Two Approximations for Effective Hamiltonians Arising from Homogenization of Hamilton-Jacobi Equations

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Abstract

Effective Hamiltonian appears in homogenization of Hamilton-Jacobi equations, semi-classical limit of Schrödinger equation, Majda-Souganidis combustion model, Hamiltonian dynamics and many other applications. We propose two new numerical methods for computing effective Hamiltonians by solving Hamilton-Jacobi equations numerically: Small-δ Method and Large-T Method. Small-δ method is based on solving an approximate cell problem directly by using a monotone numerical scheme. Large-T method is based on solving a nonlinear eigenvalue problem, which in turn reduces to evolving a pseudo-time dependent Hamilton-Jacobi equation to a large time. Numerical examples show the accuracy and efficiency of the algorithms.

Key words: Hamilton-Jacobi; homogenization; effective Hamiltonian; cell problems; monotone schemes; Small-δ method; Large-T method

1 Introduction

Consider the following homogenization problem for the first-order nonlinear Hamilton-Jacobi (HJ) equation:

$$u_t^\epsilon + H(\nabla u^\epsilon, \frac{x}{\epsilon}, t) = 0 \qquad \text{in } \mathcal{R}^d \times (0, \infty) \tag{1}$$

$$u^\epsilon = g \qquad \text{in } \mathcal{R}^d \times \{t = 0\}. \tag{2}$$

where H: $\mathcal{R}^d \times \mathcal{R}^d \times \mathcal{R} \to \mathcal{R}$ is smooth and periodic in the second variable with the period $Y = [0, 1]^d$. Here $\epsilon > 0$ characterizes the high frequency oscillating

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perturbation, therefore homogenization studies the response of a differential equation to such high frequency perturbation.

For linear differential equations, homogenization has been well studied; see [3]. For fully nonlinear equations in consideration, the homogenized equation is not obviously defined. Lions, Papanicolaou and Varadhan [23] proved the first existence result of homogenized Hamiltonian (effective Hamiltonian) by defining a so-called cell problem; later, Evans [13] recounted their results for first-order nonlinear PDEs and made further development for second-order fully nonlinear PDEs by using "perturbed test function" method. The result roughly states that under suitable assumptions, for every $\epsilon > 0$ the above equation has a unique viscosity solution u^ϵ; moreover the sequence $\{u^\epsilon\}_{\epsilon>0}$ converges to the viscosity solution u of the averaged problem

$$u_t + \bar{H}(\nabla u, t) = 0 \qquad \text{in } \mathcal{R}^d \times (0,\infty) \tag{3}$$

$$u = g \qquad \text{in } \mathcal{R}^d \times \{t = 0\}, \tag{4}$$

where the effective (averaged) Hamiltonian is defined through the cell problem: for each $p \in \mathcal{R}^d$ and $t \in (0,\infty)$, there exists a unique $\bar{H}(p,t)$ such that

$$H(p + \nabla_y v(p,t;y), y, t) = \bar{H}(p,t) \tag{5}$$

has a Y-periodic viscosity solution $v(p,t;y)$.

To see where this cell problem comes from, we give a formal derivation here. Suppose that $u^\epsilon \to u_0$ as $\epsilon \to 0$ in the uniform topology and that u^ϵ has the asymptotic expansion

$$u^\epsilon(x,t) = u_0(x,t) + \epsilon u_1(\frac{x}{\epsilon}, t) + O(\epsilon^2) \tag{6}$$

where u_1 is the first order correction term to u_0. Then by matching orders of ϵ in equation (1), we formally have

$$\frac{\partial u_0}{\partial t}(\epsilon y, t) + H(\nabla_x u_0(\epsilon y, t) + \nabla_y u_1(y,t), y, t) = O(\epsilon) \tag{7}$$

where $x = \epsilon y$. Letting $\epsilon \to 0$, we deduce that u_1 satisfies

$$H(p + \nabla_y u_1(p,t;y), y, t) = \bar{H}(p,t) \tag{8}$$

with $p = \nabla_x u_0$ and $\bar{H}(p,t) = -\frac{\partial u_0}{\partial t}$.

Such cell problems not only arise from the homogenization of HJ equations but also appear in a variety of other applications. In [24], Majda and Souganidis developed simplified effective equations for the large scale front propagation of turbulent reaction-diffusion equations; this rigorous theory is based on viscosity solutions for PDEs and homogenization theory for HJ equations. The related cell problem is of quadratic nonlinearity in the gradient variable [24,21]. In the semi-classical limit of the Schrödinger equation and quantum states [15,14], a cell problem appears in the Bloch wave form expansion and WKB approximation for the eigenstate, where in the semi-classical limit the effective Hamiltonian corresponds to the energy state associated with the eigenstate under consideration. Because a Hamilton-Jacobi equation may originate from the Calculus of Variations, there exists a natural connection between homogenization of HJ equation and the homogenization of an integral function for which the integrand is an oscillatory function; this was studied by E [11] in terms of Γ-convergence theory; see also [19,12] for more on related Aubry-Mather theory and forced Burgers equation with periodic boundary conditions. Since a Hamilton-Jacobi equation can also be formulated as a control problem by using the dynamic programming approach to optimization and differential games, the homogenization of HJ equation arises naturally if the nonlinear control system has rapidly oscillating dynamics [4]. Recently, E [12], Evans and Gomes [15,16] observed that a cell problem formally induces a generating function so that a canonical change of variables may be carried out to simplify the study of the Hamiltonian dynamics; see [14,17] for more. In the study of long time behavior of Hamilton-Jacobi equations [2], the effective Hamiltonian is also relevant.

Given that cell problems arise in so many applications and effective Hamiltonians encode a lot of information of the original systems [15,14], it is important to understand the solution behavior of the cell problem. However, in general a cell problem does not have an explicit solution; therefore, it is essential to be able to solve cell problems numerically and compute effective Hamiltonians efficiently. Since for the Majda-Souganidis model [24] the related cell problem is of quadratic nonlinearity in the gradient variable, Khouider and Bourlioux [21] differentiated the cell problem with respect to the spatial variable to eliminate the effective Hamiltonian so as to obtain a system of conservation laws; the vector solution of this system is the gradient of the eigenfunction associated with the effective Hamiltonian; finally the effective Hamiltonian is recovered through an integration process. Assuming that the Hamiltonian in the cell problem is convex in the gradient variables, the effective Hamiltonian can be represented as an inf-sup formula in the context of L^∞-calculus of variational problem [7]. Therefore, Gomes and Oberman [18] proposed to discretize the inf-sup formula directly to approximate effective Hamiltonians without computing solutions of cell problems; however, the resulting minimax optimization problem is not easy to solve.

In this paper, we propose two new methods to solve cell problems directly without assuming specific forms of Hamiltonians or convexity of Hamiltonians in the gradient variables: namely, Small-δ method and Large-T method. The Small-δ method is based on discretizing an approximate cell problem directly by using a monotone numerical Hamiltonian ([25]) and solving the resulting nonlinear system with Newton methods. The Large-T method is based on an observation by Concordel [6] that the effective Hamiltonian and an associated viscosity solution of a cell problem is related to a nonlinear additive eigenvalue problem of a time dependent operator defined by a Hamilton-Jacobi equation; we solve this nonlinear eigenvalue problem by evolving the time-dependent Hamilton-Jacobi equation to a large pseudo time; therefore this approach only requires a numerical solver for Hamilton-Jacobi equations. Because they are based upon monotone numerical schemes for Hamilton-Jacobi equations, both methods are easy to implement.

In Section 2, we present main assumptions needed and summarize various cell problems. In Section 3, we give the theoretical results for validating the two new methods for effective Hamiltonians and detail the discretization of Small-δ method and Large-T method. Section 4 gives numerical examples to demonstrate the accuracy of the two new methods. We conclude the paper with some future problems to be explored.

