Segmentation under Geometrical Conditions using Geodesic Active Contours and Interpolation using Level Set Methods.

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Abstract: Let $I : \Omega \to \Re$ be a given bounded image function, where Ω is an open and bounded domain which belongs to $\Re^n$. Let us consider $n = 2$ for the purpose of illustration. Also, let $S = (x_i)_i \in \Omega$ be a finite set of given points. We would like to find a contour $\Gamma \subset \Omega$, such that Γ is an object boundary interpolating the points from S.

We combine the ideas of the geodesic active contours {cf. Caselles Kimmel Sapiro [7] and [8]} and of interpolation of points {cf. Zhao Osher Merriman Kang [39]} in a Level Set approach developed by Osher and Sethian [32]. We present the modelling of the proposed method, theoretical results (viscosity solution) and numerical results are also given.

Key words : Geodesic Active Contour, Level Set method, viscosity solution, interpolation of points.

AMS classification: 49L25, 74G65, 68U10,

Introduction

In this paper, we aim at combining the ideas developed in the Geodesic Active Contours approach [7] with a geometrical approach including interpolation constraints. These interpolation conditions can be well data in Geophysics (see [3], [21]) or can be used to help the segmentation process when it is needed because of the image ([27]).

This will be done in the context of the Level Set Approach, developed by Osher and Sethian (Osher and Sethian [32], Sethian [33], Osher and Fedkiw [31] or in [1], [29], [30], [35], [6], [10], [9], [12], [36] , [38], [25], [24]...), which consists in considering the problem in a higher dimension, and more precisely, considering the evolving curve as the zero Level Set of a surface. Topology changes, cusps, corners are allowed in this context. Moreover, we do not have to deal with the issue of parameterization anymore since we work on a fixed rectangular grid for the discretization. It is an intrinsic representation. The process exposed consists first in minimizing an energy which contains a term in connection with the *a priori* knowledge of the image and another term linked to the geometrical constraints. Edges are assumed to be pixels where the gradient intensity function varies abruptly. The Euler-Lagrange theorem gives us the Partial Differential Equation satisfied by the surface. The equation can be seen as a mean-curvature flow like problem, except it includes a function linked to the geometrical and image information.

Position of the problem

1. Model

Let $I : \Omega \longrightarrow \Re$ be a given bounded image function, with Ω an open bounded subset of $\Re^N$. Let us consider $N = 2$ for the purpose of the illustration. As mentioned in the introduction, we plan to introduce a geometrical approach in this new method by adding interpolation constraints. Thus, let $S = (x_i)_i \in \Omega$ be a finite set of given points close to the boundary we want to determine. We would like to find a contour $\Gamma \subset \Omega$ such that Γ is an object boundary, interpolating the points from S which belong to this boundary. Let $g : [0, +\infty[\longrightarrow [0, +\infty[$ be an edge-function as in ([6], [30], [9]), such that $g(0) = 1$, g is positive, strictly decreasing and $\lim_{s \to \infty} g(s) = 0$, applied to the gradient of the image $|\nabla I(x, y)|$. An example of such a function is given by :

$$g(s) = \frac{1}{1 + s^2}$$

so,

$$g(|\nabla I(x, y)|) = \frac{1}{1 + |\nabla I(x, y)|^2} \tag{1.1}$$

Also, to the set of points S, we associate the distance function $d(x)$ from every point $x \in \Omega$ to S,

$$d(x) = \ distance(x, S) \tag{1.2}$$

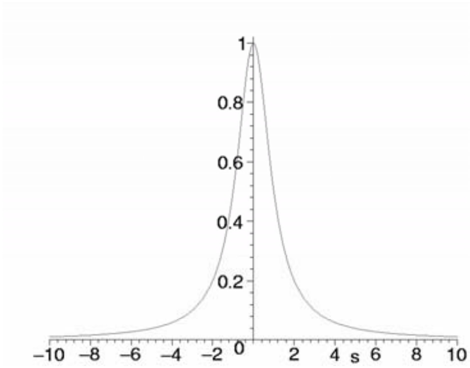

Figure 1: Representation of the function g.

Therefore, $d(x) = 0$ if and only if $x \in S$. In order to find a contour Γ such that $g \simeq 0$ or $d \simeq 0$ on Γ, we propose to minimize the following energy :

$$\mathbf{E}(\boldsymbol{\Gamma}) = \int_{\Gamma} \mathbf{d}(\mathbf{x}(\mathbf{s}))\mathbf{g}(|\boldsymbol{\nabla}\mathbf{I}(\mathbf{x}(\mathbf{s}))|)\mathbf{ds} \tag{1.3}$$

We will start with an initial guess Γ_0 and we will apply gradient descent to the energy, in a Level Set Approach. We will construct a family of curves $\Gamma(t)$ decreasing the energy as t increases.

Method

2. The Level Set Approach

The Level Set Approach ([32], [33], [31]) consists in considering the evolving active contour $\Gamma = \Gamma(t)$ as the zero level set of an hypersurface Φ, which is a Lipschitz continuous function defined by :

$$\begin{cases} \Phi : & \Omega \times [0, +\infty[\longrightarrow \Re \\ & (x,t) \longmapsto \Phi(x,t) \end{cases} \tag{2.4}$$

such that

$$\Gamma(t) = \{x, \Phi(x,t) = 0\} \text{ and } \Phi(.,t)$$

takes opposite signs on each side of $\Gamma(t)$. This is this property which enables us to re-write the energy as follows. The corresponding energy in the unknown Φ becomes :

$$F(\Phi) = \int_{\Omega} d(x) g(|\nabla I(x)|) |\nabla H(\Phi(x))| dx,$$

where H is the one-dimensional Heaviside function. By approximating this by a C^1 or C^2 regularization (Chan and Vese [10]) H_ϵ, as $\epsilon \longrightarrow 0$ and letting $\delta_\epsilon = H'_\epsilon$, the energy can be written as :

$$F_\epsilon(\Phi) = \int_{\Omega} d(x) g(|\nabla I(x)|) \delta_\epsilon(\Phi) |\nabla \Phi(x)| dx, \tag{2.5}$$

where

$$\int_{\Omega} \delta_\epsilon(\Phi)|\nabla\Phi(x)|dx$$

is an approximation of the length of the 0 level set of Φ.

For the regularization, we refer the reader to Zhao et al. [40] who propose for H_ϵ the following function:

$$H_\epsilon(z) = \begin{cases} 1 \text{ if } z > \epsilon \\ 0 \text{ if } z < -\epsilon \\ \frac{1}{2}[1 + \dfrac{z}{\epsilon} + \dfrac{1}{\pi} sin(\dfrac{\pi z}{\epsilon})] \text{ if } |z| \leq \epsilon \end{cases} \tag{2.6}$$

3. Minimization of the energy

In this section, we minimize the energy F_ϵ and we determine the associated Partial Differential Equation satisfied by Φ. In this purpose, we first use variational calculus with the extension of the classical Euler-Lagrange theorem to a function that depends on 2 variables. Then, we propose a proof based on the Gâteaux derivative.

3.1. Minimization using Euler-Lagrange theorem.

Theorem 1. *Euler-Lagrange theorem: (in 1-dimension) Suppose that $y \in C^2[x_1, x_2]$ minimizes the energy*

$$\int_{x_1}^{x_2} L(x, y, y')dx.$$

Then, y(x) satisfies the equation :

$$\tfrac{\partial L}{\partial y} - \tfrac{d}{dx}\tfrac{\partial L}{\partial y'} = 0. \text{ (Euler-Lagrange Equation)}$$

An extension to the case where y is a function of two variables is done in what follows:

Theorem 2. *Euler-Lagrange theorem: (in 2-dimensions) Suppose that $y \in C^2(\Omega)$ minimizes the energy*

$$\int_{\Omega} L\left(x_1, x_2, y, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}\right) dx_1 dx_2.$$

Then, y satisfies the equation :

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}(\frac{\partial L}{\partial y_i}) = \frac{\partial L}{\partial y}, \text{ with } y_i = \tfrac{\partial y}{\partial x_i}.$$

Proof: Let $\Psi(x_1,x_2) = y(x_1,x_2) + \epsilon\eta(x_1,x_2)$ with η such that $\eta_{/\partial\Omega} = 0$ and $(x_1,x_2) \in \Omega$. Let us denote by I the functional defined by:

$$I(\epsilon) = \int_\Omega L(x_1, x_2, \Psi, \frac{\partial \Psi}{\partial x_1}, \frac{\partial \Psi}{\partial x_2}) dx_1 dx_2.$$

If y minimizes this functional, then $\frac{dI}{d\epsilon}_{/\epsilon=0} = 0$. Thus,

$$\frac{d}{d\epsilon} \int_\Omega L(x_1, x_2, \Psi, \frac{\partial \Psi}{\partial x_1}, \frac{\partial \Psi}{\partial x_2}) dx_1 dx_{2/\epsilon=0} = 0. \tag{3.7}$$

Let denote by p and q the functions defined respectively by:

$$\begin{cases} p(x_1,x_2) = \dfrac{\partial \Psi}{\partial x_1} \\ q(x_1,x_2) = \dfrac{\partial \Psi}{\partial x_2} \end{cases}$$

Suppose that L has the regularity needed, we can deduce from (3.7) that:

$$\int_\Omega \frac{d}{d\epsilon} L(x_1, x_2, \Psi, \frac{\partial \Psi}{\partial x_1}, \frac{\partial \Psi}{\partial x_2}) dx_1 dx_{2/\epsilon=0} = 0.$$

and

$$\int_\Omega \frac{\partial L}{\partial \Psi}\eta + \frac{\partial L}{\partial p}\frac{\partial \eta}{\partial x_1} + \frac{\partial L}{\partial q}\frac{\partial \eta}{\partial x_2} dx_1 dx_{2/\epsilon=0} = 0. \tag{3.8}$$

An integration by parts with respect to the variable x_1 for the term $\frac{\partial L}{\partial p}\frac{\partial \eta}{\partial x_1}$ gives :

$$\int_\Omega \frac{\partial L}{\partial p}\frac{\partial \eta}{\partial x_1} dx_1 dx_2 = -\int_\Omega \frac{\partial}{\partial x_1}[\frac{\partial L}{\partial p}]\eta dx_1 dx_2 + \int_{\partial\Omega} \frac{\partial L}{\partial p}\eta \nu_1 d\sigma$$

where ν is the unit exterior normal. Using the fact that $\eta_{/\partial\Omega} = 0$, we deduce that :

$$\int_\Omega \frac{\partial L}{\partial p}\frac{\partial \eta}{\partial x_1} dx_1 dx_2 = -\int_\Omega \frac{\partial}{\partial x_1}[\frac{\partial L}{\partial p}]\eta dx_1 dx_2.$$

With a similar method, we get :

$$\int_\Omega \frac{\partial L}{\partial q}\frac{\partial \eta}{\partial x_2} dx_1 dx_2 = -\int_\Omega \frac{\partial}{\partial x_2}[\frac{\partial L}{\partial q}]\eta dx_1 dx_2.$$

For $\epsilon = 0$, we have the relations :

$$\left| \begin{array}{l} \dfrac{\partial L}{\partial \Psi} = \dfrac{\partial L}{\partial y}, \\ p(x,y) = \dfrac{\partial y}{\partial x_1}, \\ q(x,y) = \dfrac{\partial y}{\partial x_2}. \end{array} \right.$$

We conclude that $\forall\eta$ such that $\eta_{/\partial\Omega}=0$,

$$\int_\Omega [\frac{\partial L}{\partial y} - \frac{\partial}{\partial x_1}[\frac{\partial L}{\partial p}] - \frac{\partial}{\partial x_2}[\frac{\partial L}{\partial q}]]\eta dx_1 dx_2 = 0.$$

Then, we get the equation satisfied by y

$$\frac{\partial L}{\partial y} - \frac{\partial}{\partial x_1}[\frac{\partial L}{\partial p}] - \frac{\partial}{\partial x_2}[\frac{\partial L}{\partial q}] = 0. \blacksquare$$

In our model, we have:

$$L(x_1, x_2, \Phi, \frac{\partial \Phi}{\partial x_1}, \frac{\partial \Phi}{\partial x_2}) = \delta_\epsilon(\Phi) d(x_1, x_2) g(|\nabla I(x_1, x_2)|)|\nabla\Phi(x_1, x_2)|.$$

So, it comes

$$\begin{cases} \dfrac{\partial L}{\partial \Phi} = \delta'_\epsilon(\Phi) d(x_1, x_2) g(|\nabla I(x_1, x_2)|)|\nabla\Phi| \\ \dfrac{\partial L}{\partial p} = \delta_\epsilon(\Phi) d(x_1, x_2) g(|\nabla I(x_1, x_2)|)\dfrac{\frac{\partial \Phi}{\partial x_1}}{|\nabla\Phi|} \\ \dfrac{\partial L}{\partial q} = \delta_\epsilon(\Phi) d(x_1, x_2) g(|\nabla I(x_1, x_2)|)\dfrac{\frac{\partial \Phi}{\partial x_2}}{|\nabla\Phi|} \end{cases}$$

Thus, we get

$$\frac{\partial L}{\partial \Phi} - \frac{\partial}{\partial x_1}[\frac{\partial L}{\partial p}] - \frac{\partial}{\partial x_2}[\frac{\partial L}{\partial q}] = 0 \Longleftrightarrow \delta_\epsilon(\Phi) div(d(x_1, x_2) g(|\nabla I(x_1, x_2)|)\frac{\nabla\Phi}{|\nabla\Phi|}) = 0$$

The associated evolution equation is thus, given by:

$$\frac{\partial \Phi}{\partial t} = \boldsymbol{\delta_\epsilon(\Phi)}\mathbf{div}(\mathbf{d}(\mathbf{x}_1, \mathbf{x}_2)\mathbf{g}(|\boldsymbol{\nabla}\mathbf{I}(\mathbf{x}_1, \mathbf{x}_2)|)\frac{\nabla\Phi}{|\nabla\Phi|}) \tag{3.9}$$

We now propose a proof of this result using the Gâteaux derivative.

