed that G is a bounded function with finite support, which represents the fact that individuals interact only over some finite range. For our analysis, we can relax this assumption to

$$G \in C^1(\mathbb{R}^d) \cap W^{3,2}(\mathbb{R}^d) \cap W^{2,\infty}(\mathbb{R}^d) \tag{6.1}$$

which implies that $G \in W^{2,p}(\mathbb{R}^d)$ for any $p \in [2, \infty]$.

Note that due to $\frac{\partial^3 G}{\partial x_i x_j x_k} \in L^2(\mathbb{R}^d)$, the convolution $\frac{\partial^3 G}{\partial x_i x_j x_k} * u$ is well-defined as a function in $L^1(\mathbb{R}^d)$ due to Plancherel's Theorem and the corresponding convolution operator is continuous on $L^1(\Omega)$. Similarly, because of $\frac{\partial^2 G}{\partial x_i x_j} \in L^\infty(\mathbb{R}^d)$, the convolution $\frac{\partial^2 G}{\partial x_i x_j} * u$ is well-defined as a function in $L^1(\mathbb{R}^d)$ and the corresponding convolution operator is continuous on $L^1(\Omega)$, which can be seen from a straight-forward estimate. For a comprehensive treatment of convolution operators in L^p-spaces we refer to Champerey [5].

Function Spaces

For an open set $D \subset \mathbb{R}^N$, we denote by $C(D)$ the space of continuous functions on D and by $C^k(D)$ the space of k-times continuously differentiable functions equipped with the usual supremum-norms. Moreover, we will use the Lebesgue spaces $L^p(D)$, $1 \leq q \leq \infty$, with

$$\|u\|_{L^p(D)} = \begin{cases} \left(\int_D |u(x)|^p \, dx\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \text{ess sup}_{x \in D} |u(x)| & \text{if } p = \infty \end{cases} \tag{6.2}$$

and the Sobolev spaces $W^{k,p}(D)$, $1 \leq p \leq \infty$, $0 \leq k$ of functions with distributional derivatives up to order k in $L^p(D)$. The Sobolev space norms are defined by

$$\|u\|_{W^{k,p}(D)} = \left(\|u\|^p_{L^p(D)} + \sum_{1 \leq |\alpha| \leq k} \|\partial^\alpha u\|^p_{L^p(D)} \right)^{\frac{1}{p}} \tag{6.3}$$

for $1 \leq p < \infty$, and by

$$\|u\|_{W^{k,\infty}(D)} = \max \left\{ \|u\|_{L^\infty(D)}, \sup_{1\leq|\alpha|\leq k} \|\partial^\alpha u\|_{L^\infty(D)} \right\}. \tag{6.4}$$

Moreover, we will use the standard notations $H^k(D) = W^{k,2}(D)$ and $H_0^1(D)$ for the subspace of functions in $H^1(D)$ with vanishing trace on ∂D. For further details on the spaces $W^{k,p}(D)$ we refer to the monographs by Adams [1] and Evans [10]. The bi-dual space of $W^{1,1}(D)$ is the space of functions of bounded variation $BV(D)$, with norm

$$\|u\|_{BV(D)} = \|u\|_{L^1(D)} + |u|_{BV(D)},$$

where

$$|u|_{BV(D)} := \sup_{\mathbf{g}\in C_0^\infty(D;\mathbb{R}^d)} \int_D u \operatorname{div}\mathbf{g} \, dx.$$

is the total variation seminorm.

Finally, we need the vector valued function spaces on a real interval $I \subset \mathbb{R}$. For this sake, let $u : I \to X$ a function defined almost everywhere in I with values in some Banach space X. If u is continuous, then we say that $u \in C(I;X)$, and equip this space with the supremum norm

$$\|u\|_{C(I;X)} := \sup_{t\in I} \|u(t)\|_X.$$

In an analogous way we define the spaces $C^k(I;X)$, $L^p(I;X)$ and $W^{k,p}(I;X)$ and their norms, whose definition from the vector-valued case is obtained by changing the absolute values of $u(t)$ and its derivatives to the norm of $u(t)$ in the Banach space X. In some situations we will need a vector-valued total variation seminorm, denoted by

$$|u|_{L^\infty(I;BV(D))} := \sup_{t\in I} |u(t)|_{BV(D)}.$$

For a detailed discussion of vector valued function spaces we refer to [30].

