Nonlinear Positive Interpolation Operators for Analysis with Multilevel Grids *

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Abstract

We introduce some nonlinear positive and negative interpolation operators. The interpolation need to preserve positivity or negativity of a function. In addition, the interpolation must be pointwisely below or above the function. Some of the operators also have the pointwise monotone property over refined meshes. It is also desirable that the interpolation have the needed approximation and stability estimates. Those operators could be used in the convergence analysis for domain decomposition and multigrid methods for obstacle problems.

1 Introduction

We are interested in the convergence rate analysis of multigrid and domain decomposition methods for variational inequalities, i.e. we want to solve a convex minimization problems with some convex constraints, c.f. [3, 7]. It is well known that both domain decomposition and multigrid methods can be regarded as space decomposition and subspace correction techniques. For a given space decomposition technique, we need two constants to measure the quality of the decomposition. One constant is called the constant for the strengthened Cauchy-Schwarz inequality. The other constant is for the partition lemma, which is also called Lions's lemma. For linear problems, these constants are well established, see [12]. The concepts of using these constants to analyse the convergence rate for space decomposition techniques was extended to nonlinear problems in [6, 10, 5, 11, 9, 7]. To be more specific, we shall consider the following problem in this work:

$$\min_{v\in K} F(v). \tag{1}$$

For simplicity, we just assume that

$$K = \{v| \ v \in H_0^1(\Omega), \ v \geq 0\}, \quad F(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - fv. \tag{2}$$

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In order to use domain decomposition or multigrid methods for the above problem, we need to construct finite element or finite difference meshes that are nested and refined (the problem with non-nested mesh is much more complicated and shall not be considered here). For the partition lemma for the above problem, we need to interpolate functions from K to the different meshes or we need to interpolate functions from fine meshes to coarser meshes. The interpolation operators need to satisfy the following properties:

1. (Positivity): It shall preserve the positivity or negativity, i.e. the interpolation of a positive function shall be positive or the interpolation of a negative function shall be negative.

2. (Approximation): The interpolation shall have the needed approximation properties.

3. (Stability): The interpolation shall be stable in the needed norms.

4. (Pointwise above or below): The interpolation of a given function shall be pointwisely below or above the function.

5. (Monotonicity with mesh refinement): When interpolating a function to finer or coarser meshes, it is desirable that the interpolation over a finer mesh should be pointwisely bigger or smaller than the interpolation over a coarser mesh.

For problem (1)–(2), the standard nodal point linear Lagrangian finite element interpolation operators are not applicable in many context. In [4] and [2], some interpolation operators are given which preserve positivity. These operators are linear and satisfy the approximation and stability requirements, but do not have properties that the interpolation is below or above the interpolated functions and also do have pointwise monotonicity with respect to refined meshes. In [4], it was proved that linear positive interpolation operators may not exist if we require more than first order accuracy at extreme points. In this work, we shall introduce some operators which are not linear, but satisfy all the needed properties.

2 Some nonlinear positive interpolation operators

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of the domain $\Omega \subset R^d, d = 1,2,3$ with a mesh size h and $S_h \subset H_0^1(\Omega)$ be the corresponding piecewise linear finite element space on $\mathcal{T}_h$. In the analysis, we need to use finite element spaces with different mesh sizes. It will be assumed that h is always the smallest mesh size. For an $H > h$, we consider the case that $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$. In the following, the definition of some nonlinear interpolation operators from S_h to S_H will be given. Denote by $\mathcal{N}_H = \left\{x_0^i\right\}_{i=1}^{n_0}$ all the interior nodes for $\mathcal{T}_H$. For a given x_0^i, let ω_i be the union of the mesh elements of $\mathcal{T}_H$ having x_0^i as one of its vertexes, i.e.

$$\omega_i := \cup\{\tau \in \mathcal{T}_H, x_0^i \in \bar{\tau}\}. \tag{3}$$

Let $\{\phi_0^i\}_{i=1}^{n_0}$ be the associated nodal basis functions satisfying $\phi_0^i(x_0^k) = \delta_{ik}$, $\phi_0^i \geq 0$, $\forall i$ and $\sum_i \phi_0^i(x) = 1$. It is clear that ω_i is the support of ϕ_0^i.

In the following, standard notations for Sobolev norms will be used, i.e. $\|\cdot\|_0$ stands for the $L^2(\Omega)$ norm, $\|\cdot\|_1$ and $|\cdot|_1$ are the norms and seminorms for $H^1(\Omega)$, etc.