3.2. Gâteaux derivative.. We first recall that F is differentiable in the Gâteaux sense in $y \in X$ if the application $F'_y : x \mapsto lim_{h\to 0}\dfrac{F(y+hx)-F(y)}{h}$ is defined for all $x \in X$ and if it is linear and continuous. In this case, Riesz'theorem gives the existence of $F'(y) \in X$ such that $F'_y(x) =< F'(y), x >$, $F'(y)$ being the gradient of F in y.

Coming back to our problem and let us determine the Gâteaux derivative of the energy F_ϵ. The Gâteaux derivative of F_ϵ with respect to Φ in the Ψ direction is :

$$F'_{\epsilon_\Phi}(\Psi) = lim_{h\to 0}\frac{F_\epsilon(\Phi + h\Psi) - F_\epsilon(\Phi)}{h}$$

$$\left| \begin{array}{l} F_\epsilon(\Phi + h\Psi) - F_\epsilon(\Phi) = \\ \int_\Omega d(x_1, x_2) g(|\nabla I(x_1, x_2)|)\delta_\epsilon(\Phi + h\Psi)|\nabla\Phi + h\nabla\Psi| dx_1 dx_2 \\ - \int_\Omega d(x_1, x_2) g(|\nabla I(x_1, x_2)|)\delta_\epsilon(\Phi)|\nabla\Phi| dx_1 dx_2. \end{array} \right.$$

Then

$$\left| \begin{array}{l} F_\epsilon(\Phi+h\Psi)-F_\epsilon(\Phi)= \\ \int_\Omega d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\delta_\epsilon(\Phi+h\Psi)|\nabla\Phi|\sqrt{1+h^2\frac{|\nabla\Psi|^2}{|\nabla\Phi|^2}+2h\frac{<\nabla\Phi,\nabla\Psi>}{|\nabla\Phi|^2}}dx_1dx_2 \\ -\int_\Omega d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\delta_\epsilon(\Phi)|\nabla\Phi|dx_1dx_2 \end{array}\right.$$

and

$$\left| \begin{array}{l} \dfrac{F_\epsilon(\Phi+h\Psi)-F_\epsilon(\Phi)}{h}= \\ \dfrac{1}{h}[\displaystyle\int_\Omega d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\delta_\epsilon(\Phi+h\Psi)|\nabla\Phi|(1+h\frac{<\nabla\Phi,\nabla\Psi>}{|\nabla\Phi|^2} \\ +\dfrac{h^2}{2}\dfrac{|\nabla\Psi|^2}{|\nabla\Phi|^2}+O(h^2L(x_1,x_2)_{h\to 0}))dx_1dx_2-\displaystyle\int_\Omega d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\delta_\epsilon(\Phi)|\nabla\Phi|dx_1dx_2]. \end{array}\right.$$

Taking the limit when $h\to 0$, we get:

$$F'_{\epsilon_\Phi}(\Psi)=\int_\Omega d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\delta'_\epsilon(\Phi)\Psi|\nabla\Phi|dx_1dx_2$$
$$+\int_\Omega d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\delta_\epsilon(\Phi)\frac{<\nabla\Phi,\nabla\Psi>}{|\nabla\Phi|}dx_1dx_2.$$

So,

$$\left| \begin{array}{l} F'_{\epsilon_\Phi}(\Psi)=\int_\Omega d(x_1,x_2)g(|\nabla I(x_1,x_2)|)|\nabla\Phi|\delta'_\epsilon(\Phi)\Psi dx_1dx_2 \\ -\int_\Omega \frac{\partial}{\partial x_1}[d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\delta_\epsilon(\Phi)\frac{\frac{\partial\Phi}{\partial x_1}}{|\nabla\Phi|}]\Psi dx_1dx_2 \\ +\int_{\partial\Omega} d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\delta_\epsilon(\Phi)\frac{\frac{\partial\Phi}{\partial x_1}}{|\nabla\Phi|}\Psi\nu_{x_1}d\sigma \\ -\int_\Omega \frac{\partial}{\partial x_2}[d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\delta_\epsilon(\Phi)\frac{\frac{\partial\Phi}{\partial x_2}}{|\nabla\Phi|}]\Psi dx_1dx_2 \\ +\int_{\partial\Omega} d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\delta_\epsilon(\Phi)\frac{\frac{\partial\Phi}{\partial x_2}}{|\nabla\Phi|}\Psi\nu_{x_2}d\sigma. \end{array}\right.$$

Thus,

$$\left| \begin{array}{l} F'_{\epsilon_\Phi}(\Psi)=-\int_\Omega \delta_\epsilon(\Phi)div(d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\frac{\nabla\Phi}{|\nabla\Phi|})\Psi dx_1dx_2 \\ +\int_{\partial\Omega} d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\dfrac{\delta_\epsilon(\Phi)}{|\nabla\Phi|}\Psi(\dfrac{\partial\Phi}{\partial x_1}\nu_{x_1}d\sigma+\dfrac{\partial\Phi}{\partial x_2}\nu_{x_2}d\sigma)dx_1dx_2. \\ and \\ F'_{\epsilon_\Phi}(\Psi)=-\int_\Omega \delta_\epsilon(\Phi)div(d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\frac{\nabla\Phi}{|\nabla\Phi|})\Psi dx_1dx_2+ \\ \int_{\partial\Omega} d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\dfrac{\delta_\epsilon(\Phi)}{|\nabla\Phi|}\dfrac{\partial\Phi}{\partial\nu}\Psi d\sigma. \end{array}\right.$$

This expression must vanish for all Ψ to satisfy the Euler-Lagrange condition. We obtain therefore the following problem:

$$\begin{cases} \delta_\epsilon(\Phi)div(d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\frac{\nabla\Phi}{|\nabla\Phi|})=0. \\ \text{with the boundary conditions } \dfrac{\delta_\epsilon(\Phi)}{|\nabla\Phi|}\dfrac{\partial\Phi}{\partial\nu}=0. \end{cases}$$

The evolution problem is thus the one found in (3.9).

The quantities $\kappa = div(\frac{\nabla\Phi}{|\nabla\Phi|})$ and $\vec{n} = \frac{\nabla\Phi}{|\nabla\Phi|}$ are the Level Set representation of the mean curvature and the unit normal, the PDE can be formally rewritten by :

$$\boldsymbol{\delta_\epsilon(\Phi)(d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\kappa + < \nabla(d(x_1,x_2)g(|\nabla I(x_1,x_2)|)), \vec{n} >) = 0}. \quad (3.10)$$

As stressed by Zhao et al. [40], there is a balance between the potential force and the surface tension. A parallel can be drawn with the classical deformable models, a model which traduces an equilibrium between a regularization energy and an energy linked to the image. The flexibility in our model is all the more important as we are close to the finite set of points or on edges since in this case the expression $d(x_1,x_2)g(|\nabla I(x_1,x_2)|)$ vanishes.

Proposition 3. *The energy $F_\epsilon(\Phi)$ is decreasing with time t.*

Proof: We follow the same arguments as Zhao& *al.*[40].

$$\frac{dF_\epsilon(\Phi)}{dt} = \int_\Omega d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\delta'_\epsilon(\Phi)\Phi_t|\nabla\Phi|dx_1dx_2 + $$
$$+ \int_\Omega d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\delta_\epsilon(\Phi)\frac{<\nabla\Phi,(\nabla\Phi)_t>}{|\nabla\Phi|}dx_1dx_2.$$

and

$$\frac{dF_\epsilon(\Phi)}{dt} = -\int_\Omega \delta_\epsilon(\Phi)div(d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\frac{\nabla\Phi}{|\nabla\Phi|})\Phi_t dx_1dx_2$$
$$+ \int_{\partial\Omega} d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\delta_\epsilon(\Phi)\frac{\frac{\partial\Phi}{\partial\nu}}{|\nabla\Phi|}\Phi_t d\sigma.$$

Using the evolution equation and the boundary conditions, we get:

$$\frac{dF_\epsilon(\Phi)}{dt} = -\int_\Omega (\delta_\epsilon(\Phi)div(d(x_1,x_2)g(|\nabla I(x_1,x_2)|)\frac{\nabla\Phi}{|\nabla\Phi|}))^2 dx_1dx_2 \leq 0. \blacksquare$$

We see in (3.9) that, when a local minimum is reached, then the quantity $\frac{\partial\Phi}{\partial t}$ tends to 0, which means that the model converges. Indeed, the steady state is reached and the curve no longer evolves.

A constant α can be added to the equation (3.9) to increase the speed of convergence, which is obtained from an additional area constraint as done in Chan and Vese [10], and a rescaling can be made so that the motion is applied to all level sets by replacing δ_ϵ by $|\nabla\Phi|$. As stressed by Zhao & *al.* [39] and Alvarez & *al.* [2], it makes the flow independent of the scaling of Φ. Thus the proposed model is, for any $(x,y) \in \Omega$:

$$\begin{cases} \Phi(x,y,0) = \Phi_0(x,y) \\ \dfrac{\partial\Phi}{\partial t} = |\nabla\Phi|[div(d(x,y)g(|\nabla I(x,y)|)\dfrac{\nabla\Phi}{|\nabla\Phi|}) - \alpha] \end{cases}.$$

This model is then an active contour model based on the mean curvature flow motion, to which we have included the interpolation constraints through the function distance. The evolution equation can be formally re-written by:

$$\left\{ \frac{\partial \Phi}{\partial t} = |\boldsymbol{\nabla\Phi}|\mathbf{d}(\mathbf{x},\mathbf{y})\mathbf{g}(|\boldsymbol{\nabla}\mathbf{I}(\mathbf{x},\mathbf{y})|)\mathbf{div}(\frac{\nabla\Phi}{|\nabla\Phi|}) + < \boldsymbol{\nabla}(\mathbf{d}(\mathbf{x},\mathbf{y})\mathbf{g}(|\boldsymbol{\nabla}\mathbf{I}(\mathbf{x},\mathbf{y})|)), \boldsymbol{\nabla\Phi} > -\boldsymbol{\alpha}|\boldsymbol{\nabla\Phi}|. \right. \tag{3.11}$$

Our parabolic problem with the associated boundary conditions $\frac{\partial \Phi}{\partial \nu} = 0$ on $\partial\Omega$ (*Neumann Condition*) with ν denotes the exterior normal to the boundary of Ω is thus defined by :

$$\left\{ \begin{array}{l} \Phi(x,0) = \Phi_0(x) \\ \frac{\partial \Phi}{\partial t} = |\nabla\Phi| d(x,y) g(|\nabla I(x,y)|) div(\frac{\nabla\Phi}{|\nabla\Phi|}) + < \nabla(d(x,y)g(|\nabla I(x,y)|)), \nabla\Phi > -\alpha|\nabla\Phi|. \\ \frac{\partial \Phi}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega. \end{array} \right. \tag{3.12}$$

Some remarks can be formulated as regards the Partial Differential Equation satisfied by Φ.

1. For this following remark, we refer the reader to the works of V. Caselles, R. Kimmel and G. Sapiro [7]. For an ideal edge, $|\nabla I| = \infty$, $g = 0$ and the curve stops, which is in practice unrealistic. Here, we have refined the criterion that makes the curve stops with the term

$$< \nabla(d(x,y)g(|\nabla I(x,y)|)), \nabla\Phi >$$

   which naturally appears in the model and that uses jointly the gradient of the gradient of the image coupled with the distance function. This is more accurate that the mere gradient in particular if the variations of the gradient along a boundary are different (*cf V.Caselles, F. Catté, T. Coll and F.Dibos [6]*). Besides, it takes both, information linked with the image, *i.e* whether we are or not on a boundary and information linked with the geometrical constraints : *i.e* whether we are near or not the set of given points.

2. We have :

$$\frac{\partial \Phi}{\partial t} = |\nabla\Phi|[div(d(x,y)g(|\nabla I(x,y)|)\frac{\nabla\Phi}{|\nabla\Phi|}) - \alpha]$$

   All the Level Sets move according to :

$$\Gamma_t = d(x,y)g(|\nabla I(x,y)|)\kappa\vec{n} - < \nabla(d(x,y)g(|\nabla I(x,y)|)), \vec{n} > \vec{n} - \alpha\vec{n}.$$

with $\kappa = div(\frac{\nabla\Phi}{|\nabla\Phi|})$ and $\vec{n} = -\frac{\nabla\Phi}{|\nabla\Phi|}$ (interior normal). Indeed, we have for all level sets, $\Phi(\Gamma(t), t) = constant$.

Carrying out the differentiations with respect to the variable t we have:

$$\frac{d}{dt}[\Phi(\Gamma(t), t)] = 0 \Leftrightarrow \Phi_t + < \nabla\Phi, \Gamma_t >= 0.$$

So, using the definition of Γ_t and $\vec{n}$, we get

$$\Phi_t = d(x, y)g(|\nabla I(x, y)|)\kappa|\nabla\Phi| + < \nabla(d(x, y)g(|\nabla I(x, y)|)), \nabla\Phi > -\alpha|\nabla\Phi|.$$

3. In the right part of the PDE, $\nabla(d(x)g(|\nabla I(x)|))$ is well-defined, except at the points that are equidistant from at least two points of the finite set of given points S. Indeed, the function $d(x) = d(x, S)$ is continuous as the inf of a finite number of continuous functions but at these equidistant points the solution is non-differentiable and the gradient is not defined.
   The distance function d satisfies the Eikonal equation $|\nabla d| = 1$. In the theoretical part devoted to the existence and uniqueness of the solution of our problem, we need a certain smoothness on the distance function d. The role of curvature as a regularizing or smoothing term enables us to get the desired properties on d. What follows is taken from Sethian's book *[33]* The main conclusion that we use here is that 'a front propagating at a speed $1 - \epsilon\kappa$ for $\epsilon > 0$ does not form corners and stay smooth for all time. Furthermore, as the dependence on curvature vanishes, the limit of this motion is the entropy-satisfying solution obtained for the constant speed case.

In the next section, we aim at proving the existence and uniqueness of the solution of this parabolic problem, with the theory of viscosity.