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References

1. R.A.Adams, *Sobolev Spaces* (Academic Press, New York, London, 1975).
2. Boi, S., Capasso V., Morale, D. “Modeling the aggregative behavior of ants of the species *Polyergus rufescens.*”, Nonlinear Analysis: Real World Applications,**1**163-176 (2000).
3. M.Burger, M.Di Francesco, Y.Dolak-Struß, *Linear vs. nonlinear diffusion*, Preprint (RICAM Linz, 2005).
4. J.Carillo, *Entropy solutions for nonlinear degenerate problems*, Arch. Rat. Mech. Anal. **147** (1999), 269-361.
5. D.C.Champerey, *A Handbook of Fourier Theorems* (Cambridge Univ. Press, Cambridge, New York, 1987).
6. K.Deimling, *Nonlinear Functional Analysis* (Springer, Berlin, Heidelberg, 1985).
7. O.Diekmann, M.Gyllenberg, H.R.Thieme, *Lack of uniqueness in transport equations with a nonlocal nonlinearity*, Math. Models and Meth. in Appl. Sciences **10** (2000), 581-592
8. Y.Dolak, C.Schmeiser, *The Keller-Segel model with logistic sensitivity function and small diffusivity*, SIAM J. Appl. Math. (2005), to appear.
9. M.Escobedo, J.L.Vazquez, E.Zuazua, *Entropy solutions for diffusion-convection equations with partial diffusivity* , Trans. Amer. Math. Soc. **342** (1994), 829-842.
10. L.C.Evans, *Partial Differential Equations* (AMS, Providence, 1998).
11. S.Evje, K.H.Karlsen, N.H.Risebro, *A continuous dependence result for nonlinear degenerate parabolic equations with spatially dependent flux function*, in: H.Freistühler, G.Warnecke, eds., *Hyperbolic Problems: Theory, Numerics, Applications*, Int. Ser. Numer. Math. **140** (Birkhäuser, Basel, 2001), 337-346.
12. Forel A. *Les Fourmis de la Suisse* 2nd edition Imprimerie Coopèrative, La Chaux-de-Fonds(1920).
13. L.Gosse, G.Toscani, *Lagrangian numerical approximations to one-dimensional convolution-diffusion equations*, Preprint 2005-075, (HYKE-IHP 2005).
14. E.M. Gurtin, R.C. McCamy, *On the diffusion of biological populations*, Mathematical Biosciences, **33** (1977), 35-49.
15. D.Horstmann, *From 1970 until present: The Keller-Segel model in chemotaxis and its consequences, Part I*, Jahresbericht der DMV **105** (2003), 103-165.
16. D.Horstmann, *From 1970 until present: The Keller-Segel model in chemotaxis and its consequences, Part II*, Jahresbericht der DMV **106** (2004), 51-69.
17. K.H.Karlsen, N.H.Risebro, *On the uniqueness and stability of entropy solutions for nonlinear degenerate parabolic equations with rough coefficients*, Discrete Contin. Dyn. Syst **9** (2003), 1081-1104.
18. K.H.Karlsen, N.H.Risebro, *Convergence of finite difference schemes for viscous and inviscid conservation laws with rough coefficients*, M2AN Math. Model. Numer. Anal. **35** (2001), 239-269.
19. E.F.Keller, L.A.Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. **26** (1970), 399-415.
20. K.Kobayasi, *The equivalence of weak solutions and entropy solutions of nonlinear degenerate second-order equations*, J. Differ. Equations **189** (2003), 383-395.
21. O.A.Ladyzenskaja, V.A.Solonnikov, N.N.Uralceva, *Linear and Quasilinear Equations of Parabolic Type* (Translation of Mathematical Monographs, Volume 23, AMS, Providence, 1968).
22. G.LeFloch, *Hyperbolic systems of conservation laws* (Birkhäuser, Basel, 2002).
23. H.Li, G.Toscani, *Long-time asymptotics of kinetic models of granular flows*, Arch. Rati. Mech. Anal. **172** (2004), 407-428.
24. Mogilner A., L. Edelstein-Keshet, *A non-local model for a swarm*, J. Math. Bio. **38** (1999), 534-549.
25. D.Morale, *Cellular automata and many-particles systems modeling aggregation behaviour among populations*, Int. J. Appl. Math. & Comp. Sci.**10** (2000), 157-173.
26. D.Morale, V.Capasso, K.Ölschlaeger, *An interacting particle system modelling aggregation behavior:from individuals to populations*, J.Math. Bio.**50**, 49-66(2005).

27. T.Nagai, N.Mimura, *Asymptotic behavior for a nonlinear degenerate diffusion equation in population dynamics*, SIAM J. Appl. Math. **43** (1983), 449-464.
28. K.Ölschläger, *Large systems of interacting particles and porous medium equation*, J. Diff. Eq. **88**(1990),294-346.
29. A.Okubo, S.Levin, *Diffusion and Ecological Problems : Modern Perspectives* (Springer, Berlin, 2002).
30. R.E.Showalter, *Monotone Operators and Nonlinear Partial Differential Equations in Banach Space* (AMS Mathematical Surveys and Monographs, Providence, Rhode Island, 1997).
31. C.Topaz, A.L.Bertozzi, *Swarming patterns in a two-dimensional kinematic model for biological groups*, SIAM J. Appl. Math. **65** (2004), 152-174.