Lecture Notes

Numerical Methods for Incompressible Flow

Martin Burger

Based on Lecture Notes by René Pinnau, TU Darmstadt

Contents:

The aim of this lecture is to give an overview on modern numerical methods for the computation of incompressible flows. We start with a short introduction to fluid mechanics, including the derivation and discussion of the most important models and equations. The numerical methods discussed in the subsequent part are ordered due to the model they solve, i.e., we start with the stationary Stokes problem, a linear saddle-point problem, then proceed to stationary Navier-Stokes, which adds the complication of a nonlinear equation, and finally discuss the instationary Navier-Stokes equations, which adds time discretizations. In all cases, we shall discuss modern discretization strategies, their major properties, and the solution of the discretized equations.

Major Goals:

- Basic knowledge of fluid mechanics and associated mathematical models
- Discretization of mixed problems and their iterative solution
- Modern discretization and iteration methods for the incompressible Navier–Stokes equations
- Time integration of instationary Navier–Stokes equations
- Introduction to turbulence models

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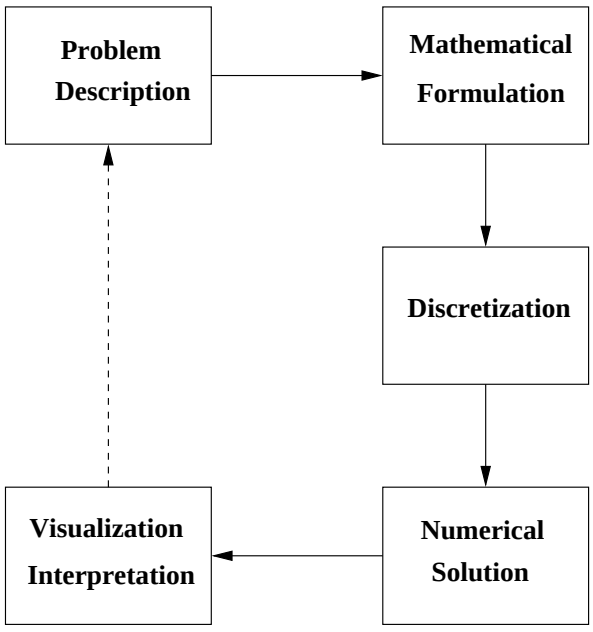
1 Introduction

The simulation of complex, dynamic processes that appear in nature or in industrial applications poses a lot of challenging mathematical problems, opening a long road from the basic problem, to the mathematical modelling, the numerical simulation, and finally to the interpretation of results. In order to achieve this goals, interdisciplinarity between applied mathematicians and experts in other fields becomes of increasing importance, since mathematical knowledge alone does not suffice in order to obtain a solution, but understanding the physics of the process is required as well.

This class is dealing with one specific and very important class of such processes, namely **Computational Fluid Dynamics** (usually abbreviated **CFD**), with a particular focus on incompressible flows.

1.1 Mathematical Modelling

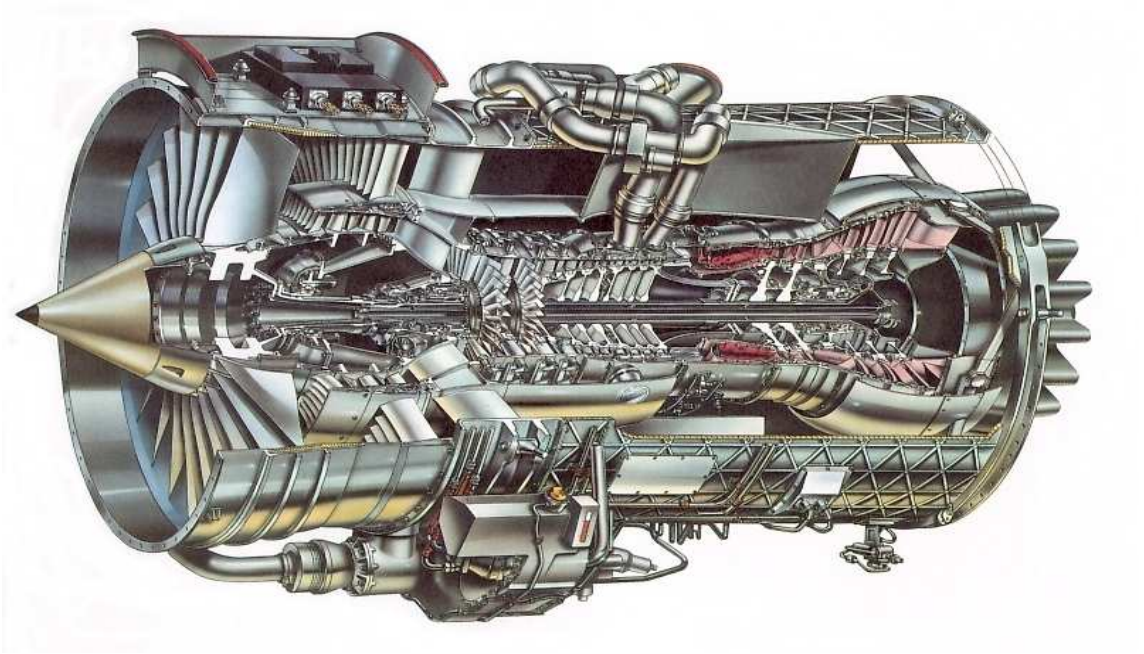
In general, the cycle of (mathematical) modelling consists of the following steps:



We will detail these steps corresponding to the design of a gas turbine combustion chamber in the following.

1.2 The Gas Turbine Combustion Chamber

If one wants to compute the flow in a gas turbine combustion chamber , one has to decide on a specific aspect one is interested in first, since the general requirements vary too much. In this case, experiments are very expensive, and due to high pressures ($p > 35$ bar) and high temperatures ($T > 1700$ K) it is questionable whether results of model experiments can be applied in practical situations. On the other hand, the time span needed for product development is strongly decreasing, and hence, CFD is of increasing importance for the design with respect to aerodynamic and combustion aspects.



Obviously, the starting point of the modelling cycle is the most difficult one for a mathematician, and input by experts in other disciplines is of high importance. However, some questions arise immediately, in particular the following fundamental one:

Which physical and chemical effects are relevant in the process ?

Here are some possible answers:

- Flow of air and the burning material
- Combustion of the burning material
- Radiation of the flame

- Creation of soot and pollutants .

These subprocesses are not independent in general, but exhibit a strong nonlinear interaction. E.g., combustion has an influence on the flow via the change of temperature and the creation of soot changes the transmission of energy by radiation.

These phenomena are only those on a coarse scale and need some refinement if one wants to achieve more accurate results. Since air and burning material are mixed in the gas turbine combustion chamber , the flow will be turbulent, i.e., effects of friction must be taken into account. The combustion combines various different molecules and chemical reactions. In such a process, one typically arrives at more than 1000 equations just to describe the reaction kinetics. If fluid burning material is used, one eventually has to simulate a reactive two-phase flow.

If all these aspects are translated into mathematical relations, one arrives at a system of partial differential equations (and usually algebraic equations for the reaction kinetics) for various field quantities, which have to be solved in a suitable domain (e.g. the whole volume of the gas turbine combustion chamber). Moreover, suitable boundary conditions are needed, taking into account physical conditions (isolation, radiation, ...).

After translating a problem into the language of mathematics, one has to transfer it into a computable form. This is usually done by generating a grid on the domain of computation (or in some associated frequency domain), on which one discretizes the equations using finite difference, finite element, or finite volume methods. The grid generation poses a challenging problem for complex 3D geometries arising in practical applications. The choice of the discretization method is a matter of taste in many cases, but it should correspond to the used grid.

After discretization, one ends up with a system of nonlinear algebraic equations (of a special structure) and has to choose appropriate solvers. The nonlinearity is usually treated by iterative methods based on succesive linearization. This leads to subproblems consisting of large systems of linear equations. Due to their size and special structure (sparsity, ...) these linear systems are again solved using iterative methods, since direct solvers are too expensive.

After obtaining a first *solution*, one needs further postprocessing. A first technique of importance is error control, which allows adaptive grid refinement and an improvement of accuracy. Another important part is the visualization of the resulting data, which enables their further interpretation. The interpretation of results is of crucial importance not only for the practitioner, but also for the mathematician, since it helps to find errors in the computations and to decide about the quality of the model. Moreover, one can compare models of different complexity and scale, whose choice is usually determined by the available computational capacity and disk memory.

If the results differ significantly from real situations, the model and/ or the numerical methods have to be refined and the cycle of mathematical modelling has to be restarted.

Finally, we mention an aspect that will not be treated in this course, namely *software engineering*, which deals with the details of implementation, like the parallelization of algorithms, the distribution of computational load to different processors, and the management of disk space. These aspects are usually determined by the specific computer architecture.

2 Derivation of models

In this chapter, we shall sketch the derivation of fundamental models of fluid dynamics. In general, we shall model a flow as a physical continuum of masses, i.e., objects in euclidean space represented by the set of their point masses. The derivation of the equations is based on some fundamental principles:

Assumptions:

- ➢ For all times $t > 0$, there exists a well-defined mass density $\varrho(x,t)$, such that the total mass $m(\Omega,t)$ in the domain Ω at time t is given by

$$m(\Omega,t) = \int_\Omega \varrho(x,t)\,\mathrm{d}V$$

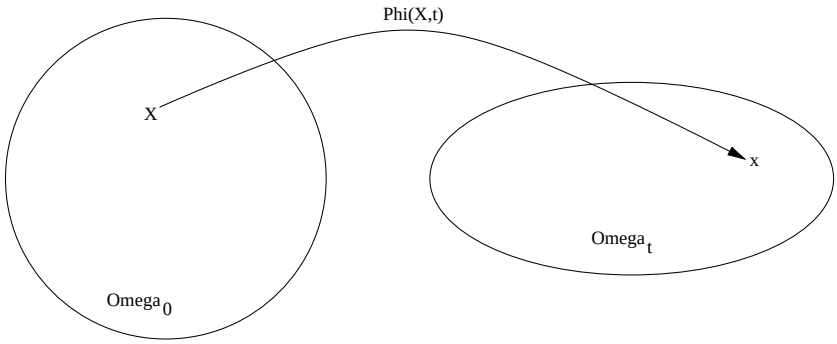
- ➢ Mass is neither produced nor disappears.
- ➢ The change of the momentum of a fluid region is equal to the applied forces (Newton's second law).
- ➢ Energy is neither produced nor disappears.

Remark 2.1. *In the language of physics, our assumptions are the hypothesis of continumm, conservation of mass, conservation of momentum, and conservation of energy.*

Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$ the region occupied by the fluid, and let $x \in \Omega$. We now consider a fluid particle X, moving through x at time t. Usually, x are called *Eulerian* coordinates for the description of the continuum of masses, and X are called *Lagrangian* or *material* coordinates.

Now let $W_0 \subset \Omega$ be a subset at the initial time $t = 0$. The function $\phi:\ W \times \mathbb{R}^+ \to \mathbb{R}^3$ describes the change of the particle position

$$W_t \stackrel{\text{def}}{=} \{\phi(X,t):\ X \in W_0\} = \phi(W_0,t).$$



For the description of the flow, the following two notions are usefull:

Definition 2.2.

- ➢ *The* trace *is the set of points* $x(X_0, t)$ *, that the particle* X_0 *covers at different times* t *.*
- ➢ *The* stream line *is the curve, whose tangent points in the direction of the actual velocity vector.*

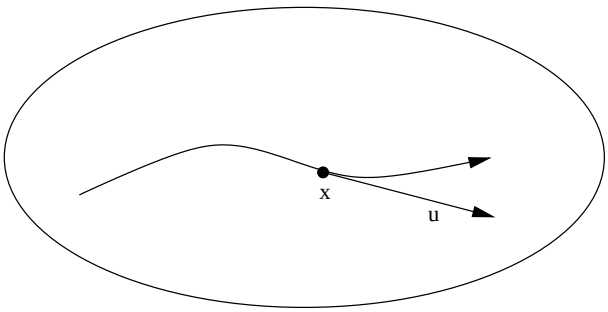
Remark 2.3. *trace and stream line are equal for stationary flows.*

We shall denote the velocity of a particle by $u(x, t)$. For fixed time t, $u(x, t)$ is a vector field on Ω. Then,

$$\begin{aligned} x: \ \mathbb{R}^+ &\to \mathbb{R}^3 \\ t &\to \phi(X, t) \end{aligned}$$

is the trajectory of the particle, and the velocity is given by

$$u(x,t) = \frac{\partial \phi}{\partial t}(X,t), \qquad \text{with } x = \phi(X,t).$$



The acceleration a of the particle can be computed using the chain rule:

$$\begin{aligned} a(x,t) = \frac{d}{dt}u(x,t) &= \frac{d}{dt}u(\phi(X,t),t) \\ &= \frac{\partial}{\partial t}u(\phi(X,t),t) + \sum_{i=1}^{3} \frac{\partial u}{\partial x_i}(\phi(X,t),t) \underbrace{\frac{\partial \phi_i}{\partial t}(X,t)}_{=u_i(x,t)} \\ &= \frac{\partial}{\partial t}u(x,t) + \sum_{i=1}^{3} u_i(x,t)\frac{\partial u}{\partial x_i}(x,t) \\ &= \frac{\partial u}{\partial t} + (u \cdot \nabla)u. \end{aligned}$$

Definition 2.4. *The symbol*

$$\frac{D}{Dt} \stackrel{\text{def}}{=} \frac{\partial}{\partial t} + u \cdot \nabla$$

is called material derivative.

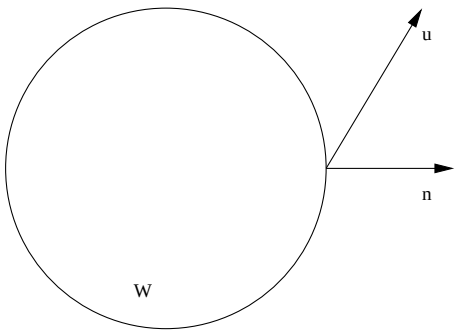
2.1 Conservation of Mass

In the following part, we shall derive conclusions from the hypothesis of continuum and conservation of mass.

For this sake, we fix a subdomain $W \subset \Omega$. The change of mass in W is given by

$$\begin{aligned} \frac{d}{dt}m(W,t) &= \frac{d}{dt}\int_W \varrho(x,t)\,\mathrm{d}V \\ &= \int_W \frac{\partial}{\partial t}\varrho(x,t)\,\mathrm{d}V \end{aligned}$$

In the following, ∂W denotes the boundary of W and n is the unit outer normal, dA is the area element on $\partial\Omega$.



The flow of volume through ∂W per unit area is given by $u \cdot n$ and the corresponding flow of mass by $\varrho u \cdot n$. Hence, the total flow of mass through ∂W is

$$\int_{\partial W} \varrho \, u \cdot n \, \mathrm{d}A$$

The principle of mass conservation implies:

> The change of mass in W equals the flow of mass over the boundary ∂W into W,

respectively in mathematical terms:

$$\boxed{\frac{d}{dt} \int_W \varrho \, \mathrm{d}V = - \int_{\partial W} \varrho \, u \cdot n \, \mathrm{d}A}$$

Using Gauss' Theorem we can rewrite this identity as

$$\int_W \left(\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho u) \right) \mathrm{d}V = 0$$

and since this identity holds for any subdomain W, we may conclude the differential version of the *continuity relation*

$$\boxed{\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho u) = 0}$$

2.1.1 The Transport Theorem

Starting with the function ϕ one can rephrase the conservation of mass as follows:

$$\int_{W_t} \varrho(x,t) \, \mathrm{d}V = \int_{W_0} \varrho(X,0) \, \mathrm{d}V$$

Since the right-hand side is independent of the time t, this yields

$$\frac{d}{dt} \int_{W_t} \varrho(x,t) \, \mathrm{d}V = 0$$

Since the domain of integration depends on time now, we cannot simply exchange differentiation and integration. In order to compute the derivative of an integral with a time-dependent domain we need the following result:

Theorem 2.5. *Let F be sufficiently smooth. Then,*

$$\begin{aligned}\frac{d}{dt}\int_{W_t} F(x,t)\,\mathrm{d}V &= \int_{W_t} \frac{DF}{Dt} + F\,\mathrm{div}\,u\,\mathrm{d}V \\ &= \int_{W_t} \frac{\partial F}{\partial t} + \mathrm{div}(F\cdot u)\,\mathrm{d}V\end{aligned}$$

Remark 2.6. *The above result is the generalization to higher dimensions of the formula for the derivative of a one-dimensional integral with parameter-dependent integrand and interval of integration.*

The proof will be carried out in the exercises, using the following lemma:

Lemma 2.7. *Let $J(X,t) := \det\left(\frac{\partial\phi}{\partial X}(X,t)\right)$. Then,*

$$\frac{\partial}{\partial t}J(X,t) = \mathrm{div}_x\, u(x,t)\cdot J(X,t)$$

with $x = \phi(X,t)$.

Using Theorem 2.5 we obtain

$$\begin{aligned}0 &= \frac{d}{dt}\int_{W_t}\varrho(x,t)\,\mathrm{d}V \\ &= \int_{W_t}\frac{\partial\varrho}{\partial t} + \mathrm{div}(\varrho u)\,\mathrm{d}V\end{aligned}$$

and since this relation holds for arbitrary $W_0 \subset \Omega$, we obtain the differential version of the continuity equation.

Remark 2.8. *In general, it depends on the regularity of solutions, whether one uses the differential or integral version of the conservation of mass. We shall assume in the following that the solutions are sufficiently smooth, such that all manipulations above are justified.*

A consequence of the conservation of mass is the following theorem, sometimes called *Reynold's transport theorem*:

Theorem 2.9. *Let $F = F(x,t)$ be a regular function. Then,*

$$\boxed{\frac{d}{dt}\int_{W_t}\varrho F\,\mathrm{d}V = \int_{W_t}\varrho\frac{DF}{Dt}\,\mathrm{d}V}$$

Fluids that leave the volume of moving subdomains constant in time, are of particular importance:

Definition 2.10. *A fluid (or its flow) is called* incompressible*, if*

$$\frac{d}{dt}\int_{W_t} \mathrm{d}V = 0$$

There are several equivalent definitions of incompressibility:

Lemma 2.11. *A fluid is incompressible if and only if one of the following conditions (and therefore all of them) holds:*

- $\operatorname{div} u = 0$,
- $J = 1$
- $\frac{D\varrho}{Dt} = 0$.

2.2 Conservation of Momentum

In the following we shall employ the conservation of momentum to derive a second equation. For the momentum of a fluid we use Newton's second law:

Change of momentum = Sum of all active forces

In general, we distinguish:

Volume forces:

$$\int_{W_t} \varrho(x,t)\, f(x,t)\, \mathrm{d}V,$$

for a force density $f = (f_1, f_2, f_3) \in \mathbb{R}^3$, e.g. gravitational forces,

and

Surface forces:

$$\int_{\partial W_t} \underbrace{n \cdot \tau(x,t)}_{=\sum_{j=1}^{3} u_j \tau_{ij}}\ \mathrm{d}A,$$

where $\tau \in \mathbb{R}^{3\times 3}$ is the corresponding stress tensor describing the internal friction and the pressure Moreover, $n \cdot \tau$ denotes the stress vector at a surface elements.

Using Newton's second law and Gauß' theorem, we deduce

$$\begin{aligned}\frac{d}{dt}\int_{W_t} \varrho u \,\mathrm{d}V &= \int_{W_t} \varrho f \,\mathrm{d}V + \int_{\partial W_t} n \cdot \tau \,\mathrm{d}A \\ &= \int_{W_t} \varrho f \,\mathrm{d}V + \int_{W_t} \operatorname{div} \tau \,\mathrm{d}V\end{aligned}$$

with the (line) divergence $(\operatorname{div}\tau)_i \stackrel{\text{def}}{=} \sum_{j=1}^{3} \frac{\partial \tau_{ji}}{\partial x_j}$.

The transport theorem implies

$$\boxed{\int_{W_t} \left(\varrho \frac{Du}{Dt} - \varrho f - \operatorname{div}\tau \right) \mathrm{d}V = 0}$$

respectively in differential form

$$\boxed{\varrho \frac{Du}{Dt} = \varrho f + \operatorname{div}\tau,}$$

i.e.,

$$\frac{D}{Dt}(\varrho u) + \varrho u \operatorname{div} u = \varrho f + \operatorname{div}\tau.$$

This relation can also be rewritten in conservation form as

$$\frac{\partial}{\partial t}(\varrho u_i) + \operatorname{div}(\varrho u_i u) = \varrho f_i + (\operatorname{div}\tau)_i \ , \quad i = 1, 2, 3$$

2.2.1 The Stress Tensor

In this section we shall further specify the stress tensor. For this sake we need the following

Assumptions:

- $\tau = -pI + \sigma, \qquad p \in \mathbb{R},\ I \in \mathbb{R}^{3\times 3},\ \sigma \in \mathbb{R}^{3\times 3}$
- σ depends linearly on ∇u .
- σ is invariant with respect to translation and rotation
- σ ist symmetric

Remark 2.12. *The first two assumptions imply spherical symmetry of the stress tensor for resting fluids. As a consequence, the pressure p always acts in direction of the unit normal. The last assumption is a conclusion of the conservation of rotational momentum, which we have not considered so far.*

From these assumptions one can conclude that σ is of the form

$$\sigma = \lambda(\operatorname{div} u)I + 2\mu D$$

where λ (volume viscosity) und μ (shear viscosity) are the viscosity coefficients (or Lamé coefficients), and $D = (D_{ij}) \in \mathbb{R}^{3\times 3}$ denotes the *deformation tensor*

$$D_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

which can be rewritten as

$$D = \tfrac{1}{2}(\nabla v + (\nabla v)^T)$$

Remark 2.13. $\mu_d = \lambda + \frac{2}{3}\mu$ *is called* pressure viscosity.

