Blind Deconvolution of Bar Code Signals

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Abstract

Bar code reconstruction involves recovering a clean signal from an observed one that is corrupted by convolution with a kernel and additive noise. The precise form of the convolution kernel is also unknown, making reconstruction harder than in the case of standard deblurring. On the other hand, bar codes are functions that have a very special form – this makes reconstruction feasible. We develop and analyze a total variation based variational model for the solution of this problem. This new technique models systematically the interaction of neighboring bars in the bar code under convolution with a kernel, as well as the estimation of the unknown parameters of the kernel from global information contained in the observed signal.

1. Introduction

We study the problem of recovering a bar code from the noisy signal detected by a bar code reader. Bar codes represent (finite) sequences of digits by (finite) sequences of dark parallel "bars" of varying thickness, separated by "white spaces" of varying size. As such, they can be conveniently modeled as characteristic functions of measurable subsets of $\mathbf{R}$ (See *Figure 1*). We therefore introduce the set

$$S = \{u(x) \in L^2(\mathbf{R}) : u(x) \in \{0,1\} \text{ a.e. } x\} \tag{1}$$

The process that converts a given bar code from the set S to the signal that is actually detected by a bar code reader depends, among other things, on the distance of the reader (the detector) to the surface where the bar code appears [10, 11, 18]. The longer the distance, the more blurred the observed signal. And depending on ambient illumination the optical sensor of the bar code reader introduces an unknown gain factor. This process has been modeled [10, 11, 18] as the convolution of the bar code $u(x)$ with a Gaussian kernel of *unknown* amplitude α and standard deviation σ, and is denoted by $T_{\alpha,\sigma}$:

$$T_{\alpha,\sigma} : u(x) \longrightarrow \alpha \cdot G_\sigma * u(x) \tag{2}$$

where α is a strictly positive constant, and

$$G_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}} \text{ with } \sigma > 0. \tag{3}$$

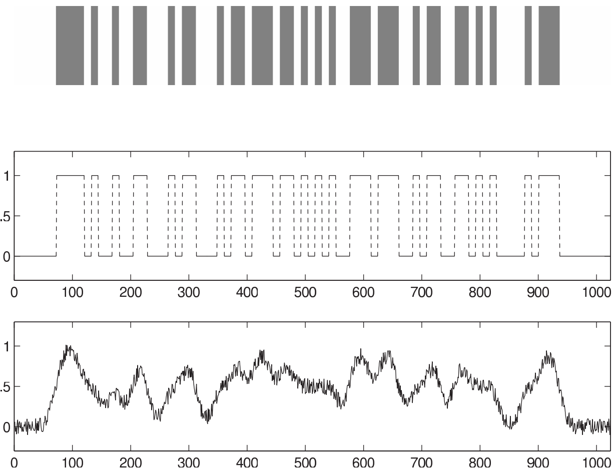

Figure 1: Top to bottom: A bar code, the function $u(x) \in S$ that represents it, and a typical observed signal $f(x)$ obtained from $u(x)$ via convolution by a Gaussian kernel followed by the addition of noise.

Furthermore, in any practical situation, the *observed* signal $f(x)$ is further corrupted by noise, which is modeled to be additive:

$$f(x) = T_{\alpha,\sigma}(u) + n(x) \tag{4}$$

Therefore, in our model it is $f(x)$ that represents the typical output of a bar code reader.

> *The goal of bar code reconstruction is to recover the bar code* $u(x)$ *from the observed signal* $f(x)$*,* **without** *any knowledge of* α *and* σ*.*

The bar code reconstruction problem thus differs from standard denoising problems of image processing in that the convolution kernel is not completely known in advance. In this regard, it is closer to the blind deconvolution problem.

2. Previous work and our approach:

Previous work on the bar code reconstruction problem is based on finding local minima and maxima in the derivative of $f(x)$ (which we called the observed signal above) using edge detector filters, and trying to relate them to the structure of the bar code they must have come from: under favorable conditions, local extrema in the derivative of the convolved signal correspond approximately to locations of discontinuities in the original bar code, and so help locate the “bars” in the code. A major difficulty with such an approach is sensitivity to noise. Moreover, locations of local maxima and minima in $f'(x)$ are difficult to relate to the true locations of edges of the "bars" in the original bar code $u(x)$ because the convolution operator involved in the observation process tends to shift their positions depending on the configuration of neighboring bars and the amount of blurring (which is another unknown of the problem); this is known as "convolution distortion" in bar code literature [18]. This effect is significant when the standard deviation of the convolution kernel becomes comparable to the smallest length scale of interest in the problem, namely the minimum of the thicknesses of bars and their distances to each other. E. Joseph and T. Pavlidis describe in [10, 11] a modification of the traditional edge detection technique to take into account this issue, but we believe it can be addressed more systematically.

Furthermore, when the standard deviation of the convolution kernel is large enough (which happens when, for example, the reader is held at a sufficiently large distance to the bar code), "edges" of the bars that make up the bar code may have no corresponding local extrema in the derivative of the convolved signal. In this situation, a traditional approach based on locating extrema in the derivative of the convolved signal would be at a significant disadvantage even in the absence of any noise.

