A Survey on Level Set Methods for Inverse Problems and Optimal Design

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The aim of this paper is to provide a survey on the recent development in level set methods in inverse problems and optimal design. We give introductions on the general features of such problems involving geometries and on the general framework of the level set method. In subsequent parts we discuss shape sensitivity analysis and its relation to level set methods, various approaches on constructing optimization algorithms based on the level set approach, and special tools needed for the application of level set based optimization methods to ill-posed problems. Furthermore, we provide a review on numerical methods important in this context, and give an overview of applications treated with level set methods.

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Finally, we provide a discussion of the most challenging and interesting open problems in this field, that might be of interest for scientists who plan to start future research in this field.

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Contents

1 Introduction

The *level set method*, originally introduced by Osher and Sethian [109] is a general framework for the computation of evolving interfaces using implicit representations. Its fundamental idea is to represent an evolution of shapes $\Omega(t) \subset \mathbb{R}^d$, $t \in \mathbb{R}^+$ via

$$\Omega(t) = \{ \ x \in \mathbb{R}^d \mid \phi(x,t) < 0 \ \}, \tag{1.1}$$

where $\phi : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ is a continuous function. The motion of the domain Ω can be formulated as a Hamilton-Jacobi equation for the level set function ϕ, as we shall see in detail below. The implicit representation of the geometry introduces the possibility to handle topological changes such as splitting and merging of connected components in an automatic way, and allows to construct efficient and accurate numerical methods. We refer to the recent monograph by Osher and Fedkiw [106] for a general introduction to the level set method and an overview of applications in several areas.

While the first decade after its invention, the level set method had enormous impact on the solution of problems in computational geometry, fluid dynamics, and materials science, it seems that due to the developments in its second decade, the level set method is becoming a standard tool for inverse problems and optimal design problems involving geometric objects as unknowns. The application of level set methods to such kinds of problems has not only increased the computational efficiency, but also opened completely new possibilities due to its flexibility of handling topological changes. Together with *linear sampling* and related methods (cf. [40, 86]), the level set method (introduced to this field by Santosa [118]) has therefore lead to a change of paradigm in inverse obstacle problems: instead of reconstructing geometric objects with strongly restricted topology under a variety of a-priori assumptions, the aim has changed to reconstruction rather general geometric objects with minimal a-priori knowledge. A similar change of paradigms has appeared in some fields of optimal design (topology optimization) already some years before due to the introduction of homogenization methods (cf. [7, 16]). While homogenization approaches may lead to microstructures or lose the clear geometric interpretation, the level set approach provides a similar topological flexibility together with a sharp interface between different materials. Therefore, the interest in level set methods is strongly increasing in topology optimization, too.

The paper is organized as follows: Section 2 starts with a basic discussion of the type of optimization problems we are interested in, and then Section 3 provides a brief introduction to the level set method. In Section 4 we discuss the basic ideas of shape calculus and its relations to the level set method. Section 5 gives a survey on recent development in the construction of level set based optimization techniques, forming somehow the core part of this paper. The aspect of regularization techniques avoiding topological restrictions for such shape optimization problems is discussed in Section 6, and Section 7 provides a brief overview of numerical methods needed in this context. Section 8 gives a survey of applications of level set based optimization techniques that have been realized so far. Finally, we give an outlook on possible further development in this field in Section 9.

Throughout this paper we shall use standard notation for partial derivatives and gradients and for Sobolev spaces (cf. [90]). Moreover we shall denote by $\mathcal{L}^d$ the d-dimensional Lebesgue measure, and by $\mathcal{H}^{d-1}$ the $d-1$-dimensional Hausdorff measure (cf. [57]).

2 Inverse and Optimization Problems Involving Geometries

There is a variety of inverse problems and optimal design problems, where the unknown variable is a geometric object, whose topology is unknown in general. The basic setup of such an optimization problem is to solve

$$J(\Omega) \to \min_{\Omega \in \mathcal{K}_{ad}}, \tag{2.1}$$

where $J : \mathcal{K}_{ad} \to \mathbb{R}$ is a suitable shape functional on a class $\mathcal{K}_{ad} \subset \mathcal{K}(D)$, where $K(D)$ is the class of compact subsets of some fixed domain D. Such *shape* or *topology optimization problems* have been studied over several decades, with the development being driven by various applications, e.g. in structural design and in fluid dynamics (cf. [16, 44, 98, 125] and the references therein). Similar problems arise in the field of inverse problems, where the aim is to reconstruct an unknown shape from indirect measurements. Here, the objective functional is usually of the form

$$J(\Omega) = \frac{1}{2}\|\mathcal{F}(\Omega) - z\|^2 \tag{2.2}$$

where $\mathcal{F} : \mathcal{K}_{ad} \to \mathcal{Z}$ is a nonlinear operator mapping to some Hilbert space $\mathcal{Z}$. Shape and topology optimization problems are challenging due to the missing vector space structure on classes of compact sets. This fact makes both theory and computations very difficult. In this survey, we focus on the latter aspect, with respect to which enormous progress has been made in the recent years due to the use of level set methods and related techniques.

In order to provide further insight to readers new to the field and for the sake of later reference, we state some model problems, representing typical cases of applications, in the following:

Model Problem 1 (Inclusion Detection). A typical inverse obstacle problem is the detection of inclusions in elastic materials or the detection of cavities (cf. [3, 4, 14, 15]). We consider the simplest model problem of anti-planar strains (cf. [15]). The inverse problem consists in the identification of $\Omega \subset D$ from displacement measurements $f_k = u_k|_M$,

$k = 1, \ldots, N$ on $M \subset \partial D$, where u_k solves

$$\Delta u_k = 0 \qquad \text{in } D \backslash \Omega \tag{2.3}$$

$$u_k = 0 \qquad \text{on } \Gamma \subset \partial D \tag{2.4}$$

$$\frac{\partial u_k}{\partial n} = g_k \qquad \text{on } \partial D \backslash \Gamma \tag{2.5}$$

$$\frac{\partial u_k}{\partial n} = 0 \qquad \text{on } \partial \Omega, \tag{2.6}$$

where g_k, $k = 1, \ldots, N$ are different applied loads.

This inverse problem can be formulated via the minimization of a least-squares functional of the form

$$J(\Omega) = \frac{1}{2} \sum_{k=1}^{N} \int_M |u_k - f_k|^2 \, d\mathcal{H}^{d-1}, \tag{2.7}$$

where $\mathcal{H}^{d-1}$ denotes the $d-1$-dimensional Hausdorff measure.

Model Problem 2 (Structural Optimization with Pressure Loads). A class of structural optimization problems of growing importance are those with design-dependent loads like pressure loads, which cannot be treated in a reasonable way by the homogenization method (cf. [9, 16, 19]). A simple model problem from linearized elasticity is given by the minimization of the compliance

$$J(\Omega) = \int_{D \backslash \Omega} f \cdot u \, dx + \int_{\partial D \backslash \Gamma} g_0 \cdot u \, d\mathcal{H}^{d-1} + \int_{\partial \Omega} g \cdot u \, d\mathcal{H}^{d-1} \tag{2.8}$$

of a material, whose stress σ and displacement u are determined by

$$- \operatorname{div} \sigma = f \qquad \text{in } D\backslash\Omega \tag{2.9}$$

$$\sigma - C : (\nabla u + \nabla u^T) = 0 \qquad \text{in } D\backslash\Omega \tag{2.10}$$

$$u = 0 \qquad \text{on } \Gamma \subset \partial D \tag{2.11}$$

$$\sigma.n = g_0 \qquad \text{on } \partial D\backslash\Gamma \tag{2.12}$$

$$\sigma.n = g \qquad \text{on } \partial\Omega. \tag{2.13}$$

Model Problem 3 (Source Reconstruction). In some applications, one has to deal with the reconstruction of a piecewise constant source term in a partial differential equation, which jumps at a material interface. The simplest model problem for this case is given by reconstructing Ω from a measurement $z = u|_M$ on a set $M \subset D$ or $M \subset \partial D$, where u is the unique solution of

$$-\Delta u - \chi_\Omega = 0 \qquad \text{in } D, \tag{2.14}$$

$$u = 0 \qquad \text{on } D, \tag{2.15}$$

where χ_Ω denotes the indicator function of $\Omega \subset D$.

The corresponding least squares problem is given by minimizing

$$J(\Omega) = \frac{1}{2} \int_M |u - z|^2 \, d\mu, \tag{2.16}$$

where μ is either the Lebesgue measure in D or the Hausdorff measure on ∂D, subject to (2.14), (2.15). We refer to Hettlich and Rundell [68, 69, 70] for a discussion of problems of this kind.

Model Problem 4 (Band Structure Design). An important case of problems are those related to minimizing the eigenvalue structure, with applications e.g. in the optimal design of photonic crystals (cf. [31, 41] for an overview). As a simple model example we consider the minimization of a functional of the form

$$J(\Omega) = \hat{J}(\Lambda(\Omega)), \tag{2.17}$$

where $\Lambda(\Omega) = (\lambda_j(\Omega))_{j\in\mathbb{N}}$ denotes the (increasingly ordered) sequence of eigenvalues related to the Helmholtz equation

$$-\Delta u = \lambda(q_0 + q_1\chi_\Omega)u \qquad \text{in } D, \tag{2.18}$$

with $u = 0$ on ∂D. Possible choices are the minimization and maximization of the first eigenvalue, respectively, i.e., $\hat{J}(\Lambda) = \pm\lambda_1$, or the maximization of a bandgap, i.e., $\hat{J}(\Lambda) = \lambda_k - \lambda_{k+1}$. This type of problems has been investigated in [108, 64].

3 Level Set Methods for Evolving Interfaces

In the following we review the (by now almost classical) level set approach to geometric motion (cf. [109, 106]). If a set $\Omega(t)$ is moving with a normal velocity V_n (note that a motion component in tangential direction does not change the shape) at its boundary, then we have due to a standard result for the derivative of parameter-dependent integrals

$$\frac{d}{dt}\int_{\Omega(t)} w \; dx = \int_{\partial\Omega(t)} w \; V_n \; d\mathcal{H}^{d-1}$$

for each smooth function w of x with compact support. Now assume that we are given a velocity V_n on $\mathbb{R}^d$ such that each level set of the continuous function ϕ is moving with normal velocity V_n, i.e., the above identity holds with $\Omega(t) = \{\phi(.,t) < \eta\}$ for each real η. Then we can take the mean value over all η and obtain from the area and co-area formula (cf. [57])

$$\int_{\mathbb{R}^d} \frac{\partial \phi}{\partial t}\, w\, dx = \frac{d}{dt} \int_{\mathbb{R}^d} \phi\, w\, dx = -\frac{d}{dt} \int_{\mathbb{R}} \int_{\{\phi(.,t)<\eta\}} w\, dx\, d\eta$$
$$= - \int_{\mathbb{R}} \int_{\partial\Omega(t)} w\, V_n\, d\mathcal{H}^{d-1}\, d\eta = - \int_{\mathbb{R}^d} |\nabla \phi|\, V_n\, w\, dx.$$

Since the test function was arbitrary, this implies that ϕ has to satisfy the so-called level set equation

$$\frac{\partial \phi}{\partial t} + V_n\, |\nabla \phi| = 0 \qquad \text{in } \mathbb{R}^d \times \mathbb{R}_+. \tag{3.1}$$

In typical applications of geometric motion, the normal velocity V_n is given from physical principles, it can depend on external fields and on geometric quantities such as the normal direction or the curvature. These quantities can be expressed in terms of the level set function, too, e.g. the unit outer normal n and the mean curvature κ are given by

$$\mathbf{n} = \frac{\nabla \phi}{|\nabla \phi|}, \qquad \kappa = \text{div } \mathbf{n} = \text{div } \left(\frac{\nabla \phi}{|\nabla \phi|} \right).$$

Note that if the velocity depends on curvature, then (3.1) becomes a fully nonlinear second order partial differential equations.