Finally, this gives the conservation of momentum as

$$\boxed{\varrho\frac{Du}{Dt} = \varrho f - \nabla p + (\lambda+\mu)\nabla(\operatorname{div} u) + \mu\Delta u}$$

2.3 Energy Balance

Up to now we have derived 4 equations (one for mass balance and one for conservation of momentum) for 5 unknowns ϱ, u, and p. In order to obtain a closed system, we need a further equation, which will be a consequence of the conservation of energy.

Let E denote the total energy, composed of kinetic and internal energy, i.e.,

$$E = \underbrace{\int_{W_t} \varrho\frac{|u|^2}{2}\,\mathrm{d}V}_{\text{kinetic energy}} + \underbrace{\int_{W_t} \varrho e\,\mathrm{d}V}_{\text{internal energy}}$$

where e denotes the *internal energy per unit of mass*. The fundamental principle governing the energy balance is

"Change of energy" = "work performed by exterior forces + heat supply"

In an analogous way to the conservation of momentum

$$\frac{d}{dt}\int_{W_t} \varrho\Big(\frac{|u|^2}{2} + e\Big)\,\mathrm{d}V = \underbrace{\int_{W_t} \varrho f\cdot u\,\mathrm{d}V}_{\text{work performed by volume forces}} \underbrace{-\int_{\partial W_t} h\,\mathrm{d}A}_{\text{heat supply over the boundary}} + \underbrace{\int_{\partial W_t} n\cdot\tau\cdot u\,\mathrm{d}A}_{\text{work performed by surface forces}}$$

Using Gauß' theorem we deduce

$$\begin{aligned}\int_{\partial W_t} n \cdot \tau \cdot u \,\mathrm{d}A &= \int_{\partial W_t} \sum_{j=1}^{3}\sum_{i=1}^{3} n_i \cdot \tau_j \cdot u_j \,\mathrm{d}A \\ &= \int_{W_t} \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{3} \tau_j \cdot u_j\Big)\,\mathrm{d}V \\ &= \int_{W_t} \operatorname{div}(\tau u)\end{aligned}$$

One can show that

$$h = n \cdot q,$$

where q denotes the *vector of heat flux density* (or just heat flux).

As a consequence of the transport theorem, we have

$$\boxed{\int_{W_t} \left(\varrho \frac{D}{Dt}\Big(\frac{|u|^2}{2} + e\Big)\right)\,\mathrm{d}V = \int_{W_t} (\varrho f u - \operatorname{div}(q - \tau u))\,\mathrm{d}V}$$

or in a differential version

$$\boxed{\varrho \frac{D}{Dt}\Big(\frac{|u|^2}{2} + e\Big) = \varrho f u - \operatorname{div}(q - \tau u)}$$

2.4 Closure Relations

Until now we have obtained 5 scalar equations for the unknowns $\varrho, p, e, q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ and $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$. Since we have 9 unknowns, we need further conditions to close the system. The remaining equations are material relations, in typical cases algebraic equations relating density, pressure, temperature, and internal energy.

A first relation of this kind is *Fourier's law* stating that the heat flux is proportional to the temperature gradient, i.e.,

$$q = -\kappa \nabla T,$$

where T is the scalar temperature and κ the so-called heat conductivity (a material dependent constant or function of temperature). In this way, we can eliminate the variables q_1, q_2, and q_3.

Further conditions are obtained from equilibrium thermodynamics, if we assume that we deal with an ideal gas and :

- An ideal gas satisfies $p = R\varrho T$, where R is the gas constant
- The assumption that the specific heat should be constant if the volume is constant leads to $c_v = \frac{de}{dT}$, and thus,

$$e = c_v T + const.$$

The above relations suffice to obtain a closed system. Depending on the specific application, different closure relations can appear, e.g. constant density or negligeable friction. Such constitutive relations have been obtained in different ways, some of them from simple axioms as above, others by fitting material laws to experimental results. In only few cases, the constitutive relations can be obtained rigorously from microscopic properties.

2.5 Summary of Equations

In the following, we sum up the equations derived in the previous sections:

- **Conservation of mass:** $\frac{D\varrho}{Dt} + \varrho \operatorname{div} u = 0$
- **Conservation of momentum:** $\varrho \frac{Du}{Dt} = \varrho f - \nabla p + (\lambda + \mu)\nabla(\operatorname{div} u) + \mu \Delta u$
- **Conservation of energy:** $\varrho \frac{D}{Dt}\left(\frac{|u|^2}{2} + e\right) = \varrho f u - \operatorname{div}(q - \tau u)$

supplied by the constitutive relations

- **Fourier's law:** $q = -\kappa \nabla T$
- **Ideal gas law:** $p = R\varrho T$
- **"Change of internal energy proportional to heat supply":** $e = c_V T$
- **Incompressible media**: $\frac{D\varrho}{Dt} = 0$
- **Fluids without friction**: $\sigma \equiv 0$ bzw. $\lambda = \mu = 0$

2.6 Exercises

Exercise 2.1. *Let u be a vector field and f a scalar function. Proof the following formulae of nabla calculus:*

- $\operatorname{div}(f\,u) = u \cdot \nabla f + f\ \operatorname{div} u\,,$
- $\operatorname{div}(\operatorname{rot} u) = 0\,,$
- $\operatorname{rot}(\nabla f) = 0\,,$
- $\operatorname{rot}(\operatorname{rot} u) = \nabla(\operatorname{div} u) - \Delta u\,.$

Exercise 2.2. *Show using Gauss' Theorem*

- *the first Green's formula*

$$\int_\Omega u\,\Delta v\ dx = -\int_\Omega \nabla u \cdot \nabla v\ dx + \int_{\partial\Omega} u(\nabla v \cdot n)\ ds,$$

- *the second Green's formula*

$$\int_\Omega u\,\Delta v\ dx = \int_\Omega \Delta u\, v\ dx + \int_{\partial\Omega} u(\nabla v \cdot n) - v(\nabla u \cdot n)\ ds$$

Exercise 2.3. *Let a fluid cover a domain $D \subset \mathbb{R}^3$, let $\Phi(x,t) \in \mathbb{R}^3$ the trajectory of a fluid particle x at time $t = 0$, and let $u(x,t) \in \mathbb{R}^3$ be the velocity of the fluid. Then,*

$$\partial_t \Phi(x,t) = u(\Phi(x,t),t).$$

Let W_0 be a subdomain of D and let $W_t, t \geq 0$ be the domain

$$W_t = \{\Phi(x,t) \in \mathbb{R}^3 | x \in W_0\}$$

Show that:

- *With $J(x,t) = |det \nabla \Phi(x,t)|$, the relation*

$$J_t(x,t) = (\operatorname{div} u(\Phi(x,t),t)) \cdot J(x,t)$$

  holds.

- *If $\operatorname{div} u = 0$ it follows that*

$$\frac{d}{dt}\int_{W_t} f(x,t)dx = \int_{W_t} [\partial_t f + u \cdot \nabla_x f](x,t)dx$$

- *The volume of W_t is constant in time.*

Exercise 2.4. *We use the notations of the previous exercise, and denote by $\rho(x,t)$ the density of the fluid. Then,*

$$\frac{d}{dt}\int_{W_t} \rho\, f\, dx = \int_{W_t} \rho\, \frac{Df}{Dt}\, dx.$$

Exercise 2.5. *Show the following formula concerning the convective part of the material derivative*

$$(u\cdot\nabla)u = \frac{1}{2}\nabla(|u|^2) - u\times \operatorname{rot} u.$$

Exercise 2.6. *Derive the conservation form of the equation of momentum conservation from its differential version.*

Exercise 2.7. *Show that the deformation tensor D given by*

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

satisfies the relation

$$\operatorname{div}(2D) = \nabla(\operatorname{div} u) + \Delta u.$$

Exercise 2.8. *Show the equivalence of the following statements:*

- *The flow is incompressible.*
- *The functional determinant J of Φ satisfies $J(x,t)\equiv 1$.*
- *The velocity satisfies $\operatorname{div} u = 0$.*
- *The density satisfies $\frac{D\rho}{Dt} = 0$.*

Exercise 2.9.

- *Show that the performed work $\int_\Omega \rho\, u\cdot f\, dx$ of the volume forces can be rewritten as*

  $$-\frac{d}{dt}\int_\Omega \rho\,\phi\, dx,$$

  if the force density f has a stationary potential ϕ ($\frac{\partial\phi}{\partial t} = 0$).
- *Unter above assumptions, the energy balance can be rewritten as*

$$\frac{d}{dt}\int_\Omega \rho\left(\frac{|u|^2}{2} + e + \phi\right)\, dx = -\int_\Omega \operatorname{div}(q - \tau\cdot u)\, dx.$$

Exercise 2.10. *Write the energy balance for an incompressible fluid in divergence form.*

Exercise 2.11. *Show for an incompressible flow that*

- *in 2d :* $\text{rot}\,((\underline{u}\cdot\nabla)\underline{u}) = (\underline{u}\cdot\nabla)\text{rot}\,\underline{u}$
- *in 3d :* $\text{rot}\,((\underline{u}\cdot\nabla)\underline{u}) = (\underline{u}\cdot\nabla)\text{rot}\,\underline{u} - ((\text{rot}\,\underline{u})\cdot\nabla)\underline{u}$

3 Interpretation

In the following we want to give some interpretations of the equations derived above and apply them to interesting specific flow situations. We will discuss equations for compressible and incompressible flow and state the appropriate boundary conditions. Finally, we discuss the proper scaling and two special situations, which allow to derive closed-form solutions.

3.1 Compressible Flows

If one considers flows, where the change of density is not negligeable, one has to use the full system of compressible equations. We have seen in the previous chapter that additional relations are needed to obtain a closed system of equations. E.g., we have

$$\begin{aligned} p &= R\,\varrho\,T, \\ e &= c_V T + \text{const} \end{aligned}$$

for an ideal gas, respectively

$$\mu_d = 0 \quad \text{bzw.} \quad \lambda = -\frac{2}{3}\mu$$

for a diluted gas.

Moreover, the heat flux is determined by Fourier's law

$$q = -\kappa \nabla T.$$

3.1.1 Compressible Navier-Stokes Equations

If the effects of frictions are not negligeable, one obtains the following system of partial differential equations

$$\begin{aligned} \frac{D\varrho}{Dt} + \varrho \operatorname{div} u &= 0 \\ \varrho \frac{Du}{Dt} &= \varrho f - R\nabla(\varrho T) + \frac{\mu}{3}\nabla(\operatorname{div} u) + \mu \Delta u \\ \varrho \frac{D}{Dt}\left(\frac{|u|^2}{2} + c_v T\right) &= \varrho f u + \operatorname{div}(\kappa \nabla T) + \mu \operatorname{div}((\nabla u + (\nabla u)^T)u) \\ &\quad -\tfrac{2}{3}\mu \operatorname{div}(u \operatorname{div} u) \end{aligned}$$

3.1.2 Boundary Conditions

In order to obtain a well-posed problem, we need additional conditions on the boundary of the computational domain. Assuming that the fluid is inside a domain $\Omega \subset \mathbb{R}^3$, describing e.g. a container, an appropriate assumption is that there is no flow through the container wall, i.e.,

$$u \cdot n = 0 \quad \text{on } \partial\Omega,$$

where n denotes the unit outer normal.

The continuity equation for the density ϱ can be considered as a linear partial differential equation of first order. The method of characteristics shows that no boundary condition for the density on $\partial\Omega$ is necessary due to $u \cdot n = 0$.

A closer look at the equation of momentum conservation shows that the condition on u does not suffice. Supposing that we had even an incompressible flow ($\operatorname{div} u = 0$). Then,

$$\varrho \frac{Du}{Dt} = \varrho f - \nabla p + \mu \Delta u$$

is a parabolic equation for u, but $u \cdot n = 0$ on $\partial\Omega$ defines only one component of the velocity vector.

The question concerning the boundary condition for the velocity of a fluid being in contact with a rigid wall is a quite delicate one. A simple and widely used condition is **adhesion condition:** $u = 0$ on $\partial\Omega$, i.e., there is no relative motion between the fluid and the wall.

Finally, we can interpret the energy balance as an equation for the temperature T, and we can use all possible boundary conditions for the heat equation, e.g.,

$$\begin{aligned} &\rightarrow \text{ fixed boundary temperature:} && T = T_b && \text{on } \partial\Omega \\ &\rightarrow \text{ isolated boundary:} && \nabla T \cdot n = 0 && \text{on } \partial\Omega \end{aligned}$$

The system is completed by specifying additional initial conditions for the density, velocity, and temperature

$$\varrho(0) = \varrho_0, \qquad u(0) = u_0, \qquad T(0) = T_0 \ \text{ in } \Omega.$$

3.1.3 Compressible Euler Equations

If viscous effects due to internal friction are negligeable, the stress tensor reduces to $\tau = -p\,I$. For such flows, compression can be so strong that discontinuities appear, so-called shocks or compression waves, e.g., for jets travelling above speed of sound.

In this case it is advantageous to consider the conservative form of the equations, which yields

$$\partial_t \rho + \operatorname{div}(\rho\, u) = 0$$
$$\partial_t(\rho\, u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \rho\, f.$$

Assuming an ideal gas, we obtain

$$p = (\gamma - 1)(\rho\, e - \frac{1}{2}\rho\, |u|^2),$$

where $\gamma \stackrel{\text{def}}{=} c_p/c_V$. The energy balance becomes

$$\partial_t(\rho\, e) + \operatorname{div}(\rho e U + p u) = \rho\, f \cdot u + \rho\, h.$$

In absence of volume and heat forces, these equations can be rewritten as a hyperbolic system of conservation laws. Defining the vector $z = (\rho, \rho u, \rho e)^T$ we have

$$\partial_t z + \operatorname{div} F(z) = 0,$$

with the flux function

$$F(z) = \begin{pmatrix} \rho u \\ \rho u \otimes u + pI \\ \rho e u + p u \end{pmatrix}.$$

Changes in density and pressure can propagate only with finite speed in a compressible fluid. We define the *speed of sound* as

$$c(p) \stackrel{\text{def}}{=} \sqrt{\frac{d\, p}{d\, \rho}}$$

and and the local Mach number

$$M \stackrel{\text{def}}{=} \frac{|u|}{c(p)}.$$

For this kind of equations on needs special numerical methods, being subject of a companion course on conservation laws. In particular, the choice of boundary conditions is difficult due to crossing characteristics.

3.2 Incompressible Flows

Contrary to gases, fluids can be compressed only using very high forces. Therefore, a fluid behaves almost like an incompressible material, i.e., $\operatorname{div} u = 0$, and the continuity equation becomes a transport equation for the density, i.e.,

$$\partial_t \rho + u \cdot \nabla \rho = 0.$$

3.2.1 Incompressible Navier-Stokes Equations

Now we state the equations whose numerical solution will be treated in detail in the following chapters. In addition to incompressibility, we assume a constant density ($\varrho = \varrho_0 \equiv 1$), which is a typical assumption for several fluids such as water or oil. Then we obtain the **incompressible Navier–Stokes Equations**

$$\left.\begin{aligned} u_t + \underbrace{(u \cdot \nabla)u}_{Convection} &= f - \frac{1}{\varrho_0}\nabla p + \underbrace{\nu \Delta u}_{Diffusion} \\ \operatorname{div} u &= 0 \end{aligned}\right\} \text{ in } \Omega$$

$$\begin{aligned} u &= 0 && \text{on} \quad \partial\Omega \\ u(0) &= u_0 && \text{in} \quad \Omega \end{aligned}$$

with $\nu = \mu/\rho_0$. Since density is constant, we do not need further relations to determine the state variables.

Since we have normalized density, the kinetic energy of the fluid is given by

$$E_{\text{kin}}(t) := \frac{1}{2}\int_\Omega |u|^2 \, \mathrm{d}x.$$

In the absence of volume forces, its time derivative is given by

$$\begin{aligned} \frac{d}{dt}E_{\text{kin}}(t) &= \int_\Omega u u_t \, \mathrm{d}x \\ &= \int_\Omega u(-(u \cdot \nabla)u - \nabla p + \nu \Delta u) \, \mathrm{d}x \\ &= -\int_\Omega u(u \cdot \nabla)u \, \mathrm{d}x - \int_\Omega u \nabla p \, \mathrm{d}x + \nu \int_\Omega u \Delta u \, \mathrm{d}x \end{aligned}$$

Using Gauss' Theorem we obtain

$$\int_\Omega u \nabla p \, \mathrm{d}x = -\int_\Omega \underbrace{\operatorname{div} u}_{=0} p \, \mathrm{d}x + \int_{\partial\Omega} p \cdot \underbrace{u \cdot n}_{=0} \, \mathrm{d}s = 0$$

and

$$\begin{aligned} \int_\Omega u(u \cdot \nabla u)u \, \mathrm{d}x &= \frac{1}{2}\int_\Omega u \nabla(|u|^2) \, \mathrm{d}x - \int_\Omega \underbrace{u \cdot (u \times \operatorname{rot} u)}_{=0} \, \mathrm{d}x \\ &= -\frac{1}{2}\int_\Omega \underbrace{\operatorname{div} u}_{=0} |u|^2 \, \mathrm{d}x + \frac{1}{2}\int_\Omega |u|^2 \underbrace{u \cdot n}_{=0} \, \mathrm{d}s \\ &= 0 \end{aligned}$$

Hence,

$$\frac{d}{dt}E_{\text{kin}}(t) = \nu \int_\Omega u\Delta u \,\mathrm{d}x$$

respectively

$$\begin{aligned}\frac{d}{dt}E_{\text{kin}}(t) &= \nu \int_\Omega u_1\Delta u_1 \,\mathrm{d}x + \nu \int_\Omega u_2\Delta u_2 \,\mathrm{d}x + \nu \int_\Omega u_3\Delta u_3 \,\mathrm{d}x \\ &= \nu \sum_{i=1}^{3} \left(- \int_\Omega |\nabla u_i|^2 \,\mathrm{d}x + \underbrace{\int_{\partial\Omega} u_i(\nabla u_i \cdot n) \,\mathrm{d}s}_{=0} \right) \\ &= -\nu \sum_{i=1}^{3} \int_\Omega |\nabla u_i|^2 \,\mathrm{d}x \\ &\leq 0\end{aligned}$$

Consequently, the kinetic energy is decreasing in time, which reflects the losses due to friction in a viscous flow.

3.2.2 Incompressible Euler Equations

If viscosity can be neglectet $(\mu = 0)$, we obtain the **incompressible Euler equations**

$$\begin{aligned}\frac{Du}{Dt} &= f - \nabla p \\ \operatorname{div} u &= 0\end{aligned}$$

Since the Euler equations are of first order, we cannot use the adhesion condition for u on $\partial\Omega$, in this case it suffices to use

$$u \cdot n = 0 \quad \text{ on } \partial\Omega.$$

The incompressible Euler equations are again of hyperbolic type and therefore enforce special methods similar to those for conservation laws, which we shall not discuss in this class.

3.2.3 Stokes Equations

If f does not depend on time, then the solution of the incompressible Navier-Stokes equations will tend to its stationary limit, given by the equation

$$\left.\begin{aligned}(u \cdot \nabla)u &= f - \frac{1}{\varrho_0}\nabla p + \nu\Delta u \\ \operatorname{div} u &= 0\end{aligned}\right\} \text{ in } \Omega$$

$$u = 0 \qquad \text{on} \quad \partial\Omega$$

This system is usually called *stationary Navier Stokes equations.*

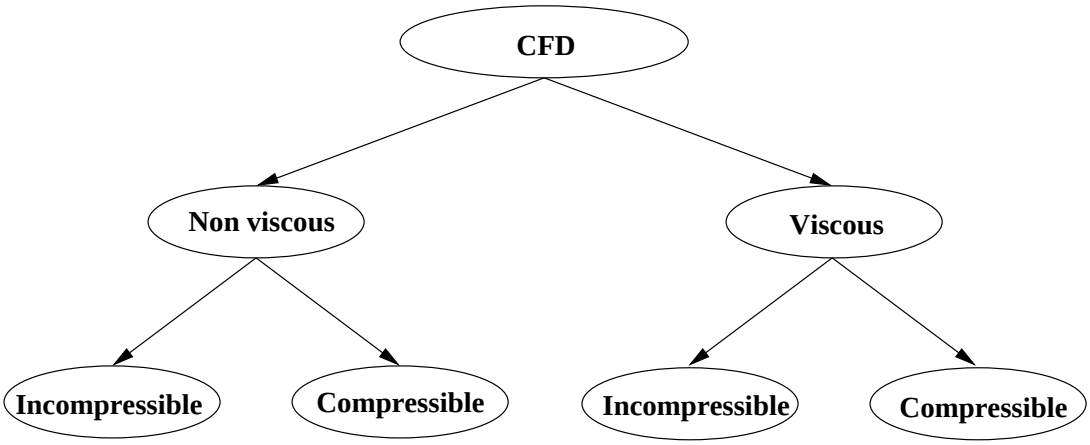
A case of particular importance, where the convergence towards the stationary situation is fast, is the one of a low *Reynolds number* , i.e., viscosity being large compared to the product of the typical speed and typical length (see the section on scaling below for details). In this case, the term on the right-hand side will dominate in the Navier-Stokes equations and we can use an appropriate rescaling of the pressure variable and the volume force to obtain the so-called *Stokes problem*

$$-\Delta u + \nabla p = f \tag{3.1}$$
$$\operatorname{div} u = 0 \tag{3.2}$$

in Ω .