2.1 A nonlinear positive interpolation operator below the function

Given a nodal point $x_0^i \in \mathcal{N}_H$ and a $v \in S_h$, let

$$I_i v = \min_{\bar{\omega}_i} v(x). \tag{4}$$

The interpolated function is then defined as

$$I_H^{\ominus} v := \sum_{x_0^i \in \mathcal{N}_H} (I_i v)\phi_0^i(x).$$

From the definition, it is easy to see that

$$I_H^{\ominus} v \leq v, \quad \forall v \in S_h, \tag{5}$$

$$I_H^{\ominus} v \geq 0, \quad \forall v \geq 0, v \in S_h. \tag{6}$$

Moreover, the interpolation for a given $v \in S_h$ on a finer mesh is always no less than the corresponding interpolation on a coarser mesh due to the fact that each coarser mesh element contains several finer mesh elements, i.e.

$$I_{h_1}^{\ominus} v \leq I_{h_2}^{\ominus} v, \quad \forall h_1 \geq h_2 \geq h, \quad \forall v \in S_h. \tag{7}$$

In addition, the interpolation operator also has the following approximation properties, c.f. p. 767 of [7].

Theorem 1 *For any $u, v \in S_h$, it is true that*

$$\|I_H^{\ominus} u - I_H^{\ominus} v - (u-v)\|_0 \leq c_d H |u-v|_1, \tag{8}$$

$$\|I_H^{\ominus} v \; - \; v\|_0 \leq c_d H |v|_1, \tag{9}$$

$$|I_H^{\ominus} u - I_H^{\ominus} v|_1 \leq c_d |u-v|_1, \tag{10}$$

where $c_d = C$ if $d=1$; $c_d = C\left(1 + \left|\log \frac{H}{h}\right|^{\frac{1}{2}}\right)$ if $d=2$ and $c_d = C\left(\frac{H}{h}\right)^{\frac{1}{2}}$ if $d=3$. Here and later, the generic constant C is used to denote constants that are independent of the mesh parameters.

2.2 A nonlinear negative interpolation operator above the function

However, if we define

$$I_i v = \max_{\bar{\omega}_i} v(x) \qquad I_H^{\oplus} v := \sum_{x_0^i \in \mathcal{N}_H} (I_i v)\phi_0^i(x). \tag{11}$$

Then it is easy to see that

$$I_H^{\oplus} v \geq v, \quad \forall v \in S_h, \qquad I_H^{\oplus} v \leq 0, \quad \forall v \leq 0, v \in S_h. \tag{12}$$

Moreover, the interpolation for a given $v \in S_h$ on a finer mesh is always no bigger than the corresponding interpolation on a coarser mesh, i.e.

$$I_{h_1}^{\oplus} v \geq I_{h_2}^{\oplus} v, \quad \forall h_1 \geq h_2 \geq h, \quad \forall v \in S_h. \tag{13}$$

From theorem 1, it is easy to see that the following is correct ([7]).

Theorem 2 *There exists an interpolation operator* $I_H^{\oplus} : S_h \mapsto S_H$ *such that*

$$\begin{aligned}
&I_H^{\oplus} v \geq v, \quad \forall v \in S_h, \\
&I_H^{\oplus} v \leq 0, \forall v \leq 0, v \in S_h, \\
&\|I_H^{\oplus} u - I_H^{\oplus} v - (u - v)\|_0 \leq c_d H |u - v|_1, \\
&\|I_H^{\oplus} v \ - \ v\|_0 \leq c_d H |v|_1, \qquad |I_H^{\oplus} u - I_H^{\oplus} v|_1 \leq c_d |u - v|_1, \forall v \in S_h.
\end{aligned}$$

2.3 A nonlinear interpolation operator above or below the function

For some cases, we need an interpolation operator which has the properties of $I_H^{\oplus}$ in some part of the domain Ω and has the properties of $I_H^{\ominus}$ in the rest of Ω. The operator we shall define in the following is a simplified version of the operator used in p.133 of [8]. For any given $v \in S_h$, we let

$$v^+(x) = max(0, v(x)), \qquad v^-(x) = min(0, v(x)).$$

It is easy to see that $v(x) = v^+(x) + v^-(x)$. The new interpolation operator is then defined as:

$$\mathcal{I}_H v := \sum_{x_0^i \in \mathcal{N}_H} (\min_{\bar{\omega}_i} v^+ + \max_{\bar{\omega}_i} v^-)\phi_0^i(x). \tag{14}$$

We have $\min_{\bar{\omega}_i} v^+ \geq 0$ and $v^-|_{\omega_i} = 0$ if $v \geq 0$ in ω_i. We have $\max_{\bar{\omega}_i} v^- \leq 0$ and $v^+|_{\omega_i} = 0$ if $v \leq 0$ in ω_i. In case that v has both negative and positive values in ω_i, then we have $\min_{\bar{\omega}_i} v^+ = 0$ and $\max_{\bar{\omega}_i} v^- = 0$. For a given v, we let

$$\Omega^+ = \{x | \ v(x) \geq 0\}, \qquad \Omega^0 = \{x | \ v(x) = 0\}, \qquad \Omega^- = \{x | \ v(x) \leq 0\}.$$