We follow a completely different path based on the variational approach to image processing: we write down an energy whose minimizer is the candidate solution. These techniques are more robust than traditional edge detectors in the presence of noise: first, they are more global in that they utilize information from all over the image domain to detect transitions in the signal, and second they avoid working with the derivative of the signal which is always an error prone quantity. Moreover, unlike previous approaches, the technique we present systematically models interactions between neighboring "bars" under convolution with a Gaussian kernel. Indeed, our model takes into account the interactions between all "bars" in the signal under convolution, and not just the neighboring few (although of course the main contribution comes from the immediate neighbors). This is accomplished by explicitly modeling the convolution process using the "fidelity" term in the energy that we propose to minimize.

Another way the model presented here differs from previous approaches is how the unknown parameter σ is handled. Our technique provides a systematic way of determining σ from the global information contained in the observed signal and avoids having to estimate it from an isolated peak or valley.

To be more specific, we introduce and study both analytically and numerically a version of the total variation based de-noising/de-blurring technique of Osher, Rudin, and Fatemi [16, 17] that is adapted to minimizing over functions that take only two values. The resulting algorithm minimizes an energy functional via gradient descent, and is thus iterative. It does not require a priori knowledge of the number of "bars" expected in the reconstructed bar code.

This new algorithm does not require pronounced local extrema in the derivative of the convolved signal in order to successfully reconstruct bar codes. It is robust under noise, even at high blur. It is therefore a promising new approach to furthering the distance from which bar codes can be reliably reconstructed by readers. Moreover, it can be extended very naturally to two dimensional signals.

Although we describe all details of the numerical implementation, our emphasis in this paper is not the speed of the algorithm but the accuracy of its reconstructions. In particular, we should point out that there are other techniques besides gradient descent for minimizing total variation based functionals that are potentially much more efficient; examples of other optimization approaches can be found in [4, 5, 6, 9, 12, 14, 19, 20]. Many previous authors studied total variation based regularization for denoising and deblurring (with a known kernel) in image processing and related applications [1, 4, 16, 17, 19]. Furthermore, application of total variation based reconstruction to blind deconvolution was previously introduced by T. F. Chan and C. K. Wong in [7]; that work deals with the situation where, unlike our setting, there is no a priori information available about the unknown function $u(x)$ and the convolution kernel.

3. Preliminary observations

We start with a uniqueness result which shows that in the absence of any noise, the signal detected by a bar code reader contains enough information to determine the bar code it resulted from:

Proposition 1 *Let $u_1, u_2 \in S$ be two bar codes such that $T_{\alpha_1,\sigma_1}(u_1) = T_{\alpha_2,\sigma_2}(u_2)$ identically. Then $u_1 = u_2$ identically. Furthermore, if $u_1 = u_2$ does not vanish identically, then $\alpha_1 = \alpha_2$ and $\sigma_1 = \sigma_2$.*

Proof: First, it is plain that $T_{\alpha_1,\sigma_1}(u_1) = T_{\alpha_2,\sigma_2}(u_2)$ vanishes if and only if $u_1 = u_2 = 0$. So let us now assume that $T_{\alpha_1,\sigma_1}(u_1) \neq 0$. Then, we can show that $\sigma_1 = \sigma_2$. To that end, suppose not and, with no loss of generality, assume that $\sigma_1 > \sigma_2$. The equality

$$\alpha_1 G_{\sigma_1} * u_1 = \alpha_2 G_{\sigma_2} * u_2 \tag{5}$$

implies:

$$u_2 = \frac{\alpha_1}{\alpha_2} G_\sigma * u_1 \text{ where } \sigma = \sqrt{\sigma_1^2 - \sigma_2^2}. \tag{6}$$

That would mean $u_2 \in C^\infty$, which is a contradiction. It follows that $\sigma_1 = \sigma_2$. With that, we get $\alpha_1 u_1 = \alpha_2 u_2$, and the conclusion of the proposition follows. □

Let us note, on the other hand, that in the presence of noise the equation $T_{\alpha,\sigma}u(x) = f(x)$ cannot in general be solved for $\alpha, \sigma, u(x)$. Indeed, for any choice of $\alpha, \sigma, u(x)$, the right hand side either takes only two values (which happens if $\sigma = 0$, a value that we allow) or is smooth. Hence, when $f(x)$ is any non-smooth function that takes more than two values, there is no corresponding solution.

Remark: Proposition 1 has an obvious higher dimensional analogue. On the other hand, there seems to be no clear way to adapt this argument to the discrete setting considered in the subsequent sections.

4. Variational models

We will study a number of total variation based models for the solution of the bar code problem. In applications, it makes more sense to pose the problem on a bounded interval I, which we will take to be $I = (0, 1)$. We then define the subset S_I of S as:

$$S_I := \{u \in S : u = 0 \text{ a.e. } x \in \mathbf{R} \setminus I\} . \tag{7}$$

We will assume that the observed signal $f(x)$ satisfies $f \in L^2(\mathbf{R})$. Also, from here onward it is more convenient to let $\alpha \geq 0$.