In general, (3.1) does not have a classical solution, but only a *viscosity solution* (cf. [91, 42] for an overview), which exists under appropriate regularity conditions on the velocity. It can be shown that the motion obtained as the zero level set of the viscosity solution is a generalization of a smooth motion in normal direction, and the motion is uniquely defined if no *fattening* occurs, i.e., if the level set $\{\phi(.,t) = 0\}$ has empty interior for all t.

By computing viscosity solutions to the level set equations, one obtains topological changes such as splitting and merging of connected components in an automatic way, since they are not even recognized by (3.1). Even if one does not expect a topological change, one usually benefits from using a level set approach instead of methods based on parametrizations, since the discretization of the parametrization does not allow to control the accuracy in the resolution of a curve or surface.

4 Level Set Methods and Shape Calculus

Shape sensitivity analysis is a classical topic in shape optimization, and defines a natural calculus on shapes. For sufficiently regular shapes (with boundary of class C^1) there are two equivalent ways of introducing shape sensitivities, namely the *deformation method* and the *speed method* (cf. [125]). Due to its relation to the level set method we shall use the latter as the basis of the following presentation.

4.1 Shape Calculus via the Speed Method

Given a set Ω, one can define a time evolution of sets $\Omega(t)$ in a velocity field $\mathbf{V}$ via

$$\Omega(t) = \{ \ y(t) \mid y(0) \in \Omega, \frac{dy}{dt}(\tau) = \mathbf{V}(y(\tau)) \text{ in } (0,t) \ \}. \tag{4.1}$$

Note that the shapes $\Omega(t)$, $t \leq T$ are well-defined due to the Picard-Lindelöf Theorem for ordinary differential equations if $\mathbf{V} \in C^{0,1}(\mathbb{R}^d)$, where $C^{0,1}(\mathbb{R}^d)$ denotes the space of Lipschitz continuous functions.

The shape sensitivity of a functional J in direction of a perturbation $\mathbf{V} \in C^{0,1}(\mathbb{R}^d)$ is then given by

$$dJ(\Omega; \mathbf{V}) = \frac{d}{dt} J(\Omega(t))|_{t=0} \tag{4.2}$$

if the derivative on the right-hand side exists, and $dJ(\Omega, .)$ is called the shape differential.

In typical cases, the shape differential can be extended to a continuous linear functional of $\mathbf{V}$ even on Banach spaces larger than $C^{0,1}(\mathbb{R}^d)$ as we shall also see in the examples below.

From the geometric intituition it is obvious that the shape variation is determined by $V_n := \mathbf{V} \cdot \mathbf{n}$ on $\partial\Omega(t)$ and consequently the shape sensitivity should not depend on other values of $\mathbf{V}$. This intuition is made rigorous in a result sometimes called "Hadamard-Zolesio structure theorem", stating that if J, Ω, and V_n are sufficiently regular, the shape differential is a linear functional of $V_n|_{\partial\Omega}$. In this case, one usually rewrites

$$dJ(\Omega; \mathbf{V}) = J'(\Omega)V_n, \tag{4.3}$$

and considers $J'(\Omega)$ as the shape sensitivity or shape gradient.

There are two prototypes of functionals J, namely *domain functionals* of the form

$$J_{dom}(\Omega) = \int_\Omega \psi \; dx \tag{4.4}$$

and *boundary functionals* of the form

$$J_{bd}(\Omega) = \int_{\partial\Omega} \psi \; d\mathcal{H}^{d-1}.$$

where ψ is a given function satisfying appropriate smoothness assumptions (to be specified below in both cases). For these two type of functionals, the following formulas for the derivatives can be deduced from the results in [44] (cf. Theorem 4.2, p. 353, and Theorem 4.3, p. 355):

- For $\psi \in W^{1,1}_{loc}(\mathbb{R}^d)$ and bounded measurable domain Ω the shape differential of J_{dom}

exists and is given by

$$dJ_{dom}(\Omega;\mathbf{V}) = \int_{\Omega} \mathrm{div}(\psi\ \mathbf{V})\ dx.$$

If Ω is an open domain with Lipschitz boundary, then (due to Gauss' Theorem) the shape gradient exists and is given by

$$J'_{dom}(\Omega)V_n = \int_{\partial\Omega} \psi\ V_n\ d\mathcal{H}^{d-1}.$$

- For $\psi \in W^{2,2}_{loc}(\mathbb{R}^d)$ and Ω being a bounded measurable domain with boundary of class C^2, the shape differential of J_{bd} exists for all $\mathbf{V} \in C^1_{loc}(\mathbb{R}^d)$ and is given by

$$\begin{aligned} dJ_{bd}(\Omega;\mathbf{V}) &= \int_{\partial\Omega} \mathrm{div}(\psi\mathbf{n})\mathbf{V}\cdot\mathbf{n}\ d\mathcal{H}^{d-1} \\ &= \int_{\partial\Omega} (\frac{\partial\psi}{\partial\mathbf{n}} + \psi\kappa)\mathbf{V}\cdot\mathbf{n}\ d\mathcal{H}^{d-1}, \end{aligned}$$

  where $\mathbf{n}$ denotes the unit outer normal and κ the mean curvature. Under this conditions, the shape gradient $J'(\Omega)$ exists and is given by

$$J'(\Omega)V_n = \int_{\partial\Omega} (\frac{\partial\psi}{\partial\mathbf{n}} + \psi\kappa)V_n\ d\mathcal{H}^{d-1}.$$

A closer inspection of the above formulas for shape sensitivities shows that under additional regularity conditions on ψ, one can extend the shape sensitivity to a continuous functional on a Hilbert space and therefore interpret $J'(\Omega)$ as an element of this space (due to the Riesz representation theorem). E.g., if $\psi \in H^1_l oc(\mathbb{R}^d)$, then the trace of ψ on $\partial\Omega$ is in the Hilbert space $H^{\frac{1}{2}}(\partial\Omega)$ and by standard theory in Sobolev spaces, $J'_{dom}(\Omega)$ can therefore be extended to a continuous linear functional in $H^{-\frac{1}{2}}\partial\Omega$, an element of which we may consider $J'_{dom}(\Omega)$ to be. This

By iterating the above definition we may also define second order shape sentivities, i.e.,

$$J''(\Omega)(V_n, W_n) = \frac{d}{dt} \left(J'(\Omega(t)) W_n \right) |_{t=0}, \tag{4.5}$$

where now $\Omega(t)$ denotes the motion of sets with normal velocity V_n. Under standard regularity conditions it is easy to see that $J''(\Omega)(.,.)$ is a symmetric bilinear form, we refer to [105] for further details on second order shape sensitivities.

4.2 Formal Computation of Shape Derivatives via Level Set Methods

Since the shape sensitivity in direction V_n is obtained from a perturbation moving the shape in normal direction with velocity V_n, we may directly relate the speed method to the level set equation (3.1), since the shape sensitivity is given by

$$J'(\phi(.,0) < 0) V_n = \frac{d}{dt} J(\phi(.,t) < 0)|_{t=0},$$

where ϕ solves (3.1).

This relation can be used to compute the shape sensitivity in a formal way directly from the level set method. E.g., for a volume functional of the form

$$J(\Omega) = \int_\Omega g(x)\ dx = \int_{\mathbb{R}^d} g(x)\ H(-\phi(x,0))\ dx,$$

where H denotes the Heaviside-function, we can compute the derivative as

$$\begin{aligned} \frac{d}{dt} J(\Omega(t)) &= \frac{d}{dt} \int_\Omega g(x)\ dx = \frac{d}{dt} \int_{\mathbb{R}^d} g(x)\ H(-\phi(x,t))\ dx \\ &= - \int_{\mathbb{R}^d} g(x)\ \delta(-\phi(x,t)) \frac{\partial \phi}{\partial t}(x,t)\ dx \\ &= \int_{\mathbb{R}^d} g(x)\ \delta(-\phi(x,t)) V_n(x)\ |\nabla \phi|(x,t)\ dx, \end{aligned}$$

with the Dirac delta distribution δ. Finally, a formal application of the co-area formula yields

$$J'(\Omega)V_n = \frac{d}{dt}J(\Omega(t))|_{t=0} = \int_{\{\phi(.,0)=0\}} g\ V_n\ d\mathcal{H}^{d-1},$$

which is the same formula as obtained with classical techniques in [125]. This formal technique of using Heaviside and delta functions has been introduced in the framework of the variational level set method by Zhao et al. [151].

We finally note that the above concept of shape sensitivities is not the only way of computing derivatives for a shape or topological derivatives. A complimentary concept concerns topological derivatives (cf. [123, 124]), which measure the variation with respect to the nucleation of holes. The combination of topological derivatives with level set methods is by far less well understood than the one of shape sensitivities, we shall discuss this aspect in Section 5.4.2. Some authors also tried a direct approach to the computation of derivatives by the level set method, namely by using a variation with respect to the level set function ϕ in some function space (cf. [35, 37, 39, 64, 87]). E.g., the formal derivative of a volume functional with respect to ϕ in a function space is given by (setting $\tilde{J}(\phi) = J(\{\phi = 0\})$)

$$\tilde{J}'(\phi)\psi = -\int_{\mathbb{R}^d} g\ \delta(-\phi)\psi\ dx, \tag{4.6}$$

which does not have a natural morphological structure, so that there is no geometric interpretation. In addition, this kind of derivative, given by $\tilde{J}'(\phi) = -g\ \delta(-\phi)$ cannot be used directly in an optimization algorithm, since it is not a function. We shall discuss this aspect in further detail below.

5 Level Set Based Shape Optimization

In the following we shall discuss the use of the level set technology to construct efficient, accurate, and flexible methods for shape optimization. The fundamental idea of this approach is the representation of the shape to be optimized as the zero level set of a continuous function ϕ and the choice of a velocity V_n that makes the shape evolve toward the optimal one.

The choice of the velocity plays the same role as the choice of the search direction in classical vector space optimization. Therefore, we also use analogous nomenclature in the categorization of level set based optimization methods:

5.1 Gradient-type Methods

The first (cf. [118]) and still most widely used optimization technique in connection with level set methods are gradient-type algorithms. The basic idea is to choose the update as a multiple of the negative gradient method and to perform a sufficiently small time step, which guarantees an update of the objective functional. The difficulty in level set based optimization is the relation between the update (the velocity) and the gradient (the shape sensitivity), since there is no inherent vector space structure.