It is easy to see that

$$\mathcal{I}_H v \geq 0 \text{ in } \Omega^+, \quad \mathcal{I}_H v \leq 0 \text{ in } \Omega^-, \quad \mathcal{I}_H v = 0 \text{ in } \Omega^0.$$

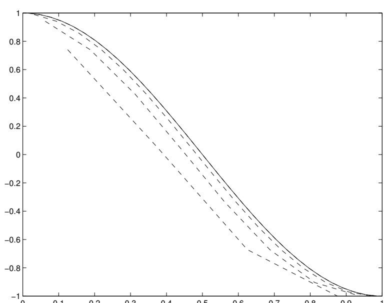
Moreover, we have that

$$\mathcal{I}_H v \leq v \text{ in } \Omega^+, \quad \mathcal{I}_H v \geq v \text{ in } \Omega^-.$$

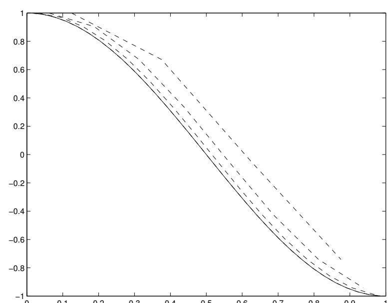
If we interpolate a function into a sequence of refined meshes, then the interpolation value is increasing on finer meshes over the region Ω^+ and the interpolation value is decreasing on finer meshes over the region Ω^-. These pointwise monotone properties are visualized in Figire 1. Similarly, the following approximation and stability properties are valid:

$$\|\mathcal{I}_H u - \mathcal{I}_H v - (u - v)\|_0 \leq c_d H |u - v|_1, \forall u, v \in S_h,$$
$$|\mathcal{I}_H u - \mathcal{I}_H v|_1 \leq c_d |u - v|_1, \forall u, v \in S_h.$$

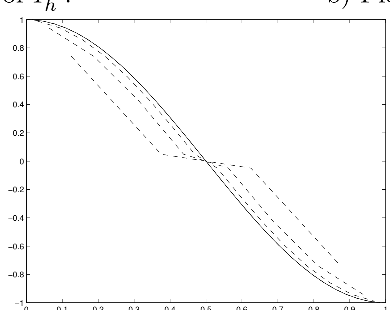
The proof for the above estimations can be done similarly as in [7].



a) Plot of $I_h^{\ominus}$.



b) Plot of $I_h^{\oplus}$.



c) Plot of $\mathcal{I}_h$.

Figure 1: Plots of the interpolation operators over a sequence of refined meshes. If the mesh is refined, the interpolation $I_H^{\ominus} v$ increases, while $I_H^{\oplus} v$ decreases. The interpolation $\mathcal{I}_H v$ increases in Ω^+ and decreases in Ω^-. $I_H^{\ominus} v$ is always below v, while $I_H^{\oplus} v$ is always above v. $\mathcal{I}_H v$ is below v in Ω^+ and above v in Ω^-.

3 Some other nonlinear interpolation operators

The interpolation operators given in §2 only preserve constants locally and this can only have first order of convergence. In this section, we will introduce an operator which preserves linear functions locally and thus it can have higher oder approximation accuracy, but we will lose the pointwise monotone property enjoyed by the operators defined in §2.

For a given $v \in S_h$, let $v_0^I = I_H v$ to be the standard nodal Lagrangian interpolation of v into S_H. For the coarser mesh S_H, let x_i^0 and ω_i be as defined in §2, c.f. (3). We shall construct a new interpolation function v_0 by defining its nodal values as

$$v_0(x_i^0) = v_0^I(x_i^0) - \max_{x\in\omega_i} \left(v_0^I(x) - v(x)\right), \quad \forall x_i^0. \tag{15}$$

For simplicity, we define $\rho_0(x) \in S_H$ to be the coarse mesh function having the nodal values

$$\rho_0(x_i^0) = \max_{x\in\omega_i} \left(v_0^I(x) - v(x)\right), \quad \forall x_i^0.$$

It is easy to see that $v_0 = v_0^I - \rho_0$. Moreover, $\rho_0(x) \geq v_0^I(x) - v(x)$, which implies

$$v_0(x) = v_0^I(x) - \rho_0(x) \leq v_0^I(x) - (v_0^I(x) - v(x)) = v(x).$$

In addition,

$$\|v_0 - v\|_0 \leq \|v_0^I - v\|_0 + \|\rho_0\|_0.$$

As $\rho_0 \in S_H$, it is known that the L^2-norm is equivalent to

$$\|\rho_0\|_0^2 = CH^d \sum_{i=1}^{n_0} |\rho_0(x_i^0)|^2.$$

Using a linear mapping to transform ω_i into a domain of unit size and applying the well-known estimate of [1], we get that

$$\|\rho_0\|_0^2 \leq CH^d \sum_{i=1}^{n_0} \|v_0^I - v\|_{0,\infty,\omega_i}^2 \leq CH^2 c_d^2 |v|_1^2.$$

