Γ-Convergence Approximation to Piecewise Constant Mumford-Shah Segmentation

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Abstract. The piecewise constant Mumford-Shah segmentation model [17] has been rediscovered by Chan and Vese [6] in their award-winning paper in the context of region based active contours. The work of Chan and Vese demonstrated many practical applications thanks to their clever numerical implementation based on the celebrated level-set approach of Osher and Sethian [18]. In the current work, we propose a Γ-convergence formulation to the piecewise constant Mumford-Shah segmentation model, and demonstrate its efficient implementation by the iterated integration of a linear Poisson equation.

1 Introduction: The Mumford-Shah Segmentation Model

The celebrated Mumford-Shah segmentation model [17] is built upon a generic mixture image model into which the edge feature is explicitly incorporated as in [10]. Consider the following image generation model:

$$\Gamma \longrightarrow u \xrightarrow{\oplus n} u_0,$$

where in the reverse order, u_0 denotes an observed image, n an additive Gaussian noise field, and u piecewise smooth (or cartoonish) image patches consistent with a given edge layout Γ.

From Bayesian point of view [10, 16], segmentation is to estimate the posterior probability

$$p(\Gamma, u \mid u_0), \quad \text{or equivalently,} \quad p(u_0 \mid u, \Gamma)p(u, \Gamma)/p(u_0).$$

In the Markovian setting [7], the joint prior can be expressed by

$$p(u, \Gamma) = p(u \mid \Gamma)p(\Gamma).$$

Thus by putting aside the constant $p(u_0)$ and working with the energy function (or the *logarithmic likelihood function*) $E = -\log p$, one obtains the structure of the Mumford-Shah model up to an ineffectual constant:

$$E[\Gamma, u \mid u] = E[\Gamma] + E[u \mid \Gamma] + E[u_0 \mid u, \Gamma].$$

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The *full* Mumford-Shah model [15, 17] is in fact explicitly expressed by:

$$E[\Gamma, u \mid u] = \sigma \text{length}(\Gamma) + \beta \int_{\Omega \setminus \Gamma} |\nabla u|^2 dx + \lambda \int_{\Omega} (u - u_0)^2 dx,$$

where $dx = dx_1 dx_2$ denotes the area element of a 2-D domain Ω.

For images made of piecewise *homogeneous* stochastic patches, only their constant averages can be identified as the cartoonish pieces, i.e.,

$$u(x) \equiv C_i, \qquad x \in \Omega_i, \quad \text{and} \quad \Omega_i \in \pi(\Omega \mid \Gamma).$$

Here the notation $\pi(\Omega \mid \Gamma)$ denotes the partitioning of the entire image domain Ω given an edge layout Γ, or the collection of *connected components* of $\Omega \setminus \Gamma$ topologically speaking. The original Mumford-Shah model is then reduced to the piecewise constant model, or simply, the *reduced* Mumford-Shah model:

$$E[(C_i' s), \Gamma \mid u_0] = \sigma \text{length}(\Gamma) + \lambda \sum_{\Omega_i \in \pi(\Omega|\Gamma)} \int_{\Omega_i} (u(x) - C_i)^2 dx.$$

Mathematically this reduced model can be obtained as a proper asymptotic limit of the full Mumford-Shah model when the parameters of the latter tend to the infinite, as discussed in the original paper of Mumford and Shah [17]. Recently in their award-winning paper [6, 21], Chan and Vese rediscovered this model in the context of region based active contours. As in [6], in the current paper, we shall mainly focus on the 2-phase model to illustrate our primary contributions:

$$E[C_+, C_-, \Gamma \mid u_0] = \sigma \text{length}(\Gamma) + \lambda \int_{\Omega_+} (u(x) - C_+)^2 dx + \lambda \int_{\Omega_-} (u(x) - C_-)^2 dx, \tag{1}$$

where Γ partitions Ω into the interior Ω_+ and exterior Ω_-. As remarkably demonstrated by Chan and Vese [6, 21], such a 2-phase model has already witnessed numerous intriguing applications in astronomy and medicine.