4. Existence, uniqueness of the solution of the problem - Viscosity theory

4.1. General Background. We may remind the reader of some elements of viscosity solutions. The general background is widely taken from the *'User's guide to viscosity solutions'* (Crandall, Ishii and Lions) [18]. This theory applies to some partial differential equations that can formally be written on the form $F(x, u, Du, D^2u)$ where Du denotes the gradient and D^2u the Hessian matrix, which is a symmetric matrix. F is generally defined by the following way :

$$F : \Re^n \times \Re \times \Re^n \times S(n) \longrightarrow \Re$$

where $S(n)$ represents the set of symmetric $(n \times n)$ matrices.
In our case, F will be defined by $F : \Omega \times \Re \times \Re^2 \times S(2) \longrightarrow \Re$. Two conditions (monotonicity condition) are necessary to apply this theory to an equation of the type $F = 0$, namely :

1. $F(x, r, p, X) \leq F(x, s, p, X)$ with $r \leq s$.
2. $F(x, r, p, Y) \leq F(x, r, p, X)$ with $Y \geq X$.

The second condition is called *degenerate ellipticity.* When both condition hold, F is said to be proper. Coming back to our problem, we plan to check if these conditions are respected in our model. We note that for parabolic problems if $(x, r, p, X) \longrightarrow F(t, x, r, p, X)$ is proper for fixed $t \in [0, T]$, then so is the associated parabolic problem :

$$u_t + F(t, x, u, Du, D^2u) = 0.$$

In our case, we have

$$\left| \begin{array}{l} \dfrac{\partial \Phi}{\partial t} - |\nabla \Phi| d(x,y) g(|\nabla I(x,y)|) div(\dfrac{\nabla \Phi}{|\nabla \Phi|}) \\ - < \nabla(d(x,y) g(|\nabla I(x,y)|)), \nabla \Phi > + \alpha |\nabla \Phi| = 0 \end{array} \right. .$$

Carrying out the differentiations yields

$$\left| \begin{array}{l} \Phi_t - \\ |\nabla \Phi| d(x,y) g\left(|\nabla I(x,y)|\right) \left(\frac{\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}}{|\nabla \Phi|} - \frac{\left(2 \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial y} \frac{\partial^2 \Phi}{\partial x \partial y} + (\frac{\partial \Phi}{\partial y})^2 \frac{\partial^2 \Phi}{\partial y^2} + (\frac{\partial \Phi}{\partial x})^2 \frac{\partial^2 \Phi}{\partial x^2}\right) |\nabla \Phi|^{-1}}{|\nabla \Phi|^2} \right) \\ - < \nabla(d(x,y)\ g(|\nabla I(x,y)|)), \nabla \Phi > + \alpha |\nabla \Phi| = 0. \end{array} \right.$$

With $\frac{p \bigotimes p}{|p|^2}$ denoting the matrix defined by

$$\frac{p \bigotimes p}{|p|^2} = \frac{1}{|p|^2} \begin{pmatrix} p_1^2 & p_1 p_2 \\ p_1 p_2 & p_2^2 \end{pmatrix}$$

with $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \Re^2$, we find the corresponding F :

$$\left| \begin{array}{l} F(x, y, r, p, X) = \\ -d(x,y) g(|\nabla I(x,y)|) trace(X) + d(x,y) g(|\nabla I(x,y)|) trace(\frac{p \bigotimes p}{|p|^2} X) \\ - < \nabla(d(x,y) g(|\nabla I(x,y)|)), p > + \alpha |p|. \end{array} \right.$$

So, we get

$$\left| \begin{array}{l} F(x, y, r, p, X) = \\ -d(x,y) g(|\nabla I(x,y)|) trace((I - \frac{p \bigotimes p}{|p|^2}) X) \\ - < \nabla(d(x,y) g(|\nabla I(x,y)|)), p > + \alpha |p|. \end{array} \right.$$

It is obvious that the first condition is satisfied since it doesn't depend explicitly on r. Let us deal with the degenerate ellipticity. We have then to compare $F(x, y, r, p, X)$ and $F(x, y, r, p, Y)$ when $Y \geq X$, that is to say compare

$$-d(x,y) g(|\nabla I(x,y)|) trace((I - \frac{p \bigotimes p}{|p|^2}) X)$$

and

$$-d(x,y)g(|\nabla I(x,y)|)trace((I - \frac{p \bigotimes p}{|p|^2})Y)$$

when the inequality holds for X and Y. The set of symmetric matrices is equipped with the usual order, which means that

$$X \leq Y \Longleftrightarrow \forall \xi \in \Re^2 - \{0_{\Re^2}\}, \xi^t X \xi \leq \xi^t Y \xi$$

Let us denote by $A(p)$ the matrix defined by

$$A(p) = I - \frac{p \bigotimes p}{|p|^2}$$

which corresponds to

$$A = \begin{pmatrix} \frac{p_2^2}{|p|^2} & -\frac{p_1 p_2}{|p|^2} \\ -\frac{p_1 p_2}{|p|^2} & \frac{p_1^2}{|p|^2} \end{pmatrix}$$

We suppose as well that $p \neq 0$ since we have a singularity for $p = 0$. The quantity $A(p)$ is positive since

$$\forall \xi \in \Re^2 - \{0_{\Re^2}\} \text{ such that } \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} :$$

$$\xi^t A(p) \xi = \frac{1}{|p|^2}(\xi_1 p_2 - \xi_2 p_1)^2 \geq 0$$

The matrix $A(p)$ is symmetric positive. Its eigen values are positive and there exists an orthonormal basis such that $D = P^t A P$, with P an orthogonal matrix and D a diagonal matrix with positive values. Thus, one can write $A = \sigma\sigma^t$, with σ the matrix defined by $\sigma = PD^{\frac{1}{2}}$. Coming back to our problem, we have, with the following notations:

$$\left| \begin{array}{l} trace(AX) = trace(\sigma\sigma^t X) = trace(\sigma^t X \sigma) \\ \text{So, } trace(AX) = \sum_{i=1}^{2} \sigma_i^t X \sigma_i, \text{with } \sigma_i \text{ the i column of } \sigma. \end{array} \right.$$

Suppose that $Y \geq X$. We can easily deduce that

$$\forall i \in \{1, 2\}, \sigma_i^t X \sigma_i \leq \sigma_i^t Y \sigma_i.$$

The functions d and g being positive, we conclude that:

$$-\mathbf{d}(\mathbf{x},\mathbf{y})\mathbf{g}(|\boldsymbol{\nabla}\mathbf{I}(\mathbf{x},\mathbf{y})|)\mathbf{trace}(\mathbf{A}(\mathbf{p})\mathbf{X}) \geq -\mathbf{d}(\mathbf{x},\mathbf{y})\mathbf{g}(|\boldsymbol{\nabla}\mathbf{I}(\mathbf{x},\mathbf{y})|)\mathbf{trace}(\mathbf{A}(\mathbf{p})\mathbf{Y}) \quad (4.13)$$

We have then proved that for $p \neq 0$, **F is degenerate elliptic**.

This is the general background in which the viscosity theory has been first introduced. We then suppose, that for all the theorems coming next, F is proper and continuous.

In the **User's guide to viscosity solutions** [18], the notion of viscosity is introduced with an example using the maximum principle, as follows : F is proper, as already said. Suppose that $u \in C^2(O)$ and that :

$$F(x, u(x), Du(x), D^2u(x)) \leq 0 \ \forall x \in \Re^n$$

Suppose that φ is also $C^2(O)$ (O being an open subset of $\Re^n$) and $\hat{x} \in O$ is a local maximum of $u - \varphi$. It implies that $D(u - \varphi) = 0$ and $D^2(u - \varphi) \leq 0$, so:

$$\begin{cases} Du = D\varphi \\ D^2u \leq D\varphi. \end{cases}$$

Using the property of degenerate ellipticity of F, we get :

$$F(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})) \leq 0$$

We can now give a definition of viscosity solutions:

Definition 4. *$u \in C(O)$ is a viscosity solution of $F = 0$ if and only if:*

> *$\forall \Phi \in C^2(O)$, if x_0 is a local maximum of $u - \Phi$, we have the relation :*
> *$F(x_0, u(x_0), D\Phi(x_0), D^2\Phi(x_0)) \leq 0$*

and

> *$\forall \Phi \in C^2(O)$, if x_0 is a local minimum of $u - \Phi$, we have the relation :*
> *$F(x_0, u(x_0), D\Phi(x_0), D^2\Phi(x_0)) \geq 0$*

If u only satisfies the first (second) inequation, then u is said to be a viscosity subsolution (viscosity supersolution).

Crandall, Ishii and Lions [18] and Barles [4] give another definition based on the notions of superjet and subjet. Regarding what has been done above, we have for x near $\hat{x}$,

$$u(x) \leq u(\hat{x}) - \Phi(\hat{x}) + \Phi(x)$$

and Taylor gives (Φ being C^2):

$$u(x) \leq u(\hat{x}) + < p, x - \hat{x} > + \frac{1}{2} < X(x - \hat{x}), x - \hat{x} > + o(|x - \hat{x}|^2), x \longrightarrow \hat{x} \qquad (4.14)$$

Thus, we say that if $u : O \longrightarrow \Re$, $\hat{x} \in O$ and (4.14) is satisfied as $O \ni x \longrightarrow \hat{x}$, $(p, X) \in J_O^{2,+}u(\hat{x})$, $J_O^{2,+}u(\hat{x})$ being the second order superjet of u at $\hat{x}$.

Definition 5. *A viscosity subsolution of $F = 0$ on O is a function $u \in USC(O)$ such that:*

$$F(x, u(x), p, X) \leq 0 \ \forall x \in O, (p, X) \in J_O^{2,+}u(x)$$

A viscosity supersolution of $F = 0$ on O is a function $v \in LSC(O)$ such that :

$$F(x, v(x), p, X) \geq 0 \ \forall x \in O, (p, X) \in J_O^{2,-}v(x)$$

u is a viscosity solution if it is both a subsolution and a supersolution. USC(O) is the set of upper semicontinuous functions on O and LSC(O) is the set of lower semicontinuous functions on O.

Consider the following parabolic problem :

$$u_t + F(t, x, u, Du, D^2u) = 0.$$

For parabolic problems, Crandall, Ishii and Lions [18] extend these definitions. Here we assume that Du and D^2u mean $D_x u(t, x)$ and $D_x^2 u(t, x)$. Instead of working on O, a locally compact subset of $\Re^n$, we work on O_T=$]0, T[\times O$. They denote by $P_O^{2,+}$ and $P_O^{2,-}$ the parabolic variants of $J_O^{2,+}$ and $J_O^{2,-}$. Thus, $P_O^{2,+}u$ is defined such that: $(a, p, X) \in \Re \times \Re^n \times S(N)$ lies in $P_O^{2,+}u(s, z)$ if $(s, z) \in O_T$ and

$$\left| \begin{array}{l} u(t, x) \leq u(s, z) + a(t-s) + < p, x-z > + \frac{1}{2} < X(x-z), x-z > \\ + o(|t-s| + |x-z|^2) \text{ as } O_T \ni (t, x) \longrightarrow (s, z) \end{array} \right.$$

Definition 6. *A subsolution of the parabolic equation on O_T is a function $u \in USC(O_T)$ such that :*

$$a + F(t, x, u(t, x), p, X) \leq 0 \text{ for } (t, x) \in O_T \text{ and } (a, p, X) \in P_O^{2,+}u(t, x)$$

A supersolution of the parabolic equation on O_T is a function $v \in LSC(O_T)$ such that:

$$a + F(t, x, v(t, x), p, X) \geq 0 \text{ for } (t, x) \in O_T \text{ and } (a, p, X) \in P_O^{2,-}v(t, x)$$

4.2. Existence and uniqueness of the solution of our problem : Proof based on Barles's work. (For more details on this part, see [28]).

In this section, we refer the reader to the following articles :

1. *Nonlinear Neumann Boundary Conditions for Quasilinear Degenerate Elliptic Equations and Applications* (G. Barles [4]).

2. *Nonlinear Oblique derivative problems for singular degenerate parabolic equations on a general domain* (H. Ishii & M-H. Sato [22]).

3. *Generalized interface evolution with Neumann boundary condition* (Y. Giga & M-H. Sato [20]).

Preliminary :

Let us denote by J, a nonempty interval of $\Re$.
We recall that if $f : J \rightarrow \Re$ is differentiable on I, we have:

f is a lipschitz function on J $\Leftrightarrow$ f' is bounded

Besides, f is a k-lipschitz function with $k = sup_J|f'|$. The function g of our problem is defined on $\Re^+$ by $g : s \rightarrow \dfrac{1}{1+s^2}$. The first derivative of g with respect to the variable s is bounded as you can see on the following figure. One has $sup|g'(s)|_{\Re^+} = \dfrac{3}{8}\sqrt{8}$, which proves that g is $\frac{3}{8}\sqrt{8}$-lipschitz.

Figure 2: Function g.

The second derivative of g with respect to the variable s is also bounded on $\Re^+$. One can easily deduce that g' is 2-lipschitz.

Proposition 7. *With,* $sup_{i,j} sup_{x\in\Omega}|\frac{\partial^2 I(x)}{\partial x_i \partial x_j}| < \infty$, $x \rightarrow g(|\nabla I(x)|)$ *is lipschitz.*

Proof : evident.

Figure 3: Function g'.

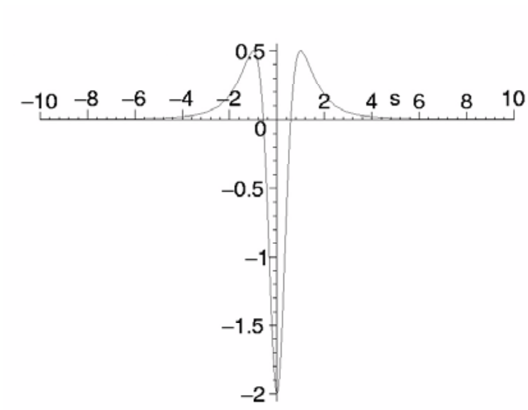

Figure 4: Function g''.