Before considering total variation based models, let us note that the following natural variational approach to the bar code problem in general does not have a solution; in other words, the approach is not well-posed:

$$\inf_{\substack{u(x)\in S_I\\ \alpha,\sigma\geq 0}} E(u, \alpha, \sigma) \text{ where } E(u, \alpha, \sigma) = \int_{\mathbf{R}} \left(\alpha \cdot G_\sigma * u - f\right)^2 dx \tag{8}$$

In fact, without some regularization on $u(x)$, even the simpler minimization problem with $\alpha \geq 0$ has no solution in general. The following claim helps explain why:

Proposition 2 *Fix $\alpha > 0$, and take any $f(x) \in L^\infty(\mathbf{R})$ such that*

$$f(x) = 0 \text{ for a.e. } x \in I^c, \text{ and } f(x) \in [0, \alpha] \text{ for a.e. } x \in I. \tag{9}$$

Then,

$$\inf_{\substack{u(x)\in S_I\\ \sigma>0}} E(u(x), \alpha, \sigma) = 0. \tag{10}$$

Proof: First, we have

$$G_\sigma * f \longrightarrow f \text{ in } L^2(\mathbf{R}) \text{ as } \sigma \to 0^+. \tag{11}$$

Also, there exists a sequence $b_n(x) \in S_I$ such that

$$\alpha b_n(x) \xrightarrow{w} f(x) \text{ weakly in } L^2(\mathbf{R}) \text{ as } n \to \infty. \tag{12}$$

The last statement follows from a well-known generalization of the Riemann–Lebesgue lemma. By compactness of the convolution operator,

$$\alpha G_\sigma * b_n \longrightarrow G_\sigma * f \text{ strongly in } L^2(\mathbf{R}) \text{ as } n \to \infty, \tag{13}$$

for any $\sigma > 0$. Taking now a sequence $\{\sigma_j\}_{j=1}^{\infty} \to 0^+$, we can take a diagonal sequence $\{b_{n_j}(x)\}$ such that

$$\alpha G_{\sigma_j} * b_{n_j} \longrightarrow f \text{ in } L^2(\mathbf{R}) \text{ as } j \to \infty, \tag{14}$$

which implies the desired conclusion. □
Combined with the uniqueness results of Proposition 1, the last proposition shows that the infemum in (8) is in general not attained, and therefore the approach is not well-posed.

As the foregoing discussion suggests, it is necessary to put some regularization on $u(x)$. With that in mind, we define the following energy:

$$E_\lambda(u, \alpha, \sigma) = \int_{\mathbf{R}} |\nabla u| + \lambda \int_{\mathbf{R}} (\alpha \cdot G_\sigma * u - f)^2 \, dx \tag{15}$$

over functions $u(x) \in S_I$, and constants $\alpha, \sigma \geq 0$. Here, λ is a positive constant. The regularization term, given by the first integral on the right hand side is defined as

$$\int_{\mathbf{R}} |\nabla u| = \sup_{\substack{\phi(x) \in C_c^1(\mathbf{R}) \\ |\phi(x)| \leq 1 \ \forall x \in \mathbf{R}}} \int u(x) \phi'(x) \, dx \tag{16}$$

and agrees with its ordinary sense whenever $u(x)$ is smooth.

A natural approach to recovering $u(x)$ from $f(x)$ would be to carry out the following minimization:

$$\inf_{\substack{u(x) \in S_I \\ \alpha, \sigma \geq 0}} E_\lambda(u, \alpha, \sigma) \tag{17}$$

However, we have the following

Proposition 3 *In general, the infemum in (17) is not attained.*

Proof: Take $f(x) := G_{\sigma_0}(x - \frac{1}{2})$ for some $\sigma_0 > 0$, and find a $\lambda > 0$ large enough so that

$$E_\lambda(0, 1, 1) = \lambda \int_{\mathbf{R}} f^2(x) \, dx > 2 \tag{18}$$

On the other hand, any non-zero function $u(x) \in S_I$ has total variation at least 2, and as in the uniqueness result of Proposition 1 we can easily see that

$$\int_{\mathbf{R}} (f(x) - \alpha \cdot G_\sigma * u)^2 \; dx > 0 \tag{19}$$

for all $u(x) \in S_I$ and $\alpha > 0, \sigma \geq 0$. Therefore,

$$E_\lambda(u, \alpha, \sigma) > 2 \tag{20}$$

for all $u(x) \in S_I$ and $\alpha, \sigma \geq 0$. Now let

$$u_j(x) = \chi_{[-\frac{1}{2j}, \frac{1}{2j}]}(x - \frac{1}{2}) \text{ for } j = 1, 2, \ldots \tag{21}$$

Consider the sequence $\{(u_j, j, \sigma_0)\}_{j=1}^{\infty}$. We have:

$$j \cdot u_j(x) \xrightarrow{j \to \infty} \delta_{\frac{1}{2}}(x) \tag{22}$$

weakly in the sense of measures. In fact,

$$j \cdot G_{\sigma_0} * u_j \xrightarrow[L^2]{j \to \infty} G_{\sigma_0}(x) = f(x) \tag{23}$$

so that:

$$\lim_{j \to \infty} E_\lambda(u_j, j, \sigma_0) = \inf_{\substack{u(x) \in S_I \\ \alpha, \sigma \geq 0}} E_\lambda(u, \alpha, \sigma) = 2 \tag{24}$$