2 Various cell problems

2.1 Stationary Hamilton-Jacobi Equations

To start with, let us first consider the homogenization for the following stationary HJ equation [13]

$$H(\nabla u^\epsilon, u^\epsilon, x, \frac{x}{\epsilon}) = 0 \text{ in } \Omega \subset \mathcal{R}^d \tag{9}$$

$$u^\epsilon = 0 \text{ in } \partial\Omega \subset \mathcal{R}^d, \tag{10}$$

where Ω denotes a bounded smooth subset of $\mathcal{R}^d$, and $H : \mathcal{R}^d \times \mathcal{R} \times \bar{\Omega} \times \mathcal{R}^d \to \mathcal{R}$ is a given smooth function.

The main hypotheses for equations (9) and (10) are

- (H1) $y \to H(p, u, x, y)$ is Y-periodic for all p, u, x;
- (H2) $H(p, u, x, y) \to \infty$ as $|p| \to \infty$ uniformly on $B(0, R) \times \Omega \times \mathcal{R}^d$ for each positive real R;

- (H3) H is Lipschitz continuous: for each positive real R there is a constant $C(R)$ such that for all $x, y, \xi, \eta \in \mathcal{R}^d$, $p, q \in B(0,R)$, $u, v \in B(0,R)$,

$$|H(p,u,x,\xi) - H(q,v,y,\eta)| \leq C(R)(|x-y| + |p-q| + |\xi-\eta| + |u-v|)$$

- (H4) $u \to H(p,u,x,y) - \mu u$ is nondecreasing for some $\mu > 0$ and all p, x, y.

Under the standing assumptions, for each $\epsilon > 0$ $u^\epsilon \in C(\Omega)$ is a viscosity solution of (9) and (10); moreover, Lions, Papanicolaou and Varadhan [23] and Evans [13] proved that $\{u^\epsilon\}_{\epsilon>0}$ converges to the unique viscosity solution u of the homogenized equation

$$\bar{H}(\nabla u, u, x) = 0 \text{ in } \Omega \subset \mathcal{R}^d \tag{11}$$

$$u = 0 \text{ in } \partial\Omega \subset \mathcal{R}^d. \tag{12}$$

where the effective Hamiltonian $\bar{H}$ is defined by the following cell problem:

Lemma 2.1 ([23,13]) *For each fixed $p \in \mathcal{R}^d$, $u \in \mathcal{R}$ and $x \in \Omega$, there exists a unique real number $\bar{H}(p,u,x)$ such that*

$$\textit{(CP1)} \quad H(\nabla_\xi v + p, u, x, \xi) = \bar{H}(p,u,x) \text{ in } \mathcal{R}^d \tag{13}$$

has a Y-periodic, Lipschitz viscosity solution $v(p,u,x;\cdot)$.

2.2 Time dependent HJ equation with Hamiltonian of no explicit time dependence

Consider periodic homogenization of

$$u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}) = 0 \qquad \text{in } \mathcal{R}^d \times (0,\infty) \tag{14}$$

$$u^\epsilon = g \qquad \text{in } \mathcal{R}^d \times \{t=0\}. \tag{15}$$

Assume that

- (H5) $H = H(p,y) : \mathcal{R}^d \times \mathcal{R}^d \to R$ is periodic in y with the period $Y = [0,1]^d$ for each $p \in \mathcal{R}^d$;
- (H6) $H(p,y) \to \infty$ as $|p| \to \infty$ uniformly for $y \in \mathcal{R}^d$;
- (H7) H is Lipschitz continuous on $B_d(0,R) \times \mathcal{R}^d$ for each positive real R and satisfies for some constant C,

$$|H(p,x) - H(p,y)| \leq C|x-y|(1+|p|)$$

for all x, y and p in $\mathcal{R}^d$.

Under these assumptions, if the initial condition g is bounded, uniformly continuous on $\mathcal{R}^d$ and $\nabla g \in L^\infty(\mathcal{R}^d)$, then for each $\epsilon > 0$ the above equation has a unique, bounded uniformly continuous viscosity solution u^ϵ ([9,8,27]). Moreover, the sequence $\{u^\epsilon\}_{\epsilon>0}$ converges as $\epsilon \to 0$ to the viscosity solution u of the averaged equation [23,6]:

$$u_t + \bar{H}(Du) = 0 \qquad \text{in } \mathcal{R}^d \times (0,\infty) \tag{16}$$

$$u = g \qquad \text{in } \mathcal{R}^d \times \{t = 0\}, \tag{17}$$

where the effective Hamiltonian $\bar{H}$ is defined by the following cell problem:

Lemma 2.2 ([6]) *For each $p \in \mathcal{R}^d$, there exists a unique real number $\bar{H}(p)$ such that the PDE*

$$\textit{(CP2)} \qquad H(p + Dv(y), y) = \bar{H}(p) \tag{18}$$

has a Y-periodic Lipschitz viscosity solution $v(p;\cdot)$.

The solution v is not unique as we can see from the structure of the Hamilton-Jacobi equation.

2.3 Time dependent HJ equation with Hamiltonians of slow variations in time

Consider the periodic homogenization of the Cauchy problem for the HJ PDE with rapidly oscillating coefficients:

$$u^\epsilon_t + H(\nabla u^\epsilon, \frac{x}{\epsilon}, t) = 0 \qquad \text{in } \mathcal{R}^d \times (0,\infty) \tag{19}$$

$$u^\epsilon = g \qquad \text{in } \mathcal{R}^d \times \{t = 0\}. \tag{20}$$

where H is Y-periodic in the second variable.

Under assumptions similar to (H1)-(H7), for each $\epsilon > 0$ there exists a unique bounded uniformly continuous viscosity solution u^ϵ. Letting $\epsilon \to 0$, Evans and Gomes [16] proved that $\{u^\epsilon\}_{\epsilon>0}$ converges to the unique viscosity solution u of

$$u_t + \bar{H}(\nabla u, t) = 0 \qquad \text{in } \mathcal{R}^d \times (0,\infty) \tag{21}$$

$$u = g \qquad \text{in } \mathcal{R}^d. \tag{22}$$

where $\bar{H}$ is defined through the following cell problem:

Lemma 2.3 ([16]) *For each fixed $p \in \mathcal{R}^d$ and t, there exists a unique real number $\bar{H}(p,t)$ such that*

$$\textit{(CP3)} \qquad H(\nabla_y v + p, y, t) = \bar{H}(p,t) \qquad \textit{in } \mathcal{R}^d \tag{23}$$

has a Y-periodic, Lipschitz viscosity solution $v(p,t;\cdot)$.

2.4 Time dependent HJ equation with Hamiltonians of fast variations in time

Consider the homogenization of

$$u_t^\epsilon + H(\nabla u^\epsilon, \frac{x}{\epsilon}, \frac{t}{\epsilon}) = 0 \qquad \text{in } \mathcal{R}^d \times (0,\infty) \tag{24}$$

$$u^\epsilon = g \qquad \text{in } \mathcal{R}^d \times \{t=0\}. \tag{25}$$

where $H = H(p,x,t)$ is Y-periodic in space variable x and [0,1]-periodic in time t.

In addition to the periodicity, assuming that $H(p,x,t)$ satisfies strict convexity in p and a gradient bound in x: there exists a constant $C > 0$ such that

$$|\nabla_x H(p,x,t)| \le C(1+|p|)$$

for all $p, x \in \mathcal{R}^d, t \in (0,\infty)$, Evans and Gomes [16] proved that $\{u^\epsilon\}_{\epsilon>0}$ converges to the unique viscosity solution u of the averaged equation

$$u_t + \hat{H}(\nabla u) = 0 \qquad \text{in } \mathcal{R}^d \times (0,\infty) \tag{26}$$

$$u = g \qquad \text{in } \mathcal{R}^d. \tag{27}$$

where $\hat{H}$ is once again defined through a cell problem:

Lemma 2.4 ([16]) *For each fixed $p \in \mathcal{R}^d$, there exists a unique real number $\hat{H}(p)$ such that*

$$\textit{(CP4)} \qquad w_t + H(\nabla_y w + p, y, t) = \hat{H}(p) \qquad \textit{in } \mathcal{R}^d \times \mathcal{R} \tag{28}$$

has a Lipschitz viscosity solution $w(p;y,t)$ which is Y-periodic in space variable y and [0,1]-periodic in time variable t.