In the above inequality, we have used the regularity of the meshes, i.e. under the minimum angle condition, the number of elements around a nodal point is always less than a constant. Using the inverse inequality, we know that $\|\rho_0\|_1 \leq CH^{-1}\|\rho_0\|_0$. In case that we want to use the H^2 norm for v, we have

$$\|\rho_0\|_0^2 \leq CH^d \sum_{i=1}^{n_0} \|v_0^I - v\|_{0,\infty,\omega_i}^2 \leq CH^2 |v|_2^2.$$

Denote v_0 by $\mathcal{I}_H^a v$. Combining these estimates with standard estimates for $v - v_0^I$, we have proved the following lemma.

Theorem 3 *Let S_H and S_h be defined as above. There exists an interpolation operator $\mathcal{I}_H^a : S_h \mapsto S_H$ such that*

$$\mathcal{I}_H^a v \leq v, \qquad \|\mathcal{I}_H^a v\|_1 \leq c_d \|v\|_1,$$
$$\|\mathcal{I}_H^a v \;-\; v\|_0 \leq c_d H |v|_1, \quad \|\mathcal{I}_H^a v \;-\; v\|_0 \leq H^2 |v|_2,$$

From the inequality

$$|\max_{\bar{\omega}_i} u - \max_{\bar{\omega}_i} v| \leq \|u - v\|_{0,\infty,\omega_i},$$

it is also easy to prove the following estimates using the techniques of [7]

Theorem 4 *For any u, v from S_h, we have*

$$\|\mathcal{I}_H^a u - \mathcal{I}_H^a v - (u-v)\|_0 \leq c_d H |u-v|_1,$$
$$\|\mathcal{I}_H^a u - \mathcal{I}_H^a v - (u-v)\|_0 \leq c H^2 |u-v|_2,$$
$$\|\mathcal{I}_H^a u - \mathcal{I}_H^a v\|_1 \leq c_d \|u-v\|_1.$$

In addition, the operator $\mathcal{I}_H^a$ have the following property which is not valid for the operators given in §2:

$$\mathcal{I}_H^a (v + v_H) = \mathcal{I}_H^a v + v_H, \quad \forall v \in S_h, v_H \in S_H, \tag{16}$$

i.e. the operator $\mathcal{I}_H^a$ is invariant for functions from the coarse mesh space S_H.

The interpolation $\mathcal{I}_H^a v$ is always below the function v, but it may not preserve positivity. It is also easy to define another operators which are always above the function or above the function in part of the domain and below the function in the rest of the domain.

4 Applications to multigrid decomposition

Assume that we have a sequence of shape regular meshes $\mathcal{T}_{h_j}$ that are produced by refining a coarse mesh. The mesh sizes $h_j, j = 1, 2, \cdots, J$ are decreasing and satisfies $c_1 \gamma^{2j} \leq h_j \leq c_2 \gamma^{2j}$ and $0 < \gamma < 1$. Let $\mathcal{M}_j$ be the piecewise linear finite element spaces over the meshes. For a given $v \geq 0$ and $v \in \mathcal{M}_J$ we shall decompose it into $v = \sum_{j=1}^J v_j$ such that $v_j \geq 0 \; \forall j$. In addition, we also need that $\|v_j\|_1 \leq c_d \|v\|_1$. Such a decomposition is needed for the proof of the partition lemma for [7] and [8]. Using the operators defined in §2, we see that the following functions v_j satisfy the needed properties:

$$v_j = I_{h_j}^{\ominus} v - I_{h_{j-1}}^{\ominus} v, \; j = 1, 2, \cdots J-1, \; v_J = v - I_{h_{J-1}}^{\ominus} v.$$

In order to show that $v_j \geq 0$ we need to use the pointwise monotone properties. In order to show the stability of v_j in H^1 we need the corresponding estimates for $I^{\ominus}$. In fact, the opeartors defined in §2 and §3 can be used in different context in the convergence analysis of domain decomposition and multigrid methods for problems like (1).

The interpolation operator $I_H^{\oplus}$ is needed for the analysis given above if we change the constraint set K given in (2) to $K = \{v | \ v \in H_0^1(\Omega), \ v \leq 0\}$. The interpolation operator $\mathcal{I}_H$ is needed if we shall work with two-obstacles, i.e. one obstacle above the solution and one obstacle below the solution.

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