Remark: This condition on I is natural. I is supposed to be the smooth function obtained from a discrete attribute function after post-processing (filtering with a gaussian filter, etc..).

$$\text{On } \Re^+, \text{ g is } \tfrac{3}{8}\sqrt{8}\text{-lipschitz} \Leftrightarrow \forall (s_1, s_2) \in (\Re^+)^2, |g(s_1) - g(s_2)| \leq \frac{3}{8}\sqrt{8}|s_1 - s_2|.$$

We easily deduce that,

$$\forall (x, y) \in \Omega \times \Omega, |g(|\nabla I(x)|) - g(|\nabla I(y)|)| \leq \frac{3}{8}\sqrt{8}||\nabla I(x)| - |\nabla I(y)||.$$

Let denote by h the function defined by

$$h(x_1, x_2) = |\nabla I(x_1, x_2)|$$

with $x = (x_1, x_2) \in \Omega$. We have

$$\frac{\partial h}{\partial x_1} = \frac{\frac{\partial I}{\partial x_1}\frac{\partial^2 I}{\partial x_1^2} + \frac{\partial I}{\partial x_2}\frac{\partial^2 I}{\partial x_2 \partial x_1}}{|\nabla I(x_1, x_2)|}$$

so, $|\frac{\partial h}{\partial x_1}| \leq sup_{x\in\Omega}(|\frac{\partial^2 I}{\partial x_1^2}|, |\frac{\partial^2 I}{\partial x_1 \partial x_2}|)\frac{|\frac{\partial I}{\partial x_1}|+|\frac{\partial I}{\partial x_2}|}{|\nabla I(x_1,x_2)|}$

And then, from Cauchy-Schwartz $|\frac{\partial h}{\partial x_1}| \leq \sqrt{2} sup_{x\in\Omega}(|\frac{\partial^2 I}{\partial x_1^2}|, |\frac{\partial^2 I}{\partial x_2 \partial x_1}|) < \infty$.

An analogous method gives

$$|\frac{\partial h}{\partial x_2}| \leq \sqrt{2} sup_{x\in\Omega}(|\frac{\partial^2 I}{\partial x_2^2}|, |\frac{\partial^2 I}{\partial x_2 \partial x_1}|) < \infty.$$

Therefore, there exists k such that:

$$||\nabla I(x)| - |\nabla I(y)|| \leq k|x - y|.$$

The functions d and $g(|\nabla I|)$ are respectively bounded on Ω, and are lipschitz functions so the product $d.g(|\nabla I|)$ is a lipschitz function. There exists k_1 such that:

$$|d(x).g(|\nabla I(x)|) - d(y).g(|\nabla I(y)|| \leq k_1|x - y|$$

As regards

$$\nabla(d.g(|\nabla I|)) = g(|\nabla I|).\nabla d + d.\nabla g(|\nabla I|), |\nabla d|$$

is bounded.

$$\nabla g(|\nabla I|) = \begin{pmatrix} \frac{\partial g(|\nabla I(x_1,x_2)|)}{\partial x_1} \\ \frac{\partial g(|\nabla I(x_1,x_2)|)}{\partial x_2} \end{pmatrix}$$

So

$$\nabla g(|\nabla I|) = \begin{pmatrix} g'(|\nabla I(x_1, x_2)|)\frac{\partial|\nabla I(x_1,x_2)|}{\partial x_1} \\ g'(|\nabla I(x_1, x_2)|)\frac{\partial|\nabla I(x_1,x_2)|}{\partial x_2} \end{pmatrix}$$

The properties on $g\prime$ and on $|\nabla I(x_1, x_2)|$ enable us to conclude that $|\nabla g|$ is bounded. Besides, with some assumptions on I, one can conclude that $x \to |\nabla(d.g(|\nabla I|))|$ is lipschitz.

In what follows, we suppose that the given image I satisfies the smoothness conditions required.

With a certain regularity on $x \to d(x)g\left(|\nabla I(x)|\right)$, one can have $x \to \left(d(x)g\left(|\nabla I(x)|\right)\right)^{1/2}$ Lipschitz (Freidlin).

In [4], G. Barles proposes an existence and uniqueness theorem for the Neumann homogeneous Boundary parabolic problem :

$$(\xi) \begin{cases} u_t + F(x, t, Du, D^2u) = 0 \text{ in } \Omega \times [0, +\infty[\\ u(x, 0) = u_0(x) \\ L(x, t, Du) = 0 \text{ on } \partial\Omega \times [0, +\infty[\end{cases} \tag{4.15}$$

based on the notion of **Equation of geometrical type**.

Definition 8. *An equation of the type* $u_t + F(x,t,Du,D^2u) = 0 \in \Omega\times]0,T[$ *is said to be of geometrical type if the function* F *satisfies :*

$$F(x,t,\lambda p,\lambda M + \mu p \bigotimes p) = \lambda F(x,t,p,M)$$

$$\forall x \in \Re^n, t \in]0,+\infty[, p \in \Re^n, M \in S(n), \lambda > 0, \mu \in \Re$$

We prove that our equation is of geometrical type.
We remind the reader that F is defined by (we omit here the term with the constant α):

$$\left| \begin{array}{l} F(x,t,p,X) = \\ -d(x)g(|\nabla I(x)|)trace((I - \frac{p\bigotimes p}{|p|^2})X) \\ - < \nabla(d(x)g(|\nabla I(x)|)), p > . \end{array} \right.$$

with $x \in \Omega$. Then, we get that

$$\left| \begin{array}{l} F(x,t,\lambda p,\lambda X + \mu p \bigotimes p) = \\ -d(x)g(|\nabla I(x)|)trace\left((I - \frac{p\bigotimes p}{|p|^2})(\lambda X + \mu p \bigotimes p)\right) \\ -\lambda \left\langle \nabla(d(x)g(|\nabla I(x)|)), p \right\rangle . \end{array} \right.$$

So,

$$\left| \begin{array}{l} F(x,t,\lambda p,\lambda X + \mu p \bigotimes p) = \\ \lambda F(x,t,p,X) \\ -\mu d(x)g(|\nabla I(x)|)trace\left((I - \frac{p\bigotimes p}{|p|^2})(p \bigotimes p)\right) . \end{array} \right.$$

But,

$$trace((I - \frac{p\bigotimes p}{|p|^2})(p \bigotimes p)) = 0$$

and it follows that

$$F(x,t,\lambda p,\lambda X + \mu p \bigotimes p) = \lambda F(x,t,p,X).$$

which proves that our equation is of geometrical type.

Theorem 9. *(Singular Case for* $p = 0$*-* F *has a discontinuity for* $Du = 0$ *but is assumed to be continuous everywhere else.) Assume that the equation considered is of geometrical type and that the following assumptions are satisfied :*

1. *Ω is a bounded domain with a $W^{3,\infty}$ boundary and $u_0 \in C(\bar{\Omega})$.*
2. *With a given t, for any $R > 0$, there exists $\nu_R > 0$ independent of t, such that for all $\lambda > 0, x \in \partial\Omega$, $-R \le v \le u \le R, p \in \Re^n$, one has:*

$$L(x,t,u,p+\lambda n(x)) - L(x,t,v,p) \ge \nu_R \lambda$$

3. *The function L is independent of u, homogeneous of degree 1 in p and there exists a constant* $C > 0$ *such that, for any* $x, y \in \bar{\Omega}$, $p, q \in \Re^n$ *and for any t:*

$$|L(x,t,p) - L(y,t,q)| \leq C[(|p| + |q|)|x-y| + |p-q|]$$

L is locally lipschitz continuous in t for every $x \in \partial\Omega$, $u \in \Re$, $p \in \Re^n$ *and for any* $R > 0$, *there exists a constant* $C_R > 0$ *such that, for all* $x \in \partial\Omega$, $0 \leq t, s < T$, $-R \leq u \leq R$, $p, q \in \Re^n$, *one has:*

$$|L(x,t,u,p) - L(x,s,u,q)| \leq C_R(1 + max(|p|,|q|))|t-s|$$

4. *For any* $R > 0$, *there exists* γ_R *such that for all* $x \in \bar{\Omega}$, $-R \leq v \leq u \leq R$, $p \in \Re^n$ *and* $M \in S(n)$, *for any t , one has:*

$$F(x,t,u,p,M) - F(x,t,v,p,M) \geq \gamma_R(u-v)$$

5. *Consider t given. For any* $R, K > 0$, *there exists a function* $m_{R,K} : \Re^+ \longrightarrow \Re$ *such that* $m_{R,K}(l) \longrightarrow 0$ *when l tends to 0 and such that for all* $\eta > 0$:

$$F(y,t,u,q,Y) - F(x,t,u,p,X) \leq m_{R,K}(\eta + |x-y|(1 + max(|p|,|q|)) + \frac{|x-y|^2}{\epsilon^2})$$

for any $x, y \in \bar{\Omega}$, $|u| \leq R$, $p, q \in \Re^n - \{0\}$ *and for any matrices* $X, Y \in S(n)$ *satisfying the following properties :*

$$|x-y| \leq K\eta\epsilon \quad (4.16)$$

$$|p-q| \leq K\epsilon(min(|p|,|q|)) \quad (4.17)$$

$$-\frac{K\eta}{\epsilon^2}\begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{K\eta}{\epsilon^2}\begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix} + K\eta\begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}. \quad (4.18)$$

Then, for any initial $u_0 \in C(\bar{\Omega})$, *there exists a unique viscosity solution* $u \in C(\bar{\Omega} \times [0, +\infty[)$ *of* ξ.

Proof. Our problem is defined by :

$$(\xi)\begin{cases} u_t + F(x,t,u,Du,D^2u) = 0 \text{ in } \Omega \times [0,+\infty[\\ u(x,0) = u_0(x) \\ L(x,t,u,Du) = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \times [0,+\infty[\end{cases} \quad (4.19)$$

with

$$\left| \begin{array}{l} F(x,t,u,p,X) = \\ -d(x)g(|\nabla I(x)|)trace((I - \frac{p\bigotimes p}{|p|^2})X) \\ - < \nabla(d(x)g(|\nabla I(x)|)), p >= 0 \text{ in } \ \Omega \times [0,+\infty[\end{array} \right. ,$$

with

$$\begin{cases} u(x,0) = u_0(x) \\ L(x,t,u,p) =< n(x), p >= 0 \text{ on } \ \partial\Omega \times [0,+\infty[\end{cases}$$

The first point of the Theorem is supposed to be satisfied.

For the second point, we have,

$$\begin{array}{l} L(x,t,u,p+\lambda n(x)) - L(x,t,v,p) =< n(x), p+\lambda n(x) > - < n(x), p > . \\ L(x,t,u,p+\lambda n(x)) - L(x,t,v,p) = \lambda \end{array}$$

We just have to take $\nu_R = 1$.

The third point is satisfied as well, since we have L independent of u, homogeneous of degree 1 in p ($L(x,\lambda p) = \lambda L(x,p)$). Besides,

$$|L(x,p) - L(y,q)| = | < n(x) - n(y), p > + < n(y), p - q > |$$

Suppose that $n(x)$ is Lipschitz, then using Cauchy-Schwarz inequality, we get:

$$\left| \begin{array}{l} |L(x,p) - L(y,q)| \leq |n(x) - n(y)||p| + |p-q| \\ \text{and n being } C^1\text{, there exists C constant such that, } \ |L(x,p) - L(y,q)| \leq C|x-y||p| + |p-q| \\ \text{So, } |L(x,p) - L(y,q)| \leq C|x-y|(|p|+|q|) + |p-q| \\ \text{And, } |L(x,p) - L(y,q)| \leq max(C,1)(|x-y|(|p|+|q|) + |p-q|) \end{array} \right.$$

Lastly, L doesn't depend on t explicitly so,

$$|L(x,t,u,p) - L(x,s,u,p)| = 0$$

The last criterion of the third point is satisfied since any constant $C_R > 0$ fits the relation.

The fourth point is obvious since F doesn't depend explicitly on u. One can choose $\gamma_R = 0$.

One can rewrite the function F by the following way:

$$F(x,t,p,X) = -trace(d(x)g(|\nabla I(x)|)(I - \frac{p\bigotimes p}{|p|^2})X) - < \nabla(d(x)g(|\nabla I(x)|)), p > .$$

Let us denote by $A(x,p)$ the positive symmetric matrix defined by:

$$A(x,p) = d(x)g(|\nabla I(x)|)(I - \frac{p\bigotimes p}{|p|^2}).$$

For the last point, the inequality (4.17) gives :

$$(Xr, r) - (Ys, s) \leq \frac{K\eta}{\epsilon^2}|r - s|^2 + K\eta(|r|^2 + |s|^2).$$

We have as well, $trace(A(x,p)X) = trace(\sigma(x,p)\sigma(x,p)^T X) = trace(\sigma(x,p)^T X\sigma(x,p))$. So,

$$trace(A(x,p)X) = \sum_{i=1}^{2}(X\sigma(x,p)e_i, \sigma(x,p)e_i),$$

with $(e_i)_i$ an orthonormal basis. Using the preceding inequality, we get :

$$\left| \begin{array}{l} trace(A(x,p)X) - trace(A(y,q)Y) \\ \\ \leq \frac{K\eta}{\epsilon^2}\sum_{i=1}^{2}|(\sigma(x,p) - \sigma(y,q))e_i|^2 + K\eta\sum_{i=1}^{2}|\sigma(x,p)e_i|^2 + |\sigma(y,q)e_i|^2 \end{array} \right. .$$

But, when $(e_i)_i$ denotes an orthonormal basis, we have $trace(Q) = \sum_{i=1}^{N}(Qe_i, e_i)$, so

$$\left| \begin{array}{l} trace(A(x,p)X) - trace(A(y,q)Y) \\ \\ \leq \frac{K\eta}{\epsilon^2}trace((\sigma(x,p) - \sigma(y,q))(\sigma(x,p) - \sigma(y,q))^T) + K\eta(trace(\sigma(x,p)\sigma(x,p)^T) \\ \\ +trace(\sigma(y,q)\sigma(y,q)^T)). \end{array} \right.$$