Moreover, Evans and Gomes [16] showed that the solution of this cell problem can be constructed as follows: first for each fixed p solve the time dependent HJ equation

$$v_t + H(p + \nabla_\xi v, \xi, t) = 0 \qquad \text{in } \mathcal{R}^d \times (0, \infty) \tag{29}$$

$$v = g \qquad \text{in } \mathcal{R}^d; \tag{30}$$

secondly, define

$$\hat{H}(p) = -\min_Y v(\cdot, 1), \tag{31}$$

$$w(x,t) = v(x,t) + t\hat{H}(p); \tag{32}$$

then it is easy to see that $w(x,t)$ solves the cell problem (CP4).

To summarize, we have presented four classes of cell problems. Because u, x in (CP1) and t in (CP3) only play the role as fixed parameters, (CP1) and (CP3) have the similar structures as (CP2) does. Therefore, it is essential to solve (CP2) efficiently. As for (CP4), according to Evans-Gomes' construction, the problem is solved numerically provided that we have efficient numerical methods to solve equations (29) and (30) ([10,26,25]). Therefore in the rest of the paper we will concentrate our efforts on designing numerical methods for solving (CP2).

3 Small-δ method and Large-T method

3.1 Small-δ method

To solve the cell problem (CP2), namely,

$$\text{(CP2)} \qquad H(p + \nabla v, y) = \bar{H}(p), \tag{33}$$

we introduce for each $\delta > 0$ an approximate cell problem (ACP)

$$\text{(ACP)} \qquad \delta v^\delta(y) + H(p + \nabla v^\delta(y), y) = 0. \tag{34}$$

Due to the assumptions (H5), (H6) and (H7) on H, there exists a unique Lipschitz viscosity solution $v^\delta(p;\cdot)$ to (ACP) for each $\delta > 0$. According to [23,13], there exists a subsequence $\{v^{\delta_j} - \min v^{\delta_j}, \delta_j v^{\delta_j}\}_{j=1}^\infty$ so that

$$v^{\delta_j} - \min v^{\delta_j} \to v \qquad \text{uniformly in } \mathcal{R}^d \tag{35}$$

$$\delta_j v^{\delta_j} \to -\bar{H}(p) \qquad \text{uniformly in } \mathcal{R}^d. \tag{36}$$

Recently, Capuzzo-Dolcetta and Ishii [4] established that

Lemma 3.1 ([4]) *Under above assumptions (H5), (H6) and (H7), there exists a constant $C > 0$ such that*

$$|\delta v^{\delta}(p; y) + \bar{H}(p)| \leq \delta C(1 + |p|). \tag{37}$$

Therefore, if $\delta > 0$ is small enough, this suggests that δv^{δ} is a good approximation to $-\bar{H}(p)$ in the uniform topology if we can solve for v^{δ} in (ACP) in the viscosity sense. We name this "Small-δ method"

To discretize (ACP) by monotone schemes which guarantee that the numerical solution converges to the viscosity solutions, we may approximate the Hamiltonian H in (ACP) by applying various monotone numerical Hamiltonians formulated in [25] in finite difference setting, or the monotone Hamiltonians [1,22] in finite volume setting.

To illustrate the approach, we apply the classical Lax-Friedrichs scheme to the following model equation

$$\delta v^{\delta}(y) + H(\nabla v^{\delta}(y)) = f(y)$$

The scheme on the uniform Cartesian grid

$$G_h = \{(x_0 + (i-1)\Delta x, y_0 + (j-1)\Delta y)\}$$

reads as follows (omit the indices δ in v_{ij} to avoid cluttered notation):

$$\begin{aligned} &\delta v_{i,j} + H\left(\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x}, \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right) \\ &\quad - \omega_x \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{\Delta x^2} - \omega_y \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta y^2} = f_{i,j}, \end{aligned} \tag{38}$$

where

$$\omega_x = \sup_{(x,y)\in Y} \frac{1}{2} \left| H_1 \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right) \right| \Delta x,$$
$$\omega_y = \sup_{(x,y)\in Y} \frac{1}{2} \left| H_2 \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right) \right| \Delta y,$$

and $H_i(p_1, p_2) = \frac{\partial H}{\partial p_i}(p_1, p_2)$ for $i = 1, 2$.

Cockburn and Qian [5] proved that a monotone numerical scheme applied to the model equation

$$u + H(\nabla u) = f$$

has a unique solution which converges to the viscosity solution as the mesh size h goes to 0.

To solve the above nonlinear system for the solution, we note that the Lax-Friedrichs numerical Hamiltonian is smooth so that we might apply Newton method. The Jacobian matrix is a circulant matrix for which we might apply Sherman-Morrison inversion formula so that the Newton update step can be speeded up significantly.

If the Hamiltonian is convex in the gradient variable, it might be possible to design fast sweeping type schemes to solve (ACP) using Godunov Hamiltonians; see related work on fast sweeping schemes for Hamilton-Jacobi equations of eikonal type [29,28,20].

3.2 Large-T method

Concordel [6] established a relation between the solution $(v, \bar{H}(p))$ of the cell problem (CP2) and an additive eigenvalue problem for a nonlinear operator.

To this end, introduce the following evolution problem [6]:

$$u_t + H(p + Du, y) = 0 \qquad \text{in } \mathcal{R}^d \times (0, \infty) \tag{39}$$
$$u = g \qquad \text{in } \mathcal{R}^d \times \{t = 0\}, \tag{40}$$

where g is a continuous, Y-periodic function in $\mathcal{R}^d$.

Under the standing assumptions, equations (39) and (40) has a unique viscosity solution in $\mathcal{R}^d \times [0, T]$ for every $T > 0$ ([27]). Therefore, for each $t > 0$, let G_t be the solution operator of the above equation acting on the space

$$\mathcal{P} = \{g : \mathcal{R}^d \to R | g \text{ is continuous and Y-periodic}\}. \tag{41}$$

Then

$$(G_t g)(y) = u(y,t) \tag{42}$$

for each $g \in \mathcal{P}$, where u is the viscosity solution of equations (39), (40). Note that the operator G_t depends on p; since p is fixed, we suppress the display of p for the sake of clarity.

The following relation holds between G_t and $\bar{H}(p)$ [6]:

Lemma 3.2 *Fix $p \in \mathcal{R}^d$, and let $(v, \bar{H}(p))$ be a solution to the cell problem (CP2), namely,*

$$H(p + Dv, y) = \bar{H}(p). \tag{43}$$

Then

$$G_t v = v - \bar{H}(p)t \tag{44}$$

for each $t > 0$.

The above Lemma essentially says that $-\bar{H}(p)t$ is an additive eigenvalue for the operator G_t and a viscosity solution of the cell problem is an associated additive eigenfunction.

The above Lemma might also be derived from the results in [2], where they studied the large time behavior of solutions of Hamilton-Jacobi equations.

Using L_∞-contraction property of the solution operator G_t [9,8,27], we may obtain the following:

Lemma 3.3 *The effective Hamiltonian may be approximated by*

$$\bar{H}(p) = -\lim_{t \to \infty} \frac{G_t g}{t}$$

in the uniform norm for any g.

Proof: Let $(v, \bar{H}(p))$ be a solution to the cell problem. Then by Lemma 3.2 we have for each $t > 0$,

$$\bar{H}(p) = \frac{v - G_t v}{t}.$$

Therefore, for any $g \in P$, we have

$$\begin{aligned}\|\bar{H}(p) + \frac{G_t g}{t}\|_\infty &= \|\frac{G_t g}{t} + \frac{v - G_t v}{t}\|_\infty \\ &\leq \frac{1}{t}(\|G_t g - G_t v\|_\infty + \|v\|_\infty) \\ &\leq \frac{1}{t}(\|g - v\|_\infty + \|v\|_\infty).\end{aligned}$$

Since g and v are continuous, periodic functions on $\mathcal{R}^d$ and independent of t, we may let $t \to \infty$ and obtain

$$\bar{H}(p) = -\lim_{t\to\infty}\left(\frac{G_t g}{t}\right). \tag{45}$$

Concordel [6] obtained the above result by assuming that the Hamiltonian is convex in the gradient variable; here we obtained the lemma without convexity of the Hamiltonian in the gradient variable.