3.3 Model Hierarchy

Altogether, we obtain the following model hierarchy:



Another standard distinction is between

$$\text{laminar} \leftrightarrow \text{turbulent}$$

Laminar flows (e.g. a slow flow out of a water tap or a flow along a channel) can be described by the Navier-Stokes equations, while new model equations are needed for turbulent flows (e.g. a fast flow out of a water tap).

Finally, there is another way of categorizing flows, namely with respect to their speed into:

$$\text{subsonic flows} \leftrightarrow \text{ transsonic flows}$$

3.4 Scaling

In the following we shall derive a dimensionless form of the incompressible Navier Stokes equations using appropriate scaling.

We start from

$$\begin{aligned} u_t + (u \cdot \nabla)u &= -\frac{1}{\varrho_0}\nabla p + \nu \Delta u \\ \operatorname{div} u &= 0 \end{aligned} \tag{3.3}$$

and choose characteristic length scale L and a characteristic speed U of the flow. This choice also determines a corresponding time scale of the flow, given by $\tau = \frac{L}{U}$.

Now we introduce new dimensionless variables

$$\tilde{x} = \frac{x}{L}, \qquad \tilde{t} = \frac{t}{\tau}, \qquad \tilde{u} = \frac{u}{U}$$

into (3.3), and conclude from the chain rule

$$\begin{aligned} \frac{U}{\tau}\tilde{u}_t + \frac{U^2}{L}(\tilde{u} \cdot \tilde{\nabla})\tilde{u} &= -\frac{1}{\varrho_0}\frac{1}{L}\tilde{\nabla} p + \frac{\nu U}{L^2}\tilde{\Delta}\tilde{u}, \\ \frac{U}{L}\tilde{\operatorname{div}}\tilde{u} &= 0 \end{aligned}$$

If we further rescale pressure to $\tilde{p} = \frac{1}{\varrho_0 U^2} p$, then multiplication by $\frac{L}{U^2}$ yields

$$\begin{aligned} u_t + (u \cdot \nabla)u &= -\nabla p + \frac{\nu}{LU}\Delta u \\ \operatorname{div} u &= 0, \end{aligned}$$

where we have dropped the tilde-notation.

The parameter

$$\boxed{\mathrm{Re} = \frac{LU}{\nu}}$$

is called *Reynolds number* and provides a measure for the viscosity of the flow.

One observes that ν being small does not imply that viscous effects can be neglected, since L and / or U may be small, too. Viscous effects can be neglected only, if $\frac{1}{\mathrm{Re}}$ is small, respectively Re large.

Furhermore, two flows can have the same Reynolds number, they are called *similar* in this case. This relation allows the well-known tests in a wind tunnel.

Example 3.1. (Flow around a sphere)

Flow 1: with radius $r = 10\mathrm{m}$, *speed* $U_\infty = 100\frac{\mathrm{km}}{\mathrm{h}}$ *and viscosity* ν

Flow 2: with radius $r = 1\mathrm{m}$, *speed* $U_\infty = 1000\frac{\mathrm{km}}{\mathrm{h}}$ *and the same viscosity.*

Then,

$$\mathrm{Re}_1 = \frac{1(\frac{\mathrm{km}^2}{\mathrm{h}})}{\nu} = \mathrm{Re}_2$$

Example 3.2. (Flow of air around a car)

$$U = 10\tfrac{\mathrm{m}}{\mathrm{s}}, \qquad L = 1\mathrm{m}, \qquad \nu_{\mathrm{Luft}} = 10^{-5}\tfrac{\mathrm{m}^2}{\mathrm{s}}$$

In this case we have

$$\mathrm{Re} = 10^6$$

Hence, one can neglect viscous effects (as long as one is not interested in the drag coefficient of the car).

Example 3.3. (Microorganisms in water)

$$U = 1\tfrac{\mathrm{mm}}{\mathrm{s}}, \qquad L = 1\mathrm{mm}, \qquad \nu_{\mathrm{water}} = 10^{-3}\tfrac{\mathrm{m}^2}{\mathrm{s}}$$

Then,

$$\mathrm{Re} = 10^{-3}$$

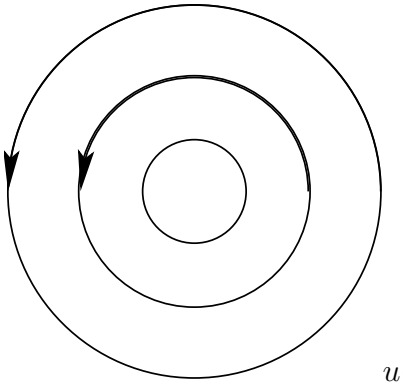
so that viscous effects dominate.

3.5 Vortex Transport Equation

Another important parameter of a flow is the

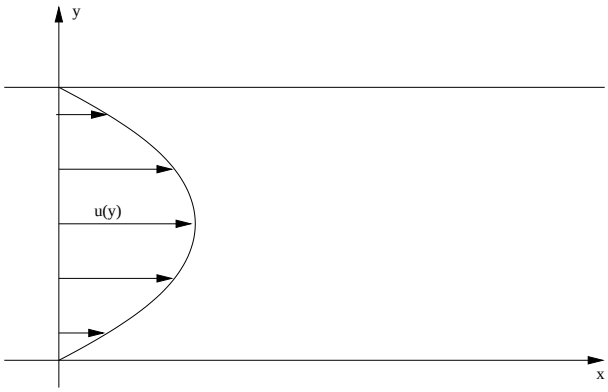
Vorticity: $\omega \overset{\mathrm{def}}{=} \mathrm{rot}\, u$

The interpretation of vorticity is not always straight forward: e.g., the flow around a sphere



$$u(x,y) = \begin{pmatrix} \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{x}{\sqrt{x^2+y^2}} \end{pmatrix}$$

is vortex free, while vorticity does not vanish for the channel flow



Using the identity

$$u \cdot \nabla u = \frac{1}{2} \nabla |u|^2 - u \times \mathrm{rot} u$$

we may conclude from the incompressible Euler equations for irrotational flows at vanishing force density that

$$\nabla \left(\frac{1}{2} \nabla |u|^2 + \frac{p}{\rho_0} \right) \equiv 0,$$

i.e., in the whole domain of the flow

$$\frac{1}{2} \nabla |u|^2 + \frac{p}{\rho_0} \equiv \mathrm{const.}$$

Hence, a local increase of speed implies a decrease of pressure.

Starting from the incompressible Navier-Stokes equations we shall now derive an equation for the vorticity ω. For this sake we apply the rot operator to

$$u_t + (u \cdot \nabla) u = -\nabla p + \frac{1}{Re} \Delta u$$

and obtain

$$\omega_t + \text{rot}\,((u \cdot \nabla)u) = \frac{1}{Re}\,\Delta(\text{rot}\,u)$$

Remark 3.4.

(1) In 2d:

$$rot\,((u \cdot \nabla)u) = (u \cdot \nabla)\omega \qquad (\omega \text{ is a scalar!})$$

(3) In 3d:

$$rot\,((u \cdot \nabla)u) = (u \cdot \nabla)\omega - (\omega \cdot \nabla)u$$

I.e., we obtain different equations depending on the space dimension.

In 2d the vortex transport equation is given by

$$\begin{aligned} \omega_t + (u \cdot \nabla\omega) &= \frac{1}{Re}\Delta\omega \\ \Leftrightarrow \quad \frac{D\omega}{Dt} &= \frac{1}{Re}\Delta\omega \end{aligned}$$

Hence, vorticity is transported through convection and diffuses. One observes that there is no diffusion of vorticity in the Euler equations $(Re \to \infty)$, i.e., it is a conserved quantity

In 3d the vortex transport equation is given by

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)u + \frac{1}{Re}\Delta\omega$$

Here, there apppears transport, stretch, and diffusion of vorticity. Note that even $\omega(t = 0) = 0$ does not imply $\omega \equiv 0$, since vorticity can be created by boundary conditions. The vortex transport equation has the advantage that pressure has been eliminated.

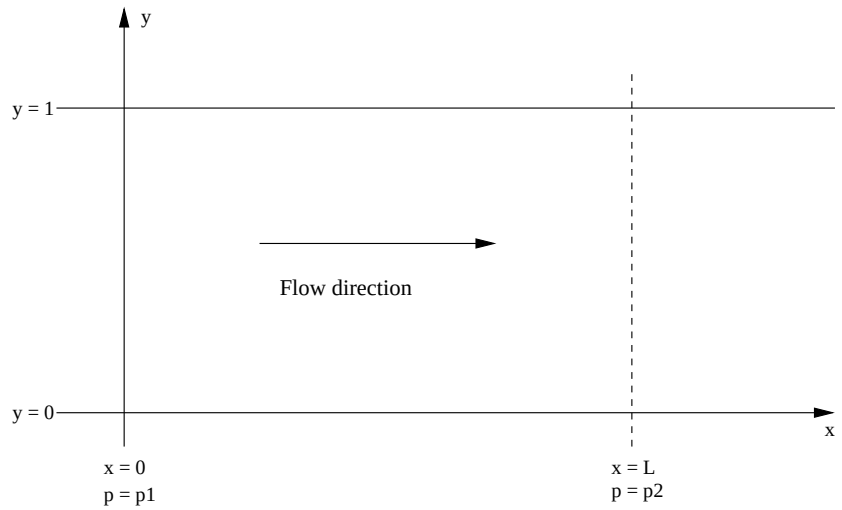
However, there is still a remaining question how to reconstruct velocity from vorticity. This is a non-trivial and lengthy problem, for which we refer to [CM84].

3.6 Closed Form Solutions

Now we shall investigate two particular situations, where the Navier-Stokes equations allow a closed form solution.

3.6.1 Viscous Channel Flow

Consider the following viscous incompressible flow between two stationary plates:



We are looking for a stationary solution of the incompressible Navier-Stokes equations of the form

$$u(x,y) = \begin{pmatrix} U(x,y) \\ 0 \end{pmatrix}, \quad p = p(x) \quad \text{with} \quad p(0) = p_1,\ p(1) = p_2.$$

The continuity equation $\operatorname{div} u = 0$ implies

$$U_x = 0 \quad \Rightarrow \quad U = U(y)$$

and conservation of momentum yields

$$0 = -UU_x - p_x + \frac{1}{Re}(U_{xx} + U_{yy})$$

with boundary values $U(x,0) = U(x,1) = 0$.

Thus,

$$p_x = \frac{1}{Re} U_{yy}.$$

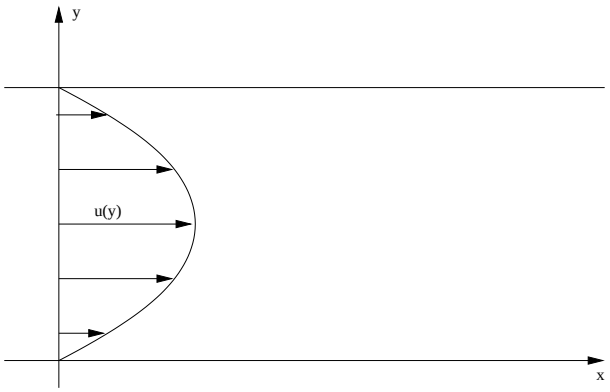
and since both sides of the equations depend on different variables, we may conclude that

$$p_x = \text{const.}, \qquad \frac{1}{Re} U_{yy} = \text{const.}$$

and consequently

$$p(x) = p_1 - \frac{p_1 - p_2}{L} x,$$
$$u(y) = y(1-y)\, Re\, \frac{p_1 - p_2}{2L}.$$

I.e., the velocity profile is a parabola.



3.6.2 Moving Plate

Assume that the half space $y > 0$ is filled with a fluid that is not moving for $t \leq 0$. Moreover, the plate at the fluid boundary starts to move at time $t = 0$ with constant velocity U in the y-direction. One may assume that the fluid starts to flow due to friction.

In this case we look for a solution of the incompressiblen Navier Stokes equations of the form

$$u = u(y,t), \quad v = w = 0.$$

Clearly, $\operatorname{div} \underline{u} = 0$ is satisfied. Conservation of momentum reduces to

$$\varrho u_t = -p_x + \mu u_{yy}, \quad p_y = p_z = 0.$$

Now p_x must be constant, because u depends on y and t only. It seems reasonable to choose $p_x = 0$. It remains to solve the diffusion equation

$$u_t = \nu u_{yy}$$

with the kinematic viscosity $\nu = \mu/\varrho$. The initial condition is given by $u(y,0) = 0$ and adhesion at the wall yields $u(0,t) = U$.

The dimensionless velocity $\tilde{u} = u/U$ solves the initial-boundary value problem

$$\begin{aligned} \tilde{u}_t &= \nu \tilde{u}_{yy}, \\ \tilde{u}(y,0) = 0, &\quad \tilde{u}(0,t) = 1. \end{aligned}$$

Since $\tilde{u}$ is dimensionless, it can only be a function of a dimensionless combination of ν, y, and t. The unit of kinematic viscosity is given by $[\nu] = \mathrm{cm}^2\mathrm{s}^{-1}$. Therefore we write $\tilde{u}$ as a function of

$$\eta = \frac{y}{2\sqrt{\nu t}}, \qquad \text{(dimensionless!)},$$

i.e., $\tilde{u} = F(\eta)$.

From the diffusion equation we obtain the boundary value problem

$$\begin{aligned} F_{\eta\eta} + 2\eta\, F(\eta) &= 0, \\ F(0) = 1, &\quad F(\infty) = 0, \end{aligned}$$

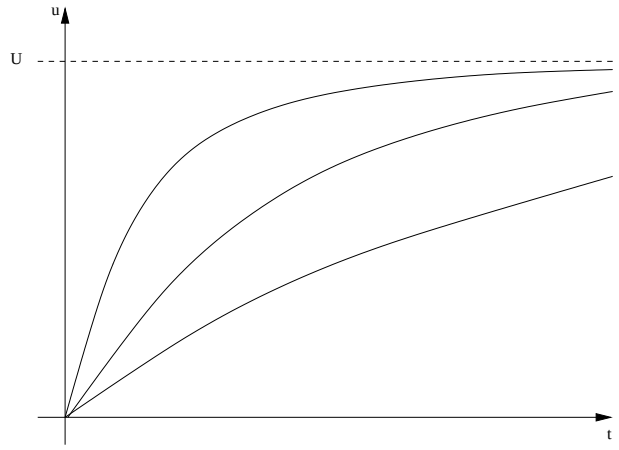
whose solution is determined via the the complimentary error function

$$F(\eta) = \operatorname{erfc}(\eta) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_\eta^\infty e^{-s^2}\, ds.$$

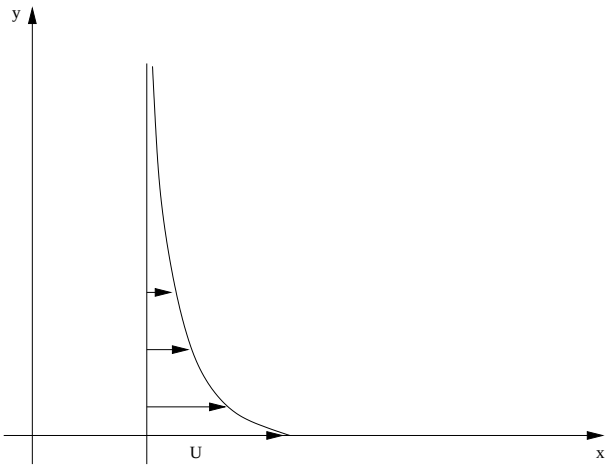
Hence, the velocity of the fluid is given by

$$u(y,t) = U \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right).$$

This means, that in a fixed distance to the plate the velocity of the fluid will converge to the velocity of the plate for $t \to \infty$.



For fixed time $t > 0$ we have



3.7 Exercises

Exercise 3.1. *Starting from the incompressible Euler equations (conservation of mass and momentum, $\lambda, \mu = 0$) compute the planar flow in a channel (i.e., between to plates) of length L , when there is a pressure difference.*

Exercise 3.2. *A very large plate is moving with a cosine law in its own plane. Above the plate there is a Newtonian fluid. Which flow appears in the fluid ?*

Exercise 3.3. *Compute the velocity and pressure profile of a stationary flow, which appears in a fluid film along a plane with slope angle α due to gravity. The height of the film orthogonal to the plane is h .*

Exercise 3.4. *A flow appearing due to viscosity between to coaxial cylinders moving with arbitrary angular velocities is called* Couette flow

- *Determine the general form of the azimuthal velocity of a stationary flow in absence of volume force (i.e., derive a formula in dependence of the radii of the cylinders and their angular velocities).*

  Hint:
  $$\frac{du_\phi}{dr} + \frac{u_\phi}{r} = \frac{1}{r}\frac{d}{dr}(r\,u_\phi), \quad \frac{du_\phi}{dr} - \frac{u_\phi}{r} = r\,\frac{d}{dr}\left(\frac{u_\phi}{r}\right)$$

- *Determine the special solution for u_ϕ und p if there is no inner cylinder (radius zero) and the outer cylinder has radius R and angular velocity ω .*

Exercise 3.5. *Show that the incompressible Navier–Stokes equations in cylindrical coordinates are given as:*

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0,$$

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\phi^2}{r} = f_r - \frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u_r}{r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\phi^2} + \frac{\partial^2 u_r}{z^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right),$$

$$\frac{\partial u_\phi}{\partial t} + u_r \frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + u_z\ ppu_\phi z - \frac{u_r\, u_\phi}{r} = f_\phi - \frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\ phi} + \nu\left(\frac{\partial^2 u_\phi}{r^2} + \frac{1}{r}\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\phi^2} + \frac{\partial^2 u_\phi}{z^2} - \frac{2}{r^2}\frac{\partial u_r}{\partial \phi}\right),$$

$$\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi} + u_z \frac{\partial u_z}{\partial z} = f_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u_z}{r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\phi^2} + \frac{\partial^2 u_z}{z^2}\right).$$

Exercise 3.6. *In order to measure viscosity, the so-called* Couette–viscosimeter *is used: The fluid is located between to coaxial cylinders. The outer one (radius R_2) is rotating with constant angular velocity ω and the inner one (radius R_1) does not move. Now the angular moment M is measured at the inner cylinder, which allows to determine the viscosity. Compute this angular moment and solve the equation for the viscosity.*

Hint: *The angular moment satisfies $M = 2\,\pi\, R_1^2\, L\tau_{r\phi}(R_1)$.*

4 Stokes Problem

In this chapter we shall discuss numerical methods for the solution of the linear Stokes problem. We shall provide some analysis in the framework of saddle-point problems, and discuss their discretization by mixed finite element methods. Moreover, we shall discuss the solution of the discretized problem.

By scaling the incompressible Navier-Stokes equations as before, and using the pressure $p = \frac{\varrho\nu U}{L}\tilde{p}$, we obtain

$$\begin{aligned} \text{Re } (u \cdot \nabla)u &= -\nabla p + \Delta u + f \\ \operatorname{div} u &= 0. \end{aligned}$$

Assuming that the the flow is rather slow ($\text{Re } \ll 1$), one can neglect the nonlinear term and obtain the linear *Stokes Problem*

$$-\Delta u + \nabla p = f \tag{4.1a}$$

$$\operatorname{div} u = 0 \tag{4.1b}$$

As discussed above, we use the boundary condition

$$u = 0 \quad \text{auf } \partial\Omega,$$

which is reasonable in particular for slow flows. A pair of functions $(u, p) \in (C^2(\Omega) \cap C^0(\bar{\Omega})) \times (C^1(\Omega))$ is called a *classical solution of the Stokes Problem*, if it satisfies (4.1). Note that pressure is determined only up to an additive constant, so that usually the normalizing condition

$$\int_\Omega p \, \mathrm{d}x = 0$$

is added to the system.

We shall now derive a variational formulation of (4.1). Let $\Omega \subset \mathbb{R}^n$, $n \leq 3$ be a bounded domain. We define the spaces

$$\begin{aligned} V &\stackrel{\text{def}}{=} [H_0^1(\Omega)]^n \\ W &\stackrel{\text{def}}{=} L_0^2(\Omega) \stackrel{\text{def}}{=} \left\{ q \in L^2(\Omega) : \int_\Omega q \, \mathrm{d}x = 0 \right\}. \end{aligned}$$

By multiplying the first equation with $v \in V$ and the second with $q \in W$, subsequent integration over Ω , and an application of Gauss' theorem, we obtain

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v \; dx - \int_\Omega \operatorname{div} v \, p \; dx &= \int_\Omega f \, v \; dx \\ \int_\Omega \operatorname{div} u \, q \; dx &= 0. \end{aligned}$$

We consequently define bilinear forms $a: V \times V \to \mathbb{R}$ and $b: V \times W \to \mathbb{R}$ via

$$a(u,v) \stackrel{\text{def}}{=} \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$$
$$b(v,w) \stackrel{\text{def}}{=} -\int_\Omega \operatorname{div} v \, w \, \mathrm{d}x.$$

This yields the *weak formulation* of (4.1):

Find $(u,p) \in V \times W$ *satisfying*

$$a(u,v) + b(v,p) = (f,v) \tag{4.2a}$$
$$b(u,q) \qquad\quad = 0 \tag{4.2b}$$

for all $(v,q) \in V \times W$. *By* $(\cdot,\cdot)$ *we denote the scalar product in* $L^2(\Omega)$.