And,

$$\left| \begin{array}{l} trace(A(x,p)X) - trace(A(y,q)Y) \leq \\ \\ \frac{K\eta}{\epsilon^2}trace((\sigma(x,p) - \sigma(y,q))(\sigma(x,p) - \sigma(y,q))^T)+ \\ \\ K\eta(trace(A(x,p)) + trace(A(y,q))). \end{array} \right.$$

With,

$$\left\{ \begin{array}{l} trace(A(x,p)) = d(x)g(|\nabla I(x)|). \\ trace((\sigma(x,p) - \sigma(y,q))(\sigma(x,p) - \sigma(y,q))^T) = |\sqrt{d(x)g(|\nabla I(x)|)}\frac{p}{|p|} - \sqrt{d(y)g(|\nabla I(y)|)}\frac{q}{|q|}|^2 \end{array} \right.$$

(The last point can be easily proved remarking that A has two eigenvalues 0 and $d(x)g(|\nabla I(x)|)$. The associated eigenvectors are respectively $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $\begin{pmatrix} -p_2 \\ p_1 \end{pmatrix}$. The orthonormal matrix P such that:

$$A(x,p) = P(p)\begin{pmatrix} 0 & 0 \\ 0 & d(x)g(|\nabla I(x)|) \end{pmatrix}P(p)^t$$

is defined by :

$$P = \begin{pmatrix} \frac{p_1}{|p|} & -\frac{p_2}{|p|} \\ \frac{p_2}{|p|} & \frac{p_1}{|p|} \end{pmatrix}$$

Thus, $\sigma(x,p) = \begin{pmatrix} 0 & -\frac{p_2}{|p|}\sqrt{d(x)g(|\nabla I(x)|)} \\ 0 & \frac{p_1}{|p|}\sqrt{d(x)g(|\nabla I(x)|)} \end{pmatrix})$.
So, we get:

$$trace(A(x,p)X) - trace(A(y,q)Y) \leq \frac{K\eta}{\epsilon^2}|\sqrt{d(x)g(|\nabla I(x)|)}\frac{p}{|p|} - \sqrt{d(y)g(|\nabla I(y)|)}\frac{q}{|q|}|^2 +$$

$$K\eta(d(x)g(|\nabla I(x)|) + d(y)g(|\nabla I(y)|)).$$

To prove the last point, we have to evaluate

$$\left| \begin{array}{l} F(y,t,u,q,Y) - F(x,t,u,p,X) = trace(A(x,p)X) - trace(A(y,q)Y) \\ \\ + < \nabla(d(x)g(|\nabla I(x)|)), p > - < \nabla(d(y)g(|\nabla I(y)|)), q > . \end{array} \right.$$

We have,

$$\left| \begin{array}{l} trace(A(x,p)X) - trace(A(y,q)Y) \\ \\ \leq \frac{K\eta}{\epsilon^2}|\sqrt{d(x)g(|\nabla I(x)|)}\frac{p}{|p|} - \sqrt{d(y)g(|\nabla I(y)|)}\frac{q}{|q|}|^2 + \\ \\ K\eta(d(x)g(|\nabla I(x)|) + d(y)g(|\nabla I(y)|)). \end{array} \right.$$

Besides, we see that:

$$\left| \begin{array}{l} |\sqrt{d(x)g(|\nabla I(x)|)}\frac{p}{|p|} - \sqrt{d(y)g(|\nabla I(y)|)}\frac{q}{|q|}|^2 \\ \\ = |(\sqrt{d(x)g(|\nabla I(x)|)} - \sqrt{d(y)g(|\nabla I(y)|)})\frac{p}{|p|} + \sqrt{d(y)g(|\nabla I(y)|)}(\frac{p}{|p|} - \frac{q}{|q|})|^2 \\ \\ \leq 2(\sqrt{d(x)g(|\nabla I(x)|)} - \sqrt{d(y)g(|\nabla I(y)|)})^2 + 2d(y)g(|\nabla I(y)|)|\frac{p}{|p|} - \frac{q}{|q|}|^2 \end{array} \right.$$

The function $x \longrightarrow d(x)g(|\nabla I(x)|)$ is bounded as seen in the *'preliminary section'* and one can find a constant θ such that $d(x)g(|\nabla I(x)|) \leq \theta$.

Using (4.16) and properties on the function $x \longrightarrow \sqrt{d(x)g(|\nabla I(x)|)}$, we conclude that:

$$\left| \begin{array}{l} trace(A(x,p)X) - trace(A(y,q)Y) \\ \\ \leq 2K\eta\zeta\frac{|x-y|^2}{\epsilon^2} + \eta(2\theta K^3 + 2\theta K). \end{array} \right.$$

We have also the relation :

$$\left| \begin{array}{l} < \nabla(d(x)g(|\nabla I(x)|)), p) > - < \nabla(d(y)g(|\nabla I(y)|)), q) >= \\ < \nabla(d(x)g(|\nabla I(x)|)) - \nabla(d(y)g(|\nabla I(y)|)), p) > + \\ < \nabla(d(y)g(|\nabla I(y)|)), p - q) > . \end{array} \right.$$

Assumptions on $\nabla(d(x)g(|\nabla I(x)|))$ enable us to find constants ρ and μ such that :

$$\left| \begin{array}{l} | < \nabla(d(x)g(|\nabla I(x)|)), p) > - < \nabla(d(y)g(|\nabla I(y)|)), q) > | \\ \\ \leq \rho |x-y| max(|p|,|q|) + \mu K \epsilon min(|p|,|q|). \end{array} \right.$$

We deduce from all that has been proved, that:

$$\left| \begin{array}{l} F(y,t,u,q,Y) - F(x,t,u,p,X) \\ \\ \leq 2K\eta\zeta\frac{|x-y|^2}{\epsilon^2} + \eta(2\theta K^3 + 2\theta K) + \\ \\ \rho|x-y|max(|p|,|q|) + \mu K\epsilon min(|p|,|q|). \end{array} \right.$$

Suppose that ϵ is small enough so that we have :

$$\mu K\epsilon min(|p|,|q|) \leq \eta$$

So,

$$\left| \begin{array}{l} F(y,t,u,q,Y) - F(x,t,u,p,X) \\ \\ \leq 2K\eta\zeta\frac{|x-y|^2}{\epsilon^2} + \eta(2\theta K^3 + 2\theta K) + \\ \\ \rho|x-y|max(|p|,|q|) + \eta. \end{array} \right.$$

and

$$\left| \begin{array}{l} F(y,t,u,q,Y) - F(x,t,u,p,X) \\ \\ \leq 2K\eta\zeta\frac{|x-y|^2}{\epsilon^2} + \eta(2\theta K^3 + 2\theta K + 1) + \\ \\ \rho|x-y|(1 + max(|p|,|q|)). \end{array} \right.$$

Let us denote by $\aleph(K)$, the element :

$$\aleph(K) = max(2K\zeta, 2\theta K^3 + 2\theta K + 1, \rho)$$

Thus,

$$F(y,t,u,q,Y) - F(x,t,u,p,X) \leq \aleph(K)(\eta + |x-y|(1 + max(|p|,|q|)) + \eta\frac{|x-y|^2}{\epsilon^2})$$

We take for $m_{R,K}$ the function defined by

$$m_{R,K}(l) = \aleph(K)l(l+1).$$

that satisfies $m_{R,K}(l) \longrightarrow 0$ when l tends to 0.
The last point is proved.■

4.3. Existence and uniqueness of the solution of our problem : Proof based on Ishii and Giga's work. Here is a first result that will be used in the following.

Proposition :

$$|\frac{p}{|p|} - \frac{q}{|q|}| \leq \frac{|p-q|}{min(|p|,|q|)}$$

Proof :

Suppose that $min(|p|,|q|) = |p|$. Let's prove that :

$$|p - \frac{|p|}{|q|}q|^2 \leq |p-q|^2$$

We then have to prove that

$$2(1 - \frac{|p|}{|q|})(p,q) \leq |q|^2 - |p|^2$$

But, $(1 - \frac{|p|}{|q|}) \in [0,1]$, and using Cauchy-Schwartz inequality, we get:

$$2(1 - \frac{|p|}{|q|})(p,q) \leq 2|p|(|q| - |p|)$$

We deduce that

$$2(1 - \frac{|p|}{|q|})(p,q) \leq |q|^2 - |p|^2$$

since $|p| \leq |q|$ and so $2|p|(|q|-|p|) \leq |q|^2 - |p|^2$. The proposition is then proved. ■

We use here the Existence Theorem of viscosity solutions introduced by H. Ishii and M-H Sato in [22]. The article inscribes in a special framework namely 'singular parabolic equations with non-linear oblique derivative boundary conditions' but we wish to apply it to our problem with homogeneous Neumann boundary conditions. As mentioned in their article, we denote by $\rho(p,q) = min(\frac{|p-q|}{min(|p|,|q|)}, 1)$. We assume that Ω is a bounded domain in $\Re^n$ with C^1 boundary. Let us first consider the following propositions.

1. $F \in C([0,T] \times \bar{\Omega} \times \Re \times (\Re^n - \{0\}) \times S^n)$ with S^n the space of $n \times n$ matrices equipped with the usual ordering.

2. There exists a constant $\gamma \in \Re$ such that for each $(t,x,p,X) \in [0,T] \times \bar{\Omega} \times (\Re^n - \{0\}) \times S(n)$, the function $u \mapsto F(t,x,u,p,X) - \gamma u$ is non decreasing on $\Re$.

3. For each $R > 0$, there exists a continuous function $w_R : [0,\infty[\longrightarrow [0,\infty[$ satisfying $w_R(0) = 0$ such that if $X, Y \in S^n$ and $\mu_1, \mu_2 \in [0,\infty[$ satisfy:

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \mu_1 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \mu_2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

then

$$\left| \begin{array}{l} F(t,x,u,p,X) - F(t,y,u,q,-Y) \\ \geq -w_R(\mu_1(|x-y|^2 + \rho(p,q)^2) + \mu_2 + |p-q| + |x-y|(1 + max(|p|,|q|)) \end{array} \right. \tag{4.20}$$

4. $B \in C(\Re^n \times \Re^n) \bigcap C^{1,1}(\Re^n \times (\Re^n - \{0\}))$

5. For each $x \in \Re^n$, the function $p \mapsto B(x,p)$ is positively homogeneous of degree one in p, ie, $B(x, \lambda p) = \lambda B(x,p)$, $\forall \lambda \geq 0, p \in \Re^n - \{0\}$.

6. There exists a positive constant θ such that $< \nu(z), D_pB(z,p) > \geq \theta$ for all $z \in \partial\Omega$ and $p \in \Re^n - \{0\}$. $\nu(z)$ denotes the unit outer normal vector of Ω at $z \in \partial\Omega$.

H.Ishii and M-H Sato assume that w_R is non-decreasing on $[0, \infty[$.

Theorem 10. *Consider the following problem :*

$$\xi \left\{ \begin{array}{l} u_t + F(t,x,u,Du,D^2u) = 0 \ in \ ]0,T[\times\Omega \\ B(x,Du) = 0 \ in \ ]0,T[\times\partial\Omega \end{array} \right. \tag{4.21}$$

satisfying $u(0,x) = g(x)$ for $x \in \bar{\Omega}$. Assume that points 1, 2, 3, 4, 5, 6 hold. Then for each $g \in C(\bar{\Omega})$ there is a unique viscosity solution $u \in C([0,T[\times\bar{\Omega})$ of ξ satisfying $u(0,x) = g(x)$ for $x \in \bar{\Omega}$.

We wish to apply this theorem to our problem.
"Our" F is defined by :

$$\left| \begin{array}{l} F(t,x,u,p,X) = \\ -trace\left(d(x)g(|\nabla I(x)|)(I - \dfrac{p \bigotimes p}{|p|^2})X \right) \\ - < \nabla(d(x)g(|\nabla I(x)|)), p >, \end{array} \right.$$

and so, by denoting by $A(x,p)$ the symmetric positive matrix defined by

$$A(x,p) = d(x)g(|\nabla I(x)|)(I - \frac{p \bigotimes p}{|p|^2})$$

we get:

$$F(t,x,u,p,X) = -trace(A(x,p)X) - < \nabla(d(x)g(|\nabla I(x)|)), p > .$$

F presents a singularity for $p = 0$ but is continuous otherwise. The first point is satisfied.

F does not depend explicitly on u so any negative constant γ satisfies the second criterion.

For the third point, the inequality gives us that for all $r, s \in \Re^n$,

$$(Xr, r) + (Ys, s) \leq \mu_1 |r - s|^2 + \mu_2(|r|^2 + |s|^2).$$

Taking successively $r = \sigma(x,p)e_i$ and $s = \sigma(y,q)e_i$ with $(e_i)_i$ an orthonormal basis (*as done in [28],* $A(x,p) = \sigma(x,p)\sigma^T(x,p)$), we get:

$$\left| \begin{array}{l} trace(A(x,p)X) + trace(A(y,q)Y) \\ \leq \mu_1 trace((\sigma(x,p) - \sigma(y,q))(\sigma(x,p) - \sigma(y,q))^T) \\ +\mu_2(d(x)g(|\nabla I(x)|) + d(y)g(|\nabla I(y)|)). \end{array} \right.$$

So

$$\left| \begin{array}{l} trace(A(x,p)X) + trace(A(y,q)Y) \\ \leq \mu_1 |\sqrt{d(x)g(|\nabla I(x)|)}\frac{p}{|p|} - \sqrt{d(y)g(|\nabla I(y)|)}\frac{q}{|q|}|^2 + 2\theta\mu_2, \end{array} \right|$$

The function $x \mapsto d(x)g(|\nabla I(x)|)$ being bounded by θ.