The above lemma provides a foundation for computing the effective Hamiltonian by solving a Hamilton-Jacobi equation to a sufficiently large time T_f. Therefore, we name it "Large-T method".

To solve the evolution problem (39), (40), we might use various popular monotone Hamilton-Jacobi solvers proposed in [10,25,1,22].

4 Numerical Experiments: 1-D case

4.1 Convex Hamiltonian: Eikonal case

We first consider the simplest case, i.e. the eikonal equation with

$$H(p, y) = \frac{1}{2}|p|^2 - V(y) \tag{46}$$

where V is a Y-periodic potential. Here the Hamiltonian is convex in the p variable.

Lions, Papanicolaou and Varadhan have obtained the following formula for the effective Hamiltonian in the 1-D case [23]:

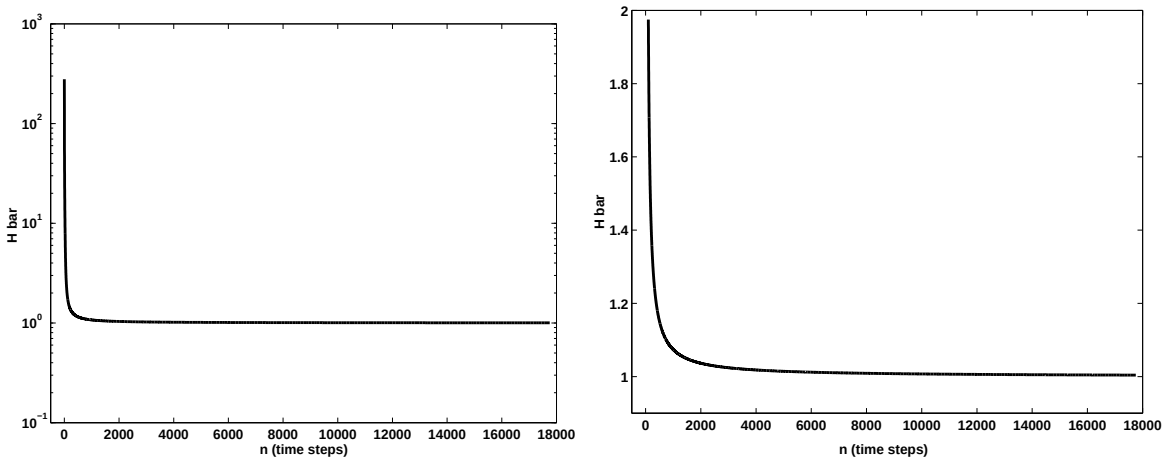

Fig. 1. Large-T Method: Convergence behavior of the algorithm on the flat part

$$\bar{H}(p) = \begin{cases} -\min V \text{ if } |p| \le p_c, \\ \lambda \qquad \text{s.t. } |p| = \int_0^1 \sqrt{2(V(y)+\lambda)}dy \text{ if } |p| > p_c \end{cases}$$

where $p_c = \int_0^1 \sqrt{2(V(y) - \min V)}dy$. We will use this formula to justify our computational result.

As an initial test, the potential function is taken as

$$V(y) = \sin 2\pi y. \tag{47}$$

Then it is easy to see that $\min V = -1$ and $p_c = \frac{4}{\pi}$.

To solve the cell problem, we apply both the Small-δ method and the Large-T method. In the case of Small-δ Method, we use the 1-D version of the Lax-Friedrichs scheme presented in equation (38). In the case of the Large-T Method, we use 1st-order Lax-Friedrichs (LxF), Local Lax-Friedrichs (LLxF) and Godunov (God) schemes [25], respectively, to solve equations (39),(40) until a large time $T = T_f$.

First let us see the convergence behavior of the algorithm for a particular p; here p =5 which is located in the non-flat region and the large time T_f=10.0 in the following computations. Figure 1(a) illustrates the convergence of the effective Hamiltonian as $t \to \infty$, and the convergence seems pretty fast. Figure 1(b) shows a zoom-in region of Figure 1(a), where we have used the upper and lower bound to measure the convergence to the limit in the uniform norm. Here the effective Hamiltonian $\bar{H}(5) = 12.5198$, and the corresponding p =5.0019 by computing the integral $\int_0^1 \sqrt{2(V(x)+\lambda)}dx$ which is a very good prediction of p =5.

Next we compute the effective Hamiltonian in an interval by the two methods.

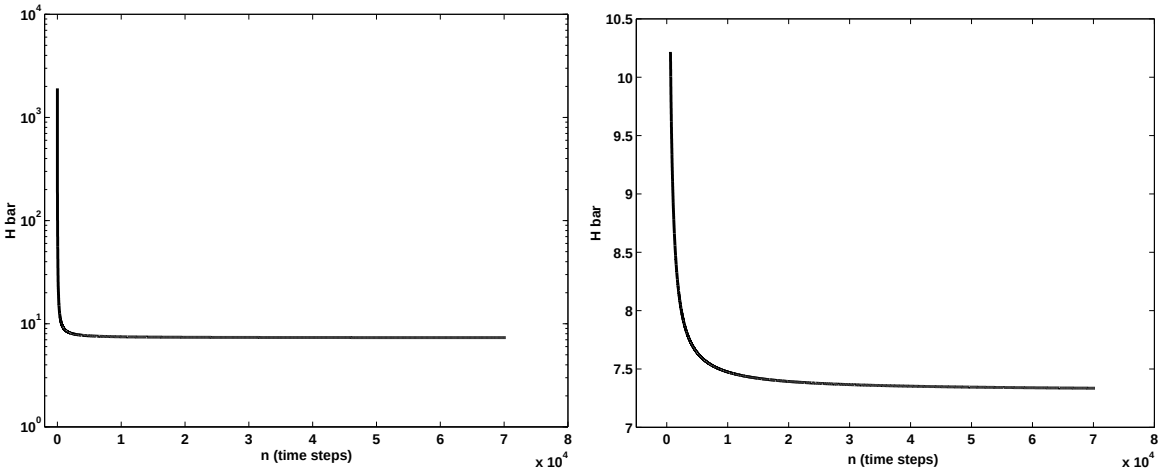

Fig. 2. Large-T Method: Convergence behavior of the algorithm on the non-flat part

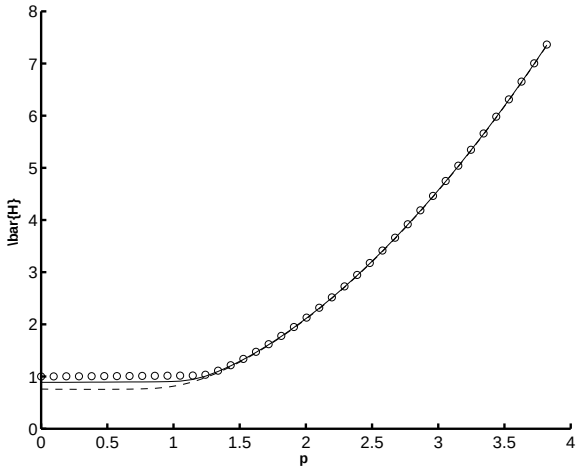

Fig. 3. Large-T Method: computed effective Hamiltonians by the LxF scheme are flat near the origin. $--$: the spatial mesh size of 21; $-$: the spatial mesh size of 41; o: the spatial mesh size of 161.

The simulation will be performed on the interval $p \in [0, \frac{12}{\pi}]$ so that the critical point $p_c = \frac{4}{\pi}$ will be on the computational mesh. Figure 3 shows the effective Hamiltonians with sampling size of 41 in p by the Large-T method with 1st order LxF as the Hamilton-Jacobi solver. In the figure, there are three curves corresponding to different mesh sizes in spatial variable x for the Hamilton-Jacobi solver; the final time is taken as T_f =1400. As we can see, the smaller the mesh size is, the more accurate the effective Hamiltonian. As predicted by the explicit formula, there is a flat part around the origin.