The first approach due to Santosa [118] (analogous to a classical approach for local shape optimization based on shape parametrization, cf. [102, 98]) started from problems, where the shape sensitivity is of the form

$$J'(\Omega)V_n = \int_{\partial\Omega} V_n \; \rho_\Omega \; d\mathcal{H}^{d-1},$$

with some density function ρ_Ω (dependent on Ω). In this case one can interpret ρ_Ω as the shape gradient and the normal velocity can be chosen as (an extension of)

$$V_n(.,t) = -\rho_{\Omega(t)} \qquad \text{on } \partial\Omega. \tag{5.1}$$

In this way, one obtains

$$\frac{d}{dt} J(\Omega(t)) = J'(\Omega(t)) V_n(.,t) = -\int_{\partial\Omega} |\rho_{\Omega(t)}|^2 \, d\mathcal{H}^{d-1},$$

and consequently the evolution decreases the objective functional and stops only if the shape gradient vanishes. This gradient evolution can be discretized in time using a forward Euler method to obtain the standard gradient method, which still decreases the objective functional if the time step is sufficiently small.

This gradient-type approach was used by several authors to different shape optimization and reconstruction problems (cf. e.g. [47, 48, 49, 79, 112, 113, 114, 144, 145]). The approach works well in these cases, but is still limited to problems allowing the above representation formula of the shape sensitivity. Since $J'(\Omega)$ is a linear functional of V_n for fixed Ω, one observes from the Riesz representation theorem, that such a representation holds indeed with some $\rho_\Omega \in L^2(\partial\Omega)$ if the shape sensitivity $J'(\Omega)$ is a continuous linear functional on the Hilbert space $L^2(\partial\Omega)$. Unfortunately, not all problems are such that the shape sensitivity is continuous on this space, but there are prominent examples where the shape sensitivity is only bounded on different spaces. E.g., for the model problem 1 related to inclusion detection, $J'(\Omega)$ is continuous on the fractional Sobolev space $H^{\frac{1}{2}}(\partial\Omega)$ (cf. [14, 25]). Numerical examples indicate that the level set method with the velocity choice (5.1) does not converge for this problem (cf. [25]). Another example of non-convergence has been obtained in image segmentation (cf. [72]).

For such cases, a more general framework was developed in [25]. The main idea of the approach is to use a generalized notion of gradient descent by allowing the choice of some (arbitrary) Hilbert space $\mathcal{V}(\Omega)$ for the normal velocity V_n and to use the weak form

$$\langle V_n(.,t), W_n \rangle_{\mathcal{V}(\Omega(t))} = -J'(\Omega(t))W_n \qquad \forall\ W_n \in \mathcal{V}(\Omega(t)). \tag{5.2}$$

This choice allows a similar descent property as above, namely

$$\frac{d}{dt} J(\Omega(t)) = J'(\Omega(t))V_n(.,t) = -\|V_n(.,t)\|^2_{\mathcal{V}(\Omega(t))}.$$

If $\mathcal{V}(\Omega)$ is chosen appropriately (such that $J'(\Omega)$ is a continuous linear functional on this space), then the velocity $V_n(.,t) \in \mathcal{V}(\Omega(t))$ is well-defined by (5.2) due to the Lax-Milgram theorem. One observes that (5.1) is a special case of (5.2) for the choice $\mathcal{V}(\Omega) = L^2(\partial\Omega)$. The gradient descent (5.2) is well-known in material science applications (cf. [32, 134, 135]), where the evolution of crystals is related to gradient flows for surface energies of the form

$$J(\Omega) = \int_{\partial\Omega} \gamma(\mathbf{n})\ d\mathcal{H}^{d-1},$$

and incorporates such important geometric motions as the *Mullins-Sekerka* or *Hele-Shaw* flow (for $\mathcal{V}(\Omega) = H^{-\frac{1}{2}}(\partial\Omega)$) and motion by *surface diffusion* (for $\mathcal{V}(\Omega) = H^1(\partial\Omega)$).

Using this framework with appropriate Hilbert spaces, the level set based gradient method can be made stable for each shape optimization problem. The numerical results in [14, 25] support this for model problem 1, when the choice $\mathcal{V}(\Omega) = H^{\frac{1}{2}}(\partial\Omega)$ is used. For some other problems the Hilbert space can be chosen even larger than $L^2(\Omega)$. E.g., for model problem 3, where the shape sensitivity is continuous on $\mathcal{V}(\Omega) = H^{-\frac{1}{2}}(\partial\Omega)$, this choice yields a speed up of the level set evolution (cf. [25]).

A general convergence proof of level set based gradient methods is still an open problem, but one can at least show that the objective functional is decreasing during the iteration and that the method can only stop in a stationary point, i.e., if $J'(\Omega) = 0$. In a special case, namely for

$$J(\Omega) = \|\mathcal{A}\chi_\Omega - z\|^2,$$

with some linear operator $\mathcal{A}$ on $L^2(D)$ and χ_Ω denoting the indicator function of $\Omega \subset D$, and z some data given in a Hilbert space. For this class of problems, a detailed analysis of convergence and regularizing properties has been carried out in [24] for a velocity choice of the form $V_n = \frac{\partial w}{\partial n}$ on $\partial\Omega$, where w solves

$$-\Delta w = (1 - 2\chi_\Omega)\mathcal{A}^*(\mathcal{A}\chi_\Omega - z),$$

with $\mathcal{A}^*$ denoting the L^2-adjoint of $\mathcal{A}$. An investigation of the mapping properties of this operator induces that this corresponds to a gradient flow in an equivalent norm on $H^{\frac{1}{4}}(\partial\Omega)$.

Model Problem 1 (Revisited). In the following we discuss the application of the above approach to the model problem of inclusion detection. The shape derivative of this problem can be computed as (cf. [15])

$$J'(\Omega)V_n = -\sum_{k=1}^{N} \int_{\partial\Omega} \nabla u_k \cdot \nabla w_k \; V_n \; d\mathcal{H}^{d-1},$$

where the functions w_k solve the adjoint problems

$$\begin{aligned} \Delta w_k &= 0 && \text{in } D\backslash\Omega \\ w_k &= 0 && \text{on } \Gamma \\ \frac{\partial w_k}{\partial n} - \chi_M(u_k - f_k) &= 0 && \text{on } \partial D\backslash\Gamma \\ \frac{\partial w_k}{\partial n} &= 0 && \text{on } \partial\Omega \end{aligned}$$

where χ_M is the indicator function of M. Note that due to the homogeneous Neumann boundary conditions for u_k and w_k on $\partial\Omega$, the expression for the shape derivative can be simplified to

$$J'(\Omega)V_n = -\sum_{k=1}^{N} \int_{\partial\Omega} \frac{\partial u_k}{\partial \tau} \cdot \frac{\partial w_k}{\partial \tau} \; V_n \; d\mathcal{H}^1,$$

for $d = 2$, where $\frac{\partial}{\partial \tau}$ denotes the tangential derivative.

As noted above, the correct choice of the Hilbert space for this problem is $\mathcal{V}(\Omega) = H^{\frac{1}{2}}(\partial\Omega)$, which can be realized by taking an extension to $H_0^1(D)$ (being the subspace of $H^1(D)$ vanishing on ∂D). Thus, (5.2) implies an equation for the extension velocity,

$$\langle V_n, W_n \rangle_{H^1(D)} = \sum_{k=1}^{N} \int_{\partial\Omega} \nabla u_k \cdot \nabla w_k \; W_n \; d\mathcal{H}^{d-1} \qquad \forall \; W_n \in H_0^1(\Omega),$$

which is a weak formulation of the boundary value problem

$$\begin{aligned} -\Delta V_n + V_n &= 0 \qquad \text{in } D \\ V_n &= 0 \qquad \text{on } \partial D \\ \left[\frac{\partial V_n}{\partial n}\right] - \sum_{k=1}^{N} \nabla u_k \cdot \nabla w_k &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where [.] denotes the jump along the boundary. Thus, the velocity V_n on $\partial\Omega$ is obtained by applying a Neumann-to-Dirichlet operator to the quantity $\sum_{k=1}^{N} \nabla u_k \cdot \nabla w_k$ derived from the shape sensitivity.

5.1.1 *Projected Gradient Methods*

In the presence of constraints of the form

$$C(\Omega) = 0, \tag{5.3}$$

with some operator C mapping to a Banach space $\mathcal{U}$, one can define a shape sensitivity of C in a analogous way to the one for shape functionals. This sensitivity can be used to modify the gradient algorithm in order to obtain an projected gradient method.

The starting point, as usual in constrained in optimization, is the Lagrangian associated to the optimization problem, i.e.,

$$\mathcal{L}(\Omega; p) = J(\Omega) + \langle p, C(\Omega) \rangle, \tag{5.4}$$

where $p \in \mathcal{U}^*$ is a dual variable. For fixed p, the shape sensitivity of the Lagrangian is given by

$$\mathcal{L}'(\Omega; p)V_n = J'(\Omega)V_n + \langle p, C'(\Omega)V_n \rangle, \tag{5.5}$$

and the derivative with respect to p just yields the contraint operator C. Since a solution of the constrained optimization problem is a saddle point of the Lagrangian, one can perform a gradient descent for the Lagrangian, i.e., replace (5.2) by

$$\langle V_n(.,t), W_n \rangle_{\mathcal{V}(\Omega(t))} = -\mathcal{L}'(\Omega(t); p(t))W_n \qquad \forall\ W_n \in \mathcal{V}(\Omega(t)). \tag{5.6}$$

Since this equation does not determine both the velocity and the Lagrange parameter, one can add a second equation. If the constraint (5.3) should be satisfied during the whole evolution, then a natural condition is obtained from

$$C'(\Omega(t))V_n(.,t) = \frac{d}{dt}C(\Omega(t)) = 0. \tag{5.7}$$

Now we can use (5.6) and (5.7) as a coupled system to determine the velocity $V_n(.,t)$ and the dual variable $p(t)$ as its solution. This indefinite linear system has a standard form and admits a unique solution if $C'(\Omega)$ satisfies appropriate regularity conditions (cf. [22]).

For the important case of a volume constraint, i.e.,

$$C(\Omega) = \int_\Omega dx - c,$$

a level set based projected gradient method was used in [108] following [116]. In this case, one obtains

$$C'(\Omega)V_n = \int_{\partial\Omega} V_n \; d\mathcal{H}^{d-1}$$

and one obtains after some calculation that

$$V_n = V_n^0 - \frac{1}{\mathcal{H}^{d-1}(\partial\Omega)} \int_{\partial\Omega} V_n^0 \; d\mathcal{H}^{d-1},$$

where V_n^0 is the "unconstrained velocity" obtained from (5.2). A detailed discussion of volume conserving geometric flows in image processing can be found in [33], see also [152].