Besides, one has:

$$\left| \begin{array}{l} |\sqrt{d(x)g(|\nabla I(x)|)}\frac{p}{|p|} - \sqrt{d(y)g(|\nabla I(y)|)}\frac{q}{|q|}|^2 = \\ \\ |(\sqrt{d(x)g(|\nabla I(x)|)} - \sqrt{d(y)g(|\nabla I(y)|)})\frac{p}{|p|} + \sqrt{d(y)g(|\nabla I(y)|)}(\frac{p}{|p|} - \frac{q}{|q|})|^2. \end{array} \right.$$

so,

$$\left| \begin{array}{l} |\sqrt{d(x)g(|\nabla I(x)|)}\frac{p}{|p|} - \sqrt{d(y)g(|\nabla I(y)|)}\frac{q}{|q|}|^2 \leq \\ \\ 2(\sqrt{d(x)g(|\nabla I(x)|)} - \sqrt{d(y)g(|\nabla I(y)|)})^2 + 2d(y)g(|\nabla I(y)|)|\frac{p}{|p|} - \frac{q}{|q|}|^2 \end{array} \right.$$

Using the preliminary and some properties on the functions $x \mapsto d(x)g(|\nabla I(x)|)$ and $x \mapsto \sqrt{d(x)g(|\nabla I(x)|)}$ as in *[28]*, we can conclude that

$$\left| \begin{array}{l} trace(A(x,p)X) + trace(A(y,q)Y) \\ \leq \mu_1(2\zeta|x-y|^2 + 2\theta\rho(p,q)^2) + 2\theta\mu_2. \end{array} \right.$$

We have to evaluate the quantity :

$$\left| \begin{array}{l} F(t,x,u,p,X) - F(t,y,u,q,-Y) = -(trace(A(x,p)X) \\ +trace(A(y,q)Y)) - (< \nabla(d(x)g(|\nabla I(x)|)), p > \\ - < \nabla(d(y)g(|\nabla I(y)|)), q >). \end{array} \right.$$

Using the same arguments as in *[28]*, we have :

$$| < \nabla(d(x)g(|\nabla I(x)|)), p > - < \nabla(d(y)g(|\nabla I(y)|)), q > | \leq \kappa|x-y|max(|p|,|q|) + C_2|p-q|.$$

From which we deduce,

$$\left| \begin{array}{l} -(F(t,x,u,p,X) - F(t,y,u,q,-Y)) \\ \leq \mu_1[2\zeta|x-y|^2 + 2\theta\rho(p,q)^2] + \\ +2\theta\mu_2 + \kappa|x-y|max(|p|,|q|) + C_2|p-q|. \end{array} \right.$$

And

$$\left| \begin{array}{l} -(F(t,x,u,p,X) - F(t,y,u,q,-Y)) \\ \leq max(2\zeta, 2\theta, C_2, \kappa))(\mu_1(\rho(p,q)^2 + |x-y|^2) \\ +\mu_2 + |p-q| + |x-y|(1 + max(|p|,|q|))). \end{array} \right.$$

We just have to take $w_R(l) = max(2\zeta, 2\theta, C_2, \kappa)l$. $w_R(0) = 0$ and w_R is non-decreasing on $[0, \infty[$.

The fourth point is fulfilled with assumptions on n.

Then, it is easy to check that B is **positively homogeneous of degree one**. For the last point, one can easily see that:

$$B(z,p) =< \nu(z), p >$$

and

$$< \nu(z), D_p B(z,p) >= |\nu(z)|^2 = 1$$

We take $\theta = 1$ and the last criterion is fulfilled.■

4.4. Proof based on Caselles & al.'s work and on Alvarez & al.'s paper. We concentrate upon the wellposedness of our problem. In this sense, we follow Caselles & *al.*(*[6]*) and Alvarez & *al.*(*[2]*) arguments. We prove existence and uniqueness of our problem in the viscosity sense for bounded Lipschitz continuous initial data. We recall that our problem can be formulated as follows, for $x \in \Re^2$:

$$\left| \begin{array}{l} \dfrac{\partial \Phi}{\partial t} - |\nabla \Phi| d(x) g(|\nabla I(x)|) div(\frac{\nabla \Phi}{|\nabla \Phi|}) \\ - < \nabla(d(x) g(|\nabla I(x)|), \nabla \Phi > +\alpha |\nabla \Phi| = 0, \quad \text{, on } [0, +\infty[\times \Omega., \end{array} \right. \tag{4.22}$$

with

$$\left\{ \begin{array}{l} \Phi(0,x) = \Phi_0(x). \\ \frac{\partial \Phi}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{array} \right. .$$

To simplify the notations, we denote by c the function defined by

$$c : x \longmapsto c(x) = d(x) g(|\nabla I(x)|)$$

With the notations $\partial_i u = \frac{\partial u}{\partial x_i}$, and the classical Einstein sommation, we reformulate our problem by:

$$(\xi) \left\{ \begin{array}{l} \frac{\partial u}{\partial t} - c(x) a_{ij}(\nabla u) \partial_{ij} u - < \nabla c(x), \nabla u > +\alpha |\nabla u| = 0 \text{ with } (t,x) \in [0, +\infty[\times \Re^2. \\ u(0,x) = u_0(x) \end{array} \right. \tag{4.23}$$

where $a_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{|p|^2}$ if $p \neq 0$ and $c(x) \geq 0$. We denote by $W^{1,\infty}(\Re^2)$ the space of bounded Lipschitz functions on $\Re^2$. For the following statement, we refer the reader to *the works of* Alvarez, Lions and Morel [2]. In the image processing framework, the equation should be solved on a domain R of $\Re^n$, so in practice on a rectangle of $\Re^2$. A natural choice is to take $[0,1] \times [0,1]$. Boundary conditions have to be determined and in this respect, the Neumann condition proves to be accurate. It corresponds indeed to the reflection of the picture across the boundary and doesn't impose any value on the boundary. As done by Alvarez & *al [2]* and in Caselles & *al [6]*, we simplify the issue by working with periodic boundary conditions, that is to say, we aim at solving ξ with solutions satisfying $u(x+2h) = u(x), \forall h \in Z^2, \forall x \in \Re^2$. That is why the Neumann condition doesn't appear in ξ. c and u_0 are extended to $\Re^2$ with the same process.

We now give the definition of viscosity solutions using the maximum principle (*as in def 1 in the subsection* **General Background**) adapted to parabolic problems.

Definition 11.

1. *$u \in C([0,T] \times \Re^2)$ for some $T < +\infty$ is a viscosity subsolution of (ξ) if : $\forall \Phi \in C^2(\Re \times \Re^2)$, if (t_0, x_0) is a local maximum of $u - \Phi$, we have the relation :*

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t_0,x_0) - c(x_0)a_{ij}(\nabla\Phi(t_0,x_0))\partial_{ij}\Phi(t_0,x_0) - < \nabla c(x_0), \nabla\Phi(t_0,x_0) > + \alpha|\nabla\Phi(t_0,x_0)| \leq 0 \\ \text{if } \nabla\Phi(t_0,x_0) \neq 0 \end{cases}$$

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t_0,x_0) - c(x_0) lim\ sup_{p \longrightarrow 0} a_{ij}(p)\partial_{ij}\Phi(t_0,x_0) \leq 0 \\ \text{if } \nabla\Phi(t_0,x_0) = 0 \end{cases}$$

2. *$u \in C([0,T] \times \Re^2)$ for some $T < +\infty$ is a viscosity supersolution of (ξ) if : $\forall \Phi \in C^2(\Re \times \Re^2)$, if (t_0, x_0) is a local minimum of $u - \Phi$, we have the relation :*

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t_0,x_0) - c(x_0)a_{ij}(\nabla\Phi(t_0,x_0))\partial_{ij}\Phi(t_0,x_0) - < \nabla c(x_0), \nabla\Phi(t_0,x_0) > + \alpha|\nabla\Phi(t_0,x_0)| \geq 0 \\ \text{if } \nabla\Phi(t_0,x_0) \neq 0 \end{cases}$$

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t_0,x_0) - c(x_0) lim\ inf_{p \longrightarrow 0} a_{ij}(p)\partial_{ij}\Phi(t_0,x_0) \geq 0 \\ \text{if } \nabla\Phi(t_0,x_0) = 0 \end{cases}$$

3. *u is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.*

With the above definitions, we can hence propose a theorem.

Theorem 12. *Suppose that* $sup_{i\in\{1,2\}} sup_{x\in\Re^2} |\frac{\partial c^{\frac{1}{2}}}{\partial x_i}| < \infty$ *and that* $sup_{i,j\in\{1,2\}} sup_{x\in\Re^2} |\frac{\partial^2 c}{\partial x_i \partial x_j}| < \infty$.

Let $u_0 \in C(R) \cap W^{1,\infty}(R)$.

1. (ξ) *admits a unique viscosity solution*

$$u \in C([0,+\infty[\times\Re^2) \cap L^\infty(0,T;W^{1,\infty}(\Re^2))\ \forall T < \infty$$

   and u satisfies $inf u_0 \leq u(t,x) \leq sup u_0$.

2. *Let v be the viscosity solution corresponding to the initial data* v_0, *then*

$$||u(t,.) - v(t,.)||_\infty \leq ||u_0 - v_0||_\infty$$

Proof :

As Caselles & *al.* *[6]* and Alvarez & *al.* [2], we first concentrate upon the uniqueness part of the claim (1) and part (2) of the theorem. We refer the reader to the *'User's guide to viscosity solutions of second order partial differential equations [18]'*, since our proof is based on their work. We consider that (t_0, x_0, y_0) is a maximum point of

$$u(t,x) - v(t,y) - (4\epsilon)^{-1}|x-y|^4 - \lambda t,\ t \in [0,T],\ x, y \in \Re^2,\ \lambda, T, \epsilon > 0.$$

The first step consists in proving that $t_0 = 0$. So, we suppose that $t_0 > 0$ and aim at rising a contradiction. Here, we refer the reader to the theorem 8.3 of the *'User's guide to viscosity solutions of second order partial differential equations'* by Crandall, Ishii, Lions *[18]*. We recall this theorem briefly.

Theorem 8.3 of the User's guide:

Let us define $u_i \in USC((0,T)\times O_i)$ for $i = 1,..,k$ where O_i is a locally compact subset of $\Re^{N_i}$. Let φ be defined on a open neighborhood of $(0,T)\times O_1 \times .... \times O_k$ and such that $(t, x_1, ..., x_k) \longrightarrow \varphi(t, x_1, ..., x_k)$ is once continuously differentiable in t and twice continuously differentiable in $(x_1, ..., x_k) \in O_1 \times .... \times O_k$. Suppose that $\hat{t} \in (0,T)$, $\hat{x}_i \in O_i$ for $i = 1,..,k$ and

$$\left| \begin{array}{l} w(t, x_1, x_2, ..., x_k) = u_1(t,x_1) + ... + u_k(t,x_k) - \varphi(t, x_1, ..., x_k) \\ \leq w(\hat{t}, \hat{x}_1, \hat{x}_2, ..., \hat{x}_k). \end{array} \right.$$

for $0 < t < T$ and $x_i \in O_i$. Assume moreover, that there is an $r > 0$ such that for every $M > 0$ there is a C such that for $i = 1,..,k$

$$\left| \begin{array}{l} b_i \leq C \text{ whenever } (b_i, q_i, X_i) \in P_O^{2,+} u_i(t,x_i) \\ |x_i - \hat{x}_i| + |t - \hat{t}| \leq r \text{ and } |u_i(t,x_i)| + |q_i| + ||X_i|| \leq M. \end{array} \right.$$

Then, for each $\epsilon > 0$, there are $X_i \in S(N_i)$ such that

$$\left|\begin{array}{l} (b_i, D_{x_i}\varphi(\hat{t}, \hat{x}_1, ..., \hat{x}_k), X_i) \in \bar{P}^{2,+}_{O_i} u_i(\hat{t}, \hat{x}_i), i \in \{1, ..., k\} \\ -(\frac{1}{\epsilon} + ||A||)I \leq \begin{pmatrix} X_1 & \dots & 0 \\ 0 & X_2 & \dots \\ \dots & \dots & \dots \\ 0 & \dots & X_k \end{pmatrix} \leq A + \epsilon A^2. \\ b_1 + ... + b_k = \varphi_t(\hat{t}, \hat{x}_1, ..., \hat{x}_k). \end{array}\right.$$

with $A = (D^2_x\varphi)(\hat{t}, \hat{x}_1, ..., \hat{x}_k)$. We apply this theorem to our problem. Obviously, φ is the function defined by

$$\varphi(t, x, y) = (4\epsilon)^{-1}|x - y|^4 + \lambda t.$$

We can find a and X such that $(a, D_x\varphi(t_0, x_0, y_0), X) \in \bar{P}^{2,+}_{O_1} u(t_0, x_0)$.
But $D_x\varphi(t_0, x_0, y_0) = (\epsilon)^{-1}|x_0 - y_0|^2(x_0 - y_0)$, so the property (1) of the theorem can be written, (considering the definition of $P^{2,+}_{O_1} u(t_0, x_0)$) as follows:

$$\left|\begin{array}{l} a - c(x_0)a_{ij}((\epsilon)^{-1}|x_0 - y_0|^2(x_0 - y_0))X_{ij} \\ - < \nabla(c(x_0)), (\epsilon)^{-1}|x_0 - y_0|^2(x_0 - y_0) > +\alpha(\epsilon)^{-1}|x_0 - y_0|^3 \leq 0. \end{array}\right. \tag{4.24}$$

Identically, there exists $-b$ and $-Y$ such that

$$(-b, D_y\varphi(t_0, x_0, y_0), -Y) \in \bar{P}^{2,+}_{O_2}(-v(t_0, y_0)).$$

and

$$D_y\varphi(t_0, x_0, y_0) = -(\epsilon)^{-1}|x_0 - y_0|^2(x_0 - y_0).$$

We recall that $\bar{P}^{2,-}_{\Omega}(v) = -\bar{P}^{2,+}_{\Omega}(-v)$, so

$$\left|\begin{array}{l} (-b, D_y\varphi(t_0, x_0, y_0), -Y) = \\ = (-b, -(\epsilon)^{-1}|x_0 - y_0|^2(x_0 - y_0), -Y) \in -\bar{P}^{2,-}_{O_2} v(t_0, y_0). \end{array}\right.$$

and

$$(b, (\epsilon)^{-1}|x_0 - y_0|^2(x_0 - y_0), Y) \in \bar{P}^{2,-}_{O_2} v(t_0, y_0).$$