However, the results in Figure 3 seem to be dissipative too much due to the numerical diffusion introduced by the Lax-Friedrichs scheme. Therefore, we turn to the 1st-order Godunov scheme. Figure 4 shows the computed effective Hamiltonian in the same setup as in the Figure 3, but the numerical Hamiltonian is changed into a Godunov Hamiltonian in the Large-T method. The

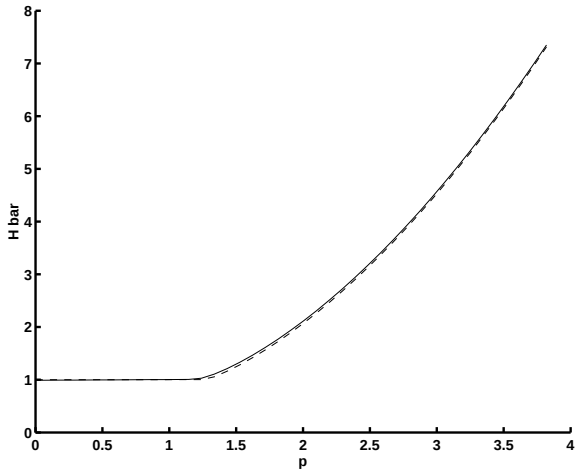

Fig. 4. Large-T Method: computed effective Hamiltonians by the 1st-order Godunov scheme are flat near the origin. $--$: the spatial mesh size of 11; $-$: the spatial mesh size of 641.

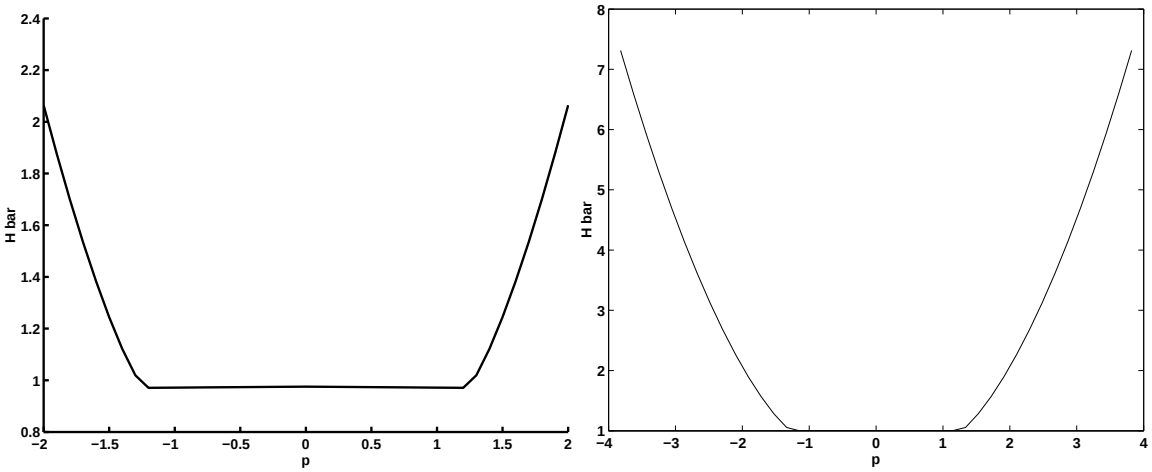

Fig. 5. (a) Effective Hamiltonians by the Small-δ Method with δ =1.e-4 and the spatial mesh size of 801; (b) Effective Hamiltonians by the Large-T method with 1st-order Godunov scheme with the spatial mesh size of 41.

results indicate that the accuracy of effective Hamiltonian is controlled by the final time T_f rather than by the spatial mesh size.

Figure 5 shows the effective Hamiltonians in the interval from $[-\frac{12}{\pi}, \frac{12}{\pi}]$ by the Small-δ Method and the Large-T Method. Note that in the flat region where $\bar{H} = 1$ the result by the Small-δ method is less accurate than that by the Large-T method with 1st-order Godunov scheme as the Hamilton-Jacobi solver.

Figure 6 shows the calibration result for $p \geq p_c$ to justify the accuracy of the Large-T method with 1st-order LxF scheme as the Hamilton-Jacobi solver. Once we know the effective Hamiltonian, we can integrate that integral in the

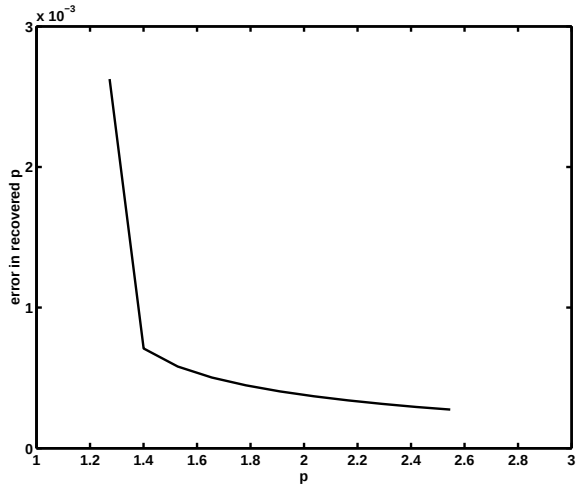

Fig. 6. Calibration for the Large-T method: the error has a jump at the critical point because the effective Hamiltonian is not smooth near the critical point $p_c = \frac{\pi}{4}$.

explicit formula to obtain the corresponding p values and compare them with the true p values.

The above 1-D example suggests that the 1st-order Godunov scheme is preferred for fast and accurate computation of the effective Hamiltonian if the Large-T method is used and if the Hamiltonian is convex in the p variable.

However, if the Hamiltonian is non-convex in the p variable, then the Godunov type schemes are not easy to code up. Therefore, we also use local Lax-Friedrichs schemes [25] in the Large-T method and apply the method to compute the effective Hamiltonians in the eikonal case. Table 1 summarizes a lot of information which is going to be explained next.

Table 1 presents the samplings of $\bar{H}$ by different schemes in the Large-T method with T =200. First, note that p =-0.3820 and p =0.7639 are in the flat part, which means that their $\bar{H}(p)$ should be equal to 1. The 2nd-order Godunov scheme with mesh size of 21 (2nd-God-m21), the 1st-order Godunov scheme with mesh size of 21 (1st-God-m21), the 1st-order Lax-Friedrichs scheme with mesh size of 321 (1st-LxF-m321) and the 1st-order Local Lax-Friedrichs scheme with mesh size of 41 (1st-LLxF-m41) all produce accurate approximations to $\bar{H}(p) = 1$. Compared to the 1st-order Lax-Friedrichs scheme with mesh size of 21 (1st-LxF-m21) with too much diffusion, the 1st-order Local Lax-Friedrichs scheme with mesh size of 21 (1st-LLxF-m21) produce decent approximations to $\bar{H}(p) = 1$. In terms of computational cost, to compute the $\bar{H}$ in the interval $[-\frac{12}{\pi}, \frac{12}{\pi}]$ with sampling size of 21, the 2nd-God-m21, the 1st-God-m21, 1st-LxF-m21, 1st-LxF-m321 and 1st-LLxF-m41 need Matlab cputime 457.2900, 93.3300, 50.9100, 1.6350e+03 and 149.7500, respectively. Therefore, in terms of the accuracy, the computational cost and the complexity of coding, the Local Lax-Friedrichs scheme is preferred in the

p	-3.8197	-0.3820	0.7639	1.9099	3.0558
2nd − God − m21	7.3216	1.0038	1.0053	1.9017	4.7045
1st − God − m21	7.3348	1.0050	1.0046	1.8992	4.7002
1st − LxF − m21	7.3148	0.7223	0.7276	1.8946	4.6983
1st − LxF − m321	7.3171	0.9843	0.9857	1.8997	4.7011
1st − LLxF − m21	7.3177	0.9200	0.9214	1.8899	4.6967
1st − LLxF − m41	7.3171	0.9818	0.9831	1.8972	4.6998

Table 1
Computed $\bar{H}(p)$ by the Large-T method (T =200) and various schemes where $m\#\#$ denotes the spatial mesh size of $\#\#$. p and its row: the sampled p values; 2nd − God: 2nd-order Godunov scheme followed by $\bar{H}(p)$ at different p's. 1st − LxF: 1st-order Lax-Friedrichs scheme followed by $\bar{H}(p)$ at different p's. 1st − LLxF: 1st-order Local Lax-Friedrichs scheme followed by $\bar{H}(p)$ at different p's.