4.1 Saddle-Point Problems

Systems of the form (4.2) are called *saddle-point problems*, since the solution (u,p) of (4.2) is also the minimizer of:

$$J(u) = \frac{1}{2}a(u,u) - (f,u) \to \min$$

subject to the constraint

$$b(u,q) = 0 \qquad \forall q \in W.$$

As usual for optimization problems with constraints, we can introduce an associated Lagrangian

$$L(u,q) = J(u) + b(u,q).$$

For each solution (u,p) of (4.2), we have

$$L(u,q) \le L(u,p) \le L(v,p) \quad \forall (v,q) \in V \times W.$$

That is, (u,p) is a saddle-point of the Lagrangian.

In this sense one may interpret the pressure $p \in W$ as a Lagrange multiplier associated to the constraint of incompressibility $\operatorname{div} u = 0$.

We shall now discuss the solution of (4.2) in a general framework. Let V, W be Hilbert spaces and $a: V \times V \to \mathbb{R}$, $b: V \times W \to \mathbb{R}$ be continuous bilinear forms. We define the space

$$Z = \{v \in V : \; b(v,w) = 0 \quad \forall w \in W\},$$

which is a closed subspace of V due to the continuity of b.

Moreover we introduce the maps $A: V \to V^*$ and $B: V \to W^*$

$$\langle Au, v\rangle = a(u,v), \quad \langle Bu, \mu\rangle = b(u,\mu)$$

as well as the dual operator $B^*: W \to V^*$

Example 4.1. *Let* $b(v,w) = -\int_\Omega \operatorname{div} v \cdot w \,\mathrm{d}x$ *. Then* $B = -\operatorname{div}$ *and* $B^* = \nabla$ *, since*

$$\langle Bu, w\rangle = -\int_\Omega \operatorname{div} v \cdot w \,\mathrm{d}x = \int_\Omega v \cdot \nabla w \,\mathrm{d}x = \langle v, B^* w\rangle$$

Hence, (4.2) is given in operator notation as

$$Au + B^* p = f \tag{4.3a}$$
$$Bu \qquad = 0 \tag{4.3b}$$

Well-posedness of this problem depends on the closedness of the image of B. For a further investigation one can use the *Closed Range Theorem* or the following result:

Lemma 4.2. *The following assertions are equivalent*

1. *There exists a* $\beta > 0$ *such that*

$$\inf_{w\in W} \sup_{v\in V} \frac{b(v,w)}{\|v\|\|w\|} \geq \beta \tag{4.4}$$

2. *The operator* $B:\ Z^\perp \to W^*$ *is an isomorphism, and*

$$\|Bv\| \geq c\|v\| \quad \forall v \in Z^\perp$$

Remark 4.3. *The condition* (4.4) *is usually called Ladyshenskaya-Babuska-Brezzi (LBB) condition or* inf-sup condition.

Theorem 4.4. *The saddle-point problem* (4.3) *has a unique solution* $(u,p) \in V \times W$ *, if*

1. *The bilinear form* a *is* V*-coercive, i.e., there exists* $\alpha > 0$ *such that*

$$a(u,u) \geq \alpha \|u\|_V^2 \quad \forall u \in Z$$

2. *The bilinear form* b *satisfies the inf-sup condition*

$$\inf_{w\in W} \sup_{v\in V} \frac{b(v,w)}{\|v\|\|w\|} \geq \beta$$

Note that V-coercivity means that the objective functional is strictly convex on the feasible set, which is clear in the case of the Stokes problem. In order to show the well-posedness of the Stokes problem it remains to verify the inf-sup condition (4.4), which turns out to be a non-trivial problem.

Theorem 4.5. *Let* $w \in L_0^2(\Omega)$ *. Then there exists* $\tilde{v} \in V$ *with* $-\operatorname{div} \tilde{v} = w$ *and a constant* $c > 0$ *such that* $\|\tilde{v}\|_V \leq c\|w\|_{L^2}$ *.*

As a consequence of the above theorem we have

$$\sup_{v\in V}\frac{b(v,w)}{\|v\|}\geq\frac{b(\tilde{v},w)}{\|\tilde{v}\|}\geq\frac{\|w\|^2}{\|\tilde{v}\|}\geq\frac{1}{c}\|w\|,$$

which implies the inf-sup condition.

Theorem 4.6. *The Stokes problem has a unique solution* $(u,p)\in V\times W$.

Proof: The coercivity of the bilinear form a ensures the uniqueness of a minimizer of

$$I(v)=\frac{1}{2}a(v,v)-f(v)\to\min!\ ,\qquad v\in Z=\{v\in V:\ b(v,w)=0\ \forall w\in W\}$$

The functional I is unbounded in radial direction, i.e., $(I(v)\to+\infty$ for $\|v\|\to\infty)$, and V is closed and convex, consequently it is weakly closed. For a minimizing sequenc $(v_n)\subset Z$ we have $\|v_n\|_V\leq M$ and hence, $v_n\to u\in Z$ (due to uniqueness of the limit, the whole sequence converges, not only a subsequence).

Because of $u\in Z$, $(u,p)\in V\times W$ solves the saddle-point problem if and only if

$$b(v,p)=f(v)-a(u,v)\quad\forall v\in Z^\perp$$

respectively

$$\langle Bv,p\rangle=f(v)-a(u,v)\quad\forall v\in Z^\perp$$

or

$$\langle v,B^*p\rangle=f(v)-a(u,v)\quad\forall v\in Z^\perp$$

which means

$$B^*p=f-Au\quad\text{in } Z^\perp$$

Since $B^*:\ W\to Z^\perp$ is an isomorphism, this equation has a unique solution $p\in W$.

The proof shows that the main difficulty is to determine the pressure. The inf-sup condition somehow guarantees that the pressure is controlled by the data.

4.2 Mixed Finite Elements

In the following we shall discuss a finite element discretization of the weak formulation of the saddle-point problem (4.2). For this sake we choose the finite element spaces $V_h\subset V$ and $W_h\subset W$. Then, the discrete variational equation is given by

Find $(u_h,p_h)\in V_h\times W_h$ *with*

$$\begin{aligned}a(u_h,v_h)+b(v_h,p_h)&=(f,v_h)\\ b(u_h,w_h)\qquad\quad&=0\end{aligned}$$

for all $(v_h, w_h) \in V_h \times W_h$.

This approach is called *mixed finite element method.*

As in the continuous case, we set

$$Z_h = \{v_h \in V_h : \ b(v_h, w_h) = 0 \ \forall w_h \in W_h\}.$$

Due to $V_h \subset V$ one has to expect $Z_h \not\subset Z$. Hence, the bilinear form a is not automatically Z_h-coercive. Nonetheless, such an approach is called *conforming* because of $V_h \subset V$. We also have to transfer the stability analysis using (4.4) to the discretized case.

Theorem 4.7. *Assume that:*

1. *The bilinear form* a *is* Z_h*-coercive, i.e., there exists* $\alpha_h > 0$ *such that*

$$a(u_h, u_h) \geq \alpha_h \|u_h\|^2, \quad \forall u_h \in Z_h.$$

2. *There exists a constant* $\beta_h > 0$ *such that*

$$\sup_{v_h \in V_h} \frac{b(v_h, w_h)}{\|v_h\|} \geq \beta_h \|w_h\| \quad \forall w_h \in Z_h^{\perp}$$

Then the solution of (4.5) *and* (4.2) *satisfies*

$$\|u - u_h\| + \|p - p_h\| \leq c \cdot \left(\inf_{v_h \in V_h} \|u - v_h\| + \inf_{w_h \in W_h} \|p - w_h\| \right)$$

Remark 4.8. *The finite element spaces* V_h *and* W_h *should not be chosen independently. In particular,* V_h *must contain enough freedom such that there are not too many restrictions on* W_h .

Example 4.9. *We discuss an example of a stable discretization of the Stokes problem. Let* Ω *be decomposed into a rectangular grid* T_h *of fineness* $h > 0$. *We choose*

$$V_h = \left\{ v_h \in [C^0(\bar{\Omega})]^2 : \ v_h\Big|_{\Omega_i} \in [Q_2(\bar{\Omega}_i)]^2, \ \Omega_i \in T_h \right\} \quad \textit{(bi-quadratic)}$$

$$W_h = \left\{ w_h \in W : \ w_h\Big|_{\Omega_i} \in P_0(\bar{\Omega}_i), \ \Omega_i \in T_h \right\} \quad \textit{(constant)}$$

Since $W_h \subset W$, *there exists* $\tilde{v} \in V$ *such that*

$$\operatorname{div} \tilde{v} = w_h \quad \textit{and} \quad \|\tilde{v}\| \leq c\|w_h\|$$

Moreover, there is a unique $\tilde{v}_h \in V_h$ *with*

$$a(\tilde{v}_h, v_h) = a(\tilde{v}, v_h) \quad \forall v_h \in V_h.$$

Using Céa's lemma we conclude

$$\|\tilde{v}_h\| \leq c\|\tilde{v}\| \Rightarrow \|\tilde{v}_h\| \leq c\|w_h\|$$

We define an interpolating element $\hat{v}_h \in V_h$ *via*

$$\begin{aligned} \hat{v}_h(p_i) &= \tilde{v}_h(p_i) \\ \int_{\Omega_j} \hat{v}_h \,\mathrm{d}x &= \int_{\Omega_j} \tilde{v} \,\mathrm{d}x \\ \int_{T_{jk}} \hat{v}_h \,\mathrm{d}s &= \int_{T_{jk}} \tilde{v} \,\mathrm{d}s \end{aligned}$$

Then we have

$$\|w_h\|^2 = \int_\Omega w_h \operatorname{div} \tilde{v} \,\mathrm{d}x = \sum_{j=1}^{N} \int_{\Omega_j} w_h \operatorname{div} \tilde{v} \,\mathrm{d}x$$

Since w_h *is constant on* Ω_j*, we may deduce with Gauß' theorem that*

$$\int_{\Omega_j} w_h \operatorname{div} \tilde{v} \,\mathrm{d}x = \int_{\Gamma_j} w_h \tilde{v} \cdot n_j \,\mathrm{d}s$$

From the interpolation condition we have

$$\begin{aligned} & \int_{\Gamma_j} w_h \tilde{v} \cdot n_j \,\mathrm{d}s = \int_{\Gamma_j} w_h \hat{v}_h \cdot n_j \,\mathrm{d}s \\ \Rightarrow \quad & \int_{\Omega_j} w_h \operatorname{div} \tilde{v} \,\mathrm{d}x = \int_{\Omega_j} w_h \operatorname{div} \hat{v}_h \,\mathrm{d}x \\ \Rightarrow \quad & \|w_h\|^2 = \int_\Omega w_h \operatorname{div} \hat{v}_h \,\mathrm{d}x \end{aligned}$$

Altogether, one obtains

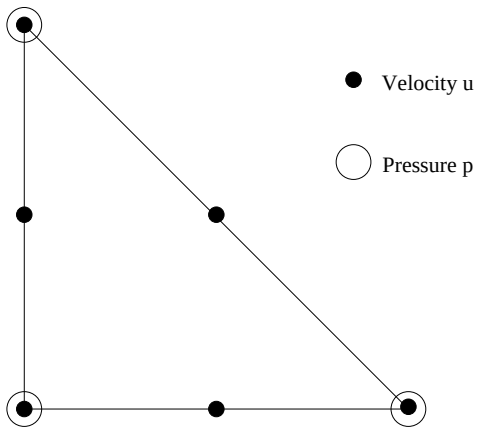
$$\sup_{v_h \in V_h} \frac{b(v_h, w_h)}{\|v_h\|} \geq \frac{b(\hat{v}_h, w_h)}{\|\hat{v}_h\|} \geq \frac{\|w_h\|^2}{\|\hat{v}_h\|} \geq c\|w_h\| \quad \forall w_h \in W_h$$

because $\|\hat{v}_h\| \leq c\|w_h\|$*. Hence, this discretization satisfies the inf-sup condition.*

Remark 4.10. *For triangularization one often uses Taylor-Hood-elements, which are based on additional nodes for the pressure in the triangle mid points. I.e., let* S_h *be the triangularization obtained by adding the midpoints of the triangles in* T *to the grid. Then, the pair*

$$\begin{aligned} V_h &= \left\{ v_h \in [C^0(\bar{\Omega})]^2 \cap [H_0^1(\Omega)]^2 : \ v_h\Big|_T \in P_2 \ \forall T \in S_h \right\} \\ W_h &= \left\{ w_h \in C^0(\Omega) \cap L_0^2(\Omega) : \ w_h\Big|_T \in P_1 \ \forall T \in T_h \right\} \end{aligned}$$

yields a stable discretization of the Stokes problem.



Another possibility of discretization is to incorporate the condition $\operatorname{div} u = 0$ into the finite element subspace, which yields non-conforming divergence-free P_1-elements and does not need to take pressure into account. Instead one could just minimize $\int |\nabla u|^2 \, \mathrm{d}x$ on this subspace. The construction of such elements however is tedious, and the associated variational equation is not easy to solve.

4.3 Schur-Complement

In the following we shall discuss an algorithm, which allows to solve the discretized saddle-point problem. Starting from

$$\begin{aligned} Au + B^* p &= f \\ Bu \qquad &= 0 \end{aligned}$$

we obtain a system of the form

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

where A is a regular matrix (corresponding to the discretization of the Laplace operator with Dirichlet boundary condition).

The linear system in this form is indefinite, i.e., it has positive as well as negative eigenvalues. Since A^{-1} exists, one can eliminate the velocity variable u and obtains the smaller linear system

$$BA^{-1}B^T p = BA^{-1} f \tag{4.6}$$

for the pressure p. Once the pressure is solved from this system we can compute the velocity vector as

$$u = A^{-1}(f - B^T p).$$

The matrix of the reduced system (4.6) is positive definite!

The only disadvantage is that $BA^{-1}B^T$ is given only in implicit form, since A^{-1} is a dense matrix in general, and the compuational effort for computing the inverse is too high for reasonable discretizations. Therefore, apppropriate iterative solution methods for (4.6) have to be constructed.

The basic form of the iteration for the pressure-Schur-complement equation is given by

1. Given p_{k-1}
2. Set $p_k = p_{k-1} - C^{-1}(BA^{-1}B^T p_{k-1} - BA^{-1}f)$

where C^{-1} is an appropriate preconditioner for $S = BA^{-1}B^T$. A simple variant of this iteration is the *Uzawa-algorithm*, where C^{-1} is a scaled version of the identity matrix:

1. Let p_0 be given.
2. For $k = 1, 2, \ldots$
   (a) Solve $Au_k = f - B^T p_{k-1}$
   (b) Set $p_k = p_{k-1} + \alpha B u_k$

If we investigate the defect

$$r_k = -Bu_k$$

and if (u, p) is solution of the saddle-point problem, then

$$\begin{aligned} r_k &= -BA^{-1}(f - B^T p_{k-1}) \\ &= BA^{-1}B^T(p_{k-1} - p) \end{aligned}$$

and hence,

$$p_k - p_{k-1} = -\alpha\, r_k = \alpha\, BA^{-1}B^T(p - p_{k-1}).$$

That is, the Uzawa-algorithm is equivalent to a gradient method with fixed time step α for the reduced problem for the pressure. In particular, the iteration converges if

$$\alpha < \frac{2}{\|BA^{-1}B^T\|}.$$

The Uzawa-algorithm is still computationally expensive, since each iteration a system with matrix A has to be solved. The number of total iterations can be reduced if the conjugate-gradient method is used instead of the gradient method. Another possibility is to choose the step size α adaptively

Remark 4.11. *Since the matrix of the saddle-point problem has condition number* $O(h^{-4})$ *appropriate preconditioning is important in order to ensure fast convergence of the cg-method*

4.4 Preconditioning

For large-scale problems, in particular in 3 d, the above strategies are very expensive, since they still enforce the exact solution of the Laplace equation. In order to reduce the computational effort one can use preconditioning.

A first possibility is the *inexact Uzawa* iteration, where one chooses a preconditioner $\hat{A}$ for A and a preconditioner $\hat{C}$ for the Schur complement $C = BA^{-1}B^T$ and then performs the following procedure:

1. Let p_0 be given.
2. For $k = 1, 2, \ldots$
   (a) Solve $\hat{A}u_k = f - B^T p_{k-1} - (A - \hat{A})u_{k-1}$
   (b) Solve $\hat{C}p_k = \hat{C}p_{k-1} + \alpha B u_k$.

As a Schur-complement preconditioner one often uses a multiple of the identity, as in the original version of the Uzawa algorithm. For preconditioning A one has several possibilities as for usual elliptic problems, one can use (block-)Jacobi, (block-)Gauss-Seidel, or multigrid methods.

A faster method is to use Krylov subspace methods for indefinite problems such as *generalized minimal residual* (GMRES) or *quasi-minimal residual* (QMR). For such an approach it is important to use an appropriate preconditioner for the saddle-point problem, for which one iteration of an inexact Uzawa method could serve.

4.5 A direct Stokes-Solver

Finally, we discuss a direct Stokes-solver, using a non-conforming approach. For this sake one needs a finite element basis in the subspace

$$X_h = \left\{ v_h \in \tilde{V}_h \times \tilde{V}_h : \ \operatorname{div} v_h = 0 \right\}$$

where

$$\tilde{V}_h = \left\{ v_h \in L^2(\Omega) : v_h\Big|_T \text{ linear } \forall T \in T_h, \ v_h \text{ continuous in } Q \in S_h \right\}.$$

Here, $S_h \stackrel{\text{def}}{=} \{Q_1, \ldots, Q_m\}$ is the set of edge centers of the triangle $T \in T_h$. This approach is a non-conforming finite element method, because

$$\tilde{V}_h \not\subset H^1(\Omega),$$

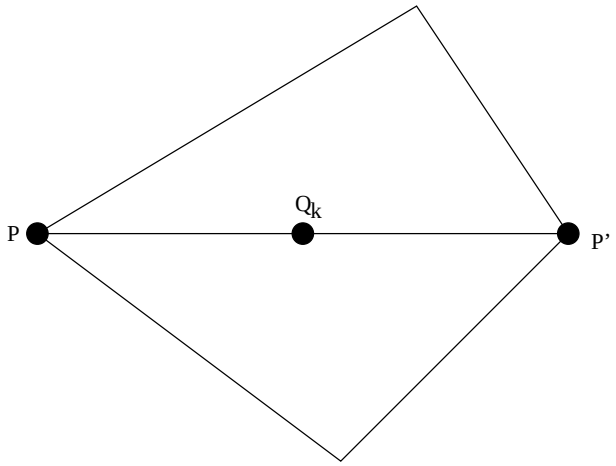
due to $v_h \notin C^0(\bar{\Omega})$. For functions $v_h \in \tilde{V}_h$ we can enforce $\operatorname{div} u_h$ in each triangle $T \in T_h$. We efine

$$(\operatorname{div} v_h)\Big|_T = \operatorname{div}(v_h\Big|_T) \; \forall v_h \in \tilde{V}_h, \; T \in \bar{S}_h.$$

Then, there is one degree of freedom left for the pressure in $T \in T_h$:

$$p_h \in Q_h = \left\{ q_h \in L^2(\Omega) : \; q_h\Big|_T \equiv \text{const.} \; \forall T \in T_h \right\}.$$

Now let $\tilde{w}_k \in \tilde{V}_h$ be the basis function associated with Q_k.