This can be formulated by the inequality:

$$\left|\begin{array}{l} b - c(y_0)a_{ij}((\epsilon)^{-1}|x_0 - y_0|^2(x_0 - y_0))Y_{ij} \\ - < \nabla(c(y_0)), (\epsilon)^{-1}|x_0 - y_0|^2(x_0 - y_0) > +\alpha(\epsilon)^{-1}|x_0 - y_0|^3 \geq 0. \end{array}\right. \tag{4.25}$$

The second point of the theorem says that for each $\gamma > 0$, we have :

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A' + \gamma A'^2$$

with $A' = (D^2_X\varphi)(t_0, x_0, y_0)$. In fact, it can formally be written with a block matrix defined by:

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} B + \mu B^2 & -B - \mu B^2 \\ -B - \mu B^2 & B + \mu B^2 \end{pmatrix}$$

with $\mu = 2\gamma$ and B the matrix defined by

$$B = (\epsilon^{-1})|x_0 - y_0|^2 I_2 + 2(\epsilon^{-1})(x_0 - y_0) \bigotimes (x_0 - y_0).$$

and B^2,

$$B^2 = (\epsilon^{-2})|x_0 - y_0|^4 I_2 + 8(\epsilon^{-2})|x_0 - y_0|^2 (x_0 - y_0) \bigotimes (x_0 - y_0).$$

The third point immediatly gives $a - b = \lambda$. We prove now that $x_0 \neq y_0$. Suppose that $x_0 = y_0$. It implies that $B = 0$ and we have then to use the definition of viscosity solutions in the singular case, which means that,

$$a - c(x_0) \lim sup_{p \longrightarrow 0} \; a_{ij}(p) X_{ij} \;\leq\; 0. \quad (4.26)$$

$$b - c(y_0) \lim inf_{p \longrightarrow 0} \; a_{ij}(p) Y_{ij} \;\geq\; 0. \quad (4.27)$$

Moreover, we easily deduce that $X \leq 0$ and $Y \geq 0$. Noting that for all p, $a_{ij}(p)X_{ij} = \xi^T X \xi$ with $\xi = \begin{pmatrix} \frac{p_2}{|p|} \\ -\frac{p_1}{|p|} \end{pmatrix}$, we can conclude from the definition of viscosity solutions in the singular case (4.26) and (4.27) that

$$a \leq 0$$
$$b \geq 0$$

Thus, $a - b \leq 0$, which contradicts the third point of the theorem $a - b = \lambda > 0$. Therefore, $x_0 \neq y_0$. We now choose to take $\mu = \epsilon |x_0 - y_0|^{-2}$. We then get the inequality:

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 2(\epsilon)^{-1} \begin{pmatrix} C & -C \\ -C & C \end{pmatrix}$$

with

$$\left| \begin{array}{l} B + \epsilon|x_0 - y_0|^{-2} B^2 = 2(\epsilon)^{-1}|x_0 - y_0|^2 I_2 + 10(\epsilon)^{-1}(x_0 - y_0) \bigotimes (x_0 - y_0). \\ \text{Thus, } \; C = |x_0 - y_0|^2 I_2 + 5(x_0 - y_0) \bigotimes (x_0 - y_0) \end{array} \right.$$

Let us denote by G the matrix defined by:

$$\begin{pmatrix} c(x_0)A & (c(x_0)c(y_0))^{\frac{1}{2}} A \\ (c(x_0)c(y_0))^{\frac{1}{2}} A & c(y_0)A \end{pmatrix}$$

with

$$A = (a_{ij}((\epsilon)^{-1}|x_0 - y_0|^2 (x_0 - y_0)))_{ij}.$$

G is symmetric and non-negative since one can observe that for all $\xi = (\xi_1, \xi_2)^T \in \Re^4$,

$$\left| \begin{array}{l} \xi^T G \xi = \\ c(x_0)\xi_1^T A \xi_1 + (c(x_0)c(y_0))^{\frac{1}{2}} \xi_1^T A \xi_2 + (c(x_0)c(y_0))^{\frac{1}{2}} \xi_2^T A \xi_1 \\ +c(y_0)\xi_2^T A \xi_2. \end{array} \right.$$

But, A is a symmetric matrix, non-negative and one can decompose A into $A = \sigma\sigma^T$. It implies that:

$$\left| \begin{array}{l} \xi^T G\xi = \\ c(x_0)\xi_1^T\sigma\sigma^T\xi_1 + (c(x_0)c(y_0))^{\frac{1}{2}}\xi_1^T\sigma\sigma^T\xi_2 + (c(x_0)c(y_0))^{\frac{1}{2}}\xi_2^T\sigma\sigma^T\xi_1 \\ +c(y_0)\xi_2^T\sigma\sigma^T\xi_2. \end{array} \right.$$

We deduce (using that $c \geq 0$) that:

$$\xi^T G\xi = |\sqrt{c(x_0)}\sigma^T\xi_1 + \sqrt{c(y_0)}\sigma^T\xi_2|^2 \geq 0.$$

As G is a symmetric non-negative matrix, one can decompose it into $G = \chi\chi^T$. It implies that:

$$trace(G\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}) = trace(\chi\chi^T\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}).$$

So

$$trace(G\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}) = trace(\chi^T\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}\chi)$$

and

$$trace(G\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}) = \sum_i (\chi_i)^T\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}\chi_i, \text{with } \chi_i \text{ the i th column of } \chi.$$

The last inequality with C gives:

$$\left| \begin{array}{l} \sum_i (\chi_i)^T\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}\chi_i \leq 2(\epsilon)^{-1}\sum_i (\chi_i)^T\begin{pmatrix} C & -C \\ -C & C \end{pmatrix}\chi_i \\ \text{So, } trace(G\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}) \leq 2(\epsilon)^{-1}trace(G\begin{pmatrix} C & -C \\ -C & C \end{pmatrix}) \end{array} \right.$$

which leads to the relation:

$$\left| \begin{array}{l} c(x_0)a_{ij}X_{ij} - c(y_0)a_{ij}Y_{ij} \\ \leq 2(\epsilon)^{-1}(c(x_0)^{\frac{1}{2}} - c(y_0)^{\frac{1}{2}})^2 trace(AC). \end{array} \right.$$

In fact, one can observe that $trace(AC) = |x_0 - y_0|^2$. We get,

$$c(x_0)a_{ij}X_{ij} - c(y_0)a_{ij}Y_{ij} \leq 2(\epsilon)^{-1}(c(x_0)^{\frac{1}{2}} - c(y_0)^{\frac{1}{2}})^2|x_0 - y_0|^2.$$

Using the two inequalities obtained above (4.26) and (4.27), we obtain using Cauchy-Schwartz inequality that:

$$\left| \begin{array}{l} a - b = \lambda \\ \leq \\ c(x_0)a_{ij}X_{ij} - c(y_0)a_{ij}Y_{ij} \\ + < \nabla(c(x_0)), (\epsilon^{-1})|x_0 - y_0|^2(x_0 - y_0) > \\ - < \nabla(c(y_0)), (\epsilon^{-1})|x_0 - y_0|^2(x_0 - y_0) > \\ \leq \\ 2(\epsilon)^{-1}(c(x_0)^{\frac{1}{2}} - c(y_0)^{\frac{1}{2}})^2|x_0 - y_0|^2 \\ + (\epsilon^{-1})|\nabla(c(x_0)) - \nabla(c(y_0))||x_0 - y_0|^3. \end{array} \right.$$

With the assumptions on the function c and $c^{\frac{1}{2}}$, we can find constants K_1 and K_2 such that:

$$a - b \leq K_1(\epsilon)^{-1}|x_0 - y_0|^4 + K_2(\epsilon^{-1})|x_0 - y_0|^4.$$

And,

$$\lambda \leq K(\epsilon)^{-1}|x_0 - y_0|^4 \tag{4.28}$$

Following Caselles *& al.*'s proof and Alvarez *& al.*'s one, we now want to estimate $|x_0 - y_0|$. We recall the reader that we aim at proving that $t_0 = 0$ and in this purpose we have supposed untrue this statement. As (t_0, x_0, y_0) is a maximum point for

$$u(t,x) - v(t,y) - (4\epsilon)^{-1}|x - y|^4 - \lambda t,$$

if we take $x = y = y_0$ and $t = t_0$, we get:

$$u(t_0, y_0) - v(t_0, y_0) - \lambda t_0 \leq u(t_0, x_0) - v(t_0, y_0) - (4\epsilon)^{-1}|x_0 - y_0|^4 - \lambda t_0.$$

From this last inequality, we deduce that:

$$(4\epsilon)^{-1}|x_0 - y_0|^4 \leq u(t_0, x_0) - u(t_0, y_0)$$

Using the fact that $u \in L^\infty(0, T; W^{1,\infty}(\Re^2))$, denoting by L the Lipschitz constant associated to $u(t_0, .)$, we have:

$$(4\epsilon)^{-1}|x_0 - y_0|^4 \leq u(t_0, x_0) - u(t_0, y_0) \leq L|x_0 - y_0|.$$

Thus, $|x_0 - y_0| \leq (4\epsilon L)^{\frac{1}{3}}$. From (4.28), we deduce that :

$$\lambda \leq M\epsilon^{\frac{1}{3}}L^{\frac{4}{3}} \tag{4.29}$$

We may suppose that $sup_{[0,T]\times\Re^2}|u - v| \neq 0$ otherwise it is easy to conclude and get the second point of the theorem. We choose to take:

$$\begin{array}{l} \epsilon^{\frac{1}{3}} = \delta sup_{[0,T]\times\Re^2}|u - v|, \\ \lambda = 2M\delta sup_{[0,T]\times\Re^2}|u - v|L^{\frac{4}{3}} \end{array}$$

with $\delta > 0$ determined later on. This choice contradicts (4.29), which proves that $t_0 = 0$.

The second main step of the proof consists in estimating $sup_{[0,T]\times\Re^2}|u - v|$. Since $t_0 = 0$, we have:

$$\left| \begin{array}{l} u(t,x) - v(t,y) - (4\epsilon)^{-1}|x - y|^4 - \lambda t \\ \leq sup_{x,y\in\Re^2\times\Re^2}(u_0(x) - v_0(y) - (4\epsilon)^{-1}|x - y|^4). \end{array} \right.$$

But,

$$sup_{x,y\in\Re^2\times\Re^2}(u_0(x) - v_0(y) - (4\epsilon)^{-1}|x - y|^4) =$$

$$sup_{x,y\in\Re^2\times\Re^2}(u_0(y) - v_0(y) + u_0(x) - u_0(y) - (4\epsilon)^{-1}|x - y|^4).$$

Using the fact that $|u_0(x) - u_0(y)| \leq L|x-y|$, we can deduce that

$$\left| \begin{array}{l} sup_{x,y\in\Re^2}(u_0(x) - v_0(y) - (4\epsilon)^{-1}|x-y|^4) \\ \leq sup_{\Re^2}|u_0(y) - v_0(y)| + sup_{x,y\in\Re^2\times\Re^2}(L|x-y| - (4\epsilon)^{-1}|x-y|^4). \\ \leq sup_{\Re^2}|u_0(y) - v_0(y)| + sup_{r\geq 0}(Lr - (4\epsilon)^{-1}r^4). \end{array} \right.$$

In the inequality (4.29), taking $x = y$ leads to the relation :

$$-\lambda t + sup_{[0,T]\times\Re^2}(u-v) \leq sup_{\Re^2}|u_0 - v_0| + \frac{3}{4}L^{\frac{4}{3}}\epsilon^{\frac{1}{3}}.$$

Inserting the expressions of $\epsilon^{\frac{1}{3}}$ and λ into (4.27), we get:

$$\left| \begin{array}{l} sup_{[0,T]\times\Re^2}(u-v) \\ \leq \\ sup_{\Re^2}|u_0 - v_0| + \frac{3}{4}L^{\frac{4}{3}}\delta sup_{[0,T]\times\Re^2}|u-v| \\ +2M\delta L^{\frac{4}{3}}T sup_{[0,T]\times\Re^2}|u-v|. \end{array} \right.$$

Exchanging the role of u and v, and taking $\delta \longrightarrow 0$, we obtain:

$$sup_{[0,T]\times\Re^2}|u-v| \leq sup_{\Re^2}|u_0 - v_0|.$$

This proves the second part of the theorem and the uniqueness claim of part one.

Let's deal now with the existence of a viscosity solution. As Caselles *& al.* and Alvarez *& al.*, we first remark that if u is a solution :

$$inf_{\Re^2}u_0 - \delta t \leq u \leq sup_{\Re^2}u_0 + \delta t, \forall\delta > 0, \text{ on } [0,\infty[\times\Re^2 .$$

Therefore,

$$inf_{\Re^2}u_0 \leq u \leq sup_{\Re^2}u_0.$$

To prove that , consider the function $\Phi(t,x) = sup_{\Re^2}u_0 + \delta t$, $\delta > 0$, and suppose that $u - \Phi$ has a local maximum at a point (t_0, x_0) with $t_0 > 0$. Using the definition of subsolution, we have:

$$\left| \begin{array}{l} \frac{\partial\Phi}{\partial t}(t_0,x_0) - c(x_0) lim\ sup_{p\longrightarrow 0}a_{ij}(p)\partial_{ij}\Phi(t_0,x_0) \leq 0. \\ \text{It implies that } \frac{\partial\Phi}{\partial t}(t_0,x_0) \leq 0. \\ \Rightarrow \delta \leq 0, \text{ which contradicts } \delta > 0. \end{array} \right.$$

Thus, $t_0 = 0$. Following Caselles *& al.*'s proof [6] and Alvarez *& al.*'s one [2], we now want to give an *a priori* estimate of $|\nabla u|$. We consider a smooth solution of the problem (ξ). a_{ij} is supposed to be smooth as well. We aim at establishing a parabolic equation satisfied by $|\nabla u|^2$. In this purpose, we differentiate our equation with respect to the variable x_I.

