# A LEVEL SET FORMULATION FOR VISIBILITY AND ITS DISCRETIZATION

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ABSTRACT. We study an implicit visibility formulation and show that the corresponding closed form formula satisfies a dynamic programming principle, and is the viscosity solution of a Hamilton-Jacobi equation involving jump discontinuities in the Hamiltonian. We derive the corresponding discretization in multi-dimensions and prove convergence in the one dimensional case. Finally, we introduce a generalization of the original Hamilton-Jacobi equation and the corresponding efficient numerical algorithms so that visibility of an observer in non-constant media can be computed. We also introduce a specialization of the algorithms for environments in which occluders are described by the graph of a function.

## 1. INTRODUCTION

We are interested in constructing a representation of what an observer can see in a bounded domain under the presence of opaque obstacles that obstruct the "lines-of-sight" of the observer. Alternatively, this problem can be interpreted as finding the shadowed region for given point light source. We assume that the obstacles are much larger than the wavelength of visible lights and that no reflection takes place. In this perspective, we are interested in a simplified high frequency wave propagation problem of solving the wave equation in the frequency domain

$$\Delta w + \left(\frac{\omega}{c}\right)^2 w = 0,$$

with Dirichlet boundary condition at the light source $x_0$ (observing location), absorbing boundary condition on the surface of obstacle, and a single frequency solution in the form $w(\mathbf{x}) = A(\mathbf{x})\exp(i\omega S(\mathbf{x}))$. In this setting, $c(\mathbf{x})$ is the wave speed and is assumed to be 0 inside the obstacles; $\omega$ denotes the frequency of light and is assumed to be very large so that the geometrical optics theory [14] gives good approximation. In the theory of geometrical optics, to leading order as $\omega^{-1} \to 0$, the shadowed region is bounded by the family of rays that tangent the non-reflecting obstacles and the obstacles themselves. Outside of the shadowed region the eikonal equation for the phase is derived:

$$c(\mathbf{x})|\nabla S| = 1,\ S|_{\mathbf{x}_0} = 0. \tag{1.1}$$

The rays emanating from $\mathbf{x}_0$ travel along the gradient of the phase $r(\mathbf{x}) := \pm\nabla S/|\nabla S|$. If $c(\mathbf{x})$ is constant outside of the obstacle, the ray that passes through a point $\mathbf{x}$ is simply the straight half line that starts out from $\mathbf{x}_0$ and reaches $\mathbf{x}$.

We are interested in solving the wave propagation problem in a simplified setting compared to the context described in the previous paragraph. We are concerned with efficient ways of approximating the shadowed region accurately on Cartesian grids. In [22], the authors proposed a level set algorithm to do so, assuming that $c(\mathbf{x})$ is constant and the obstacles are implicitly represented by a continuous function. Their algorithm can be interpreted as solving a Hamilton-Jacobi equation that has discontinuous dependence on its solution. The

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zero level set of the solution would then correspond to the shadow boundary. In this paper, we show that this algorithm constructs the viscosity solution of the non-standard Hamilton-Jacobi equation involved, discuss the corresponding numerical methods, and generalizations in several contexts that include the case of variable wave speed.

## 2. Computing Visibility

Computing visibility has been an important task in many scientific problems. Examples include high frequency scattering problems, computing creaping waves that propagate along the scattering surfaces, and deposition problems that appear in materials science. In many of these problems, more general and complex numerical algorithms are implemented to solve equations that are more closedly related to the underlying physics. For example, Maxwell's equations can be solved, in principle, with high frequency initial data over a domain with complicated geometry. But in practice, it might not be computationally economical to do so. Some of these problems ay becomes easier to manage if the shadowed regions are precomputed. It is possible to design much simpler and robust algorithms for this simpler objective of finding the visible region, without other information such as the amount of illumination. this is what we mean by computing visibility in this paper.

Another class of prominent applications requiring visibility computtion lies in scientific visualization. In this context, the basic form of visibility computation is believed to be mostly solved, and selected algorithms have been implemented on hardware . Nevertheless, as the demand for more detailed rendering increases, the challenges posed by large datasets and near real-time computation necessitate the development of new techniques. A number of authors have addressed these challenges [7], a detailed review of which can be found in [10]. Almost all of these methods use explicit representations (such as triangles) and a variety of computational geometry techniques such as hierarchical, ordered data structures and related algorithms to reduce the number of primitives considered, e.g., [3, 2, 16][20][11][19].