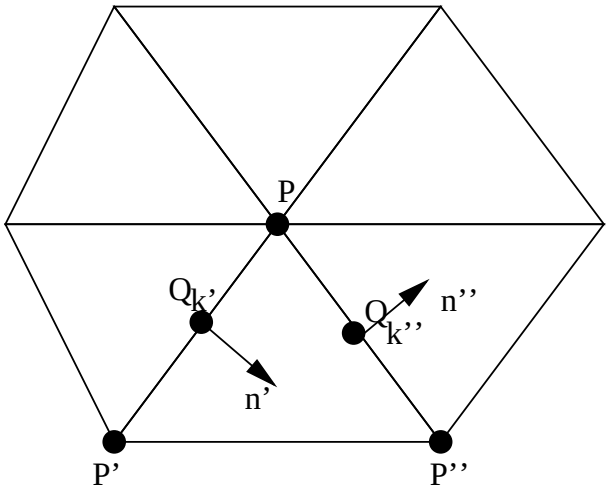


We set

$$\vec{w_k}(x) \stackrel{\text{def}}{=} \frac{P - P'}{|P - P'|} \tilde{w}_k(x)$$

Then $\vec{w}_k$ is constant along $P - P'$ and therefore $\operatorname{div}_h \vec{w}_k = 0$. The basis function is accordingly associated to each triangle edge (one speaks of "edge element" or "edge-based" discretizations, opposed to standard "nodal elements"). Moreover, we associate a basis function $\vec{w}_p$ associated to the node P via

$$\vec{w_p}(x) = + \frac{n'}{|P - P'|} w_{k'}(x) + \frac{n''}{|P - P''|} w_{k''}(x), \quad x \in T$$



Then the tangential component is vanishing for each edge, and

$$\vec{w}_p \cdot n' = \frac{1}{|P - P'|}, \quad \vec{w}_p \cdot n'' = \frac{1}{|P - P''|} \ .$$

One can show that these functions are divergence free and via a dimensional argument it is easy to see that $\vec{w}_k$ and $\vec{w}_p$ form a basis of X_h. Now let $X_h = \text{span}\,\{\vec{w}_i\}$. Then the discretized solution of the Stokes problem can be written as

$$u_h = \sum_i u^i \, \vec{w}_i.$$

Using this form in the saddle-point problem, one obtains the linear system

$$Au = \hat{f}$$

with $A_{ij} = (a(\vec{w}_i, \vec{w}_j))_{ij}, \quad \hat{f}_i = (f, \vec{w})_i$

The matrix A is symmetric positive definite, so that standard methods like cg can be used for its solution, as well as usual preconditioners.

Remark 4.12. *This approach has some advantages, in particular the solution is always divergence free, even if one uses iterative solvers. However, it also suffers from some disadvantages:*

1. *One arrives at few unknowns, but stiffness matrices with less zero entries.*
2. *The matrix A is ill-conditioned for fine discretizations, i.e.* $\text{cond}(A) = \mathcal{O}(h^{-4})$.
3. *The extension to 3d is complicated, but possible. However, the algorithm becomes very slow for large problems.*

4.6 Exercises

Exercise 4.1. *Let* $FF^T = BB^T$ *be a Cholesky decomposition of* BB^T *. Using* F *compute a triangular decomposition of*

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

- *if* $A = I$ *,*
- *if* $A = L^T L$ *.*

Exercise 4.2. *Let* A *be a symmetric positive definite matrix and* B *be a matrix of maximal rank. Show that*

$$\mathrm{cond}(BA^{-1}B^T) \leq \mathrm{cond}(A^{-1})\,\mathrm{cond}(BB^T).$$

Exercise 4.3. *Show that*

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ B & I \end{pmatrix} \begin{pmatrix} A{-}1 & 0 \\ 0 & -BA^{-1}B^T \end{pmatrix} \begin{pmatrix} A & B^T \\ 0 & I \end{pmatrix}.$$

What is the connection between this factorization and the reduced equation ?

Exercise 4.4. *Discretize the* $1\,d$ *Stokes problem using piecewise linear finite elements for the velocity* u *and piecewise linear elements for the pressure* p *. Does this discretization yield a stable system matrix ?*

Exercise 4.5. *Discretize the* $1\,d$ *Stokes problem using piecewise linear finite elements for the velocity* u *and piecewise constant elements for the pressure* p *. Does this discretization satisfy the LBB-condition ? Use the boundary conditions* $u(0) = 1$ *,* $u(0) = 0$ *, and the right-hand side* $f = \sin(2 * pi * x)$ *.*

Exercise 4.6. *Use the exact and inexact Uzawa iteration to solve the* $1\,d$ *Stokes problem. For the inexact Uzawa iteration, use a Jacobi and a Gauss-Seidel preconditioner for the Laplace operator. Compare the results.*

Exercise 4.7. *Use the cg iteration for the Schur complement to solve the* $1\,d$ *Stokes problem with the above discretization.*

Exercise 4.8. *Rewrite the* $2\,d$ *Stokes problem in polar coordinates.*

Exercise 4.9. *Consider a* $2\,d$ *flow between two rotating spheres with rotational symmetry. Implement a finite element discretization of this problem. Use* $R_1 = 0.3$ *and* $R_2 = 1$ *, the boundary condition* $u(R_1) = -0.5$ *,* $u(R_2) = 1$ *, and the right-hand side* $f = 0$ *. (Hint: Use the previous exercise and the rotational symmetry, which yields a one-dimensional problem in terms of the radius. The boundary conditions at the radius of the inner and outer sphere are given by the angular velocity of the spheres. Use the same discretization strategy as for* $1\,d$ *Stokes).*

5 Stationary Navier-Stokes Equations

In the following let $\Omega \subset \mathbb{R}^n$, $n \leq 3$ be a bounded domain. We are looking for a solution of the stationary incompressible Navier-Stokes equations

$$\left.\begin{aligned} \operatorname{div} u &= 0 \quad \text{in } \Omega \\ -\nu\Delta u + (u\cdot\nabla)u + \nabla p &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{aligned}\right\} \tag{NSG}$$

where $\Gamma = \partial\Omega$ denotes the boundary of the domain.

Definition 5.1. *A pair (u,p) is called classical solution of the Navier-Stokes problem, if $u \in C^2(\Omega)\cap C^0(\bar\Omega)$ and $p \in C^1(\Omega)\cap C^0(\bar\Omega)$ satisfy* (NSG) *in Ω.*

5.1 Existence and Uniqueness

Since classical solutions of the Navier-Stokes equations can be obtained under very restrictive assumptions only, we shall now derive a weak formulation.

For this sake we introduce the function spaces

$$V \overset{\text{def}}{=} \{u \in [H_0^1(\Omega)]^n : \operatorname{div} u = 0\}$$

and

$$W = L_0^2(\Omega).$$

We define the bilinear form $a:\ V\times V \to \mathbb{R}$,

$$a(u,v) \overset{\text{def}}{=} \nu\int_\Omega \nabla u\nabla v\,\mathrm{d}x = \nu\int_\Omega \sum_{i=1}^3 \nabla u_i \nabla v_i\,\mathrm{d}x$$

and the trilinear form $b:\ H^1(\Omega)\times H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$,

$$b(u,v,w) \overset{\text{def}}{=} \sum_{i,j=1}^n \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i\,\mathrm{d}x = \int_\Omega [(u\cdot\nabla)v]\cdot w\,\mathrm{d}x$$

By multiplying the second equation in (NSG) with $v \in V$ and an application of Gauss' theorem, we obtain

$$\nu\int_\Omega \nabla u\nabla v\,\mathrm{d}x - \underbrace{\int_{\partial\Omega}(\nabla u\cdot u)v\,\mathrm{d}s}_{=0} + \int_\Omega (u\cdot\nabla)u\cdot v\,\mathrm{d}x$$

$$- \underbrace{\int_\Omega p\,\operatorname{div} v\,\mathrm{d}x}_{=0} + \underbrace{\int_{\partial\Omega} pv\,\mathrm{d}s}_{=0} = \int_\Omega fv\,\mathrm{d}x$$

$$\Leftrightarrow \quad a(u,v) + b(u,u,v) = (f,v)$$

Lemma 5.2. *The trilinear form b is continuous.*

Proof: We have $u_i, v_i, w_i \in H^1(\Omega)$ and $H^1(\Omega) \hookrightarrow L^4(\Omega)$, hence $u_i, v_i, w_i \in L^4(\Omega)$. Thus, $u_j w_i \partial_j v_i \in L^1(\Omega)$. Consequently b is well-defined.
Moreover, we have

$$\begin{aligned}\left|\int_\Omega u_j \partial_j v_i w_i \,\mathrm{d}x\right| &\le \int_\Omega |u_j \partial_j v_i w_i| \,\mathrm{d}x \\ &\le \Big(\int_\Omega (u_j w_i)^2 \,\mathrm{d}x\Big)^{1/2} \Big(\int_\Omega \Big|\frac{\partial v_i}{\partial x_j}\Big|^2 \,\mathrm{d}x\Big)^{1/2} \\ &\le \Big(\int_\Omega u_j^4 \,\mathrm{d}x\Big)^{1/4} \Big(\int_\Omega w_i^4 \,\mathrm{d}x\Big)^{1/4} \Big(\int_\Omega \Big|\frac{\partial v_i}{\partial x_j}\Big|^2 \,\mathrm{d}x\Big)^{1/2} \\ &\le \|u_j\|_{L^4} \|w_i\|_{L^4} \|v_i\|_{H^1} \\ &\le c \|u_j\|_{H^1} \|w_i\|_{H^1} \|v_i\|_{H^1},\end{aligned}$$

due to the Cauchy-Schwarz inequality and continuous embedding. Summation over i, j yields continuity.

Lemma 5.3. *The trilinear form b has the following properties:*

1. $b(u, v, v) = 0 \quad \forall u \in V,\ \forall v \in H^1(\Omega)$
2. $b(u, v, w) = -b(u, w, v) \quad \forall u \in V,\ \forall v, w \in H^1(\Omega)$

Proof:

1.
$$\begin{aligned}b(u, v, v) = \sum_{i,j=1}^{u} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} v_i \,\mathrm{d}x &= \sum_{i,j=1}^{n} \int_\Omega u_j \frac{1}{2} \frac{\partial}{\partial x_j}(v_i^2) \,\mathrm{d}x \\ &= -\sum_{i,j} \frac{1}{2} \int_\Omega \frac{\partial u_j}{\partial x_j} v_i^2 \,\mathrm{d}x \\ &= -\sum_{i} \frac{1}{2} \int_\Omega v_i^2 \operatorname{div} u \,\mathrm{d}x \\ &= 0\end{aligned}$$

2. follows from 1., since
$$\begin{aligned}0 = b(u, v+w, v+w) &= \underbrace{b(u, v, v)}_{=0} + b(u, v, w) + b(u, w, v) + \underbrace{b(u, w, w)}_{=0} \\ &= b(u, v, w) + b(u, w, v)\end{aligned}$$

Definition 5.4. *Let $\nu > 0$ and $f \in L^2(\Omega)$. A function u is called weak solution of the Navier-Stokes problem* (NSG)*, if $u \in V$ and*

$$a(u,v) + b(u,u,v) = (f,v) \quad \forall v \in V$$

Theorem 5.5. *There exists a weak solution u of the Navier-Stokes problem and a constant $c > 0$ such that*

$$\|u\|_{H^1} \leq \frac{c}{\nu}\|f\|_{L^2} \ .$$

Proof: The proof is carried out using the Galerkin method: choose a sequence of subspaces $V_k \subset V$ with $\dim V_k = k$, such that $\bigcup_{k=1}^{\infty} V_k$ is dense in V. Let $V_k = \text{span}\,(w^1, \ldots, w^k)$. Consider the Galerkin equations

$$k \in \mathbb{N}: \qquad a(u^k, w^i) + b(u^k, u^k, w^i) = (f, w^i) \quad \forall i = 1, \ldots, k$$

Then the solution of this nonlinear algebraic system can be written as

$$u^k = \sum_{j=1}^{k} \xi_j^k w^j, \quad \xi_j^k \in \mathbb{R}$$

Hence, one has to solve a nonlinear system for the unknowns $\xi_1^k, \ldots, \xi_k^k$. One can show that this system has a solution. Moreover, the sequence (u^k) is bounded in V, since

$$a(u^k, v) + b(u^k, u^k, v) = (f, v) \quad \forall v \in V_k$$

and in particular, for $v = u_k$,

$$\nu\|\nabla u^k\|_{L^2}^2 + \underbrace{b(u^k, u^k, u^k)}_{=0} = (f, u^k) \leq \|f\|_{L^2}\|u^k\|_{L^2}$$

$$\Rightarrow \qquad \|\nabla u^k\|_{L^2} \leq \frac{c}{\nu}\|f\|_{L^2}$$

which implies together with the Poincaré inequality

$$\|u^k\|_{H^1} \leq \frac{c}{\nu}\|f\|_{L^2}$$

Hence, there exists a weakly convergent subsequence (u^k) such that

$$u^k \rightharpoonup u \quad \text{in } V \quad (k \to \infty)$$

and due to the compactness of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, we may conclude

$$u^k \to u \quad \text{in } L^2(\Omega) \quad (k \to \infty)$$

This implies further

$$a(u^k, w^i) \to a(u, w^i)$$

and

$$b(u^k, u^k, w^i) \to b(u, u, w^i)$$

Thus, one may pass to the limit in the equation and obtains

$$a(u, w^i) + b(u, u, w^i) = (f, w^i) \quad \forall i \in \mathbb{N}$$

and due to the density of the subspace

$$a(u, v) + b(u, u, v) = (f, v) \quad \forall v \in V,$$

i.e., $u \in V$ is a weak solution.

After ensuring uniqueness of a weak solution, the next obvious question is uniqueness, which is not easy and not completely investigated until now. There are several examples of nonuniqueness, if the data are large. For small data (relative to the viscosity $\Rightarrow$ small Reynolds number, small forces) uniqueness can be shown:

Theorem 5.6. *There exists a constant $c = c(\Omega) > 0$, such that the solution of the Navier-Stokes equation is unique, if*

$$\nu^2 \geq c\|f\|_{L^2}. \tag{5.1}$$

Proof: Let u_1, u_2 be two solutions, then

$$a(u_i, v) + b(u_i, u_i, v) = (f, v) \quad \forall v \in V, \quad i = 1, 2.$$

With $v = u_i$, we obtain

$$\|\nabla u_i\|_{L^2} \leq \frac{c_1}{\nu}\|f\|_{L^2}$$

Using $w = u_1 - u_2$ and subtracting the above equations yields

$$\begin{aligned} 0 &= a(u_1, v) + b(u_1, u_1, v) - a(u_2, v) - b(u_2, u_2, v) \\ &= a(w, v) + b(u_1, w, v) + b(w, u_2, v) \quad \forall v \in V \end{aligned}$$

In particular, we obtain for $v = w$

$$a(w, w) = -b(w, u_2, w)$$

and hence,

$$\begin{aligned} \nu\|\nabla w\|_{L^2}^2 = a(w, w) &\leq c_2\|w\|_{H^1}^2\|u_2\|_{H^1} \\ &\leq \frac{c_1 c_2}{\nu}\|f\|_{L^2}\|\nabla w\|_{L^2}^2 \\ \Rightarrow \qquad \|\nabla w\|_{L^2}^2(\nu - \frac{c}{\nu}\|f\|_{L^2}) &\leq 0. \end{aligned}$$

Therefore, (5.1) implies

$$\|w\|_{H^1} = 0 \quad \Leftrightarrow \quad u_1 = u_2.$$

Remark 5.7. *The following type of regularity can be obtained for weak solutions:*

$$\partial\Omega \in C^{k+2}, \ f \in W^{k,r}(\Omega), \ k \geq -1, \ r > 1, \ \Rightarrow \ u \in W^{k+2,r}(\Omega).$$

5.2 Nonlinear Iterative Methods

Since the incompressible Navier-Stokes equations are a nonlinear system of equations, one needs iterative methods for the solution in general. The literature on incompressible flows contains a variety of different methods, which are all based on a (different) linearization of the equations. We shall present and discuss some of the most frequently used methods in the following.

5.2.1 Oseen–Iteration

The Oseen–Iteration is a *secant modulus* method, where the equations are linearized by freezing the nonlinearity.

Algorithm 5.8 (Oseen–Iteration).

For $k = 0, 1, 2, \ldots$

Given $u_k \in V$ *, solve*

$$u^{k+1} \in V: \quad a(u^{k+1}, v) + b(u^k, u^{k+1}, v) = (f, v) \quad \forall v \in V.$$

Hence, the first term in the nonlinear part is evaluated at the last iterate. Consequently, one has to solve a linear equation in each step of the iteration, the so-called *Oseen equation*, which corresponds to the linear system

$$\operatorname{div} u = 0 \quad \text{in } \Omega \tag{5.2a}$$

$$-\nu\Delta u + (w \cdot \nabla)u + \nabla p = f \quad \text{in } \Omega \tag{5.2b}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{5.2c}$$

given w, or, in weak form

$$u \in V: \quad a(u, v) + b(w, u, v) = (f, v) \quad \forall v \in V.$$

Using the Lax–Milgram lemma one can show existence and uniqueness of a weak solution:

Theorem 5.9. *Let* $f \in L^2(\Omega)$ *and* $w \in V$*. Then,* (5.2) *has a unique weak solution* $u \in V$*.*

Note that the above result does not depend on the size of the data. However, if the data are larger the numerical solution of the Oseen equation is difficult (similar to the Navier–Stokes equations with large Reynolds number) and it is not guaranteed that the nonlinear iteration converges. The convergence of the Oseen–Iteration can be shown for small data using the Banach fixed point theorem:

Theorem 5.10. *If* $\nu^2 \geq c\,\|f\|_{L^2}$*, the iterates* u^k *obtained from the Oseen method converge to the unique solution* $u \in V$ *of the Navier–Stokes problem.*

Remark 5.11. *In general, one can only expect a linear rate of convergence of the Oseen iteration.*

5.2.2 Newton–Method

In order to obtain quadratic convergence, one can use the classical Newton method. For this sake one can discretize the system first and then apply a Newton iteration at the finite-dimensional level, or alternatively, one can formulate a Newton iteration for the system of partial differential equations and discretize subsequently. Since it is easier to compute derivatives at the functional level, we shall follow the latter approach. For this sake we write the Navier-Stokes problem in operator form as

$$u \in V : \quad Au + B(u) - F = 0$$

with

$$\begin{aligned} \langle Au, v\rangle &= a(u, v) \\ \langle B(u), v\rangle &= b(u, u, v) \\ \langle F, v\rangle &= (f, v) \end{aligned}$$

for all $v \in V$.

Now we define

$$G(u) \stackrel{\text{def}}{=} Au + B(u) - F$$

and compute the linearization of G in direction $\phi \in V$:

$$G'(u)[\phi] = A\phi + B'(u)[\phi]$$

where

$$\langle B'(u)[\phi], v\rangle \stackrel{\text{def}}{=} b(u, \phi, v) + b(\phi, u, v) \quad \forall v \in V.$$

The Newton iteration is then given by

Algorithm 5.12 (Newton–Method).

For $k = 0, 1, 2, \ldots$

1. *Given* $u_k \in V$, *solve*

$$\delta u_k \in V : \quad G'(u_k)[\delta u_k] = -G(u_k).$$

2. *Update* $u_{k+1} = u_k + \delta u_k$.

The assembly of the matrix corresponding to the discretization of $G'(u_k)$ is computationally expensive. Moreover, the system has the disadvantage of not being symmetric, so that suitable solvers have to be used.

Therefore, one often uses *Quasi–Newton method* instead, which need an approximation of $G'(u_k)$ only. One possibility is a so-called frozen Newton method, where the

matrix $G'(u_k)$ or an approximation is used for ℓ steps, i.e., for computing the iterates $u_{k+1}, \ldots, u_{k+\ell}$. Then the Newton matrix $G'(u_{k+\ell})$ will be computed again and used for the next ℓ iterations. Of course, using a Quasi–Newton method will result in a lower than quadratic, but still superlinear rate of convergence.

Another important aspect for Newton-type methods is the starting value, since one only obtains local convergence in general. One possibility is to use some steps of an Oseen–iteration first, and to use the result as a starting value for a Newton method. An alternative is to use globalization strategies based on the minimization of the residual.

5.2.3 Least–Squares Method

Let $u \in V$. We define $\xi = \xi(u) \in V$ via the defect

$$a(\xi, v) = a(u, v) + b(u, u, v) - (f, v) \quad \forall v \in V.$$

The bilinear form a defines an equivalent norm on $H_0^1(\Omega)$ via

$$\|\xi\|_a^2 \stackrel{\text{def}}{=} a(\xi, \xi)$$

and $u \in V$ is a solution of the Navier–Stokes equation iff it is a minimizer of

$$J(w) \stackrel{\text{def}}{=} \frac{1}{2} \|\xi(w)\|_a^2$$

i.e.,

$$J(u) = \min_{w \in V} J(w) = 0.$$

In a least–squares method, the computation of u is based on minimizing the functional J, e.g. using a cg–iteration. In this case, each iteration step enforces the solution of some linear Stokes problems.

5.2.4 Peaceman–Racheford Method

The Peaceman-Racheford method is an iteration based on alternating directions for the Navier–Stokes equations

$$\begin{aligned} -\nu \Delta u + (u \cdot \nabla) u + \nabla p &= f \quad \text{in } \Omega \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{aligned}$$

where the sequence of approximations (u_n, p_n) is constructed through the following algorithm:

Algorithm 5.13.

1. *Choose an initial value* (u_0, p_0) .
2. *For* $k = 0, 1, 2, \ldots$
   (a) *Solve*

$$\begin{aligned} -\nu\Delta u_{k+1/2} + r_k\, u_{k+1/2} + \nabla p_{k+1/2} &= f - (u_k \cdot \nabla)u_k + r_k\, u_k \\ \operatorname{div} u_{k+1/2} &= 0 \\ u_{k+1/2}|_\Gamma &= 0 \end{aligned}$$

   for $(u_{k+1/2}, p_{k+1/2})$.
   (b) *Use the result to solve*

$$\begin{aligned} -\nu\Delta u_{k+1} + (u_{k+1} \cdot \nabla)u_{k+1} + r_k\, u_{k+1} &= f - \nabla p_{k+1/2} + r_k\, u_{k+1/2} \\ u_{k+1}|_\Gamma &= 0 \end{aligned}$$

   for (u_{k+1}, p_{k+1}) .

with suitable parameters $r_k \in \mathbb{R}^+$.