Large-T method and we will use the scheme if the Godunov scheme is hard to implement.

4.2 A Non-convex Hamiltonian

Next we consider a non-convex Hamiltonian

$$H(p, y) = \frac{1}{2}(|p|^2 - 1)^2 - V(y) \tag{48}$$

where V is a Y-periodic potential. Here the Hamiltonian is non-convex in the p variable. We also define

$$f(p) = \frac{1}{2}(|p|^2 - 1)^2. \tag{49}$$

Similar to the eikonal case [23], we might obtain the following formula for the effective Hamiltonian (see Appendix for derivation):

$$\bar{H}(p) = \begin{cases} -\min V \text{ if } |p| \le p_c, \\ \lambda \qquad \text{s.t. } |p| = \int_0^1 \sqrt{1 + \sqrt{2(V(y) + \lambda)}} dy \text{ if } |p| > p_c. \end{cases}$$

where

p_{true}	$\bar{H}(p)$	p_{int}
1.4000	0.9993	1.4019
1.6000	1.3602	1.6009
1.8000	2.5809	1.8006
2.0000	4.5430	2.0004

Table 2
p_{true}: the sampled p values; $\bar{H}(p)$: the computed effective Hamiltonian; p_{int}: the integration of $\int_0^1 \sqrt{1+\sqrt{2(V(y)+\bar{H}(p))}}dy$, where $V = \sin 2\pi y$.

$$p_c := \int_0^1 \sqrt{1+\sqrt{2(V(y)-\min V)}}dy.$$

We will use this formula to justify our computational result.

As a first test, the potential function is taken as

$$V(y) = \sin 2\pi y. \tag{50}$$

Then it is easy to see that $\min V = -1$. For this example, we will apply Large-T method with both Lax-Friedrichs and local Lax-Friedrichs schemes as the Hamilton-Jacobi solvers for (39), (40). The result is shown in Figure 7. First, the formula predicts that there is a flat part in $\bar{H}$ and $\bar{H} = 1$ in that region, and the computational results confirm that. In the nonflat part, the predicted p values by computing $\int_0^1 \sqrt{1+\sqrt{2(V(y)+\bar{H}(p))}}dy$ with $\bar{H}(p)$ taken to be computed values (Local Lax Friedrichs) match with the true p values very well; see Table 2. Second, the figures in Figure 7 also show that the Local Lax-Friedrichs scheme is less dissipative than the Lax-Friedrichs scheme.

Figure 8 shows turning points of the effective Hamiltonian from the non flat region to the flat region at the location predicted by the formula, where the potential function is $V(y) = \sin 2\pi y$ and $p_c \approx 1.4918$ by the formula. The figure shows that the computed turning point is around 1.5 with sampling size of 0.1 in $p \in [-2, 2]$; compare to the results in Figure 7, where the sampling size in $p \in [-2, 2]$ is 0.2.

Figure 9 shows the effective Hamiltonians with different potential functions; if the magnitude of the potential is decreased, then the flat region is shortened and the effective Hamiltonian $\bar{H}$ in the region becomes smaller too.

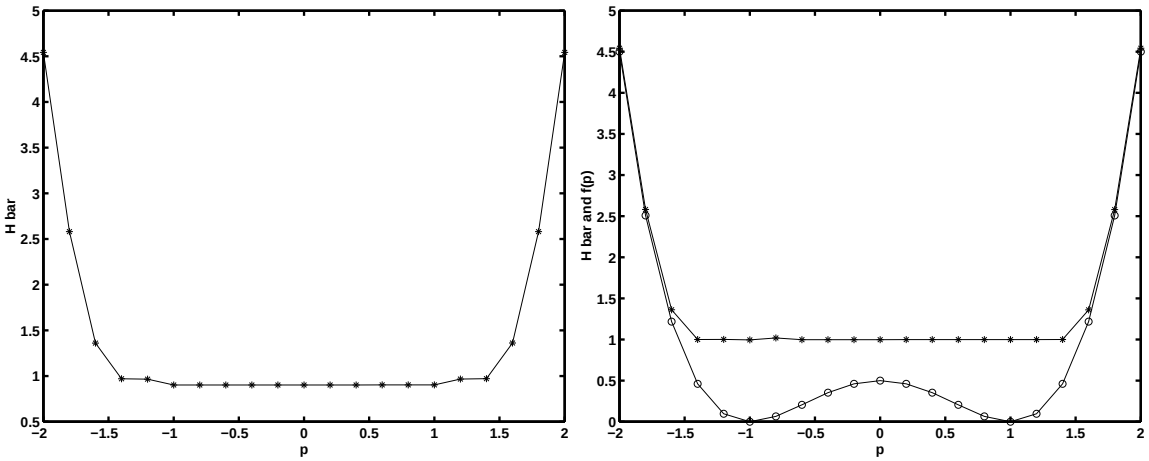

Fig. 7. Effective Hamiltonians by the Large-T Method with T =200. (a) Lax-Friedrichs scheme with the spatial mesh size of 321; (b) Local Lax Friedrichs scheme with the spatial mesh size of 81.

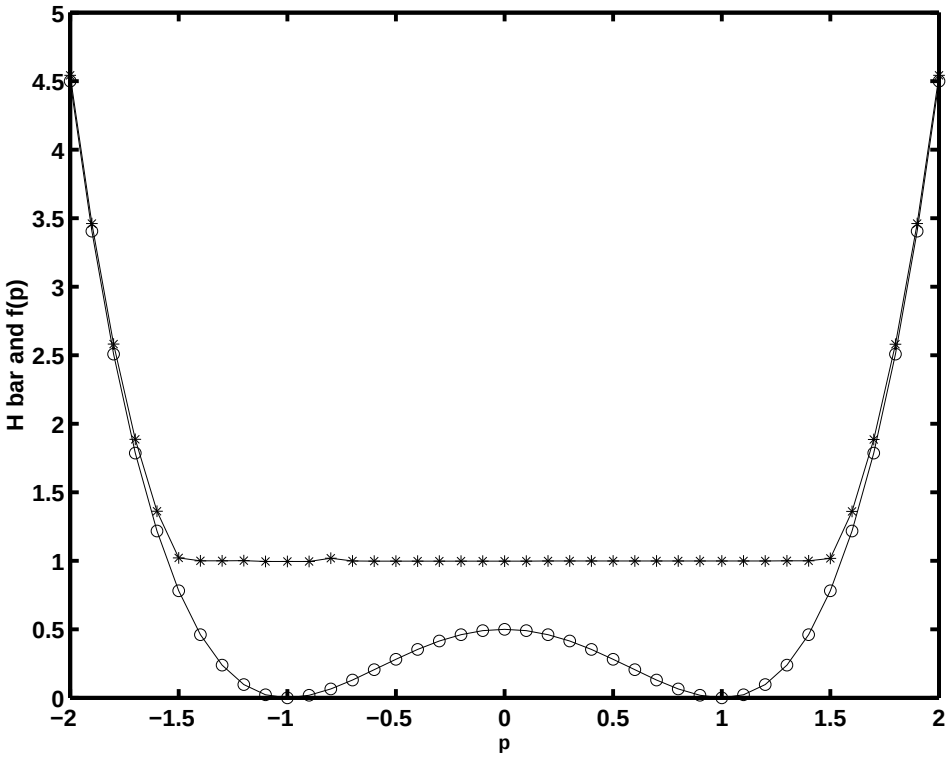

Fig. 8. *-: computed Effective Hamiltonian; o-: $f(p)$. Effective Hamiltonians by the Large-T Method with T =200 and local Lax-Friedrichs scheme with the spatial mesh size of 81. The number of samples for $p \in [-2, 2]$ is 41; $p_c \approx 1.4918$.