Model Problem 2 (Revisited). We illustrate the application of the projected gradient method for this model problem in the case $\hat{J}(\Lambda) = -\lambda_1$, i.e., for the goal of maximizing the first eigenvalue. From the weak form

$$\int_D \nabla u_1 \cdot \nabla v \; dx = \lambda_1 \int_D (q_0 + q_1 \chi_\Omega) u_1 v \; dx$$

we obtain (for $v = u_1$) the following formula for the first eigenvalue

$$\lambda_1 = \frac{\int_D |\nabla u_1|^2 \; dx}{\int_D (q_0 + q_1 \chi_\Omega) u_1^2 \; dx},$$

with u_1 being the first eigenvector, normalized via

$$\int_D (q_0 + q_1 \chi_\Omega) u_1^2 \; dx = 1.$$

Thus, the shape sensitivity is given by

$$
\begin{aligned}
J'(\Omega)V_n &= -\lambda_1'(\Omega)V_n \\
&= -\frac{2\int_D \nabla u_1 \cdot \nabla u_1' \; dx}{\int_D (q_0 + q_1\chi_\Omega)u_1^2 \; dx} \\
&\quad +2\frac{\int_D |\nabla u_1|^2 \; dx}{(\int_D (q_0 + q_1\chi_\Omega)u_1^2 \; dx)^2} \int_D (q_0 + q_1\chi_\Omega)u_1 u_1' \; dx \\
&\quad +q_1 \frac{\int_D |\nabla u_1|^2 \; dx}{(\int_D (q_0 + q_1\chi_\Omega)u_1^2 \; dx)^2} \int_{\partial\Omega} V_n u_1^2 \; d\mathcal{H}^{d-1}
\end{aligned}
$$

where u_1' is the sensitivity of the eigenfunction with respect to the shape. Inserting the weak form of the Helmholtz equation for u_1 and the normalization of the eigenfunction, the first to terms on the right-hand side cancel and consequently

$$J'(\Omega)V_n = q_1 \int_{\partial\Omega} V_n u_1^2 \; d\mathcal{H}^{d-1} \int_D |\nabla u_1|^2 \; dx.$$

Choosing an L^2-gradient flow we obtain

$$V_n^0 = -q_1 V_n u_1^2 \int_D |\nabla u_1|^2 \; dx,$$

and consequently, together with the gradient projection for the volume constraint

$$V_n = -q_1 \int_D |\nabla u_1|^2 \; dx \left(V_n u_1^2 - \frac{1}{\mathcal{H}^{d-1}(\partial\Omega)} \int_{\mathcal{H}^{d-1}(\partial\Omega)} V_n u_1^2 \; d\mathcal{H}^{d-1} \right).$$

5.1.2 *Step Size Selection*

We have seen above that the time-continuous gradient evolution yields a descent of the objective functional. In practice, one cannot perform a continuous evolution, but rather an evolution with small time steps, i.e., starting with a shape $\Omega_k = \Omega(t_k)$ one computes a descent direction via (5.2) at $t = t_k$ and then uses the (stationary) velocity $V_n(., t_k)$ in the time interval $(t_k, t_k + \tau_k]$ to obtain the next shape at $t_{k+1} = t_k + \tau_k$.

In order to determine the time step τ_k such that a decrease of the objective functional is obtained, one can now use standard step size selection methods like the Armijo-Goldstein or Wolfe rules (cf. [104]), if the appearing derivatives are replaced by shape sensitivities. However, some numerical results indicate that it might be advantageous to violate the step size rules in some times and even to allow for an increase in the objective functional, since it is often followed by a strong decrease in following steps (cf. [14]).

5.2 Newton-type Methods

If the shape functional admits second order shape sensitivities, one can attempt to use a level set based Newton method. In an analogous way to the standard Newton method in metric spaces we can define a Newton step as the minimization of a quadratic approximation (now in the sense of shape calculcus) with respect to the update (now the velocity), i.e..

$$\frac{1}{2}J''(\Omega)(V_n, V_n) + J'(\Omega)V_n + J(\Omega) \to \min_{V_n \in \mathcal{V}(\Omega)} . \tag{5.8}$$

From this quadratic variational problem one obtains the Newton equation

$$J''(\Omega)(V_n, W_n) = -J'(\Omega)W_n \qquad \forall\ W_n \in \mathcal{V}(\Omega) \tag{5.9}$$

for the velocity V_n. As for the classical Newton method, this yields a descent direction if the bilinear form J'' is positive definite, since in this case

$$\frac{d}{dt}J(\Omega(t)) = J'(\Omega(t))V_n = -J''(\Omega)(V_n, V_n) \leq \beta \|V_n\|^2 \tag{5.10}$$

with $\beta < 0$.

In general, $J''(\Omega)$ is not positive definite globally so that approximations have to be used in order to obtain a descent method, usually called truncated Newton methods. Their main idea is to use only the positive definite part of $J''(\Omega)$ in the Newton equation for the velocity. This approach has been used for image segmentation problems in [71, 72] and leads to robust and fast algorithms.

5.3 Gauss-Newton Methods for Least-Squares Problems

For functionals of least-squares type (2.2) one can use Gauss-Newton type methods such as the Levenberg-Marquardt method. The main idea of the Gauss-Newton technique is a special sequential quadratic approximation of the objective functional by disregarding second derivatives of the operator $\mathcal{F}$ (and thus, Gauss-Newton methods can be interpreted as a special type of a truncated Newton method).

For a functional of the form (2.2), the first two shape sensitivities are given by

$$J'(\Omega)V_n = \langle \mathcal{F}'(\Omega)V_n, \mathcal{F}(\Omega) - z\rangle,$$

$$J''(\Omega)(V_n, W_n) = \langle \mathcal{F}'(\Omega)V_n, \mathcal{F}'(\Omega)W_n\rangle + \langle \mathcal{F}''(\Omega)(V_n, W_n), \mathcal{F}(\Omega) - z\rangle.$$

The velocity in a Gauss-Newton method is obtained by disregarding the second derivatives of $\mathcal{F}$ in the Newton equation, i.e.,

$$\langle \mathcal{F}'(\Omega)V_n, \mathcal{F}'(\Omega)W_n\rangle = -\langle \mathcal{F}'(\Omega)W_n, \mathcal{F}(\Omega) - z\rangle \qquad \forall\, W_n \in \mathcal{V}(\Omega). \tag{5.11}$$

Such a Newton-type approach for inverse obstacle problems was proposed already by Santosa [118], but hardly used afterwards. In order to stabilize the iteration and to avoid ill-conditioning of the linear system, one can use a Levenberg-Marquardt strategy, i.e., add an additional quadratic penalization term on the update of the form $\alpha\|V_n\|^2_{\mathcal{V}(\Omega)}$ to the sequential quadratic problems. This leads to the the linear system

$$\langle \mathcal{F}'(\Omega)V_n, \mathcal{F}'(\Omega)W_n\rangle + \alpha\langle V_n, W_n\rangle_{\mathcal{V}(\Omega)} = -\langle \mathcal{F}'(\Omega)W_n, \mathcal{F}(\Omega) - z\rangle \tag{5.12}$$

for the update. For $\alpha > 0$ this system is automatically well-posed and one can verify a descent property (cf. [26]).

In general, one can expect a Gauss-Newton type method to significantly decrease the number of iterations, but this does not imply that the overall computational effort will decrease, since the solution of (5.11) or (5.12), respectively, may be expensive. The efficient solution of this linear system for the model problems 1 and 3 has been investigated in [26] using an *all-at-once approach* (cf. [29, 30, 65, 66]), which leads to a large, sparse indefinite linear system for the velocity, a linearized state variable and a dual variable. With appropriate preconditioning, the solution of this linear system can be realized with a computational effort compareable to one or two steps of a gradient method, and the resulting overall method clearly outperforms the gradient-type methods (cf. [26]).

Model Problem 3 (Revisited). In the following we discuss the application of a Levenberg-Marquardt type method to source reconstruction. The sensitivity of $\mathcal{F}(\Omega) = u|_M$ with respect to the shape is given by $\mathcal{F}'(\Omega)V_n = u'|_M$, where $u' \in H_0^1(D)$ is the weak solution of

$$\int_D \nabla u' \cdot \nabla v \; dx = \int_{\partial\Omega} V_n d\mathcal{H}^{d-1}, \qquad \forall v \in H_0^1(D).$$

A straightforward calculation shows that

$$\mathcal{F}^*(\Omega)(\mathcal{F}(\Omega) + \mathcal{F}'(\Omega)V_n - z) = w|_{\partial\Omega},$$

in $L^2(\partial\Omega)$, where w is the solution of the adjoint problem

$$\int_D \nabla w \cdot \nabla v \; dx = \int_M (u + u' - z) \; v \; dx, \qquad \forall v \in H_0^1(D).$$

Thus, the equation determining the velocity in the level set based Levenberg-Marquardt method in $L^2(\partial\Omega)$ is given by

$$\alpha V_n + w = 0,$$

subject to the above equations for w and u', i.e., one has to solve a coupled linear system of three equations for V_n, w, and u' to obtain the velocity.

5.4 Methods for Special Features

For several purposes such as topological restrictions or multiple phases, the above level set approach can be modified. We shall review some recent developments in the following.

5.4.1 *Preserving Topology*

In some applications it may be of interest to preserve the topology of a design, e.g., in a the optimization of microstructured optical fibres (cf. [5]). Since this is not guaranteed in the level set framework, special treatment is needed if one wants to use the benefits of the level set methods nonetheless.

An automatic way to incorporate this additional property has been proposed recently by Alexandrov and Santosa [6] via an interior point approach. The penalized functional to be minimized for small $\epsilon > 0$ is given by

$$J_\epsilon(\Omega) := J(\Omega) + \epsilon H(\Omega), \tag{5.13}$$

where

$$H(\Omega) = -\int_{\partial\Omega} (\log[d_\Omega(x + \sigma\nabla d_\Omega(x))] + \log[-d_\Omega(x - \sigma\nabla d_\Omega(x))]) \; d\mathcal{H}^{d-1} \tag{5.14}$$

for some small constant $\sigma > 0$. Here and below, d_Ω denotes the signed distance function to $\partial\Omega$, i.e., $d_\Omega(x)$ is equal to the positive distance of x to $\partial\Omega$ if $x \in \mathbb{R}^d\backslash\Omega$ and to the negative distance if $x \in \Omega$.. The reasoning behind this term is as follows: first of all, ∇d_Ω is the unit outer normal on $\partial\Omega$. Hence, the first term ensures that $d_\Omega(x + \sigma \nabla d_\Omega(x))$ is positive, which implies that the minimal distance between two connected components of $\{d_\Omega < 0\}$ is at least σ. Similary, the second term implies that a lower bound on the minimal distance of two connected components of $\{d_\Omega > 0\}$, and consequently, no change of topology can arise.

5.4.2 *Nucleating Holes*

In some applications one observes that the level set method does not lead to enough topological changes (cf. [9, 28]), in particular the level set methods presented above cannot reconstruct inner contours like a ring-type structure in an automatic way.