Chan and Vese have successfully implemented the above model using the celebrated level-set computing technology as invented and continuously advanced by Stan Osher and James Sethian [18]. Multiphase computational frameworks have also been developed by Chan and Vese [21], and lately by Lie, Lysaker, and Tai [13].

The current work is complementary to the above level-set approach. Inspired by the Γ-convergence approximation to the full Mumford-Shah model developed by Ambrosio and Tortorelli [1], we propose a new Γ-convergence formulation of the reduced Mumford-Shah model, and its robust and fast computational implementation. As in the work [1], our approach overcomes the fundamental theoretical and computational difficulties resulting from the free-boundary nature of the Mumford-Shah model (both the full and the reduced). The computation is reduced to the iterated integration of a linear Poisson equation, which can be easily and efficiently implemented in Matlab in a uniform code, without extra intermediate processing steps (e.g., normal extension and reinitialization) [21].

The organization of the paper goes as follows. Section 2 briefly reviews the essence of the Γ-convergence approximation to the full Mumford-Shah model. In Section 3, we introduce our new Γ-convergence approximate model to the reduced (i.e., piecewise constant) Mumford-Shah model. Efficient computational schemes and examples of generic test images are presented in Section 4.

2 Γ-Convergence Approximation to the Full M.-S. Model

Γ-convergence has its rigorous mathematical definition in metric spaces [1]. For our application, the plainest intuition behind could be best revealed by phase-field modelling of superconductors, as in the the works of Nobel Laureates Ginzburg and Landau [11]. In this section we shall briefly explain the core idea in terms of approximation theory.

In the Γ-convergence setting [1, 8, 14], a curve Γ (in 2-D) is instead represented by a 2-D function $z = z_\epsilon(x_1, x_2) \in [0, 1]$, depending upon a small scale parameter ϵ. The energy associated with such a *phase field* z is defined as

$$L_\epsilon[z] = \int_\Omega \epsilon |\nabla z|^2 dx + \int_\Omega \frac{(1-z)^2}{4\epsilon} dx.$$

Since $\epsilon \ll 1$, under any finite energy bound, the second term demands the phase field $z = z_\epsilon(x_1, x_2)$ to be as close to 1 as possible almost everywhere on the image domain Ω.

In addition, suppose that along certain narrow bands (intended to be the ϵ-neighborhoods of a curve Γ) the field z sharply drops down to zero. Then physically the graph of z looks more like a canyon. Denote the medial line of the canyon by Γ. Then the entire Γ-convergence machinery is built upon the following remarkable approximation result:

$$L_\epsilon[z] \simeq \text{length}(\Gamma). \tag{2}$$

Rigorous mathematical analysis is more involved but a qualitative glimpse is not very far beyond the level of Advanced Calculus as presented below.

Applying the generic inequality $2AB \leq A^2 + B^2$, one has

$$L_\epsilon[z] \geq \int_\Omega |\nabla z||z-1| dx = \frac{1}{2} \int_\Omega |\nabla w|, \qquad w = (1-z)^2,$$

where the graph of $w = (1-z)^2$ looks like a set of walls. Most contributions to the integral come from a narrow band along Γ since w is flat away from it. Assuming that Γ is smooth, the narrow tubular neighborhood can then be parameterized by the tangential (arc length) and normal coordinates s and n. Since w remains almost constant along the tangential direction, we have $|\nabla w(s,n)| \simeq |\partial w / \partial n|$, and

$$\frac{1}{2} \int_\Omega |\nabla w| \simeq \int_\Gamma \int_{-\epsilon}^{\epsilon} \frac{1}{2} \left| \frac{\partial w}{\partial n} \right| dnds = \int_\Gamma \frac{1}{2} \text{TV}(w(s, \cdot)) ds.$$

For any fixed s, the total variation $\mathrm{TV}(w(s,\cdot)$ along the normal direction is ideally 2, since each shoulder of the wall contributes 1 (by ascending from 0 to 1 and then descending from 1 to 0). Hence we have shown qualitatively that

$$L_\epsilon[z] \geq \mathrm{length}(\Gamma).$$

Assisted with a suitable ordinary differential equation [1], one can further show that the lower bound can indeed be approached by some sequence of z's.