5 Numerical Experiments: 2-D Eikonal Equations

We apply the Large-T method to compute the effective Hamiltonian on the domain $[-\frac{4.4}{\pi}, \frac{4.4}{\pi}] \times [-\frac{4.4}{\pi}, \frac{4.4}{\pi}]$ for

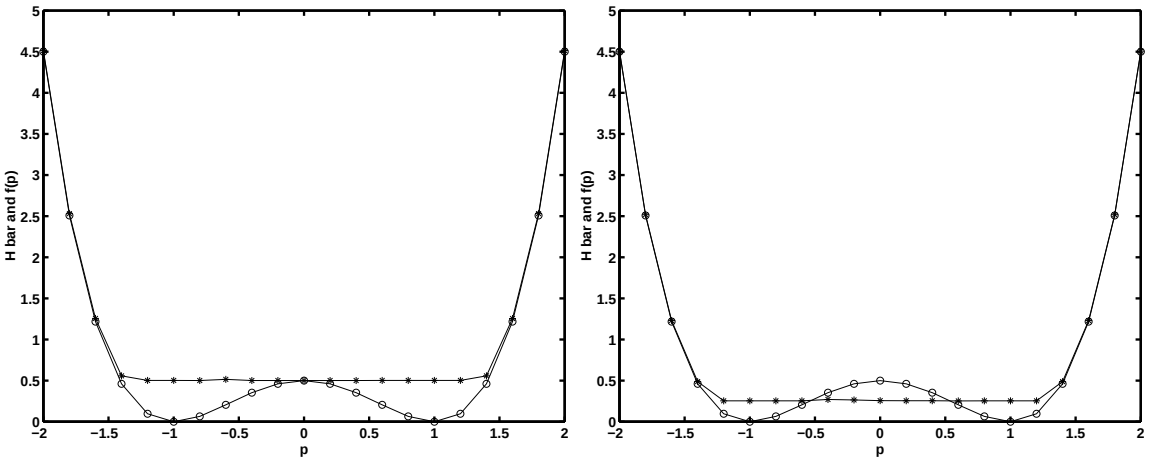

Fig. 9. *-: computed Effective Hamiltonian; o-: $f(p)$. Large-T Method with T =200 and the spatial mesh size of 81; local Lax-Friedrichs scheme used here. (a) $V = 0.5 \sin 2\pi x$; (b) $V = 0.25 \sin 2\pi x$.

$$H(p_1, p_2, x_1, x_2) = \frac{1}{2}|p|^2 - V(x_1, x_2), \tag{51}$$

where V is a Y-periodic potential to be specified. Here the Hamiltonian is convex in p, we mainly apply 1st-order Godunov scheme to solve equations (39), (40).

As a first example, we consider

$$V(x_1, x_2) = \cos 2\pi x_1 + \cos 2\pi x_2, \tag{52}$$

which corresponds to two uncoupled pendulums.

The sampling size in p is 11×11, and the unit cube in the spatial space is discretized into 21×21 uniform mesh. The final time T_f is 1000, and the Matlab cputime is 6233.3 sec. to finish the computation. Figure 10 shows the surface and contour of the effective Hamiltonian $\bar{H}$. In this case the exact solution is known. In the flat region, the exact solution $\bar{H}(0,0)$ is 2.0, and the computed solution is 2.0006, where the error is 0.0006. In the non-flat region, by the above one-dimensional solver the solution $\bar{H}(1.4006)$ is 1.1233 and $\bar{H}(1.1205)$ is 1.0008, so the solution $\bar{H}(1.4006, 1.1205)$ should be around 2.1241; the computed effective Hamiltonian by the 2-D Godunov 1st-order scheme is 2.1235, therefore the error is 0.0006.

As a second example, we take

$$V(x_1, x_2) = \sin 2\pi x_1 \sin 2\pi x_2. \tag{53}$$

The sampling size in p is 11×11, and the unit cube in the spatial space is

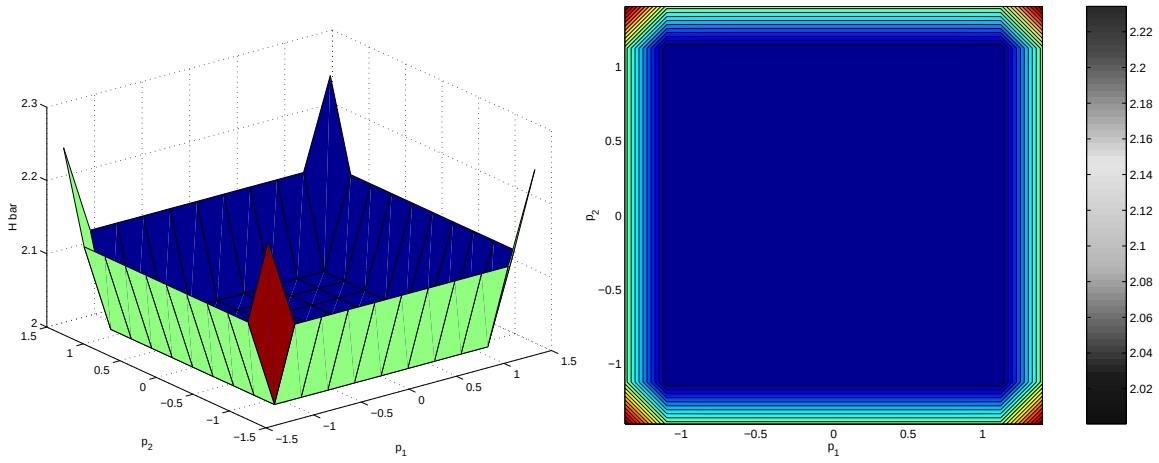

Fig. 10. 2-D effective Hamiltonian. a) Surface; b) Contour

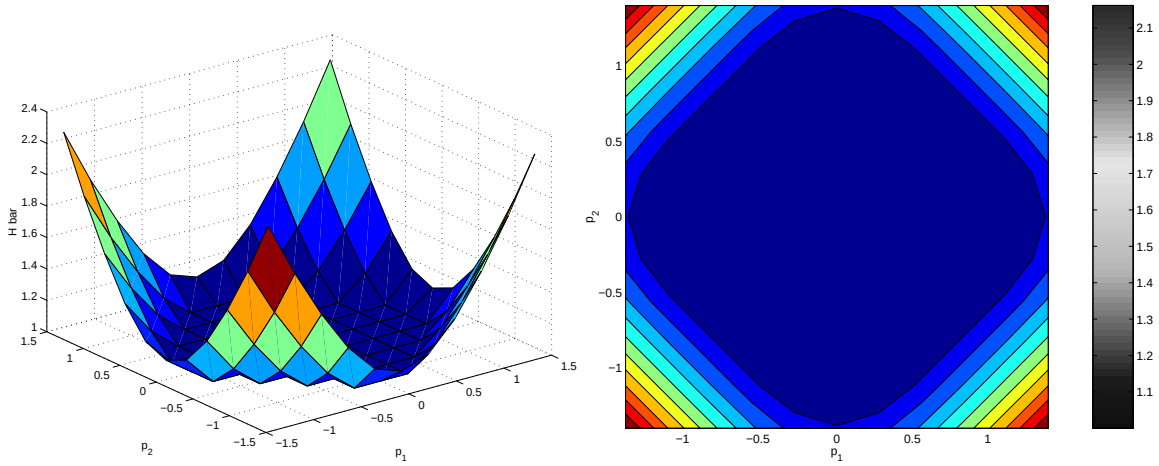

Fig. 11. 2-D effective Hamiltonian. a) Surface; b) Contour

discretized into 21×21 uniform mesh. The final time T_f is 1000, and the Matlab cputime is 5947.0 sec. to finish the computation. The result is shown in Figure 11.