In order to force the nucleation of inner holes, one can use the concept of the topological derivative (cf. [123, 124]), which measures the variation with respect to the nucleation of an inifinitesimal hole. Since the shape of an infinitesimal hole should not matter, one can restrict the variations to spherical shapes. The topological derivative with respect to a spherical perturbation at $\overline{x} \in D$ is given by

$$d_\mathcal{T} J(\Omega; \overline{x}) = \lim_{R \downarrow 0} \frac{J(\Omega \backslash B_R(\overline{x})) - J(\Omega)}{|B_R(\overline{x}) \cap D|}$$

if the limit on the right-hand side exists. Here and below $B_R(x)$ denotes the ball of radius R centered at x. Since $\overline{x} \in D$ is arbitrary, we can consider the topological derivative for fixed Ω as a function $g(x) := d_\mathcal{T} J(\Omega; x)$ indicating whether a nucleation at $x \in D$ is favorable. If $g(x)$ is negative, then a nucleation at x will decrease the objective functional, the largest decrease will be obtained for a nucleation at the minimizer of g.

Hence, the topological derivative can be used as an additional criterion to nucleate new holes, either in alternation with the optimization using shape sensitivities, or by adding a term to the level set evolution. Such a modification was proposed in [28] as

$$\frac{\partial \phi}{\partial t} + V_n |\nabla \phi| = -g,$$

where V_n is the velocity obtained from the shape gradient as above. The source term on the right-hand side is motivated as follows: If $g(x)$ is negative for $x \in \Omega = \{\phi < 0\}$, then it is favourable to nucleate a hole, i.e., to increase the level set function, which is guaranteed by the positive source term $-g(x)$. Vice versa, if $g(x)$ is positive, then one should not nucleate a hole at such a position and since the source term $-g(x)$ is negative there, a nucleation will not happen.

In an analogous way, one can define a topological derivative for the nucleation of new phases outside Ω via

$$d_{\mathcal{T}} J(\Omega; \overline{x}) = \lim_{R \downarrow 0} \frac{J(\Omega \cup B_R(\overline{x})) - J(\Omega)}{|B_R(\overline{x}) \cap D|}$$

and use it analogously with a source term in the level set equation.

5.4.3 *Special Shapes*

A third type of restriction is those to optimal designs of special shapes like ellipses or circles (e.g. caused due to manufacturing restrictions), while one wants to keep the topology flexible. A possible way to deal with such shapes has been proposed recently by Miller [97], called "parametric level set method". Though this nomenclature seems contradictory to the level set paradigm at a first glance, it actually turns out to be a rather natural approach of problems. The basic idea is to represent not the interface but the level set function by a given parametric family, e.g. classes of multivariate polynomials.

The rationale behind this approach is the following: E.g., an ellipse is the zero level set of a second order polynomial, and therefore two disjoint ellipses can be represented by a polynomial of order four (as the product of the two second order polynomials representing the individual ellipses). Proceeding this way one may conclude that all unions of N disjoint ellipses are included in the set of polynomials of order $2N$. In this case, optimization based on the parametric level set approach reduces to the finite-dimensional problem of minimizing with respect to the $2N$ coefficients of the polynomials. Since this problem can be solved even with global optimization techniques if N is small, one may use this approach also to obtain a starting value for a more general level set based optimization technique.

In order to obtain other classes of shapes one can use spline parametrizations of the level set function, an approach also used recently in CAD (cf. e.g. [82]).

5.4.4 *Multiple Phases*

In some shape optimization problems one has to deal with multiple phases, i.e., the problem is of the form

$$J(\Omega_1, \ldots, \Omega_K) \to \min_{\Omega_i \cap \Omega_j = \emptyset} . \tag{5.15}$$

In such a case one needs multiple level set functions to represent the geometries. The easiest way is to use a level set function for each phase, i.e.,

$$\Omega_j = \{\phi_j < 0\}.$$

An alternative way to represent multiple phases has been introduced in [143], which allows to represent 2^m level sets by m different level set functions. For each combination $i \neq j$ one obtains four different phases, namely

$$\Omega_{ij}^{++} = \{\phi_i > 0\} \cap \{\phi_j > 0\}, \qquad \Omega_{ij}^{--} = \{\phi_i < 0\} \cap \{\phi_j < 0\},$$

$$\Omega_{ij}^{+-} = \{\phi_i > 0\} \cap \{\phi_j < 0\}, \qquad \Omega_{ij}^{-+} = \{\phi_i < 0\} \cap \{\phi_j > 0\},$$

and the total number of phases is obtained by taking all combinations $i < j$.

The velocity for a level set based optimization technique can be chosen as above from the shape sensitivity with respect to each phase Ω_j (cf. [151]). The main difficulty in the numerical realization is the possible appearance of a "vacuum phase" (except with n level sets for cases where the number of phases is exactly $2n$) and therefore one has to incorporate the constraint that each point belongs to one of the phases Ω_j, $j = 1, \ldots, K$. Zhao et. al. proposed to use the constraint

$$\sum_{j=1}^{K} H(\phi_j(x,t)) = 1, \qquad \forall x, t \tag{5.16}$$

in a representation with K level set functions, where H denotes the Heaviside function. This constraint has been incorporated using a projected gradient method.

5.5 Related Methods

In the following we discuss two different approaches for shape optimization and inverse problems, which are closely connected to level set methods.

5.5.1 *Phase-Field Methods*

Phase-field methods are closely related to the level set approach, the unknown geometry is usually obtained as the level set $\{\psi > \frac{1}{2}\}$ of a continuous function ψ. The main difference to the level set method (which somehow approximates the signed distance funtion) is that ψ approximates the indicator function χ_Ω of the set Ω.

The phase field approach is of particular interest for functionals of the form $J(\Omega) = \tilde{J}(\chi_\Omega)$ and is based on a relaxation of the problem by minimizing

$$\tilde{J}(\psi) + \alpha(\epsilon) \int_D \left(\epsilon |\nabla \psi|^2 + \frac{1}{\epsilon} W(\epsilon) \right) \, dx \to \min_\psi$$

with W being a double-well potential, e.g., $W(s) = s^2 \, (1-s)^2$. As $\epsilon \to 0$ and $\alpha(\epsilon) \to 0$, this relaxation converges to the original shape optimization problem under appropriate conditions on $\tilde{J}$ in the sense of Γ-convergence (cf. [43, 21]). If $\alpha(\epsilon)$ does not tend to zero, the problem converges to a perimeter-regularized version of the original shape optimization problem.

If one considers gradient-flows for ψ corresponding to the above optimization problems, then one ends up with parameter-dependent reaction-diffusion equations. Using formal asymptotic expansions and in some cases rigorous analysis one can show that these gradient flows converge to the same motion of sets as obtained with the level set approach (cf. e.g. [12]). Thus, the optimization techniques obtained with a phase-field approach are quite similar to those obtained with level set methods. The main differences are that on the one hand a continuation in ϵ might improve the convergence (cf. [20]), on the other hand too small values of ϵ enforce a very fine discretization of ψ (cf. [96]). In general, it might be a good idea to start the optimization with a phase-field method for large ϵ, do some continuation decreasing ϵ, and then switch to a level set method for the local shape optimization. We refer to [19, 150] for phase field methods in structural optimization, and to [20, 115] for phase-field methods in inverse obstacle problems.

5.5.2 *Regularization by Hamilton-Jacobi Equations*

An approach for the computation of discontinuous solutions of inverse problems closely related to inverse problems has been introduced by Kindermann [83] recently. The problem under investigation is the solution of a linear equation of the form

$$\mathcal{K}u = f,$$

where $\mathcal{K}$ is a linear operator mapping $L^2(D)$ to some Hilbert space. Starting point of the approach is an idea introduced by Neubauer and Scherzer [103], who considered the graph of a discontinuous one-dimensional function as a curve in the plane and added a regularization term related to the length of the curve. This approach turns out to be similar to bounded variation regularization (cf. [116]), but allows a more detailed analysis with respect to convergence rates and convergence properties of the graph of u. The approach was later generalized to multi-dimensional problems (cf. [84, 85]), but it leads to a nonlinear variational problem to be solved. As a possible solution method, a gradient flow was proposed in [83], which leads to the Hamilton-Jacobi equations of the form

$$\frac{\partial u}{\partial t} + (\mathcal{K}^*(\mathcal{K}u - f))\sqrt{1 + |\nabla u|^2} - \alpha\sqrt{1 + |\nabla u|^2}\ \mathrm{div}\ \left(\frac{\nabla}{\sqrt{1 + |\nabla u|^2}}\right) = 0, \tag{5.17}$$

with the L^2-adjoint $\mathcal{K}^*$. The relation of this approach to level set methods is twofold: First of all, discontinuous solutions of Hamilton-Jacobi equations can be defined, analyzed and computed via a level set approach similar to the original idea of this regularization methods, namely by representing the graph of u as the zero level set of a higher-dimensional function (cf. [60, 140]). Secondly, by introducing a weight-parameter σ between the vertical and horizontal lengths of the graph, the above equation changes to

$$\frac{\partial u}{\partial t} + \sqrt{1-\sigma+\sigma|\nabla u|^2}\left[\mathcal{K}^*(\mathcal{K}u - f) - \alpha \text{ div } \left(\frac{\nabla u}{\sqrt{1-\sigma+\sigma|\nabla u|^2}}\right)\right] = 0,$$

for $\sigma \in [0,1]$ and this equation converges to a gradient flow analogous to those in level set methods for $\sigma \to 1$.

6 Level Set Methods and Ill-Posed Problems

Many shape optimization problems, in particular those arising from inverse obstacle problems, are ill-posed, i.e., either there exists no solution and/or they do not depend on the data in a stable way. Therefore, regularization methods have to be used in order to compute a stable approximation of the minimizer (cf. [53] for details on regularization of inverse problems).

If the aim is to compute regularized solutions of a rather general topological structure, the regularization technique must reflect this paradigm. In particular, one cannot use the regularizations of parametrizations, an approach that has been used frequently in this context. We shall discuss two different approaches to regularization that have been used succesfully in this context, though a rigorous analysis is still missing in most cases.

6.1 Geometric Variational Regularization

A first direct approach consists in a direct regularization of the minimization problem either by adding a penalty or a constraint. For this sake one needs an appropriate regularization functional $\mathcal{R} : \mathcal{K}_{ad} \to \mathbb{R}$ and a regularization parameter $\alpha > 0$ (usually small). A minimal assumption on $\mathcal{R}$ is lower semicontinuity and the compactness of the level sets $\{\mathcal{R} \leq C\}$ for $C \in \mathbb{R}$ in an appropriate topology on $\mathcal{K}_{ad}$.