Hence, the first step consists in solving a linear stokes-type equation, while the second step is the solution of a nonlinear partial differential equation of the form

$$-\nu\Delta y + (y \cdot \nabla)y + c\, y = f \tag{5.3a}$$

$$y|_\Gamma = 0 \tag{5.3b}$$

with $c \geq 0$, in absence of the incompressibility condition.

For the first step we can use the same methods as discussed above for the Stokes problem, but it still remains to solve the nonlinear problem (5.3). We investigate a least–squares method, using the notation $X = [H_0^1(\Omega)]^n$ and

$$A \stackrel{\text{def}}{=} -\nu\Delta + c\, I$$

$$G(y) \stackrel{\text{def}}{=} (y \cdot \nabla)y - f.$$

Sei X^* der Dualraum von X . Then we have $A \in \mathcal{L}(X, X^*)$ and

$$\langle Ax, y\rangle_{X^*,X} = \langle Ay, x\rangle_{X^*,X} \quad \forall x, y \in X$$

as well as

$$\exists \gamma > 0 : \quad \langle Ax, x\rangle_{X^*,X} \geq \gamma \, \|x\|_X^2 \quad \forall x \in X.$$

The nonlinear operator G has the following derivatives:

Lemma 5.14. *$G : X \to X^*$ is twice Fréchet–differentiable with*

$$G'(y)[u] = (y \cdot \nabla)u + (u \cdot \nabla)y,$$
$$G''(y)[u, v] = (v \cdot \nabla)u + (u \cdot \nabla)v.$$

A solution $x \in X$ of (5.3) is found from

$$0 = F(x) \stackrel{\text{def}}{=} Ax + G(x).$$

Since G is not symmetric, it is not a derivative of a nonlinear functional on X, and consequently this equation is not the optimality condition for some optimization problem. Therefore we rather consider the associated least–squares functional

$$J(x) = \frac{1}{2} \|F(x)\|_A^2$$

with

$$\|f\|_A \stackrel{\text{def}}{=} \sqrt{\langle f, A^{-1} f \rangle_{X^*,X}}.$$

Lemma 5.15. *The map $f \to \sqrt{\langle f, A^{-1} f \rangle_{X^*,X}}$ defines a norm on X^*, which is equivalent to the dual norm on X^*.*

Theorem 5.16. *Let $x \in X$ be a solution of $F(x) =$, with $F'(x) : X \to X^*$ being a homeomorphism. Then the functional*

$$J(x) = \frac{1}{2} \left\langle F(x), A^{-1} F(x) \right\rangle_{X^*,X}$$

is strictly convex in a neighbourhood of x.

Proof: We compute the first two variations of J using the symmetry of A:

$$J'(x)[v] = \left\langle F(x), A^{-1} F'(x)[v] \right\rangle_{X^*,X},$$
$$J''(x)[v, w] = \left\langle F'(x)[w], A^{-1} F'(x)[v] \right\rangle_{X^*,X} + \left\langle F(x), A^{-1} F''(x)[v, w] \right\rangle_{X^*,X}.$$

Let $F(x) = 0$ with $F'(x) : X \to X^*$ regular. Then

$$\begin{aligned} J''(x)[v, v] &= \left\langle F'(x)[v], A^{-1} F'(x)[v] \right\rangle_{X^*,X} \\ &\geq \delta \|F'(x)[v]\|_A^2 \\ &\geq \varepsilon \|v\|_X^2 \quad \forall v \in X. \end{aligned}$$

Since J is twice Fréchet–differentiable, there exists a neighbourhood $B(x)$ of x with

$$J''(y)[v, v] \geq \frac{\varepsilon}{2} \|v\|_X^2 \quad \forall (y, v) \in B(x) \times X.$$

Hence, J is strictly convex in $B(x)$.

We can now use a gradient method for the minimization of J. In order to compute the gradient, we use the linear functional $v \mapsto J'(y)v$ for $v \in X$ and compute its Riesz–representation in X. In X we use the scalar product

$$(x, y)_A \stackrel{\text{def}}{=} \langle Ax, y \rangle_{X^*,X} .$$

For the representing element $g(y)$ for $J'(y)$ in X we have

$$\begin{aligned}(g(y), v)_A &= J'(y)[v] \\ &= \big\langle F(y), A^{-1}F'(y)[v] \big\rangle_{X^*,X} \\ &= \big\langle F'(y)^* A^{-1} F(y), v \big\rangle_{X^*,X} \\ &= \big\langle A\, A^{-1} F'(y)^* A^{-1} F(y), v \big\rangle_{X^*,X} \\ &= (A^{-1}F'(y)^* A^{-1} F(y), v)_A\end{aligned}$$

and hence,

$$g(y) = A^{-1}F'(y)^* A^{-1} F(y).$$

Consequently, the gradient method is given by

Algorithm 5.17.

1. *Choose* $x_0 \in X$.
2. *For* $k = 0, 1, 2, \ldots$
   (a) $z_k = A^{-1}F'(x_k)$, $g_k = A^{-1}F'(x_k)^* z_k$
   (b) $\alpha_k = \operatorname{argmin}_{\alpha \geq 0} J(x_k - \alpha\, g_k)$
   (c) $x_{k+1} = x_k - \alpha_k\, g_k$

In each step of the iteration we have to solve two linear equations and obtain a step size solving a one-dimensional equation. Taking into account that $J(x_k - \alpha\, g_k)$ is a polynomial of order four in α, then one can determine the optimal α exactly solving a third order algebraic equation.

5.3 Discretization

Having discussed nonlinear iteration methods for the original system of partial differential equations, we shall now investigate the discretization of the Navier–Stokes equations, respectively of the subproblems arising in the nonlinear iteration schemes.

5.3.1 Existence of Discrete Solutions

Let $\Omega \subset \mathbb{R}^2$ be a polygonales Gebiet and $\mathcal{T}_h$ a regular triangularization. For the discretization of

$$u \in V: \quad a(u,v) + b(u,u,v) = (f,v) \quad \forall v \in V$$

one can use the non-conforming finite element subspaces

$$X_h \stackrel{\text{def}}{=} \left\{ v_h \in L^2(\Omega): \quad V_h|_T \in P_1 \; \forall T \in \mathcal{T}_h \text{ und } v_h \text{ stetig in den Seitenmittelpunkten} \right\}$$
$$V_h \stackrel{\text{def}}{=} \left\{ v_h \in X_h^2: \quad \operatorname{div}(v_h|_T) = 0 \text{ und } v_h(Q_j) = 0 \; \forall Q_j \in \partial\Omega \right\}$$

within the variational formulation

$$u_h \in V_h: \quad a_h(u_h, v_h) + b_h(u_h, u_h, v_h) = (f, v_h)_h \quad \forall v_h \in V_h, \tag{5.4}$$

where

$$a_h(u_h, v_h) \stackrel{\text{def}}{=} \sum_{T \in \mathcal{T}_h} \int_T \nabla u_h \cdot \nabla v_h \, dx$$
$$b_h(u_h, v_h, w_h) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_T \sum_{i,j=1}^{2} \left(u_{hj} \frac{\partial v_{hi}}{\partial x_j} w_{hi} - u_{hj} v_{hi} \frac{\partial w_{hi}}{\partial x_j} \right) dx$$
$$(f, v_h)_h \stackrel{\text{def}}{=} \int_\Omega f \, v_h \, dx.$$

Since the space $V_h \not\subset [H^1(\Omega)]^2$ is non-conforming, summation has to be carried out over single triangles.

Remark 5.18. *For $u, v \in [H^1(\Omega)]^2$ with $\operatorname{div} u = 0$ and $w \in [H_0^1(\Omega)]^2$ we have*

$$b(u,v,w) = b_h(u,v,w).$$

In this sense the trilinearform is approximated in a consistent way.

Remark 5.19. *For $v_h \in V_h$ gilt $(1, \operatorname{div} v_h)_h = \sum_{T \in \mathcal{T}_h} \int_T \operatorname{div} v_h \, dx = 0$.*

Theorem 5.20. *The discrete Navier–Stokes problem* (5.4) *has at least a solution $u_h \in V_h$ and there exists a corresponding pressure $p_h \in Q_h \stackrel{\text{def}}{=} \{q_h \in L^2(\Omega): \quad q_h|_T \in P_0 \; \forall T \in \mathcal{T}_h\}$ such that*

$$a_h(u_h, v_h) + b_h(u_h, u_h, v_h) = (f, v_h)_h + (p_h, \operatorname{div} v_h)_h \quad \forall v_h \in X_{h0}. \tag{5.5}$$

Remark 5.21. *In an analogous way one can study the existence of discrete solutions for conforming finite element subspaces.*

Existence of discrete solutions is a first fundamental properties, but obviously for a reasonable numerical methods we should obtain convergence towards the original solution, which is guaranteed by the following result:

Theorem 5.22. *Let*

1. Ω *be sufficiently smooth*
2. $\mathcal{T}_h$ *be a regular triangularization*
3. $f \in [L^2(\Omega)]^2$, $\nu > 0$
4. $(u_h, p_h) \in V_h \times Q_h|_{\mathbb{R}}$ *solves* (5.5) *for* $h \in (0, h_0)$

Then there exists a subsequence $h \to 0$ *and* $(u, p) \in V \times L^2(\Omega)|_{\mathbb{R}}$ *, such that*

$$\begin{aligned} u_h &\to u \quad \text{in } [L^2(\Omega)]^2 \\ \nabla u_h &\to \nabla u \quad \text{in } [L^2(\Omega)]^4 \\ p_h &\rightharpoonup p \quad \text{weakly in } L^2(\Omega) \end{aligned}$$

and (u, p) *is a weak solution of the Navier–Stokes problem.*

If $(u, p) \in [H^2(\Omega)]^2 \times H^1(\Omega)$ *then*

$$p_h \to p \quad \text{in } L^2(\Omega).$$

Unfortunately, this result is reflected in practice often only by convergence for very fine discretization size h , in particular for large Reynolds numbers the discrete solution does not need to be close to the original solution. One reason for this behaviour is the dominating convective term $(u \cdot \nabla)u$. We shall discuss this effect and its treatment for a simple model problem in the following.

5.4 Singularly Perturbed Equations

When we discretize e.g. the Oseen iteration, each step leads to a*lineare* equation of the form

$$\begin{aligned} -\varepsilon \Delta u + b \nabla u + c\, u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

For simplicity we only consider one component of the velocity u only and $c \geq 0$. For large Reynolds numbers we have $0 < \varepsilon \ll 1$ and $\|b\|_{L^\infty} \approx 1$. Such problems are called *singularly perturbed*, since the stabilizing elliptic part $-\Delta u$ is multiplied by the small parameter ε , and for $\varepsilon \to 0$ one obtains a non-elliptic first-order problem. As a consequence, one has to expect that $u_\varepsilon \not\to u_0$, where u_0 solves the so-called *reduced problem* arising for $\varepsilon = 0$.

Example 5.23. *Consider the equation*

$$-\varepsilon u'' - u' = 0, \quad u(0) = 0, \quad u(1) = 1$$

which has the exact solution

$$u_\varepsilon(x) = \frac{1 - e^{x/\varepsilon}}{1 - e^{1/\varepsilon}}.$$

A standard finite difference discretization yields

$$-\varepsilon D^+ D^- u_i - D^0 u_i = 0, \quad u_0 = 0, \quad u_N = 1$$

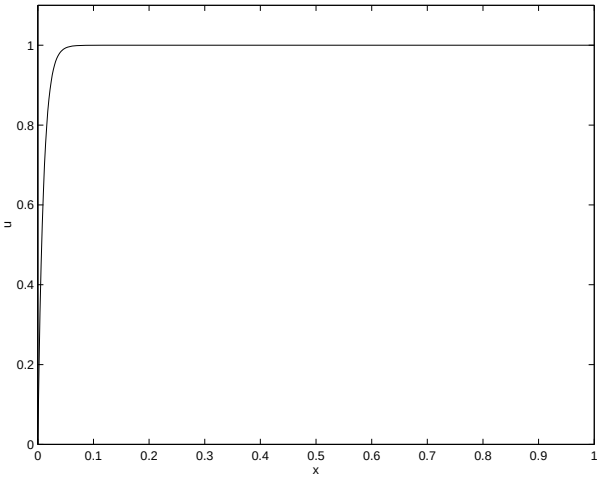
and

$$u_i = \frac{1 - r^i}{1 - r^N}, \qquad \text{with } r = \frac{2\varepsilon - h}{2\varepsilon + h}.$$

As a consequence on obtains a bound on the grid size $h < 2\varepsilon$, in order to obtain an M–matrix, which is needed for a stable discretization and in order to obtain a solution without artificial oscillations. For small ε, this is a unpleasant complication, since it leads to a very large problem. But even with $h = \varepsilon$ we have

$$\lim_{h \to 0} u_1 = \frac{2}{3}, \qquad \text{but} \qquad \lim_{h \to 0} u(x_1) = 1 - \frac{1}{e}.$$

The reason for this behaviour is not only the numerical algorithm, but the boundary layer *that appearsin the solution at $x = 0$.*



5.4.1 Upwind Method

In order to avoid a bound on the grid size, one has to replace the central difference quotient $b_i D^0 u_i$ by a one-sided difference operator. For such a choice we obviously

have to possibilities - forward or backward. The correct one (the upwind discretization) is somehow determined by the direction of the flow, which means in our model example by the sign of the coefficient b:

$$\begin{aligned} b_i D^+ u_i, &\quad \text{falls } b_i < 0 \\ b_i D^- u_i, &\quad \text{falls } b_i > 0 \end{aligned}$$

This yields the so-called *Upwind method:*

$$-\varepsilon D^+ D^- u_i + \max\{b_i, 0\} D^- u_i + \min\{b_i, 0\} D^+ u_i + c_i u_i = f_i, \quad u_0 = u_N = 0.$$

Assume for simplicity that $b > 0$. Then each line of the discretization matrix is of the form

$$\begin{pmatrix} 0 & \cdots & 0 & -\frac{\varepsilon}{h^2} - \frac{b_i}{h} & \frac{2\varepsilon}{h^2} + \frac{b_i}{h} + c_i & -\frac{\varepsilon}{h^2} & 0 & \cdots & 0. \end{pmatrix}$$

Consequently, we obtain an L–Matrix, which turns out to be even an M–Matrix, and one can show that:

Lemma 5.24. *The Upwind–method is uniformly stable in ε, i.e.*

$$\|u_h\|_\infty \leq M \|f_h\|_\infty$$

with $M > 0$ independent of h and ε.

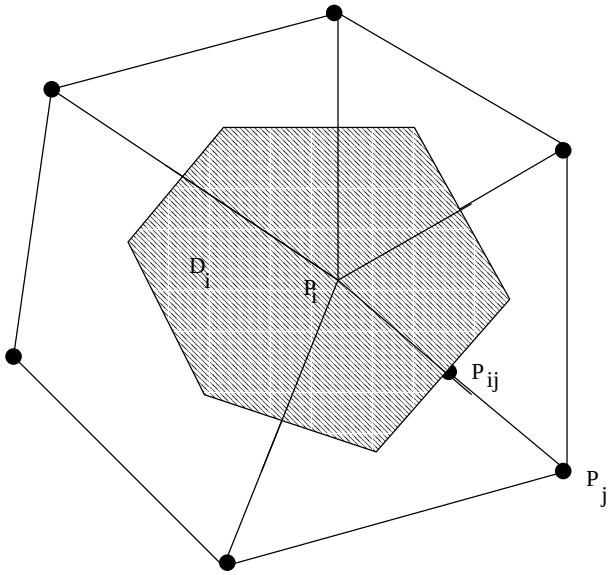
The same upwind approach can be applied in the two-dimensional case: In order to discretize a term of the form $b\nabla u = b^1 u_x + b^2 u_y$ on a grid with points x_i, y_j, we can use

$$(b\nabla u)_{ij} \approx \max\{b^1_{ij}, 0\} D^-_x u_{ij} + \min\{b^1_{ij}, 0\} D^+_x u_{ij} + \max\{b^2_{ij}, 0\} D^-_y u_{ij} + \min\{b^2_{ij}, 0\} D^+_y u_{ij}.$$

We shall transfer the idea to 2d finite elemente discretizations. We consider again

$$\begin{aligned} -\varepsilon \Delta u + b\nabla u + c\, u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with polygonal domain $\Omega \subset \mathbb{R}^2$ and $c - \frac{1}{2} \operatorname{div} b \geq 0$. Sei $\mathcal{T}_h$ being a regular triangularization of Ω. We generate a *secondary grid* as follows:



We first define

$$\begin{aligned}\Lambda_i &= \{j : \quad \exists T \in \mathcal{T}_h \text{ mit } P_i, P_j \in T\}\\ m_{ij} &= |\Gamma_{ij}|\\ l_{ij} &= |P_i P_j|\,.\end{aligned}$$

Discretizing $-\Delta u$ using piecewise linear finite elements, we obtain a finite difference scheme for the nodal values of the form

$$\sum_{j\in\Lambda_i} \frac{m_{ij}}{l_{ij}}(u(P_i) - u(P_j)),$$

which again yields an M–matrix.

Now we consider the discretization of the convective term $(b\cdot\nabla u_h, v_h)$. We have

$$(b\cdot\nabla u_h, v_h) = (\mathrm{div}(u_h b), v_h) - ((\mathrm{div}\, b)u_h, v_h).$$

and for the first term we obtain

$$\begin{aligned}(\mathrm{div}(u_h b), v_h) &= \sum_i \int_{D_i} \mathrm{div}(u_h b)\, v_h\, dx\\ &\approx \sum_i v_h(P_i) \int_{D_i} \mathrm{div}(u_h b)\, dx\end{aligned}$$

and Gauss' Theorem implies

$$\begin{aligned}&= \sum_i v_h(P_i) \int_{\partial D_i} (b\cdot\nu_i) u_h\, d\Gamma_i\\ &= \sum_i v_h(P_i) \sum_{j\in\Lambda_i} \int_{\Gamma_{ij}} (b\cdot\nu_{ij}) u_h d\Gamma_{ij}.\end{aligned}$$

We use an integration rule

$$\int_{\Gamma_{ij}} (b \cdot \nu_{ij}) u_h d\Gamma_{ij} = b(P_{ij}) \cdot \nu_{ij} \, |\Gamma_{ij}| \, (\lambda_{ij} u_h(P_i) + (1 - \lambda_{ij}) u_h(P_j))$$

with weights $\lambda_{ij} \geq 0$ to be determined. Then,

$$(\mathrm{div}(u_h b), v_h) \approx \sum_i v_h(P_i) \sum_{j \in \Lambda_i} b(P_{ij}) \cdot \nu_{ij} \, |\Gamma_{ij}| \, (\lambda_{ij} u_h(P_i) + (1 - \lambda_{ij}) u_h(P_j)).$$

In a similar way we can approximate the second term

$$\begin{aligned}
((\mathrm{div}\, b) u_h, v_h) &= \sum_i \int_{D_i} (\mathrm{div}\, b) u_h \, v_h \, dx \\
&\approx \sum_i u_h(P_i) v_h(P_j) \int_{D_i} \mathrm{div}\, b \, dx \\
&= \sum_i u_h(P_i) v_h(P_j) \sum_{j \in \Lambda_i} (b(P_{ij}) \cdot \nu_{ij}) m_{ij}.
\end{aligned}$$

Thus, the approximation of the convective term becomes

$$(b \cdot \nabla u_h, v_h) \approx \sum_i v_h(P_i) \sum_{j \in \Lambda_i} (b(P_{ij}) \cdot \nu_{ij}) m_{ij} ((\lambda_{ij} - 1) u_h(P_i) + (1 - \lambda_{ij}) u_h(P_j)),$$

which yields a discretization matrix B with entries

$$\begin{aligned}
B_{kk} &= \sum_{j \in \Lambda_k} (b(P_{kj}) \cdot \nu_{kj}) m_{kj} (\lambda_{kj} - 1), \\
B_{kl} &= (b(P_{kl}) \cdot \nu_{kl}) m_{kl} (1 - \lambda_{kl}), \qquad \text{for } l \in \Lambda_k, \\
B_{kl} &= 0, \qquad \text{sonst.}
\end{aligned}$$

Finally, the choice

$$\lambda_{kl} = \begin{cases} 1, & b(P_{kl}) \cdot \nu_{kl} \geq 0, \\ 0, & b(P_{kl}) \cdot \nu_{kl} < 0 \end{cases}$$

guarantees that B is an M–matrix.

The discretization of the remaining term $cu - f$ can be performed in a standard way using piecewise linear finite elements, one can also apply mass lumping.

The discrete problem can now be summarized as

$$u_h \in H_h : \quad a_h(u_h, v_h) = (f, v_h)_h \quad \forall v_h \in H_h.$$

If we use the special norm

$$\|v_h\|_\varepsilon \stackrel{\mathrm{def}}{=} \sqrt{\left(\varepsilon \, |v_h|^2_{H^1} + \|v_h\|^2_h\right)}$$

with

$$\|v_h\|_h^2 = \sum_i v_h^2(P_i)\, meas(D_i),$$

then:

Theorem 5.25.

- $\|u - u_h\|_\varepsilon \leq C\varepsilon^{-\frac{1}{2}}\, h\, (\|u\|_{H^2} + \|f\|_{W^{1,p}})$
- $\|u - u_h\|_\varepsilon \leq C\, h\, (\|u\|_{H^2} + \|f\|_{W^{1,p}})$, *if we have a regular triangularization on* Ω.