As a third example, we take

$$V(x_1, x_2) = \cos 2\pi x_1 + \cos 2\pi x_2 + \cos 2\pi(x_1 - x_2). \tag{54}$$

The sampling size in p is 11×11, and the unit cube in the spatial space is discretized into 11×11 uniform mesh. The final time T_f is 1000, and the Matlab cputime is 2442.4 sec. to finish the computation. In this example, we also compute $\bar{H}$ at the point $(0, \frac{4.4}{\pi})$ using different spatial mesh sizes. The computed $\bar{H}$ are 2.0133, 2.0373 and 2.0519 with spatial mesh sizes 17, 33 and 65, respectively, which implies that the computed $\bar{H}$ is convergent. The result is shown in Figure 12.

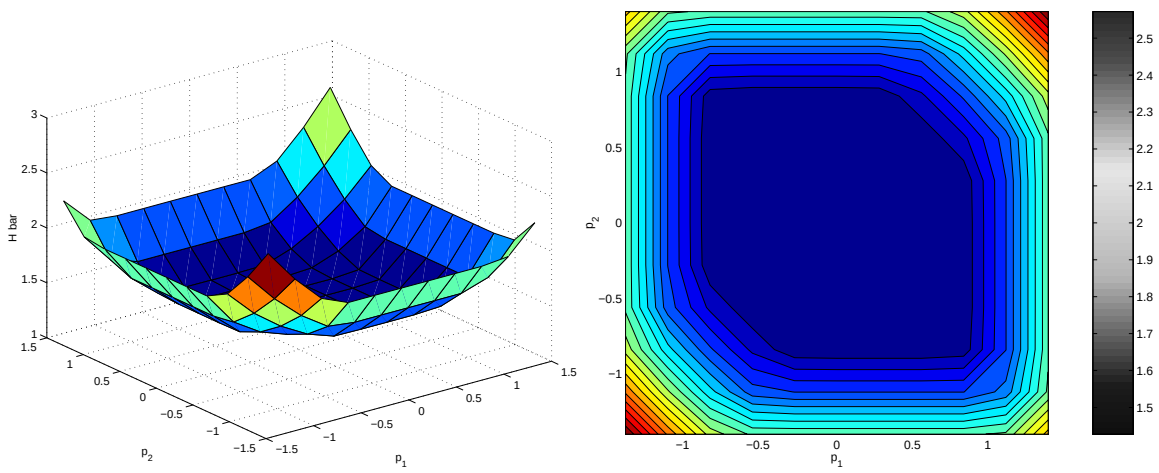

Fig. 12. 2-D effective Hamiltonian. a) Surface; b) Contour

6 Conclusion

We have presented two simple approaches to computing effective Hamiltonians which only require Hamilton-Jacobi solvers. Numerical examples have shown that the approaches are accurate and efficient; thus the approach may be used in understanding Hamiltonian dynamics and computing Aubry-Mather measure via solving a transport equation in weak KAM theory [14].

Future work might include designing fast sweeping type schemes for cell problems based on the related approximate cell problems and implementing Sherman-Morrison inversion formula to solve the approximate cell problems more efficiently.

A Appendix The formula (50) for $\bar{H}$

Consider the 1-D cell problem

$$H(p+u_y,y)=\bar{H}(p) \tag{A.1}$$

where the nonconvex Hamiltonian is taken as

$$H(p,y)=\frac{1}{2}(|p|^2-1)^2-V(y) \tag{A.2}$$

with V a Y=[0,1]-periodic potential. Here the Hamiltonian is nonconvex in the p variable.

We would like to derive a formula for $\bar{H}$ in this case. From the defintion of the cell problem we have

$$\frac{1}{2}(|p+u_y|^2-1)^2 - V(y) = \bar{H}(p). \tag{A.3}$$

Therefore,

$$(|p+u_y|^2-1)^2 = 2(\bar{H}(p)+V(y)). \tag{A.4}$$

This implies that

$$\bar{H}(p) \geq -\min_Y V(y) \tag{A.5}$$

and

$$|p+u_y|^2 = 1 + s_1(y)\sqrt{2(\bar{H}(p)+V(y))}, \tag{A.6}$$

where $|s_1(y)|$=1. Furthermore, we have

$$u_y = -p + s(y)\sqrt{1+s_1(y)\sqrt{2(\bar{H}(p)+V(y))}}, \tag{A.7}$$

$$u = \int_0^x -p + s(y)\sqrt{1+s_1(y)\sqrt{2(\bar{H}(p)+V(y))}}dy + C, \tag{A.8}$$

where $0 \leq x \leq 1$, $|s(y)| = 1$ and C is a constant. By periodicity, $u(0) = u(1)$, which implies that

$$p = \int_0^1 s(y)\sqrt{1+s_1(y)\sqrt{2(\bar{H}(p)+V(y))}}dy. \tag{A.9}$$

Since p may be arbitrary, there is no λ such that

$$|p| = \int_0^1 \sqrt{1-\sqrt{2(\lambda+V(y))}}dy \tag{A.10}$$

if p is large enough. Therefore, we have

$$u = \int_0^x -p + s(y)\sqrt{1+\sqrt{2(\bar{H}(p)+V(y))}}dy + C, \tag{A.11}$$

a piecewise-C^1 function.

If $\bar{H}(p) > -\min_Y V(y)$, then p and $\bar{H}(p)$ satisfies

$$p = \pm \int_0^1 \sqrt{1+\sqrt{2(\bar{H}(p)+V(y))}}dy \tag{A.12}$$

$$|p| = \int_0^1 \sqrt{1+\sqrt{2(\bar{H}(p)+V(y))}}dy. \tag{A.13}$$

Moreover, if

$$|p| \geq p_c := \int_0^1 \sqrt{1+\sqrt{2(V(y)-\min V)}}dy, \tag{A.14}$$

then

$$|p| = \int_0^1 \sqrt{1+\sqrt{2(\lambda+V(y))}}dy \tag{A.15}$$

always has a solution λ, which is defined as $\bar{H}(p)$. Otherwise, if $|p| \leq p_c$, then

$$\bar{H}(p) = -\min_Y V(y). \tag{A.16}$$

Now let us look at the solution u constructed above, and we would like to verify that u is a viscosity solution of the cell problem. By Y-periodicity, if $p > p_c$, then

$$u = \int_0^x -p + \sqrt{1+\sqrt{2(\bar{H}(p)+V(y))}}dy + C$$

is a C^1 solution; if $p < -p_c$, then

$$u = \int_0^x -p - \sqrt{1+\sqrt{2(\bar{H}(p)+V(y))}}dy + C$$

is also a C^1 solution; if $-p_c \leq p \leq p_c$, then u is a piecewise C^1 solution:

$$u = \int_0^x -p + s(y)\sqrt{1+\sqrt{2(\bar{H}(p)+V(y))}}dy + C$$

where $|s(y)| = 1$; i.e., there is a point y_0 such that $s(y)$ changes sign at y_0 and the following equation holds

$$\int_0^1 s(y)\sqrt{1+\sqrt{2(\bar{H}(p)+V(y))}}dy = p.$$

Since u is also a viscosity solution for Hamilton-Jacobi equation

$$|p+u_y|^2 = 1+\sqrt{2(\bar{H}(p)+V(y))}$$

which has a convex Hamiltonian, the only allowed discontinuity is the one that satisfies $u_x(x^-) - u_x(x^+) > 0$. Based on this fact, if $-p_c \leq p \leq p_c$, then $s(y)$ can change sign from positive to negative at some point y_0, and $u_x(y_0^-) - u_x(y_0^+) > 0$. Furthermore, Theorem 1.3 in [8] implies that u is a piecewise C^1 viscosity solution for the cell problem.

Overall, the formula for the effective Hamiltonian is

$$\bar{H}(p) = \begin{cases} -\min V \text{ if } |p| \leq p_c, \\ \lambda \qquad \text{s.t. } |p| = \int_0^1 \sqrt{1+\sqrt{2(V(y)+\lambda)}}dy \text{ if } |p| > p_c. \end{cases}$$

where

$$p_c := \int_0^1 \sqrt{1+\sqrt{2(V(y)-\min V)}}dy.$$

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