This establishes the proposition. $\square$

Perhaps the variational problem (17) has solutions for *most* choices of the original signal $f(x)$. Nevertheless, the proposition given above makes a strong argument against using it to solve the bar code problem. We therefore look for alternative variational models. One possibility is to put restrictions on the constant $\alpha \geq 0$. To that end, let K be a compact subset of positive real numbers, and consider instead of (17) the following variational problem:

$$\inf_{\substack{u(x) \in S_I \\ \alpha \in K \text{ and } \sigma \geq 0}} E_\lambda(u, \alpha, \sigma) \tag{25}$$

We then have:

Proposition 4 *Problem (25) has a solution (infemum is attained).*

Proof: Let $\{(u_j, \alpha_j, \sigma_j)\}_j$ be a minimizing sequence for (25). The sequence $\{u_j\}_j$ lies in S_I and has bounded total variation. Therefore, by the standard compactness result (see e.g. [8]) for functions with bounded variation, and the fact that α_j all lie in a compact set K, we can assume by passing to a subsequence if necessary that u_j and α_j converge to $u_\infty(x)$ and $\alpha_\infty \in K$ respectively. Since S_I is a closed subset of $L^2(\mathbf{R})$, we have that $u_\infty(x) \in S_I$.

If it happens that $\liminf_{j\to\infty} \sigma_j = \infty$, then one can easily see that $(u \equiv 0, \alpha = 1, \sigma = 1)$ is a minimizer. If $\liminf_{j\to\infty} \sigma_j < \infty$, we pass to a further subsequence so that $\sigma_j \to \sigma_\infty \geq 0$. It is possible that $\sigma_\infty = 0$, but no matter: in any case we have

$$\alpha_j G_{\sigma_j} * u_j \longrightarrow \alpha_\infty G_{\sigma_\infty} * u_\infty \text{ in } L^2 \tag{26}$$

even when $\sigma_\infty = 0$. Existence of a minimizer $(u_\infty, \alpha_\infty, \sigma_\infty)$ follows from this continuity property, and the lower semicontinuity of the total variation norm (first term). □

Remark: We allow $\sigma = 0$. This is to be interpreted to mean that:

$$\inf_{\substack{u(x)\in S_I \\ \alpha\in K \text{ and } \sigma\geq 0}} E_\lambda(u, \alpha, \sigma) = \int_{\mathbf{R}} |\nabla v| + \lambda \int_{\mathbf{R}} (\alpha v - f)^2 \, dx \tag{27}$$

for some $v(x) \in S_I$ and $\alpha \in K$.

We now list some possible variational models whose well posedness follows immediately from Proposition 4. Which model among these is more applicable depends on what kind of additional information is available in practice.

Corollary 1 *The following minimization problems have solutions:*

1. *Given a fixed α_0,*

$$\inf_{\substack{u(x)\in S_I \\ \sigma\geq 0}} E_\lambda(u, \alpha_0, \sigma) \tag{28}$$

2. *Given a continuous, strictly positive function $\psi(x)$ such that $\psi(x) \to \infty$ as $|x| \to \infty$,*

$$\inf_{\substack{u(x)\in S_I \\ \alpha,\sigma\geq 0}} \Big(E_\lambda(u, \alpha, \sigma) + \psi(\alpha) \Big) \tag{29}$$

3. *Given $\varepsilon > 0$,*

$$\inf_{\substack{u(x)\in S_I^\varepsilon \\ \alpha,\sigma\geq 0}} E_\lambda(u, \alpha, \sigma) \tag{30}$$

where

$$S_I^\varepsilon := \Big\{ u \in S_I : u(x) = 1 \Rightarrow u(y) = 1 \ \forall y \in [x - \beta_1, x + \beta_2]$$
$$\textit{for some} \beta_1, \beta_2 \geq 0 \textit{ with } \beta_1 + \beta_2 \geq \varepsilon \Big\} \quad (31)$$

4. $\inf_{\substack{u(x) \in S_I \\ \alpha, \sigma \geq 0}} E_\lambda(\alpha u, 1, \sigma)$.

Proof: Problem 1 is the same as the problem given in (25) with $K = \{\alpha_0\}$, so has a solution by Proposition 4. To see that Problem 2 has a solution, observe that the minimum energy is bounded from above by $M := \lambda \int f^2 \, dx + \psi(0)$; that means, in particular, we only consider α such that $\psi(\alpha) \leq M$. By hypothesis on ψ, that constrains α to a compact subset of **R**, so that reasoning of Proposition 4 applies. For Problem 3, the minimum energy is clearly bounded by $\lambda \int f^2 \, dx$. For any $u \in S_I^\varepsilon, u \not\equiv 0$ this restricts relevant α to a compact set (independent of u) so that once again the reasoning of Proposition 4 can be applied to find a minimizer. Finally, in Problem 4 the energy bounds the total variation of the function $\alpha u(x)$ so that the compactness and continuity argument of Proposition 4 can be applied to a minimizing sequence of these functions. □

Remarks: 1. The first variational problem in the corollary above is useful when the amplitude of the kernel can be estimated a priori. We present a number of numerical experiments with this approach in the subsequent sections.