The regularized problem then consists either in minimizing the penalized problem

$$J_\alpha(\Omega) := J(\Omega) + \alpha\mathcal{R}(\Omega) \to \min_{\Omega \in \mathcal{K}_{ad}}, \tag{6.1}$$

or the constrained problem

$$J(\Omega) \to \min_{\Omega \in \mathcal{K}_{ad}}, \tag{6.2}$$

$$\text{subject to } \alpha\mathcal{R}(\Omega) \leq 1. \tag{6.3}$$

If J is lower semicontinuous, then the properties of $\mathcal{R}$ imply in both cases that the minimizer lies in a compact set and one can easily conclude the existence of a minimizer using the fundamental theorem of optimization. Moreover, for inverse obstacle problems one can derive weak stability results in dependence on the data $z \in \mathcal{Z}$ by standard techniques (cf. [53] for weak stability results for inverse problems) and convergence as $\alpha \to 0$ if there exists a solution of the limit problem.

In order to apply this strategy to practical problems, we need specific topologies on classes of compact sets and appropriate regularization functionals that allow for general topologies. A widely used regularization functional is the perimeter, i.e. $\mathcal{R}(\Omega) = \mathcal{H}^{d-1}(\partial\Omega)$, where $\mathcal{H}^{d-1}$ denotes the $d-1$-dimensional Hausdorff measure. The perimeter always favours solutions with a shape close to the circle (which is the minimizer of perimeter at fixed volume, the so-called Wulff shape). If one wants to incorporate different a-priori ideas about the shape one can use an anisotropic version (similar to surface energies in materials science)

$$\mathcal{R}(\Omega) = \int_{\partial\Omega} \gamma(\nu)\ d\mathcal{H}^{d-1},$$

with a positive homogeneous function γ of the orientation.

The perimeter is lower semicontinuous and its level sets are compact in the L^1-topology (i.e., the topology induced by the indicator functions of the shapes in $L^1(D)$), so that well-posedness of the regularized problem can be guaranteed if J is lower semicontinuous in this topology (which is the case for several applications). Since the perimeter coincides with the total variation of the indicator function, the analysis can be carried out in an analogous way to the one for total variation regularization (cf. [1, 34, 116]). Some applications, e.g. inclusion detection problems, are not lower semicontinuous in the L^1-topology, but with respect to the Hausdorff metric (cf. [44, 57, 99] for a detailed definition). The perimeter functional satisfies the above assumptions only if $d = 2$ and if the number of connected components of Ω is finite.

Another possibility to choose a regularization functional is to use a curvature-dependent energy (cf. [62, 63]) of the form

$$\mathcal{R}(\Omega) = \int_{\partial\Omega} \kappa^2\ d\mathcal{H}^{d-1},$$

where κ denotes the mean curvature of the boundary. This regularization functional (and the corresponding Willmore flow) have recently been used by several authors for image impainting (cf. [10, 36, 95]). For a level set method implementing the gradient flow for the Willmore functional we refer to [52].

6.2 Regularizing Level Set Methods

Alternatively to the direct regularization of the shape optimization problem, one can use the paradigm of iterative regularization (cf. [53]) and construct regularizing level set methods. If the level set evolution is smoothing like e.g. in gradient-type or Levenberg-Marquardt evolutions, the iteration has a regularizing effect itself if an appropriate stopping criterion is used.

For inverse problems, a widely used stopping criterion is the *discrepancy principle*, which consists in terminating the iteration process the first time when the residual $\|\mathcal{F}(\Omega) - z\|$ is of the same order as the data noise level. The main advantage of this approach for inverse problems is that there is no additional parameter (as α above) to be tuned, which might allow to save computational effort.

For an inclusion detection problem, the two approaches have been compared in several numerical examples in [14] and lead to similar results. The analysis of this regularization approach in general is completely open, but for a special case a detailed analysis in the L^1 topology was carried out in [24].

A variational approach to regularization by level set methods was proposed by Leitao and Scherzer [87], who added a penalty of the form $\alpha\|\nabla\phi\|^2$. For this way it is possible to carry out a convergence analysis of the level set method as $\alpha \to 0$ in a weak sense (cf. [59]). The arising functional has been minimized using a gradient-type algorithm, which lead to a rather high number of iterations. A disadvantage of this approach is that the invariance of the method on the choice of the level set function ϕ is lost and the regularization term will favour a flat ϕ, which might make the reconstruction of the zero level set difficult.

7 Numerical Issues

In the following we give a short overview of numerical methods needed for the implementation of level set based optimization methods. We shall not discuss the methods in detail, but mainly provide links to literature for the interested readers.

7.1 Numerical Solution of Hamilton-Jacobi Equations

For the numerical computation of shape evolution via the level set method one has to solve the Hamilton-Jacobi equation (3.1). We shall present numerical methods for this issue in this section.

7.1.1 *First-Order Equations*

In the important special case where V_n in (3.1) is a function only of x, t, and the normal n, (3.1) becomes a first-order Hamilton-Jacobi equation whose (viscosity) solutions generally develop kinks (jumps in derivatives). The appearance of these singularities in the solution means that special, but not terribly complicated, numerical methods have to be used, usually on uniform Cartesian grids. This was first discussed in [109] and numerical schemes developed there were generalized in [80],[110]. The key ideas involve monotonicity, upwind differencing, essentially nonoscillatory (ENO) schemes, and weighted essentially nonoscillatory (WENO) schemes.

Usually, numerical methods are developed for general nonlinear Hamilton-Jacobi equations of the form

$$\phi_t + H(x,t;\nabla\phi) = 0, \tag{7.1}$$

but they can easily be translated to the case of level set methods by choosing

$$H(x,t,\nabla\phi) = V_n(x,t,\frac{\nabla\phi}{|\nabla\phi|})|\nabla\phi|.$$

A Hamilton-Jacobi equation like (3.1) is usually discretized in two steps:

(1) Approximate the nonlinear term using ideas developed in [109],[110], borrowed from their origins in the numerical solution of conservation laws [67],[120],[121].

(2) Use total variation diminishing (TVD) Runge-Kutta schemes, derived in [120] to do the time discretization.

The starting point of finite difference methods are monotone schemes based on

$$\frac{\partial\phi}{\partial t}(x,t) = -\hat{H}(x,t,D_1^+\phi(x,t),D_1^-\phi(x,t),\ldots,D_d^+\phi(x,t),D_d^-\phi(x,t)),$$

where $D_j^\pm$ denote forward and backward difference quotients given by

$$D_j^\pm\phi(x,t) = \pm\frac{\phi(x\pm he_j,t)-\phi(x,t)}{h}$$

with e_j being the j-th unit vector. The numerical flux $\hat{H}$ (a Lipschitz-continuous function) satisfies two fundamental conditions:

(1) Consistency: $\hat{H}(x,t,u_1,u_1,\ldots,u_d,u_d) = H(x,t,u_1,\ldots,u_d)$

(2) Monotonicity: $\hat{H}(x,t,\downarrow,\uparrow,\ldots,\downarrow,\uparrow)$, i.e. $\hat{H}$ is nonincreasing in the arguments in-

cluding positive difference quotients and nondecreasing in those including negative difference quotients.

Possible choices for the numerical flux are the Godunov flux based on upwinding, the (local) Lax-Friedrichs flux based on introducing numerical diffusion, or the Roe flux (cf. [106] for details). A monotone scheme as introduced above is of first-order accuracy with respect to the grid size only, so that usually so-called *essentially non-oscillatory (ENO) schemes* are used to increase the order. The basic idea of an ENO-scheme of order m is to use $m-1$ further grid points and a polynomial of degree m interpolating these points. Among all such polynomials the one with the smallest second to m-th derivative is chosen (in an inductive construction), and the derivative of this polynomial at x gives the ENO-approximation of the derivative. We refer to [67, 110] for a detailed discussion of ENO interpolation and its use for Hamilton-Jacobi equation. In [93] and then later in [81, 80] the idea was generalized to taking an appropriate weighted combination of polynomials to approximate $\nabla\phi$, leading to so-called *weighted-essentially non-oscillatory (WENO) schemes* , which can even increase the order at the same number of grid points used for the approximation of the derivative. For triangular grids, ENO schemes can be constructed using discontinuous Galerkin methods (cf. [77]).

Finally, one can use a first, second, or third order *total variation diminishing (TVD) Runge-Kutta schemes*, devised in [120] to perform the time integration in an explicit way. If we have a semidiscrete approximation of the form

$$\frac{\partial \Phi}{\partial t} = L(\Phi),$$

where Φ is a vector representing the values of ϕ at the grid points, then the 1^{st}, 2^{nd} and

3^{rd} order TVD Runge-Kutta methods are

$$\Phi^{j+1} = \Phi^j + \tau L(\Phi^j) \qquad \text{(1st order)} \tag{7.2}$$

$$\begin{cases} \Phi^{j+\frac{1}{2}} &= \Phi^j + \tau L(\Phi^j) \qquad \text{(2nd order)} \\ \Phi^{j+1} &= \frac{\Phi^{j+\frac{1}{2}}+\Phi^j}{2} + \frac{\tau}{2} L(\Phi^{j+\frac{1}{2}}) \end{cases}$$

$$\begin{cases} \Phi^{j+\frac{1}{3}} &= \Phi^j + \tau L(\Phi^j) \\ \Phi^{j+\frac{2}{3}} &= \frac{3}{4}\Phi^j + \frac{1}{4}\Phi^{j+\frac{1}{3}} + \frac{1}{4}\tau L(\Phi^{j+\frac{1}{3}}) \qquad \text{(3rd order)} \\ \Phi^{j+1} &= \frac{1}{3}\Phi^j + \frac{2}{3}\Phi^{j+\frac{2}{3}} + \frac{2}{3}\tau L(\Phi^{j+\frac{2}{3}}) \end{cases}$$

We finally note that stability of such explicit schemes is only obtained under some restriction on the time step τ in dependence on the grid size h, the so-called *Courant-Friedrichs-Levy (CFL) condition*

$$\max |V_n| \tau < h.$$

In the context of TVD Runge-Kutta methods, the schemes have been optimized to obtain a stability bound on the time step as large as possible (cf. [126]).

7.1.2 *Mean Curvature-type Equations*

In several cases, in particular for gradient methods with perimeter regularization, the velocity is of the form $V_n = V_n^0 + \kappa$, where V_n^0 is a given function of the location and κ denotes the mean-curvature. In such a case one has to deal with a degenerate parabolic equation of second order of the form

$$\frac{\partial \phi}{\partial t} + V_n^0 \; |\nabla \phi| - |\nabla \phi| \; \text{div} \left(\frac{\nabla \phi}{|\nabla \phi|} \right) = 0.$$

It has been become standard to use a *lagged-diffusivity approximation* for the time discretization, which amounts to using a semi-implicit scheme of the form

$$\phi^{j+1} - \phi^j + \tau V_n^0 \ |\nabla \phi^j| - \tau |\nabla \phi^j| \ \mathrm{div}\left(\frac{\nabla \phi^{j+1}}{|\nabla \phi^j|}\right) = 0.$$

The spatial discretization can be performed by finite difference or by finite element methods, such as proposed by Deckelnick and Dziuk [46], who used piecewise linear element methods for the weak formulation

$$\int_D \frac{\phi^{j+1} - \phi^j}{Q} \ v \ dx + \int_D \frac{\nabla \phi^{j+1} \cdot \nabla v}{Q} \ dx = - \int_D \tau V_n^0 \ v \ dx, \qquad \forall \ v.,$$

where $Q = \sqrt{\epsilon + |\nabla \phi^j|^2}$.