Notice that the above analysis crucially relies upon the assumption that z does touch down to the zero along Γ. But the energy form $L_\epsilon[z]$ alone does not guarantee it. Thus in Ambrosio and Tortorelli's approximation [1], it is explicitly enforced through the second term of the Mumford-Shah model:

$$E_\epsilon[z] = \sigma\left(\int_\Omega \epsilon|\nabla z|^2 dx + \int_\Omega \frac{(1-z)^2}{4\epsilon} dx\right) + \beta\int_\Omega z^2|\nabla u|^2 dx + \lambda\int_\Omega (u-u_0)^2 dx.$$

Along the jump (edge) set Γ, ∇u is not classically defined, or remains very large (or expensive) even after discrete sampling or continuous blurring. Thus the second term forces z to touch down to zero along Γ to bound the total energy.

3 Γ-Convergence Form of the Reduced M.-S. Model

For the reduced (piecewise constant) Mumford-Shah model, the lack of the gradient term loses the control factor that forces the field z to drop near edges. In the current paper, therefore, we propose a proper variation of Ambrosio and Tortorelli's original formulation for the full Mumford-Shah model [1]. As in Chan and Vese [6], we shall primarily focus on the 2-phase model, and multiphase extensions can be similarly accomplished as in Vese and Chan [21], and in particular, in the recent work of Lie, Lysaker, and Tai [13].

To explicitly enforce the 2-phase separation without turning to the gradient information ∇u, we propose to replace the original phase field energy by

$$L_\epsilon[z] = \int_\Omega \left(9\epsilon|\nabla z|^2 + \frac{(1-z^2)^2}{64\epsilon}\right) dx.$$

The range of z is restricted within $[-1,1]$. Since $\epsilon \ll 1$, a bounded energy will force $z=1$ or $z=-1$ almost everywhere. Following the similar inequality in the preceding section, one has

$$L_\epsilon[z] \geq \frac{3}{4}\int_\Omega |\nabla z||1-z^2| dx = \frac{3}{4}\int_\Omega \left|\nabla\left(z-\frac{z^3}{3}\right)\right| dx \simeq \frac{3}{4}\int_\Gamma\int_{-\epsilon}^{\epsilon} \mathrm{TV}(w) dn ds,$$

where $w = w(z) = z(1-z^2/3)$ is a monotone function on $z \in [-1,1]$, and the local curvilinear coordinates have been applied along the transition medial line (where $z=0$), as in the preceding section. Since $w(-1) = -2/3$ and $w(1) = 2/3$, one has $\mathrm{TV}(w(z(s,\cdot))) = 4/3$ locally along each s-normal line. Thus we have qualitatively established the lower bound:

$$L_\epsilon[z] \geq \mathrm{length}(\Gamma).$$

Further elaborate study shows that the hyperbolic tangent transition:

$$z(s,n) = \tanh\left(\frac{n}{24\epsilon}\right)$$

can approach the lower bound in an exponential rate. Thus $L_\epsilon[z]$ is indeed a good approximation to the length of Γ.

In the ideal scenario of two *pure* phases, one then defines their associated regions separately:

$$\Omega_\pm = \{x \in \Omega \mid z = \pm 1\}.$$

The associated indicator functions are ideally given by

$$1_+(x) = \left(\frac{1+z}{2}\right)^2, \qquad 1_-(x) = \left(\frac{1-z}{2}\right)^2.$$

(The square is mainly for computational stability in case that z strays away from $[-1,1]$.) Then,

$$\int_{\Omega_\pm} (u_0 - C_\pm)^2 dx = \int_\Omega \left(\frac{1 \pm z}{2}\right)^2 (u_0 - C_\pm)^2 dx.$$