2. The third variational problem is over bar codes in which the thickness of each "bar" is bounded below by $\varepsilon > 0$. This is indeed the case with bar code signals in practice. On the other hand, S_I^ε can be described by a finite number of parameters, and penalizing the total variation of the signal might be unnecessary.

We next consider whether solving the variational problem (25) indeed leads to the correct bar code if the observed signal is noise free. To be precise, fix a $b(x) \in S_I$ such that $b(x) \neq 0$. Let $f(x)$, the observed signal, be given by

$$f(x) = \alpha_0 G_{\sigma_0} * b(x) \text{ for some } \alpha_0, \sigma_0 > 0. \quad (32)$$

The following proposition shows that indeed for every large enough choice of the parameter λ the solution (minimizer) of the variational problem (25) recovers the correct bar code $b(x)$ and the parameters α_0, σ_0 from an observed signal of the form (32).

Proposition 5 *If $\alpha_0 \in K$ and $f(x)$ is given by (32), then there exists a λ_* such that whenever $\lambda \geq \lambda_*$ the unique minimizer of (25) is precisely $(b(x), \alpha_0, \sigma_0)$.*

Proof: Begin by noting that

$$E_\lambda(b(x), \alpha_0, \sigma_0) = \int_{\mathbf{R}} |\nabla b| = 2N \tag{33}$$

where $N \in \mathbf{N}$. Therefore, the claim is proved if we show that for any $\alpha, \sigma \geq 0$ and any $u(x) \in S_I$ with $\int |\nabla u| \leq 2N$ and $u(x) \neq b(x)$ we have $E_\lambda(u, \alpha, \sigma) > 2N$.

Case 1: $\int |\nabla u| = 2N$.

Then, by the uniqueness result,

$$\int (\alpha G_\sigma * u - f)^2 \, dx = 0 \text{ only if } (u(x), \alpha, \sigma) = (b(x), \alpha_0, \sigma_0). \tag{34}$$

Therefore, for any other $u(x)$, we have

$$E_\lambda(u, \alpha, \sigma) = 2N + \lambda \int (\alpha G_\sigma * u - f)^2 \, dx > 2N. \tag{35}$$

Case 2: $\int |\nabla u| < 2N$.

The set of all such $u(x)$ lie in a finite dimensional space; indeed, any such $u(x)$ can be written as

$$u(x) = \sum_{j=1}^{N-1} \chi_{(a_j, b_j)}(x) \text{ where } a_j \leq b_j < a_{j+1} \text{ for all } j, \tag{36}$$

and $a_j, b_j \in [0, 1]$. The function F of $2N$ variables defined as

$$F\left(a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}; \alpha; \sigma\right) \equiv E_\lambda \left(\sum_j \chi_{(a_j, b_j)}(x), \alpha, \sigma \right) \tag{37}$$

is continuous for $\alpha \in K$ and $\sigma \geq 0$. Moreover,

$$\lim_{\sigma \to \infty} \alpha G_\sigma * u = 0 \text{ in } L^2 \tag{38}$$

uniformly in $u \in S_I$ and $\alpha \in K$. Recalling once again that by the uniqueness result $E_\lambda(u, \alpha, \sigma) > 0$ for all u that fall under this case, these imply:

$$m \equiv \inf_{\substack{u \in S_I \text{ and } \int |\nabla u| < 2N \\ \alpha \in K \text{ and } \sigma \geq 0}} \|\alpha G_\sigma * u - f\|_{L^2}^2 > 0. \tag{39}$$

Choosing $\lambda = \frac{2N}{m}$, the conclusion of the proposition follows. $\square$

5. Computation

We start by describing a numerical technique for minimizing the following special instance of (25):

$$\inf_{\substack{u(x)\in S_I\\ \sigma\geq 0}} E_\lambda(u,\alpha,\sigma) \tag{40}$$

where $\alpha > 0$ is kept fixed. Our approach is to approximate this problem by a "diffuse interface" model that involves a small parameter. This allows us, among other things, to impose the two-valuedness constraint on $u(x)$ via a penalty term in the energy and minimize over all functions instead. To that end, we introduce the following energies:

$$\begin{aligned} E_{\varepsilon,\lambda}(u,\sigma) := \int_{\mathbf{R}} \left(\varepsilon\big(u'(x)\big)^2 + \frac{1}{\varepsilon} W\big(u(x)\big) \right)\, dx \\ + \lambda \int_{\mathbf{R}} \Big(\alpha G_\sigma * u(x) - f(x)\Big)^2 dx \end{aligned} \tag{41}$$

Here, ε is the positive small parameter, and $W(x)$ is the "double-well" function $W(x) = x^2(1-x)^2$. Then, the following minimization problem:

$$\inf_{\substack{u(x)=0 \text{ for a.e. } x\in I^c\\ \sigma\geq 0}} E_{\varepsilon,\lambda}(u,\sigma) \tag{42}$$

approximates (40) as $\varepsilon \to 0^+$. The idea for this well known approximation technique comes from the work of Modica and Mortola [13], which deals with the more interesting setting of $(n-1)$-dimensional interfaces in $\mathbf{R}^n$. (See [2] and [15] for other examples of such approximations in different contexts).