5.4.2 Stream-line Diffusion

The Upwind method discussed in the previous section cannot be applied to finite element subspaces using higher order polynomials, since the M–matrix property and the stability of the scheme will be lost.

An suitable alternative is the *stream-line diffusion method*, which achieves a stabilization by an appropriate choice of test functions. We start from the weak formulation

$$u_h \in V_h: \quad \varepsilon(\nabla u_h, \nabla w_h) + (b\nabla u_h + c\, u_h, w_h) = 0 \quad \forall w_h \in W_h.$$

The choice of test functions W_h needs not necessarily be equal for the space V_h, which is usually called a *Petrov–Galerkin method*.

The idea of streamline diffusion methods consists in choosing test functions of the form

$$w_h \stackrel{\text{def}}{=} v_h + \beta b\nabla v_h, \quad v_h \in V_h$$

with $\beta \in \mathbb{R}$. Note that for C^0–elements we have $v_h + \beta b\nabla v_h \notin H^1(\Omega)$, and hence one has to be careful with defining the associated bilinear form. We define

$$A(u, v) = \varepsilon \sum_T \int_T \nabla u \nabla v\, dx + \int_\Omega (b\nabla u + cu)v\, dx.$$

Then, A is well-defined since $v|_T$ is polynomial.

The weak formulation is obtained as

$$u_h \in V_h: \quad A(u_h, v_h + \beta b\nabla v_h) = (f, v_h + \beta b\nabla v_h) \quad \forall w_h \in V_h.$$

Hence, we modify the right-hand side and obtain the additional terms

$$(b\nabla u_h + cu_h, \beta b\nabla v_h)$$

and

$$\varepsilon \sum_T \int_T \nabla u_h \Delta v_h \, dx$$

on the leftt-hand side. The latter vanishes for example for piecewise linear elements. For $v_h = u_h$ we obtain an additional term $\beta \|b\nabla u_h\|_{L^2}^2$ in the coercivity estimate, i.e., we obtain artifical viscosity in direction of b.

We shall now investigate the streamline diffusion method using piecewise linear triangular elements, assuming

$$c \in \mathbb{R}^+, \quad \operatorname{div} b = 0.$$

We denote the error by $e = u - u_h$ and by $\eta = I_h u - u$ the difference between interpolation and exact solution. From the Galerkin orthogonality $A(uev_h) = 0$ for all $v_h \in V_h$ we obtain

$$A(e, e) = A(e, \eta).$$

Hence, after applying Young's inequality we obtain the estimate

$$ab \leq \frac{\alpha}{2} a^2 + \frac{1}{2\alpha} b^2, \quad \alpha > 0$$

and together with the standard estimate

$$\begin{aligned} \|\eta\|_{L^2} &\leq C\, h^2 \|u\|_{H^2} \\ \|\nabla\eta\|_{L^2} &\leq C\, h \|u\|_{H^2}, \end{aligned}$$

for the interpolation error one may conclude

$$\begin{aligned} \varepsilon \|\nabla e\|_{L^2}^2 + \beta \|b\nabla e\|_{L^2}^2 + c \|e\|_{L^2}^2 \\ \leq C \|u\|_{H^2} \left\{ \beta\varepsilon(\varepsilon + h) + h^2(\varepsilon + \beta + \beta^2) + (c + \beta)h^4 \right\} \end{aligned}$$

Choosing $\beta = \beta^* h$ we may conlude:

Theorem 5.26. *Let $\beta = \beta^* h$ and $\varepsilon < h$. Moreover, let $c \in \mathbb{R}^+$ and $\operatorname{div} b = 0$. Then the streamline diffusion method with piecewise linear finite elements allows the error estimate*

$$\begin{aligned} \|u - u_h\|_{L^2} &\leq C\, h^{3/2} \|u\|_{H^2} \\ \varepsilon^{1/2} \|u - u_h\|_{H^1} &\leq C\, h^{3/2} \|u\|_{H^2} \\ \|b\nabla(u - u_h)\|_{L^2} &\leq C\, h \|u\|_{H^2} \end{aligned}$$

Remark 5.27. *Note that all these estimates depend on ε, since $\|u\|_{H^2} \to \infty$ as $\varepsilon \to 0$.*

Compared to the Upwind method one obtains a significantly improved convergence order, in particular for the gradient in stream line direction. The reason is the non-homogeneous distribution of the artifical viscosity. Another advantage is the straightforward generalization to higher order finite elements. The disadvantage is the loss of the M–matrix property.

5.5 Exercises

Exercise 5.1. *Implement the Oseen-iteration for the stationary Navier-Stokes equation in the same setup as Exercise 4.5. Discretize the additional convective term using the Upwind method. Vary the viscosity ν and observe the convergence behaviour.*

Exercise 5.2. *Implement the Newton-iteration for the stationary Navier-Stokes equation in the setup of the previous exercise. Use a starting value obtained from few Oseen-iterations.*

Exercise 5.3. *Implement a least-squares / gradient method for the stationary Navier-Stokes equation in the setup of the previous exercise. Vary the viscosity ν and observe the convergence behaviour.*

6 Instationary Navier–Stokes Equations

After the analytical and numerical investigation of the incompressible Navier–Stokes equations in the stationary case, we shall now turn our attention to the instationary case, where we have to deal with additional time dependence. We shall first start with the basics in the analysis and then discuss different discretization strategies.

In the instationary case we are looking for a velocity field $u(x,t) : \bar{\Omega} \times [0,T] \to \mathbb{R}^N$ and a pressure $p(x,t) : \bar{\Omega} \times [0,T] \to \mathbb{R}$, determined by the initial-boundary value problem

$$\operatorname{div} u = 0 \quad \text{in } Q \stackrel{\text{def}}{=} \Omega \times (0,T) \tag{6.1a}$$

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f \quad \text{in } Q \tag{6.1b}$$

$$u = 0 \quad \text{on } \Sigma \stackrel{\text{def}}{=} \partial\Omega \times (0,T) \tag{6.1c}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega \tag{6.1d}$$

Definition 6.1. *A pair* (u,p) *is called* classical solution *of the instationary Navier–Stokes problem, if it satisfies* (6.1) *and* $u \in C^2(\bar{Q})$, $p \in C^1(\bar{Q})$.

As usual, the existence of classical solutions can be obtained only under restrictive assumptions. The study of classical solutions of the instationary Navier–Stokes equations and their regularity is still a somewhat open topic, it has been formulated as one of the "Mathematical Millennium Problems" by the Clay Math Institute in 2000.

We shall consider weak solutions in the following. For this sake we need some tools from functional analysis:

Definition 6.2. *Let* X *be a Banach space. Then the space* $L^p(0,T;X)$ *(shortly* $L^p(X)$*) is the class of all measurable (vector-valued) functions* $v : (0,T) \to X$, *such that*

$$\|v\|_{L^p(X)} \stackrel{\text{def}}{=} \left(\int_0^T \|v(t)\|_X^p \; dt \right)^{1/p}$$

is finite.

Remark 6.3. *For* $p = \infty$ *we define*

$$L^\infty(X) \stackrel{\text{def}}{=} \left\{ v : (0,T) \to X : \quad \sup_{t \in (0,T)} \|v(t)\|_X < +\infty \right\}.$$

Remark 6.4. $L^p(0,T;X)$ *is again a Banach space. If* X *is reflexive and separable, with dual space* X^*, *then* $L^q(0,T;X)^*$ *is the dual space of* $L^p(0,T;X)$ *for* $1 < p < \infty$, *where* $q = \frac{p}{p-1}$.

Definition 6.5. *A* Gelfand triple *is given by*

$$X \subset H \subset X^*$$

with

- X *being a separable, reflexive Banach space, with dual* X^*
- H *being a separable Hilbert space*
- X *being dense in* H

Example 6.6. $X := H_0^1(\Omega)$, $H := L^2(\Omega)$, $V^* := H^{-1}(\Omega) = (H_0^1(\Omega))^*$. *Then, obviously* $H_0^1(\Omega) \subset L^2(\Omega)$, *but* $L^2(\Omega) \subset H^{-1}(\Omega)$ *is not trivial, since*

$$g \in L^2(\Omega)$$

has to be identified with a linear functional

$$T_g(f) \stackrel{\text{def}}{=} \int_\Omega f\, g\, dx \quad \forall f \in H_0^1(\Omega)$$

on $H_0^1(\Omega)$.

Definition 6.7. *The function* $u \in L^2(0,T;X)$ *has the generalized time derivative* $u' \in L^2(0,T;X^*)$, *iff*

$$\int_0^T \phi'(t)\, u(t)\, dt = -\int_0^T \phi(t)\, u'(t)\, dt \quad \forall \phi \in C_0^\infty(0,T).$$

Definition 6.8. $W(0,T;X) \stackrel{\text{def}}{=} \{v \in L^2(0,T;X) : \quad v' \in L^2(0,T;X^*)\}$

6.1 Existence and Uniqueness

Let in the following

$$\begin{aligned} \mathcal{V} &\stackrel{\text{def}}{=} \{v \in C_0^\infty(\Omega) : \quad \operatorname{div} v = 0\}, \\ V &\stackrel{\text{def}}{=} \overline{\mathcal{V}}^{H_0^1(\Omega)}, \\ H &\stackrel{\text{def}}{=} \overline{\mathcal{V}}^{L^2(\Omega)}. \end{aligned}$$

Lemma 6.9. *Let* $H^\perp \stackrel{\text{def}}{=} \{u \in L^2(\Omega) : \quad (u,v) = 0 \quad \forall v \in H\}$. *Then,*

$$H^\perp = \left\{u \in L^2(\Omega) : \quad u = \nabla p, \quad p \in H^1(\Omega)\right\}.$$

Definition 6.10. *Let* $\nu, T > 0$, $f \in L^2(0,T;V^*)$, $u_0 \in H$. *Then* u *is called* weak solution *of the instationary Navier–Stokes problem* (6.1)*, iff*

1. $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$
2. $$\frac{d}{dt}(u,v) + a(u,v) + b(u,u,v) = \langle f, v\rangle \quad \forall v \in V \tag{6.2}$$
3. $u(0) = u_0$

The equation (6.2) has to be understood in the sense of scalar distributions, i.e.,

$$-\int_0^T (u(t),v)\phi'(t)\,dt + \int_0^T \left[a(u(t),v) + b(u(t),u(t),v)\right]\phi(t)\,dt = \int_0^T \langle f(t),v\rangle\,\phi(t)\,dx \quad \forall \phi \in C_0^\infty(0,T).$$

Remark 6.11. *The condition* $u \in L^\infty(0,T;H)$ *seems to be restrictive, but (3) would not be well-defined for* $u \in L^2(0,T;H)$ *only.*

The initial condition is satisfied in the following sense:

Lemma 6.12. *If* $u \in L^2(0,T;V)$ *satisfies the equation* (6.2)*, then* $u \in C^0([0,T];V^*)$.

The regularity for the time derivative depends on the spatial dimension, which is one of the reasons for the difficulties in the analysis of the system.

Theorem 6.13. *Let* u *be a weak solution of* (6.1)*. Then,*

1. $u' \in L^1(0,T;V^*)$ *for* $N \in \mathbb{N}$
2. $u' \in L^2(0,T;V^*)$ *for* $N = 2$
3. $u' \in L^{4/3}(0,T;V^*)$ *for* $N = 3$

With these preliminaries we can now state the main existence result:

Theorem 6.14. *Let* $\nu, T > 0$, $f \in L^2(0,T;V^*)$, *and* $u_0 \in H$. *Then there exists a weak solution of* (6.1).

The proof of this result would exceed the time frame of this course, we just comment on the basic steps:

- First we apply an implicit semi-discretization in time (e.g. implicit Euler), which yields a sequence of nonlinear stationary problems (Rothe method).

- Since the stationary problems are similar to the stationary Navier–Stokes problem, only including an additional stabilization term from the time derivative, one can conclude existence of a solution for each stationary problem by analogous reasoning as for the stationary Navier–Stokes equations.
- The next step is to derive appropriate a-priori estimates independent of the time step, which turns out to be the most difficult part.
- Finally, one can use compactness arguments to extract a subsequence converging to the solution of the instationary problem as the time step tends to zero.

To verify uniqueness of the solution is even more complicated, the results also depend on the spatial dimension.

Theorem 6.15. *Let* $N = 2$. *Then there exists a unique weak solution of* (6.1).

The basic ingredient of the proof is to use

$$\frac{d}{dt}(u(t), u(t)) = 2\,\langle u'(t), u(t)\rangle$$

and to apply Gronwall's lemma.

In three spatial dimensions, no uniqueness result is available so far in the class of functions we use for weak solutions. Uniqueness can be obtained by considering classes of solutions of higher regularity. On the other hand it is not clear if solutions of such higher regularity exist at all.

Theorem 6.16. *Let* $N = 3$. *Then there exists a unique weak solution of* (6.1) *with*

$$u \in L^8\left(0, T; \left[L^4(\Omega)\right]^3\right).$$

The question of uniqueness and regularity of solutions of the instationary Navier–Stokes equations are fundamental from a physical viewpoint, too, since they have a strong relation to the appearance of turbulence phenomena.

6.2 Time Discretizations

Approximations of the instationary Navier–Stokes problem are usually based on a semidiscretization in time, followed by a spatial discretization at each time step, e.g. using finite elements.

In the following we present some possibilities for the time discretization. We shall always consider a time discretization into the times $t_k = k\tau$, for a positive time step τ. The solution at step k is abbreviated by $(u^k, p^k) = (u(t_k), p(t_k))$.

6.2.1 Explicit Euler Method

The explicit Euler method is probably the simplest time discretization using a forward differencing for the time derivative. This would yield the explicit formula

$$\frac{u^{k+1} - u^k}{\tau} = -(u^k \cdot \nabla)u^k + \nu\Delta u^k - \nabla p^k + f^k$$

for the velocity u^{k+1}, given u^k and $f^k(x) = f(x, k\tau)$. This method has several disadvantages, in particular for incompressible flows. First of all, any explicit method for a second order problem enforces $\tau \leq Ch^2$ for some constant C which results in a very high number of time steps. The second disadvantage, which is even more serious, is the fact that the incompressibility constraint cannot be satisfied exactly and the error in this equation will increase linearly with the number of time steps. Therefore it isusually not a good idea to use an explicit scheme for incompressible flows.

The explicit time discretization may make sense if one uses a discrete subspace that exactly incorporates the incompressibility condition and if the viscosity ν is small. In this case the restriction of the time step (νh^2) is not too small, and the solution of a stationary problem in an implicit method is expensive.

6.2.2 Implicit Euler Method

The implicit Euler method is a straight-forward time discretization based on backward time differencing, which yields the nonlinear stationary problem

$$\frac{u^{k+1} - u^k}{\tau} + (u^{k+1} \cdot \nabla)u^{k+1} - \nu\Delta u^{k+1} + \nabla p^{k+1} = f^{k+1}$$
$$\operatorname{div} u^{k+1} = 0,$$

with $f^{k+1}(x) = f(x, (k+1)\tau)$.

Thus, in each time step we have to solve a modified *stationary* Navier–Stokes problem. For this sake one can use e.g. the Newton iteration or the Oseen iteration. If the time step is not too large the solution u^k at the previous time step should provide a good starting value for the nonlinear iteration, which consequently should converge fast. As for ordinary differential equations, one can only expect first order convergence in time, i.e., the error between the solution of the instationary Navier–Stokes problem and the solution of the semi-discrete problem is of order τ.

An advantage of the implicit Euler method is the unconditional stability, which can be seen by using the weak formulation of the time discrete problem with test function u^{k+1}, implying

$$\langle u^{k+1}, u^{k+1}\rangle + \tau\nu\langle\nabla u^{k+1}, \nabla u^{k+1}\rangle = \langle u^k, u^{k+1}\rangle + \tau\langle f^{k+1}, u^{k+1}\rangle.$$

Using the Cauchy-Schwarz inequality we can estimate

$$\langle u^k, u^{k+1}\rangle \le \left\|u^k\right\| \left\|u^{k+1}\right\| \le \frac{1}{2}\left\|u^k\right\|^2 + \frac{1}{2}\left\|u^{k+1}\right\|^2$$

and

$$\langle f^{k+1}, u^{k+1}\rangle \le \left\|\nabla\Delta^{-1} f^{k+1}\right\| \left\|\nabla u^{k+1}\right\| \le \frac{1}{2\nu}\left\|\nabla\Delta^{-1} f^{k+1}\right\|^2 + \frac{\nu}{2}\left\|\nabla u^{k+1}\right\|^2.$$

Thus,

$$\frac{1}{2}\left\|u^{k+1}\right\|^2 + \frac{\tau\nu}{2}\left\|\nabla u^{k+1}\right\|^2 \le \frac{1}{2}\left\|u^k\right\|^2 + \frac{\tau}{2\nu}\left\|\nabla\Delta^{-1} f^{k+1}\right\|^2.$$

Summing up over time, we obtain

$$\frac{1}{2}\left\|u^k\right\|^2 + \frac{\tau\nu}{2}\sum_{j=0}^{k}\left\|\nabla u^j\right\|^2 \le \frac{1}{2}\left\|u_0\right\|^2 + \sum_{j=0}^{k}\frac{\tau}{2\nu}\left\|\nabla\Delta^{-1} f^j\right\|^2.$$

which is a time-discrete version of

$$\|u(t)\|^2 + \frac{\nu}{2}\int_0^t \|\nabla u(s)\|^2 \; ds \le \|u_0\|^2 + \frac{1}{2\nu}\int_0^t \left\|\nabla\Delta^{-1} f(s)\right\|^2 \; ds.$$

In particular, the norm of the velocity remains bounded.

In order to investigate the order of convergence with respect to the time step τ, we consider the error $v^k := u^k - u(t_k)$ and $q^k = p^k - p(t_k)$, where (u,p) is the solution of the full instationary Navier–Stokes equations. Then, these variables satisfy

$$\frac{v^{k+1}-v^k}{\tau} + (u^{k+1}\cdot\nabla)v^{k+1} + (v^{k+1}\cdot\nabla)u(t_{k+1}) - \nu\Delta v^{k+1} + \nabla q^{k+1} = R_k$$
$$\operatorname{div} v^{k+1} = 0,$$

with

$$R_k = \frac{\partial u}{\partial t}(t_{k+1}) - \frac{u(t_{k+1})-u(t_k)}{\tau}.$$

Multiplying the first equation by v^{k+1} and integration over Ω as usual we obtain

$$\frac{1}{\tau}\langle v^{k+1}-v^k, v^{k+1}\rangle + b(u(t_{k+1}), v^{k+1}, v^{k+1}) + b(v^{k+1}, u(t_{k+1}), v^{k+1}) + \nu\langle\nabla v^{k+1}, \nabla v^{k+1}\rangle = \langle R_k, v^{k+1}\rangle$$

Due to the usual properties of the trilinear form in the Navier–Stokes problem we obtain

$$\langle (u^{k+1}\cdot\nabla)v^{k+1}, v^{k+1}\rangle = 0.$$

If u is sufficiently regular we may find a constant $c>0$ such that $\|\nabla u\|_\infty < c$ and $\|R_k\| \le c\tau$. Consequently, we have

$$b(v^{k+1}, u(t_{k+1}), v^{k+1}) = \langle v^{k+1}\cdot\nabla u(t_{k+1}), v^{k+1}\rangle \ge -c\left\|v^{k+1}\right\|^2$$

and

$$\langle R_k, v^{k+1} \rangle \leq c\tau \left\| v^{k+1} \right\| \leq \frac{c}{2}\tau^2 + \frac{c}{2} \left\| v^{k+1} \right\|^2 .$$

The first term can be estimated using the Cauchy-Schwarz inequality, which implies

$$\frac{1}{\tau}\langle v^{k+1} - v^k, v^{k+1} \rangle \geq \frac{1}{2}(\left\| v^{k+1} \right\|^2 - \left\| v^k \right\|^2).$$

Altogether, this implies the estimate

$$(1 - 3c\tau) \left\| v^{k+1} \right\|^2 + 2\tau\nu \left\| \nabla v^{k+1} \right\|^2 \leq 2c\tau^3 + \left\| v^k \right\|^2 .$$

Now let $q = \frac{1}{1-3c\tau}$, which is positive if τ is sufficiently small, and define $a_k := q^{-k} \left\| v^{k+1} \right\|^2$. Then,

$$a_{k+1} \leq a_k + 2c\tau^3 q^{-(k+1)}.$$

By summation we finally obtain

$$\left\| v^{k+1} \right\|^2 \leq q^{k+1} \frac{1 - q^{-k-1}}{1 - q} 2c\tau^3 = \frac{2}{3}c\tau^2(q^{k+1} - 1) \leq \frac{2}{3}c\tau^2(q^N - 1),$$

where N is the total number of time steps, i.e., $T = N\tau$. From a standard inequality, we obtain

$$q^N = \left(1 + \frac{3cTq}{N}\right)^N \leq e^{3cTq},$$

and thus,

$$\left\| v^{k+1} \right\|^2 \leq \frac{2}{3}c\tau^2(e^{3cTq} - 1).$$

For $\tau \to 0$, we have $q \to 1$ and $e^{3cTq} \to e^{3cT}$ and thus,

$$\left\| v^{k+1} \right\| = \mathcal{O}(\tau).$$

6.2.3 Crank–Nicholson Method

The Crank–Nicholson method is a *semi-implicit method* based on taking the "average" of the explicit and the implicit Euler method in the parabolic equation, while enforcing the incompressibility condition exactly:

$$\frac{u^{k+1} - u^k}{\tau} + (u^{k+1/2} \cdot \nabla)u^{k+1/2} - \nu\Delta u^{k+1/2} + \nabla p^{k+1/2} = f^{k+1/2}$$
$$\operatorname{div} u^{k+1} = 0,$$

with

$$u^{k+1/2} = \frac{1}{2}(u^{k+1} + u^k), \quad u^{k+1/2} = \frac{1}{2}(p^{k+1} + p^k), \quad f^{k+1/2} = \frac{1}{2}(f^{k+1} + f^k).$$

Again, each step enforces the solution of a modified stationary Navier–Stokes problem, and therefore needs the same computational effort as the implicit Euler method. Why should one then use Crank–Nicholson ? The main reason is the higher order of convergence, the Crank-Nicholson method achieves second order in the time step.