In combination, we thus propose to approximate the reduced Mumford-Shah model (1) by the following Γ-convergence energy:

$$\begin{aligned} E_\epsilon[z, C_+, C_- \mid u_0] = \sigma \int_\Omega \left(9\epsilon|\nabla z|^2 + \tfrac{(1-z^2)^2}{64\epsilon}\right) dx + \\ \lambda \int_\Omega \left(\tfrac{1+z}{2}\right)^2 (u_0 - C_+)^2 dx + \lambda \int_\Omega \left(\tfrac{1-z}{2}\right)^2 (u_0 - C_-)^2 dx. \end{aligned} \tag{3}$$

One intends to minimize the energy by some optimal phase field z and means $C_\pm$'s.

Notice that all the four terms involve the field function z, but only the last two contain the mean fields $C_\pm$'s. Denote the sum of the last two terms by the "conditional" energy $E[C_+, C_- \mid u_0, z]$ given any z. Then the standard property of weighted least square approximation explicitly yields the conditional optima.

Theorem 1 (Optimal Means). *Given any square integrable phase field z on a finite domain Ω, as long as z is not constant, the optimal means $C_\pm$'s to a given image u_0 in terms of $E[C_+, C_- \mid u_0, z]$ are given by:*

$$C_\pm = C_\pm[z] = \frac{\int_\Omega (1 \pm z(x))^2 u_0(x) dx}{\int_\Omega (1 \pm z(x))^2 dx}. \tag{4}$$

On the other hand, by the direct method of Calculus of Variations based on minimizing sequences [9], one can establish the existence of minimizers to (3).

Theorem 2 (Existence of Optimal Phase Fields). *Let u_0 be a square integrable image given on a bounded domain Ω. Then there exists an optimal triple (z^*, C_+^*, C_-^*) which achieves the minimum energy of $E_\epsilon[z, C_+, C_- \mid u_0]$ among the admissible class of Sobolev phase fields [9].*

To compute an optimal minimizer, one could apply the conditional mean field formulae (4) to reduce the triple energy $E_\epsilon[z, C_+, C_- \mid u_0]$ to an energy solely depending upon z:

$$E_\epsilon[z \mid u_0] = E_\epsilon[z, C_+[z], C_-[z] \mid u_0].$$

But this energy is no longer quadratic in z and complexities multiply due to the denominators involving z.

Thus in practice, one employs the *alternating minimization* technique prevailing in multivariable optimization problems [8, 20]. For given z^n at step n, one computes the optimal means $C_\pm^n = C_\pm[z^n]$ by the formulae (4), and then updates z^n to z^{n+1} by treating $C_\pm$'s as known and minimizing

$$\begin{aligned} E_\epsilon[z \mid u_0, C_+, C_-] = \sigma \int_\Omega \left(9\epsilon|\nabla z|^2 + \tfrac{(1-z^2)^2}{64\epsilon}\right) dx + \\ \lambda \int_\Omega \left(\tfrac{1+z}{2}\right)^2 (u_0 - C_+)^2 dx + \lambda \int_\Omega \left(\tfrac{1-z}{2}\right)^2 (u_0 - C_-)^2 dx. \end{aligned} \tag{5}$$

4 Fast and Robust Numerical Implementation; Examples

Computationally, the optimization problem (5) is solved via its Euler-Lagrange equation. Write $e_\pm = u_0 - C_\pm$ as the residuals on $\Omega_\pm$, which are independent of z since $C_\pm$ are given. Let $\mu = \lambda/(4\sigma)$. Then the Euler-Lagrange equation of $E_\epsilon[z \mid u_0, C_+, C_-]$ is given by

$$0 = -9\epsilon\Delta z - \frac{(1-z^2)z}{32\epsilon} + \mu e_+^2(1+z) - \mu e_-^2(1-z), \tag{6}$$

with the Neumann adiabatic boundary condition. One further rewrites it to:

$$-9\epsilon\Delta z + \left(\frac{z^2}{32\epsilon} + \mu(e_+^2 + e_-^2)\right) z = \mu e_-^2 - \mu e_+^2 + \frac{z}{32\epsilon},$$