We propose to minimize the approximate energies $E_{\varepsilon,\lambda}$ via gradient descent, starting with an initial guess for $u(x)$ and σ. In particular, we initialize $u(x)$ so that it has as many interfaces as can be resolved, and let it coarsen as the descent progresses.

The Euler-Lagrange equations for these energies lead to the following gradient descent equations:

$$u_t = 2\varepsilon u_{xx} - \frac{1}{\varepsilon} W'(u) - 2\lambda\alpha G_\sigma * \big(\alpha G_\sigma * u - f\big) \tag{43}$$

and

$$\sigma_t = -\alpha \int_{\mathbf{R}} \big(\alpha G_\sigma * u - f\big)\Big(\big(\frac{\partial G_\sigma}{\partial \sigma}\big) * u\Big)\, dx. \tag{44}$$

If we minimize also over α, the optimal choice for it at any given $t \geq 0$ would be given by

$$\alpha(t) = \left(\int f(x)\big(G_\sigma * u\big)\, dx\right)\left(\int \big(G_\sigma * u\big)^2 dx\right)^{-1} \tag{45}$$

for as long as $\alpha(t) \in K$.

Numerically, these equations are discretized and solved for a large value of t. The important feature is how the variable u is handled. Let $u^n(x)$ be the numerical solution at the n-th time step (i.e. at $t = n\delta t$, where δt is the time step size). The time discretization used for u is as follows:

$$u^{n+1} - 2(\delta t)\Big\{\varepsilon u^{n+1}_{xx} - \lambda\alpha^2 G_\sigma * G_\sigma * u^{n+1}\Big\} = u^n - (\delta t)\Big(\frac{1}{\varepsilon}W'(u^n) - 2\lambda\alpha G_\sigma * f\Big) \quad (46)$$

The right hand side of this equation is completely in terms of u^n, while the left hand side is given by a linear operator acting on u^{n+1}. We discretize this equation also in space, and solve the resulting fully discrete equation via the fast Fourier transform. Note that the nonlocal term in the left hand side of course has the pleasant expression

$$\big(\hat{G}_\sigma\big)^2(\xi)\hat{u}^{n+1}(\xi) \quad (47)$$

in terms of the Fourier transform of the quantities involved.

6. Numerical results

We applied the numerical technique described in the previous section to a number of bar codes. One crucial point in such numerical experiments is that the constants involved (such as the fidelity weight constant λ) should be kept fixed; in other words, once a good value for the constants have been chosen, they should work for **all** (or at least a wide range of) test signals without any change.

We therefore fixed the constants as follows:

Variable		Value
ε	=	0.00107
λ	=	7500

These values were used verbatim for all our experiments.

In each experiment, a "clean" bar code signal was corrupted by convolving it with a kernel of known standard deviation, followed by the addition of some noise. Then, the algorithm of this paper was carried out to see if it could recover the correct bar code and the correct standard deviation of the convolution kernel. In the experiment of *Figure 6*, α was also treated as an unknown and recovered by the algorithm. We experimented with three different choices of standard deviation for the kernel, which were approximately 0.012, 0.013, and 0.014. These values were chosen because $\sigma = 0.014$ seems to be about when the technique starts yielding erroneous results at signal to noise ratios of about 20 dB; for values of the standard deviation much smaller than these, the reconstruction is almost perfect at moderate noise levels. *Figures 2* and *3* present results from two of these experiments, where the former is an example of completely successful reconstruction, and the latter an example of how the algorithm begins to fail when the noise level and/or the blur factor get too high.

The initial guess for the bar code (required by the gradient descent algorithm) was taken to be an alternating sequence of 1's and 0's. We made the following **empirical observation** concerning the initial guess for the variance:

> *It is better to start with a low initial guess for the standard deviation of the convolution kernel.*

This prevents the initial guess for the signal from coarsening too rapidly and ending up with very few "bars": presumably, there are local minima associated with large values of the standard deviation. In the experiments presented, we took 0.0079 or 0.0088 as our initial guess. Starting with even lower initial values seems to make no difference in the final result; it only takes the algorithm longer to reach it.

Figure 4 makes an important point. Its top row shows the derivative of the bar code signal used in *Figures 2* and *3* after convolution with a kernel of standard deviation $\sigma = 0.01414$. Some of the discontinuities in the original bar code have **no corresponding local extrema** in the derivative of the convolved signal; for example, see the thin bars located near the 495th and 530th grid points. Hence, a traditional reconstruction technique that attempts to locate extrema of the signal's derivative would be at a very significant disadvantage at this level of blur even in the absence of any noise. The middle row shows the observed signal, which is obtained from the convolved signal by adding a moderate amount of noise. The last (bottom) row shows the reconstruction obtained by the algorithm of this paper (under the assumption that α is known) from the corrupted signal shown in the middle row.