A different semi-implicit approach based on operator splitting has been proposed by Smereka [122].

7.1.3 *Velocity Extension*

As we have seen below, level set based optimization methods define a velocity on the interface $\partial\Omega$ only, but an extension to a larger domain is needed to solve the Hamilton-Jacobi equation (3.1). A natural way of extending the velocity is a constant extension in normal direction. Since the normal direction to a level set is parallel to $\nabla\phi$, this can be formulated as the computation of a viscosity solution of the linear first-order equation

$$\nabla V_n . \nabla \phi = 0, \qquad \text{in } \mathbb{R}^d \backslash \partial\Omega,$$

where the extension velocity V_n has to be equal to the given velocity $\hat{V}_n$ on $\partial\Omega$. The solution of this equation can either be obtained by a marching scheme (cf. [2]) or by computing the large time limit $w(.,s)$ as $s \to \infty$ of the linear transport equation

$$\frac{\partial w}{\partial s} + \text{ sign}(V_n^0)(\nabla w . \nabla \phi) = 0,$$

where V_n^0 is an arbitrary extension of $\hat{V}_n$ (cf. [132]).

7.1.4 *Redistancing*

During the level set evolution, the function ϕ can become very flat or steep, which is an undesirable effect and can lead to high numerical errors either in the reconstruction of the zero level set or in the numerical schemes. Therefore, *redistancing* is performed usually after some (or even each) time step, which means that the level set function is reinitialized to become close to the signed distance function of the actual shape Ω.

A basic observation that enables the efficient computation of distance functions on grids is that the signed distance function d_Ω is a viscosity solution of

$$|\nabla d_\Omega| = 1 \qquad \text{in } \mathbb{R}^d \backslash \partial\Omega,$$

with the boundary condition $d_\Omega = 0$ on $\partial\Omega$. This offers the possibility to compute the signed distance function as the large-time limit $s \to 0$ of the corresponding evolution equation

$$\frac{\partial \psi}{\partial s} - \text{ sign}(\phi)(|\nabla \psi| - 1) = 0,$$

where ϕ is the starting level set function (cf. [132]). This equation is again a first-order Hamilton-Jacobi equation and can be solved numerically using methods as discussed above. Several schemes for this task have been introduced, differing in particular in the way the sign-function is approximated (cf. [117, 128, 131, 132]). A finite element method for solving the redistancing equation has been introduced in [136]. Usually it suffices to compute few time steps, since the convergence towards the signed distance function is very fast locally around the zero level set and the form of the level set function away from the zero level set is not important.

7.2 Numerical Solution of PDEs with Interfaces

In typical applications in inverse problems and optimal design it does not suffice to solve the level set equation numerically, but one also needs solvers for partial differential equations with discontinuous coefficients and / or interfaces. Since the discontinuity sets and interfaces are changing during the optimization process, it is important that the solution method are adapted to the level set method, since in typical applications it is too expensive to resolve the interface by the grid at each iteration of the shape optimization procedure.

7.2.1 *Local Averaging Methods*

The basic idea of a local averaging method is to approximate interface terms by local weighted averages over level sets, i.e.,

$$\int_\Omega g \; dx = \int_{\{\phi<0\}} g \; dx \approx \int_{\mathbb{R}} w_\epsilon(\eta) \int_{\{\phi<\eta\}} g \; dx \; d\eta,$$
$$\int_{\partial\Omega} g \; d\mathcal{H}^{d-1} = \int_{\{\phi=0\}} g \; d\mathcal{H}^{d-1} \approx \int_{\mathbb{R}} w_\epsilon(\eta) \int_{\{\phi=\eta\}} g \; d\mathcal{H}^{d-1} \; d\eta,$$

where the weight function w_ϵ is a smooth approximation of the Dirac δ-distribution (and converges to this distribution as $\epsilon \to 0$). Due to the area and co-area formula the resulting terms can be rewritten as

$$\int_{\mathbb{R}} w_\epsilon(\eta) \int_{\{\phi<\eta\}} g \; dx \; d\eta = - \int_D H_\epsilon(\phi) \; g \; dx,$$
$$\int_{\mathbb{R}} w_\epsilon(\eta) \int_{\{\phi=\eta\}} g \; d\mathcal{H}^{d-1} \; d\eta = \int_D w_\epsilon(\phi) \; |\nabla \phi| \; g \; dx,$$

where $H_\epsilon = \int_{-\infty}^{\eta} w_\epsilon(\rho) \; d\rho$ is an approximation of the Heaviside function.

In this way, all interface terms become coefficients in the partial differential equations, which can then be treated by standard methods like finite differences or finite elements. In order to ensure a reasonable accuracy it is important to choose appropriate functions w_ϵ (cf. [54, 137, 138]) or to use adaptive refinement (which can be based on standard a-posteriori estimation). One also observes that the interface need not be reconstructed explicitely for this approach, but it suffices to evaluate the level set function and its gradient on the grid, which increases the efficiency, in particular in 3d applications.

7.2.2 *Extended Finite Element and Immersed Interface Methods*

Extended finited element methods (x-fem) are a special version of the partition of unity method by Babuska and Melenk [11] for interface problems. The main idea is to keep the (triangular) finite element grid fixed, but to add additional basis functions ("enrichment functions") that allow for jumps of the solution or its derivative at the interface. The enrichment functions can again be formulated via Heaviside functions applied to ϕ, the interface need not be reconstructed explicitely in this case either, but one only has to determine those elements that include a part of interface in order to place enrichment functions there. We refer to Moes et. al. [100, 130] for extended finite element methods in different geometric situations.

The immersed interface method is somehow the finite difference equivalence of the extended finite element method, it is based on local corrections to the finite difference stencil in cells including parts of the interface. By introducing new variables modelling jump heights along the interface it is possible to include possible discontinuities, and one ends up with an indefinite system for the original variables at the grid points and the smaller vector of jump heights. As for extended finite element methods it suffices to determine those cells including part of the interface, which fits well to the level set approach. We refer to [78, 88, 89, 119] for further details.

7.3 Numerical Solution of PDEs on Interfaces

In several applications one has to solve equations on interfaces, e.g. in Newton methods for problems with perimeter term, since the second shape sensitivity of the perimeter includes the Laplace-Beltrami operator (cf. [72]). An obvious option for the numerical solution of such equations is the reconstruction of the zero level set and its triangulation, but this causes an enormous computational effort, in particular in 3d.

A more genuine level set approach consists in using the actual level set function ϕ for the formulation of the differential equation on each level set, and using subsequent averaging to obtain an equation on the whole domain (cf. [17, 18]). E.g., the Laplace-Beltrami operator can be rewritten as

$$\Delta_S u = \frac{1}{|\nabla\phi|}\text{div }\left(|\nabla\phi|\ P\nabla u\right) \qquad \text{with } P = I - \frac{\nabla\phi\nabla\phi^T}{|\nabla\phi|^2}.$$

It can be shown that such an extension of an equation on an interface to the whole domain has the same solution (cf. [27]). The arising equation can be discretized by standard methods such as finite elements, only some care has to be taken in the iterative solution of the discretized equation due to the anisotropy and degeneracy of the arising operator.

8 Applications

In this section we provide an overview of recent applications of the level set method to optimal design and inverse obstacle problems.

8.1 Structural Optimization

Recently, several authors started to apply the level set method to problems in structural optimization, in particular to those that cannot be modeled by the homogenization method (cf. [7]). Sethian and Wiegmann [119] solved a rather classical compliance minimization problem by choosing the velocity as a function of the shape gradient.

Using the a level set based gradient method as presented above, Allaire et. al. [8, 9] and Wang et. al. [144, 145, 147, 148] were able to solve standard compliance minimization problems in 2d and 3d, as well as problems with more general shape functionals. In addition, they applied the approach to problems design-dependent loads. The paper of Allaire et. al. [9] also includes a comparison of results obtained by the level set based algorithms to those obtained with the homogenization method, and some experiments concerning the influence of the initial value for the optimization, which was found to be significant for this type of problems. A generalization to structural optimization problems with multiple materials has been carried out recently by Wang and Wang [146, 149].

The phase-field approach was applied to the design of a dam (also a problem with design-dependent loads) by Bourdin and Chambolle [19]. They also provide a detailed convergence analysis of the phase-field method and the approximations used for the elasticity problem in the framework of Γ-convergence. A recent paper of Wang and Zhou [150] includes further applications to structural optimization problems, and a comparison of the results obtained with the phase-field, the level set, and the homogenization method.

8.2 Band Structure Design and Photonic Crystals

A problem of growing technological importance is the optimal design of photonic crystals, which consists of several challenging subproblems (cf. [31, 41]). The fundamental problem is the optimization of the shape of air holes in a periodic structure such that a certain bandstructure is obtained, e.g., a bandgap being maximized. Model problem 4 is a simplified version of this problem, it has been treated by a level set based projected gradient method in [108], and by a Gauss-Newton type approach using approximate eigenvalue calculations in [64].

8.3 Inclusion Detection

With a level set based optimization approach, inclusion detection in the version of the simple model problem 1 from a single measurement ($N = 1$) has been treated in [14, 25]. Though theoretical results indicate that it is possible to identify inclusions with an arbitrary number of connected components in this case (cf. [15]), it turned out that the level set evolution does not split in this case in presence of noise.