6.2.4 Semi-implicit Oseen Method

In order to reduce the computational effort at each time step, it seems reasonable to replace the nonlinear stationary problems by linear ones in a similar way as in the Oseen iteration, which yields a modified Oseen equation at each step:

$$\frac{u^{k+1} - u^k}{\tau} + (u^k \cdot \nabla)u^{k+1} - \nu\Delta u^{k+1} + \nabla p^{k+1} = f^{k+1}$$
$$\operatorname{div} u^{k+1} = 0.$$

6.2.5 Semi-implicit Oseen–Crank–Nicolson Method

In an analogous way as above we can obtain a successive linearization of the Crank-Nicholson method, which leads to

$$\frac{u^{k+1} - u^k}{\tau} + (u^k \cdot \nabla)u^{k+1/2} - \nu\Delta u^{k+1/2} + \nabla p^{k+1/2} = f^{k+1/2}$$
$$\operatorname{div} u^{k+1} = 0.$$

6.2.6 Analysis of Time Discretizations

In general, the analysis of time discretizations is guided by the questions of convergence and stability. We start by considering the simple linear test equation

$$x' = -\lambda x, \quad t \geq 0,$$

with $\operatorname{Re} \lambda > 0$. For fixed time step τ the behaviour of the method for large t depends on the factor $\omega = \omega(\lambda\tau)$. In particular, the one step methods we have discussed so far yield

$$x_k = \omega^k x_0.$$

A look at the theory of ordinary differential equations shows that the following properties should be maintained by each scheme:

- *Local Stability:* $|\omega(\lambda\tau)| \leq 1$
- *Global Regularity:* $\lim_{\mathrm{Re}\ \lambda\to\infty} |\omega(\lambda\tau)| \leq 1 - \mathcal{O}(\tau)$
- *Strong A–Stability:* $lim_{\mathrm{Re}\ \lambda\to\infty} |\omega(\lambda\tau)| \leq 1 - \delta < 1$
- *Weak Dissipation:* $|\omega(\lambda\tau)| \approx 1$ for $\mathrm{Re}\ \lambda = 0$

For classical one-step-θ-methods, i.e., discretizations of the form

$$x_{k+1} - x_k = -\lambda\tau(\theta x_{k+1} + (1-\theta)x_k)$$

the factor ω satisfies

$$\omega(z) = \frac{1-(1-\theta)z}{1+\theta z}.$$

Example 6.17.

1. *Explicit Euler* $(\theta = 0)$*: Conditional stability only for* $\tau < 1/\lambda$
2. *Implicit Euler* $(\theta = 1)$*: Strongly A-stable (*$|\omega(z)| \to 0$ *for* $\mathrm{Re}\ \lambda \to \infty$*), but dissipative (*$|\omega(i\tau)| < 1$*, e.g.* $|\omega| = 0.995$ *for* $\tau = 0.1$*)*
3. *Crank–Nicolson Method* $(\theta = 1/2)$*: only A–stable* $|\omega(z)| \to 1$ *for* $\mathrm{Re}\ \lambda \to \infty$*), but not dissipative (*$|\omega(i\tau)| = 1$*)*

6.2.7 Fractional–Step–θ–Method

With the exception of the explicit Euler method, the methods discussed above are (semi–) implicit schemes. For schemes of higher order of convergence, like the Runge–Kutta method, it is not clear if the regularity for the nonlinear Navier–Stokes problem is sufficient to guarantee a high order of convergence. The Crank–Nicholson method achieves a second order convergence, but instabilities can arise since the method is only A–stable.

We shall therefore consider a different class of method, which is of second order and strongly A–stable, with few dissiplation: the so-called *fractional–step–θ–methods.* The idea of such schemes is an operator splitting, which has a *macro time step* $\tau = t_{k+1} - t_k$ that is decomposed into three *micro time steps* s_i via $\tau = s_1 + s_2 + s_3$. The advantage of this approach is that different algorithms can be used for each micro time step. One possibility is presented in the following scheme:

Let

$$\theta = 1 - \frac{\sqrt{2}}{2}, \quad \theta' = 1 - 2\theta$$
$$\alpha = \frac{1-2\theta}{1-\theta}, \quad \beta = 1 - \alpha$$
$$\tilde{\theta} = \alpha\theta\tau - \beta\theta'\tau.$$

A macro time step is then composed by the following micro time steps

1.
$$\frac{u^{k+\theta}-u^k}{\theta\tau}+\alpha(u^{k+\theta}\cdot\nabla)u^{k+\theta}-\alpha\nu\Delta u^{k+\theta}+\nabla p^{k+\theta}=f^{k+\theta}+\beta\nu\Delta u^k-\beta(u^k\cdot\nabla)u^k$$
$$\operatorname{div} u^{k+\theta}=0,$$

2.
$$\frac{u^{k+1-\theta}-u^{k+\theta}}{\theta'\tau}+\beta(u^{k+1-\theta}\cdot\nabla)u^{k+1-\theta}-\beta\nu\Delta u^{k+1-\theta}+\nabla p^{k+1-\theta}=$$
$$=f^{k+1-\theta}+\alpha\nu\Delta u^{k+\theta}-\alpha(u^{k+\theta}\cdot\nabla)u^{k+\theta}$$
$$\operatorname{div} u^{k+1-\theta}=0,$$

3.
$$\frac{u^{k+1}-u^{k+1-\theta}}{\theta\tau}+\alpha(u^{k+1}\cdot\nabla)u^{k+1}-\alpha\nu\Delta u^{k+1}+\nabla p^{k+1}=$$
$$=f^{k+1}+\beta\nu\Delta u^{k+1-\theta}-\beta(u^{k+1-\theta}\cdot\nabla)u^{k+1-\theta}$$
$$\operatorname{div} u^{k+1}=0.$$

Lemma 6.18. *The above fractional–step–θ –method satisfies*

$$\omega(z)=\frac{(1-\beta\theta z)^2(1-\alpha\theta' z)}{1+\alpha\theta z)^2(1+\beta\theta' z)}$$

and using the parameters given above we have:

- *Strong A–stability (*$\lim_{\mathrm{Re}\ \lambda\to\infty}|\omega(z)|=\frac{\beta}{\alpha}\approx 0.7$ *)*
- *Second Order Convergence*
- *Good Smoothing Property (*$|\omega(i\tau)|\approx 0.9998$ *for* $\tau=0.8$ *).*

6.2.8 Projection Methods

Another class of time discrete schemes are so-called *projection methods*, which have the following form in general:

Algorithm 6.19.

1. *Choose* u^0, p^0 .

2. *For* $k = 0, 1, 2, \ldots$

   (a) *Solve*
   $$u^{k+1/2} + N(u^{k+1/2}) = u^k + \tau \left[M(u^k) + f^k - \nabla p_{old} \right].$$

   (b) *Solve*
   $$\begin{aligned} \Delta q^{k+1/2} &= \frac{1}{\tau} \operatorname{div} u^{k+1/2} \qquad in\ \Omega \\ \nabla q^{k+1/2} \cdot n &= \frac{1}{\tau} u^{k+1/2} \cdot n \qquad auf\ \partial\Omega \end{aligned}$$
   .

   (c) *Update*
   $$p^{k+1} = p_{old} + q^{k+1/2}, \quad u^{k+1} = u^{k+1/2} - \tau q^{k+1/2}.$$

There is still a certain freedom in choosing M, N and p_{old}, which leads to a variety different schemes:

- $N(v) = 0$, $M(v) = \nu\Delta v - (v \cdot \nabla)v$,
- $N(v) = -\nu\Delta v$, $M(v) = -(v \cdot \nabla)v$,
- $N(v) = (v \cdot \nabla)v$, $M(v) = \nu\Delta v$.

For the old value of the pressure p_{old} one has e.g. the following choicess

$$p_{old} = 0, \quad p_{old} = p^k, \quad p_{old} = 2p^k - p^{k-1}.$$

The main idea of this method is first to compute a velocity field without taking into account incompressibility, and then perform a pressure correction, which is a projection back to the subspace of divergence free vector fields. Note that the step (b) is equivalent to the minimization of
$$\int_\Omega |\tau\nabla q - u^{k+1/2}|^2 \, dx$$
over q, i.e., $\tau q^{k+1/2}$ corresponds to the projection of $u^{k+1/2}$ to the subspace of gradient vector fields, which is the orthogonal complement of the space of divergence-free vector fields. For the different choices of M and N, problems of different complexity are obtained in each iteration step. The Neumann problem for the pressure correction can be solved using fast standard methods.

The choice of p_{old} mainly influences the order of convergence of the scheme.

Remark 6.20. *The boundary conditions $q^{k+1/2}$ are obtained from a compatibility relation for the Neumann problem:*

$$0 = \int_\Omega \operatorname{div} u^{k+1}\, dx = \int_\Omega \operatorname{div} u^{k+1/2}\, dx - \tau \int_\Omega \Delta q^{k+1/2}\, dx$$

and hence,

$$\int_{\partial\Omega} \nabla q^{k+1/2} \cdot n\, ds = \frac{1}{\tau} \int_\Omega \operatorname{div} u^{k+1/2}\, dx = \frac{1}{\tau} \int_{\partial\Omega} u^{k+1/2} \cdot n\, ds.$$

7 Turbulence

In the following we provide a short introduction into turbulence models derived from the instationary Navier–Stokes model.

Turbulent flows have the following properties:

- Irregular instationary behaviour
- High vorticity
- Dissipativity

Therefore, turbulent flows are three-dimensional, full of losses and irreversible, which makes there numerical simulation difficult. The irreversibility appears in the phenomenon of vortices breaking up into smaller and smaller vortices until complete dissipation is reached. Moreover, many models consider the force term f to be rather of stochastic than derministic nature. The small spatial scales appearing in turbulent flows need extremely fine grids and therefore an enormous computational effort and memory consumption.

A way to (partially) overcome the difficulties arising in turbulent flows is to derive simplified models, which do not incorporate all detailed effects as the nonlinear Navier–Stokes model, but focus on specific important parts of the behaviour. For stationary incompressible flows with constant viscosity one can derive the *Reynolds–Averaged–Navier–Stokes equations (RANS)*, which we shall sketch in the following.

The main idea of the averaging is to decompose the velocity u into a deterministic mean velocity $\bar{u}$ and a (stochastic) perturbation u' with mean zero, i.e.,

$$u_i = \bar{u}_i + u_i', \qquad i = 1, 2, 3.$$

In the same way we can decompose the pressure into $p = \bar{p} + p'$.

Since the mean value $\langle u' \rangle$ vanishes, and since we can interchange differentiation and averaging, we obtain that

$$\begin{aligned} \langle \Delta u' \rangle &= \Delta \langle u' \rangle = 0 \\ \langle \bar{u}.\nabla u' \rangle &= \bar{u}.\nabla \langle u' \rangle = 0 \\ \langle u'.\nabla \bar{u} \rangle &= \langle u' \rangle.\nabla \bar{u} = 0 \\ \langle \operatorname{div} u' \rangle &= \operatorname{div} \langle u' \rangle = 0 \\ \langle \nabla p' \rangle &= \nabla \langle p' \rangle = 0. \end{aligned}$$

Hence, if we apply averaging over the stochastic perturbations to the stationary Navier–Stokes equations we obtain the Reynolds-averaged Navier–Stokes equations

$$\begin{aligned} \bar{u}.\nabla \bar{u} - \nu \Delta \bar{u} + \nabla \bar{p} &= \langle f \rangle - \langle u'.\nabla u' \rangle, \\ \operatorname{div} \bar{u} &= 0. \end{aligned}$$

By averaging we obtain a Navier–Stokes system for the mean value, with an additional source term, that can also be rewritten as

$$-\langle u'.\nabla u'\rangle = -\operatorname{div}\langle u'\otimes u'\rangle.$$

Hence, it acts like an additional stress tensor, called Reynolds stress tensor.

Since we have to treat the tensor $u'\otimes u'$ as an additional variable, we need additional equations to determine the Reynolds stress tensor. A common approach is the *vortex viscosity model*

$$-\langle u'\otimes u'\rangle = \tau_t = \nu_t\left(\nabla\bar{u}+\nabla\bar{u}^T\right)-\frac{2}{3}k_t I,$$

where ν_t denotes the turbulent viscosity and k_t the turbulent kinetic energy, given by

$$k_t = \frac{1}{2}\langle|u'|^2\rangle.$$

In the following we shall discuss three possible models:

7.1 Prandtl Model

In an analogous way to the kinetic theory of gases, in which single molecules have a mean free path, one can formulate an algebraic one-dimensional turbulence model based on a characteristic length ℓ. This length can be interpreted in two different ways, either as a diameter of a fluid ball, or the way a fluid ball is moving until mixture effects cause its dissipation. In order to connect these possibilities one can assume that these two quantities (radius and mean free path) are proportional.

In the following we assume that the turbulence effect enters in the y-direction. Then, for a fluid ball entering into the layer y from $y+\ell$ we have a velocity difference

$$\delta\bar{u}_1 = \bar{u}_1(y)-\bar{u}_1(y+\ell)$$

with respect to the mean velocity $\bar{u}_1$. Hence, the perturbation velocity u_1' can be approximated to first order as

$$u_1' = k_1\ell\frac{\partial\bar{u}_1}{\partial y}.$$

If a fluid ball with perturbation velocity u_1' is displacing another one into the layer below, this means for the second ball that its perturbation velocity v_1' in direction y is given by

$$v_1' = k_2\ell\frac{\partial\bar{v}_1}{\partial y}.$$

Hence, we obtain the Reynolds stresses (with $k_1=k_2=1$)

$$|\tau_t| \approx |u_1'\otimes v_1'| = \ell^2\frac{\partial\bar{u}_1}{\partial y}\frac{\partial\bar{v}_1}{\partial y},$$

i.e., this gives another nonlinear term in the Navier–Stokes equations.

Note that in a purely one-dimensional model this corresponds to

$$\nu_t = \ell^2 \left| \frac{\partial \bar{u}_1}{\partial y} \right| .$$

7.2 Baldwin-Lomax Model

The Baldwin-Lomax model is a generalization of the Prandtl mixture model to multiple dimensions by using

$$\nu_t = \ell^2 |\omega|$$

with the vorticity

$$\omega = \frac{1}{2} \nabla \times u.$$

Note that for a flow between two plates in the absence of a pressure gradient the Baldwin-Lomax model reduces to the Prandtl model since $\omega = (0, 0, \frac{\partial \bar{u}_1}{\partial y})$.

7.3 $k - \epsilon$ Model

The most prominent and probably most important turbulence model is the *k-ϵ model*. The name of this model arises from the fact that one solves two additional differential equations, one for the dissipation rate ϵ and the second for the turbulent kinetic energy k. The turbulent viscosity is given by

$$\nu_t = C \sqrt{k} \ell = C \frac{k^2}{\epsilon}.$$

Hence, after computing k and ϵ, we can use them to obtain the Reynolds stress tensor

$$\tau_t = C \frac{k^2}{\epsilon} \left(\nabla \bar{u} + \nabla \bar{u}^T \right) - \frac{2}{3} k I.$$

The equation for the dissipation is given by

$$\frac{D\epsilon}{Dt} = \frac{\partial \epsilon}{\partial t} + \operatorname{div}(\epsilon \bar{u}) = C_{\epsilon,1} \frac{\epsilon}{k} (\tau_t : \nabla \bar{u}) - C_{\epsilon,2} \frac{\epsilon^2}{k} + \nu_t \Delta \epsilon$$

and the one for the kinetic energy by

$$\frac{Dk}{Dt} = \frac{\partial k}{\partial t} + \operatorname{div}(k \bar{u}) = \operatorname{div} \langle \frac{1}{2} \tau_t u' - p' u' \rangle + \tau_t : \nabla \bar{u} - \nu_t \langle |\nabla u'|^2 \rangle + \nu_t \Delta k.$$

The terms on the right-hand side model the following effects:

- The first term $\operatorname{div}\langle\frac{1}{2}\tau_t u' - p'u'\rangle$ is the turbulent diffusion of the kinetic energy.
- The second term $\tau_t : \nabla\bar{u}$ models the production of turbulent kinetic energy due to deformation.
- The third term $-\nu_t\langle|\nabla u'|^2\rangle = -2\nu_t k$ models dissipation effects.
- The final term $\nu_t\Delta k$ is a standard diffusion effect.

Obviously, in the above form the equations for k and ϵ do not suffice to obtain a closed system, since it still depends on u' and even p' via the turbulent diffusion of kinetic energy and the production of turbulent kinetic energy due to deformation. Therefore we need further simplifications, which we shall discuss in the following.

Since the turbulent diffusion of the kinetic energy is usually modeled as a standard diffusion term of k, i.e.,

$$\operatorname{div}\langle\frac{1}{2}\tau_t u' - p'u'\rangle \approx \frac{\nu_t}{\sigma_k}\Delta k.$$

The constant σ_k is called Prandtl number.

In the second term, the Reynolds stress is replaced by the mean stress (i.e., the one created by the mean velocity $\bar{u}$), which leads to the approximation

$$\tau_t : \nabla\bar{u} \approx \nu_t\left(\nabla\bar{u} + \nabla\bar{u}^T\right) : \nabla\bar{u} =: P(\nabla\bar{u})$$

If we introduce all these approximations into the above two equations as well as into the Reynolds averaged Navier–Stokes equations, we obtain the following system (using the short notation u for the mean velocity $\bar{u}$):

$$\begin{aligned}
\frac{\partial u}{\partial t} + u.\nabla u - \nu\Delta u &= f - \nabla p + \nu_t \operatorname{div}\left(\nabla u + \nabla u^T\right) - \frac{2}{3}\nabla k + \nabla\nu_t\left(\nabla u + \nabla u^T\right)\\
\frac{\partial \epsilon}{\partial t} + \operatorname{div}(\epsilon u) - \nu_t\Delta\epsilon &= C_{\epsilon,1}\frac{\epsilon}{k}P(\nabla u) - C_{\epsilon,2}\frac{\epsilon^2}{k}\\
\frac{\partial k}{\partial t} + \operatorname{div}(ku) - \nu_t\Delta k &= \frac{\nu_t}{\sigma_k}\Delta k + P(\nabla u) - 2\nu_t k\\
\operatorname{div} u &= 0.
\end{aligned}$$

Collecting all diffusion terms on the right-hand side and using the relation for ν_T and $\operatorname{div}\nabla u^T = \nabla\operatorname{div} = 0$, we obtain

$$\begin{aligned}
\frac{\partial u}{\partial t} + u.\nabla u - (\nu + C\frac{k^2}{\epsilon})\Delta u &= f - \nabla\left(p - \frac{2}{3}k\right) + C\nabla\left(\frac{k^2}{\epsilon}\right)\left(\nabla u + \nabla u^T\right)\\
\frac{\partial \epsilon}{\partial t} + \operatorname{div}(\epsilon u) - C\frac{k^2}{\epsilon}\Delta\epsilon &= C_{\epsilon,1}\frac{\epsilon}{k}P(\nabla u) - C_{\epsilon,2}\frac{\epsilon^2}{k}\\
\frac{\partial k}{\partial t} + \operatorname{div}(ku) - C\frac{k^2}{\epsilon}\left(1 + \frac{1}{\sigma_k}\right)\Delta k + 2C\frac{k^3}{\epsilon} &= P(\nabla u)\\
\operatorname{div} u &= 0.
\end{aligned}$$

The numerical solution of the k-ϵ is a challenging problem for several reasons: First of all, the number of equations increases with respect to the Navier–Stokes model. Moreover, the new equations include additional nonlinearities and couplings which increase the difficulty.

In the final version of the k-ϵ model, we need five additional constants: $C, C_{\epsilon,1}, C_{\epsilon,2}, \sigma_k$, and σ_ϵ. For a flow between plates the constants below can be used, which are usually the default values in simulation tools for the k-ϵ model:

C	$C_{\epsilon,1}$	$C_{\epsilon,2}$	σ_k	σ_ϵ
0.09	1.44	1.92	1.0	1.3

Another delicate part is the formulation of suitable boundary conditions for the k-ϵ model, which we omit here. For a detailed exposition and analysis of the k-ϵ model we refer to the monography by Mohammadi and Pironneau [MP94].

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