We recall the equation of our problem.

$$\frac{\partial u}{\partial t} - c(x)a_{ij}(\nabla u)\partial_{ij}u - < \nabla c(x), \nabla u > + \alpha|\nabla u| = 0$$

Carrying out the differentiation, the notation u_{ij} denoting $\partial_{ij}u$,and H being the operator defined by

$$H(p) = \sqrt{|p|^2 + \epsilon}$$

with $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \Re^2$, we obtain :

$$\left| \begin{array}{l} \dfrac{\partial u_I}{\partial t} - \dfrac{\partial c(x)}{\partial I}a_{ij}(\nabla u)\partial_{ij}u - c(x)a_{ij}(\nabla u)\partial_{ijI}u - c(x)\dfrac{\partial a_{ij}(\nabla u)}{\partial p_k}\partial_{kI}u\partial_{ij}u \\ -\partial_{Ii}c(x)\partial_i u(x) - \partial_i c(x)\partial_{Ii}u(x) + \alpha\dfrac{\partial H}{\partial p_k}(\nabla u)u_{kI} = 0. \end{array} \right. \tag{4.30}$$

By multiplying by $2u_I$ equation (4.30) and adding on the I, and after remarking that :

1. $\sum_I 2u_I \frac{\partial u_I}{\partial t} = \frac{\partial |\nabla u|^2}{\partial t}$.

2. $\sum_I 2u_I c(x)a_{ij}(\nabla u)\partial_{ijI}u = c(x)a_{ij}(\nabla u)\partial_{ij}|\nabla u|^2 - 2\sum_I c(x)a_{ij}(\nabla u)\partial_{iI}u\partial_{jI}u$.

3. $\sum_I 2\alpha\frac{\partial H}{\partial p_k}(\nabla u)u_{kI}u_I = \alpha\frac{\partial H}{\partial p_k}\partial_k(|\nabla u|^2)$.

4. $\sum_I 2\partial_i c(x)\partial_{Ii}u(x)u_I = \partial_i c(x)\partial_i(|\nabla u|^2)$.

5. $\sum_I 2c(x)\frac{\partial a_{ij}(\nabla u)}{\partial p_k}\partial_{kI}u\partial_{ij}uu_I = c(x)\frac{\partial a_{ij}(\nabla u)}{\partial p_k}\partial_{ij}u\ \partial_k(|\nabla u|^2)$.

we get, still using Einstein sommation convention:

$$\left| \begin{array}{l} \frac{\partial|\nabla u|^2}{\partial t} - c(x)\frac{\partial a_{ij}}{\partial p_k}\partial_{ij}u\partial_k(|\nabla u|^2) - c(x)a_{ij}(\nabla u)\partial_{ij}|\nabla u|^2 \\ -\partial_i c(x)\partial_i(|\nabla u|^2) + \alpha\frac{\partial H}{\partial p_k}\partial_k(|\nabla u|^2) = 2\frac{\partial C}{\partial I}[a_{ij}(\nabla u)u_I u_{ij}] \\ +2\partial_{Ii}c(x)\partial_i u(x)\partial_I u - 2c(x)a_{ij}(\nabla u)\partial_{iI}u\partial_{jI}u. \end{array} \right.$$

We have, for I fixed, :

$$\left| \begin{array}{l} \frac{\partial c^{\frac{1}{2}}(x)}{\partial I} = \frac{1}{2}c(x)^{-\frac{1}{2}}\frac{\partial c}{\partial I} \leq sup_i sup_{x\in\Re^2}|\frac{\partial c^{\frac{1}{2}}}{\partial x_i}| \\ \text{We easily deduce from that the relation that for I fixed,} \\ \frac{\partial c}{\partial I} \leq 2 sup_i sup_{x\in\Re^2}|\frac{\partial c^{\frac{1}{2}}}{\partial x_i}|c(x)^{\frac{1}{2}}. \end{array} \right. \tag{4.31}$$

Besides, following [2], and remarking that $a_{ij}\xi_i\xi_j$ is non-negative, as prooved previously, we have the purely algebraic relation:

$$|a_{ij}(\nabla u)u_{ij}| \leq K_2(a_{ij}(\nabla u)u_{Ii}u_{Ij})^{\frac{1}{2}}.$$

Using Cauchy-Schwartz inequality, we deduce with (4.30) and (4.31) :

$$\frac{\partial C}{\partial I}[a_{ij}(\nabla u)u_I u_{ij}] \leq K_1K_2|\nabla u|(c(x)(a_{ij}u_{Ii}u_{Ij}))^{\frac{1}{2}}. \tag{4.32}$$

and

$$2|\partial_{Ii}c(x)\partial_i u(x)\partial_I u| \leq 4sup_{i,j}sup_x|\frac{\partial^2 c}{\partial x_i \partial x_j}||\nabla u|^2. \tag{4.33}$$

Thus, using these relations, we have the inequality,

$$\left|\begin{array}{l} \frac{\partial|\nabla u|^2}{\partial t} - c(x)\frac{\partial a_{ij}}{\partial p_k}\partial_{ij}u\partial_k(|\nabla u|^2) - c(x)a_{ij}(\nabla u)\partial_{ij}|\nabla u|^2 \\ -\partial_i c(x)\partial_i(|\nabla u|^2) + \alpha\frac{\partial H}{\partial p_k}\partial_k(|\nabla u|^2) \leq 4sup_{i,j}sup_x|\frac{\partial^2 c}{\partial x_i\partial x_j}||\nabla u|^2 \\ +K_1K_2|\nabla u|(c(x)a_{ij}u_{Ii}u_{Ij})^{\frac{1}{2}} - 2c(x)a_{ij}(\nabla u)\partial_{Ii}u\partial_{Ij}u. \end{array}\right.$$

If we consider only the term $-2c(x)a_{ij}u_{Ii}u_{Ij} + K_1K_2|\nabla u|(c(x)a_{ij}u_{Ii}u_{Ij})^{\frac{1}{2}}$, one can see it as a second order polynomial expression with respect to $(c(x)a_{ij}u_{Ii}u_{Ij})^{\frac{1}{2}}$, whose coefficient of higher degree term is negative. The maximum is then reached for the value $\frac{K_1K_2|\nabla u|}{4}$. Thus, a majoration can be established , taking the value of 'our polynomial expression' in the right part of the inequality for $\frac{K_1K_2|\nabla u|}{4}$. So,

$$\left|\begin{array}{l} \frac{\partial|\nabla u|^2}{\partial t} - c(x)\frac{\partial a_{ij}}{\partial p_k}\partial_{ij}u\partial_k(|\nabla u|^2) - c(x)a_{ij}(\nabla u)\partial_{ij}|\nabla u|^2 \\ -\partial_i c(x)\partial_i(|\nabla u|^2) + \alpha\frac{\partial H}{\partial p_k}\partial_k(|\nabla u|^2) \leq 4sup_{i,j}sup_x|\frac{\partial^2 c}{\partial x_i\partial x_j}||\nabla u|^2 \\ +\frac{K_1^2K_2^2}{8}|\nabla u|^2. \end{array}\right.$$

and so

$$\left|\begin{array}{l} \frac{\partial|\nabla u|^2}{\partial t} - c(x)\frac{\partial a_{ij}}{\partial p_k}\partial_{ij}u\partial_k(|\nabla u|^2) - c(x)a_{ij}(\nabla u)\partial_{ij}|\nabla u|^2 - \\ \partial_i c(x)\partial_i(|\nabla u|^2) + \alpha\frac{\partial H}{\partial p_k}\partial_k(|\nabla u|^2) \leq K_3|\nabla u|^2. \end{array}\right.$$

K_3 is a constant that only depends on

$$sup_i sup_{x\in\Omega}|\frac{\partial c^{\frac{1}{2}}}{\partial x_i}|, sup_p|a_{ij}(p)|, sup_{i,j}sup_x|\frac{\partial^2 c}{\partial x_i\partial x_j}|.$$

Then, by applying the Maximum Principle, we deduce that:

$$||\nabla u(t,x)||_{L^\infty(\Omega)} \leq e^{K_3 t}||\nabla u_0||_{L^\infty(\Re^2)}.$$

Now, following Alavarez & al. 's proof, we approximate our initial problem by an equation for which we are able to produce smooth solutions. In this purpose, we denote by $u_0^\epsilon \in C^\infty(\Re^2)$ the periodic function such that $u_0^\epsilon \longrightarrow u_0$ uniformly, $||\nabla u_0^\epsilon||_{L^\infty(\Re^2)} \leq$

$||\nabla u_0||_{L^\infty(\Re^2)}$, $||u_0^\epsilon||_{L^\infty(\Re^2)} \leq ||u_0||_{L^\infty(\Re^2)}$. c_ϵ is defined by $c_\epsilon = c + \epsilon$, $H_\epsilon(p) = (|p|^2 + \epsilon)^{\frac{1}{2}}$, $a_{ij}^\epsilon = \epsilon\delta_{ij} + \alpha_{ij}^\epsilon$ with $\alpha_{ij}^\epsilon = \delta_{ij} - \frac{p_i p_j}{|p|^2+\epsilon}$.

The general theory of quasilinear parabolic equations (*cf. Ladyzhenskaja and Solonnikov [26]*), enables us to conclude that there exists u^ϵ smooth solution of

$$\frac{\partial u^\epsilon}{\partial t} - c_\epsilon a_{ij}^\epsilon(\nabla u^\epsilon) u_{ij}^\epsilon - < \nabla c^\epsilon, \nabla u^\epsilon > + \alpha H_\epsilon(\nabla u^\epsilon) = 0.$$

with $u_0^\epsilon(0,x) = u_0^\epsilon(x)$.

Using the estimation that has been made above, we get:

$$||\nabla u^\epsilon(t,.)||_{L^\infty(\Re^2)} \leq e^{K_3 t} ||\nabla u_0^\epsilon||_{L^\infty(\Re^2)}.$$

We easily deduce that:

$$|u^\epsilon(t,x) - u^\epsilon(t,y)| \leq e^{K_3 t} ||\nabla u_0^\epsilon||_{L^\infty(\Re^2)} |x-y|.$$

for all $x \in \Re^2$ and $t \in [0,T]$. This can formally be rewritten by :

$$|u^\epsilon(t,x) - u^\epsilon(t,y)| \leq C_T |x-y|.$$

Then, we have :

$$|u^\epsilon(s,x) - u^\epsilon(t,x)| \leq C_T |t-s|^{\frac{1}{2}}$$

By combining these two last inequalities, we can conclude, thanks to Arzela-Ascoli theorem, that a subsequence of u^ϵ uniformly converges on $[0,T] \times R$ to a function

$$u \in C([0,T] \times \Re^2) \bigcap L^\infty(0,T; W^{1,\infty}(\Re^2)). \ \square$$

Lastly, the criterion that makes the algorithm stop can be either a preset number of iterations or a check that the solution is stationary.

5. Numerical results

In order to illustrate the proposed method , we give a numerical example. The interpolation conditions are useful in order to help the process when some image data are missing as shown in [27]. But the interpolation conditions also permit to choose an initial condition which is far from the final result as shown in the following numerical example (image coming from *Matlab*).

So, to help the process, we give some interpolation conditions (6 points):

We use the method proposed in this paper knowing that :

- The criterion that makes the algorithm stop can be either a preset number of iterations or a check that the solution is stationary.

- The distance is computed using the Fast Marching Method (see for instance [33]).

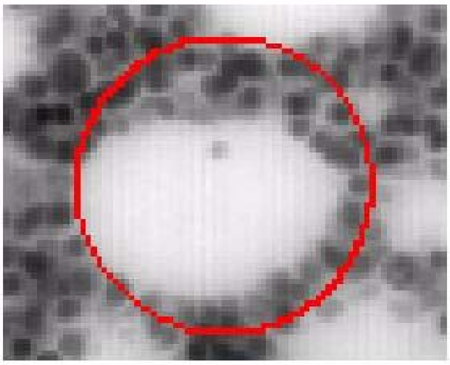

Figure 5: Initial condition.

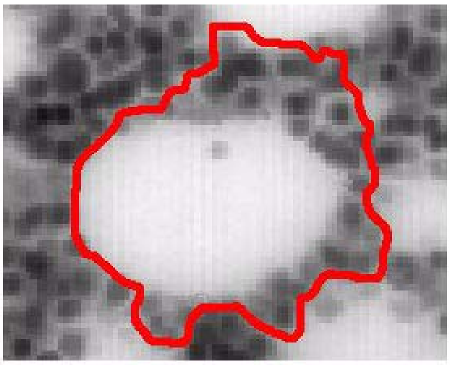

Figure 6: Using classical tool (snakes), we obtain the following result.

- The distance is normalized in order to have a same weight between a priori informations of the image and geometrical constraints.

- The discretization is made using finite differences as done in Chan and Vese [10].

- We have taken $\delta t = 0.1$. The regularization term is equal to 0.8. The number of iterations is 20 in this numerical example.

We obtain the following result:

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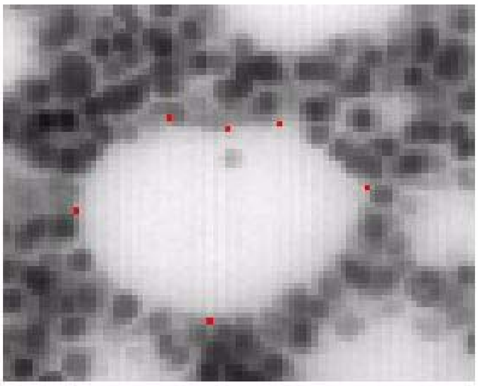

Figure 7: Interpolation conditions (5 points).

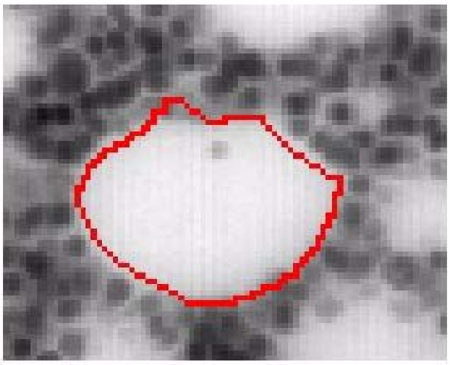

Figure 8: Final result using interpolation conditions.

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