or simply $-9\epsilon\Delta z + R(z)z = f(z)$ with R and f denoting the corresponding terms. The latter can be solved iteratively by having the z's in R and f frozen:

$$z_m \to z_{m+1} : \quad -9\epsilon\Delta z_{m+1} + R_m z_{m+1} = f_m, \tag{7}$$

where $R_m = R(z_m) \geq 0$ and $f_m = f(z_m)$. Thus at each step it suffices to solve this linear Poisson equation on $z_{m+1}(x_1, x_2)$, which can be implemented efficiently in Matlab due to many fast elliptic solvers. Our computational experiments show that even ordinary Guass-Jacobi type of iteration schemes [12] lead to fast and robust convergence, including starting from any random initial guess.

The following flow summarizes our entire algorithm:

$$\hookrightarrow z^n \overset{\text{by}(4)}{\longrightarrow} [C_+^n, C_-^n] \to \left[z_m^{n+1} \overset{\text{by}(7)}{\hookrightarrow} z_{m+1}^{n+1} \right] \to z^{n+1} \to$$

The examples in the next section have all been generated from this algorithm. Below we briefly discuss how to properly choose the parameters in the model.

(a) The Γ-convergence parameter ϵ should be in the order of $O(h)$, where h denotes the grid scale of a discrete image domain, for example $\epsilon = 4h$.

(b) Generally σ (or the tension parameter) is of order $O(1)$, while the fitting Lagrange multiplier λ should be inversely proportional to the variance of the Gaussian noise embedded in the observed image u_0 [2–5, 19].

In Figure 4, we have demonstrated the performance of our new model and algorithm on three generic test images: peppers, the Milky Way, and the Pathfinder on the Mars by NASA (USA). For the images of peppers and the Pathfinder, we have shown the Γ-convergence output z's, while for the Milky Way in the middle, the zero level curve (i.e., the sharp transition curve) of the output z has been superimposed upon the original image u_0. (The associated Movies and Matlab codes are available from the author upon request.)

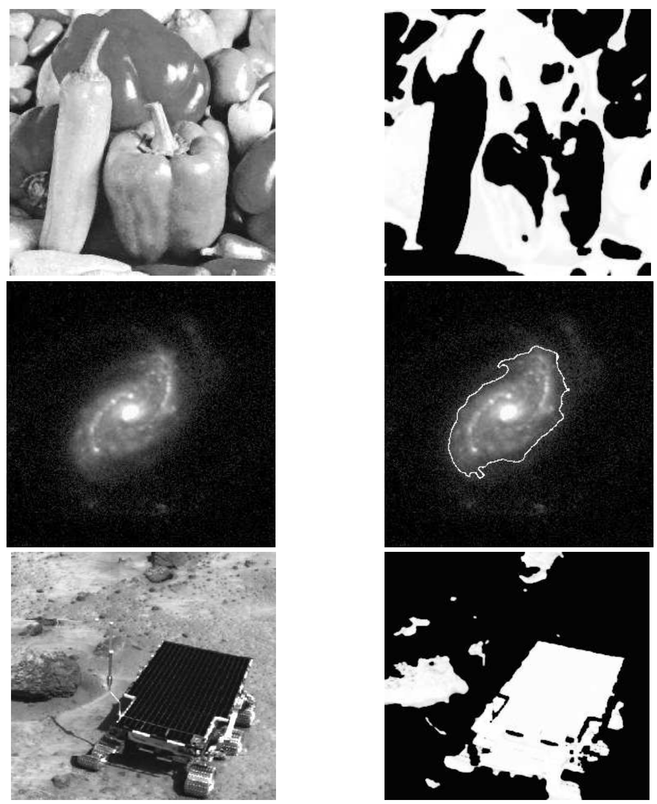

Fig. 1. Left: three generic images u_0's: peppers, the Milky Way, and the Pathfinder landed on the Mars (NASA, USA); Right: the output z's or their zero-level curves.

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