Naturally, the level of noise that the algorithm can tolerate depends on the level of blur: as the standard deviation of the convolution kernel gets large, the amplitude of noise that can be tolerated for a reasonable reconstruction decreases. For example, when the parameters ε, λ are casually selected to have the values quoted above, and when σ is about 0.01, the algorithm yields accurate reconstructions at signal to noise ratios as low as 13 dB. When σ is about 0.013, the algorithm starts to fail, as shown in *Figure 3*, at signal to noise ratios below 16 dB. The failure in *Figure 3* seems to be typical: one or more of the thinner bars do not get reconstructed. It is also possible to end up with extraneous thin bars. These are the types of error that would adversely affect recognition; otherwise, tiny discrepancies in the location of

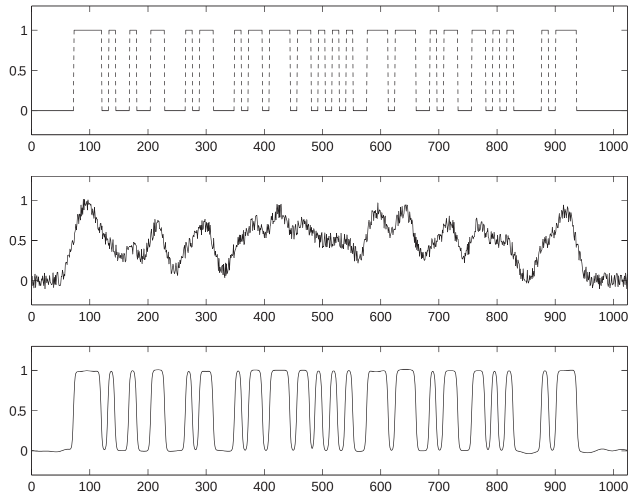

Figure 2: Top to bottom: The bar code, corresponding corrupted signal, and the reconstruction found by the algorithm of this paper, under the assumption that kernel amplitude α is known. Standard deviation of the kernel used to generate the corrupted signal was $\sigma = 0.0118$. The algorithm found $\sigma = 0.0115$. Root mean square signal to root mean square noise ratio was about 19.3 dB.

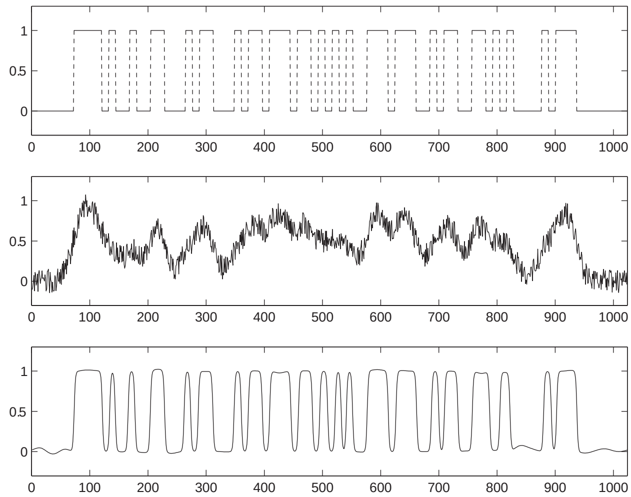

Figure 3: Top to bottom: The bar code, corresponding corrupted signal, and the reconstruction found by the algorithm of this paper, under the assumption that kernel amplitude α is known. Notice that it failed to reconstruct one of the bars, located at around the 800th grid point. Standard deviation of the kernel used to generate the corrupted signal was $\sigma = 0.0129$. The algorithm found $\sigma = 0.0124$. Root mean square signal to root mean square noise ratio was about 15.7 dB.

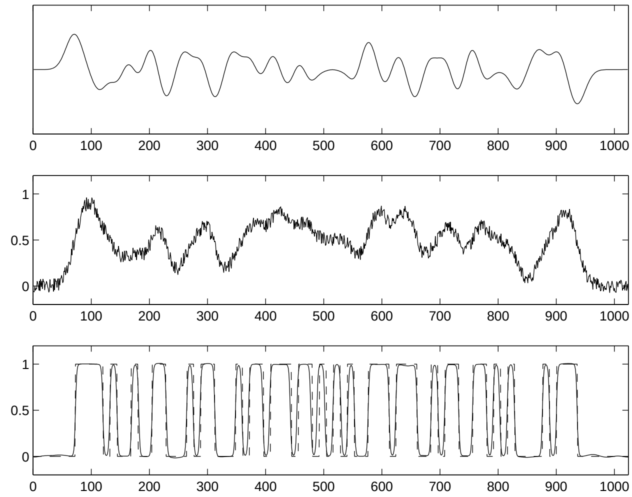

Figure 4: Top to bottom: Derivative of a convolved bar code signal, the convolved signal after addition of noise, and the bar code reconstructed by the algorithm, under the assumption that α is known. Standard deviation of the kernel used was $\sigma = 0.01414$. The algorithm found $\sigma = 0.01408$. Root mean square signal to root mean square noise ratio was about 21.55 dB.