A analogous problem for the identification of cavities (with the Laplace equation replaced by an anisotropic elliptic equation) for three measurements ($N = 3$) was investigated using a Levenberg-Marquardt type approach in [26]. In this case it turned out

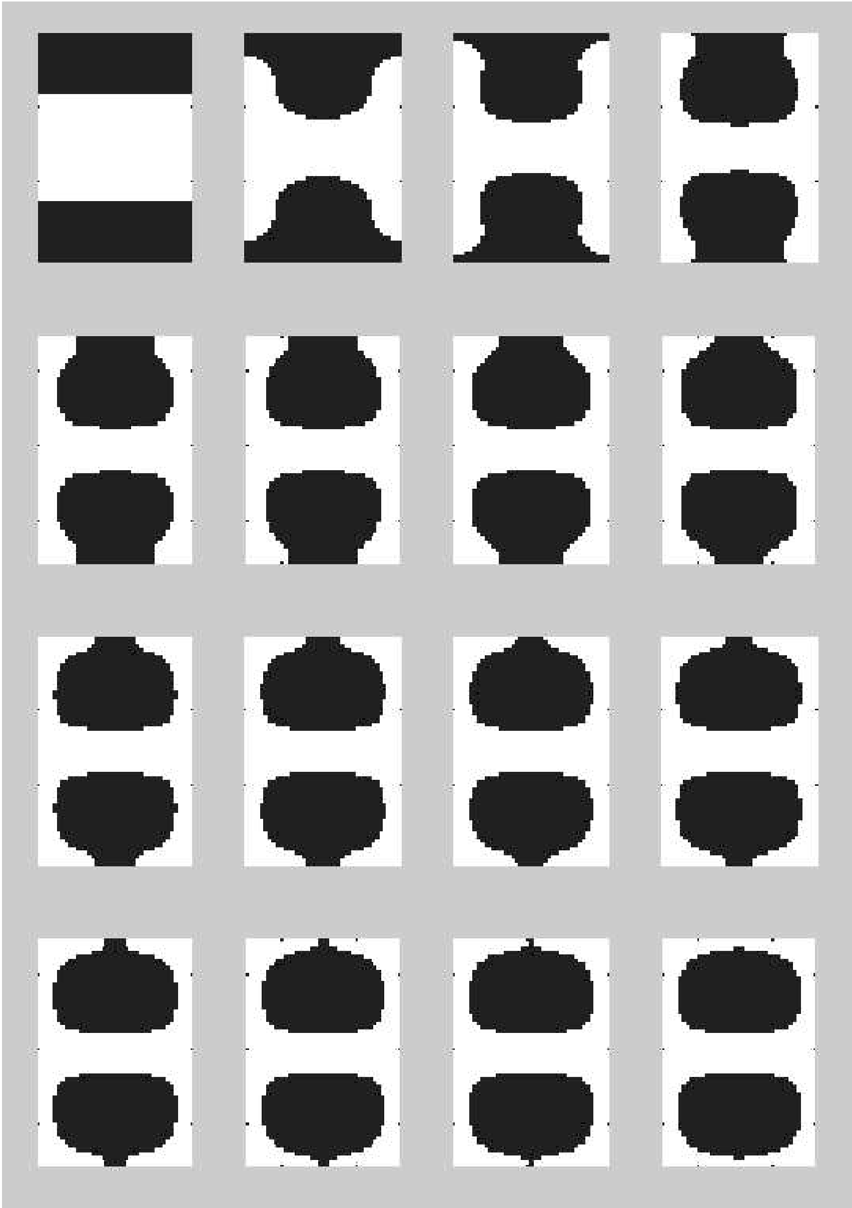

FIGURE 1. Maximization of the bandgap $\lambda_2 - \lambda_1$ in model problem 4, the figure shows the evolution of the level set during the iteration (from [108]).

that the level set based optimization allows for a change of topology and successfully reconstructs multiply connected inclusions if the noise level is reasonably small. The corresponding evolution of the level set is illustrated in Figure 2 Moreover, a comparison

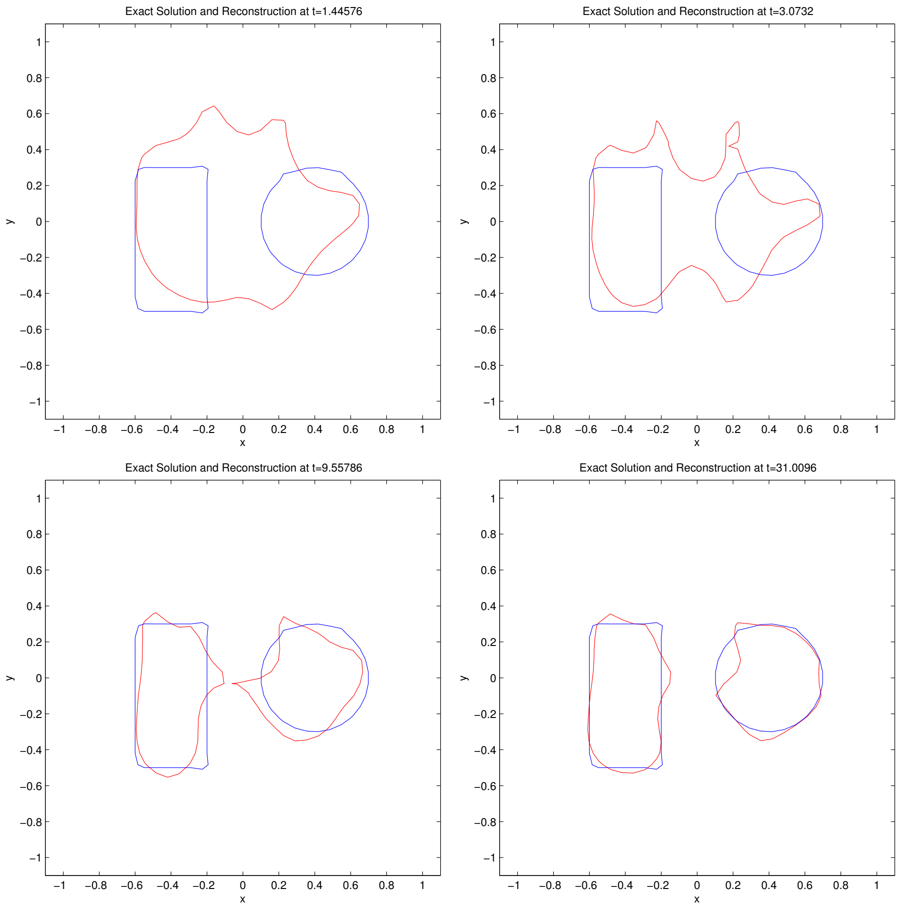

FIGURE 2. Exact solution (blue) and shapes (red) obtained from the LMLS method at iterates 5, 10 (above), 15, and 20 (below) (from [26]).

with a gradient-type level set method showed a significant reduction of computational effort.

A related problem with Dirichlet condition on the interface, i.e., replacing the Neumann condition (2.6) by $u = 0$ on the interface has been treated by a gradient-type method in [79]. Since the set M where measurements are taken was an open subset of D in this was case, the level set method was able to reconstruct multiply connected obstacles.

Fang and Ito [55] applied a level set based gradient method to the identification of semiconductor contact regions, which amounts to the identification of a discontinuity in a lower order coefficient of an elliptic partial differential equation, a problem exhibiting similarities with model problem 3.

8.4 Scattering and Tomography Problems

One of the first application of the level set method was electromagnetic scattering, where several problems have been solved using gradient-type methods by Litman et. al. [92], and later by Ramananjaona et. al. [112, 113, 114]. Dorn et. al. [47, 49, 48] applied level set based gradient methods to electromagnetic tomography in different situations. Hou et al. [76] combined the level set approach with location estimation by time-reversal techniques for applications to radar and sonar.

Rondi and Santosa applied and analyzed phase-field method to the reconstruction of piecewise constant conductivities in eletrical impedance tomography [115]. Recently, the solution of impedance tomography problems by level set methods has been investigated by Griesmair [61] and by Chung et. al. [45].

8.5 Image Processing and Segmentation

Already before the level set method became attractive for inverse and optimal design problems, it was used for image processing problems, many of them being of a variational structure like image segmentation via the Mumford-Shah functional (cf. [101, 99], and [38, 39, 72] for level set based solution methods).

However, since many objectives in imaging are different from those in inverse and optimal design problems, and due to the large amoung of work on level set methods in this area we shall not give a detailed survey of the development here, but refer to the monograph by Osher and Paragios [107] and the survey paper by Tsai and Osher [141].

8.6 Medical Imaging

Several problems on the boundary between image processing and inverse obstacle problems appear in medical imaging. The segmentation of medical images by level set methods has been investigated by several authors (cf. [50, 75, 94, 111, 139]). Recently level set based optimization methods have been used for the morphological registration of medical images by Droske and Rumpf [51, 129] and by Vemuri et. al. [142], where objective functionals similar to elastic energies has been minimized using level set based gradient methods. Inverse obstacle problems related to EEG and MEG imaging have been solved by Faugeras et. al. [56].

8.7 State-Constrained Optimal Control

A rather unusual application was investigated by Hintermüller and Ring [73, 74], whose aim was a robust solution method for a state-constrained optimal control problem related to an elliptic partial differential equation. They explicitely modeled the active set of the state constraint as a set and reformulated the original optimal control problem as a shape optimization problem for the active set. This resulting shape optimization problem was solved using a gradient-type level set method. Since state constrained optimal control of partial differential equations is an important subject, there seems to be a large variety of possible applications of this approach.

9 Open and Future Problems

In this concluding section we want to state some open problems and possible further developments in level set based optimization.

9.1 Analysis of Level Set Based Optimization Methods

A challenging task for future mathematical research is the analysis of level set based optimization methods. Though the understanding and applications of such methods have dramatically increased in the recent years, there is still a lack of rigour with respect to many aspects such as e.g.

- **Well-posedness of the level set evolution:** For many of the evolutions obtained by level set based optimization algorithms, in particular for the model problems presented here (and their more general versions), there are no rigorous results guaranteeing existence and uniqueness. Such an analysis is similar to the well-posedness of moving boundary problems under general conditions, which is also a subject full of open questions.
- **Regularity of the level sets:** Since in many applications, boundary conditions for a partial differential equation are posed on the zero level set. In order to justify such models, some regularity of the level set is needed.
- **Convergence:** An obvious need is a rigorous convergence analysis of level set based optimization methods. So far, a detailed analysis has been carried out only in a single case (cf. [24]), for more general methods and problems only partial results such as the decrease of the objective functional are known.

9.2 Quasi-Newton and SQP-type Level Set Methods

Quasi-Newton methods such as the BFGS algorighm, and sequential quadratic programming (SQP) methods are standard tools in modern nonlinear optimization. Due to their advantageous properties it seems of interest to develop analogous methods for shape optimization problems, in connection with the level set approach. This is not a trivial task, since both methods need comparisons of quantities like the velocity on two consecutive shapes $\Omega(t_k)$ and $\Omega(t_{k-1})$, i.e., objects in different vector spaces. The understanding and construction of Quasi-Newton and SQP-type method remains as a challenging task for future research.

9.3 Nucleation

As discussed above, the nucleation of holes is not always automatic in level set based optimization methods. The use of topological derivatives can certainly help in this respect as the preliminary results in [28] show, but it is still not clear if this way is best and most efficient one.

9.4 Crack Detection

A challenging future problem is the detection of cracks by elastic or electromagnetic measurements (cf. [13, 23]). With respect, to its mathematical structure, this problem is analogous to inclusion detection (see model problem 1), but with the difference that a crack is an open curve in 2d or an open surface in 3d. Consequently, a crack cannot be represented by a single level set function. In the framework of crack propagation, the representation was carried out by using two or more level set functions, a first one whose zero level set includes the crack and further ones to cut the curve at the crack tips (cf. [127]). However, with this approach it is not straight forward to handle topological changes in a straight forward way and further research is necessary in future.

Another difficulty arising for level set based optimization methods is the need to extend the velocity along level sets (and not in normal direction as discussed earlier), since the shape calculus will only yield normal velocities of the crack and tangential velocities for the crack tip (cf. [58]).

Finally, care has to be taken in theory and numerics due to the low regularity of fractured domains (and the subsequent low regularity of solutions of partial differential equations on such domains). For the numerical solution, enriched finite element methods with additional basis functions for incorporating singularities at the crack tips turned out to be a good choice (cf. [127]).

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