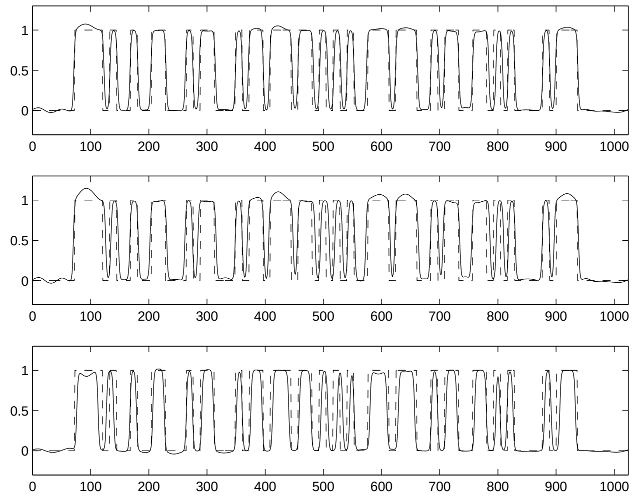

Figure 5: Top to bottom: Bar codes reconstructed by the algorithm under the assumption that kernel amplitude α is known, when the assumed amplitude is off by 10%, 20%, and 25%, respectively, from the true value. The solid line corresponds to the reconstruction, and the dashed line to the original (correct) bar code. The noise level and variance of the kernel used to corrupt the original bar code was the same as in *Figure 2*.

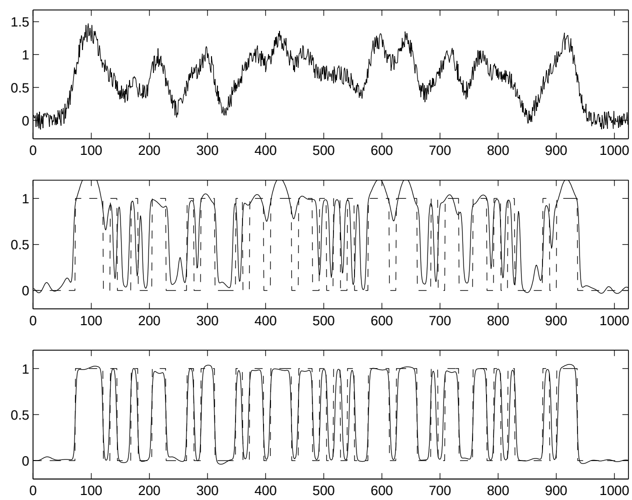

Figure 6: Top graph shows the original noisy signal obtained using a kernel whose standard deviation and amplitude were about 0.0118 and 1.4 respectively. The middle graphs shows the result of bar code reconstruction with amplitude of kernel assumed known to be 1. The standard deviation found was 0.0072. In contrast, the bottom graph shows the reconstruction obtained when the amplitude of the signal is assumed unknown and is minimized over starting from an initial guess of 1. The standard deviation found by the algorithm was 0.0116, and the amplitude found was 1.41.

bars are inconsequential, since they are easy to compensate for by using additional a priori information about bar codes. At a blur level of about $\sigma = 0.014$, a signal to noise ratio of about 21 dB seems to be the threshold for reliable reconstructions. With a more careful tuning of the parameters ε, λ, we expect that the performance of the algorithm can be improved.

Most of our numerical experiments correspond to Model 1 of Corollary 1, where the amplitude α of the kernel is assumed known a priori, and therefore is not minimized over (however, see *Figure 6*). We chose this simplest model for the bulk of our experiments because at this stage we would like to leave open the issue of how to treat minimization over α, since this decision involves knowing what kind of additional information is available in practice about this parameter. Right now, it is not clear which model among Models 2 through 4 of Corollary 1 would be the most appropriate, and so we do not wish to commit to any one in particular.

In general, we of course cannot expect to have a perfect estimate for the kernel amplitude α (but it is easy to get *some* estimate: for instance, the amplitude of the observed signal). In *Figure 5* we investigate the sensitivity of the results to errors in the assumed value of this parameter. When the width (standard deviation) of the convolution kernel and the noise level are moderate, we see that the results are quite stable under perturbations in α: the method seems to be resilient up to about 25% error. However, results get more sensitive to such perturbations if the original signal is corrupted by a wider kernel. This is to be expected. Once we allow α to be updated (i.e. minimize also with respect to α), we would expect these results to improve.

Indeed, *Figure 6* shows a sample computation where amplitude of the moderately degraded signal is treated as another unknown, and the initial estimate for it is off by about 40%. Therefore, minimization is now carried out also over α, as explained in the previous section: the optimal choice of the constant α is prescribed at every time step according to formula (45). The results show the improvement obtained by minimizing over α versus keeping α fixed (at a wrong value) as in the experiments of *Figure 5*.

7. Conclusion

In this paper we developed a variational approach for blind deconvolution of bar code signals. It is based on the total variation restoration model of image processing. The analysis we presented motivates and substantiates our approach, and suggests several related variational models that can be used for the solution of the bar code problem. These models require no prior information about the number of bars expected in the code, and are well suited for recovering bar codes from very blurred and noisy observed signals.

By concentrating on a specific one of our proposed models, we demonstrated how these variational models can be implemented in practice. This involves, among other things, using approximate functionals that are numerically more easily treated. Results of some of the numerical experiments, along with all details of how they were obtained, are included. These suggest that our technique can recover bar codes from highly degraded signals.

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