Under most of the algorithms referenced above, light rays are straight lines emanating from the observing position, and a point is either visible or invisible to a given observer. This makes it difficult to represent and manipulate the occluded region on computers. In particular, this type of representation is inadequate if one needs to compute numerically the sensitivity of visibility under certain changes in the setting (e.g. observer positions). Sensitivity of visibility information can be crucial to applications of vision based surveillance and robotic planning. In [6], the authors considered a class of optimization problems involving certain functionals of the visibility information. There, optimality conditions are derived and solved numerically on the grid; the optimality equations contain "derivatives of the visibility" with respect to the observer location. It is not hard to think of situations in which rays bend due to inhomogeneity in the velocity field. Therefore, it is desirable to have a notion of visibility that is smooth enough so it can be differentiated easily by robust numerical methods, and that it can be generalized to the case of non-straight rays. This is the motivation of our current work. We will see that by considering the visibility problem under a suitable PDE approach, we can obtain several generalizations naturally and derive the corresponding numerical methods based on some new robust methods for solving Hamilton-Jacobi equations.

*Visibility in an implicit setting.* Throughout this paper, the obstacles are implicitly defined by the negative part of a function $\phi$; i,e, the location of the obstacles corresponds to the set $\{\phi \leq 0\}$. $\phi$ will be taken as the signed distance function in most applications.

A natural approach is to define visibility by the difference of the so-called geodesic distance function $u_1$, which solves (1.1) with $c(\mathbf{x}) = 0$ on $\{\phi \leq 0\}$ (where obstacles situate) and $c(\mathbf{x}) = 1$ on $\{\phi > 0\}$, and the Euclidean distance function $u_2$ that solves the eikonal equation (1.1) with $c(\mathbf{x}) \equiv 1$ on the whole domain. A point $\mathbf{y}$ is occluded from the vantage point $\mathbf{x}_0$ is $u_1(\mathbf{y}) > u_2(\mathbf{y})$, and visible if $u_1(\mathbf{y}) = u_2(\mathbf{y})$. This was the approach adopted in [1]. The shadow boundary corresponds to the boundary of the set $\{u_1 = u_2\}$ which is hard to locate to high numerical accuracy since $u_1 - u_2$ is not differentiable there. Furthermore, numerically solving eikonal equations to a desired quality on a grid with infinite index of refraction, $c^{-1}$, is a delicate problem.

In [22], the authors propose to allow rays to propagate through the entire domain as if there was no obstacle. The visibility information is instead encoded by a continuous function $\psi$ defined by

$$\psi(\mathbf{x};\mathbf{x}_o) := \min_{t\in[0,1]} \phi\left(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0)\right). \tag{2.1}$$

This formula prescribes $\psi(\mathbf{x};\mathbf{x}_o)$ to be the minimum value of $\phi$ along the line segment connecting $\mathbf{x}$ and $\mathbf{x}_o$. A point is occluded if this function is negative there, and vice versa. This function is obviously Lipschitz continuous and hence it is smooth almost everywhere, including, in particular, over the shadow boundaries. The problem of handling discontinuities and singularities in solving eikonal equations on the grid is circumvented. We can thereby differentiate and integrate visibility with suitable accuracy by robust numerical methods developed for the Level Set Method [18, 17]. For instance, the distance function to the shadow boundaries can easily be constructed by an application of the fast sweeping method [23] as a useful byproduct. In fact, in the following, we propose a Hamilton-Jacobi equation and show that $\psi$ defined in (2.1) is the corresponding viscosity solution of the propsed equation; we show that $\psi$ can be constructed by a fast sweeping method.

In this paper, we discuss some properties of the notion of visibility defined in (2.1) and the corresponding numerical schemes. We show that $\psi$ satisfies a dynamic programming principle, and is the unique solution to an integral equation as well as the viscosity solution to a Hamilton-Jacobi equation with discontinuous dependence on the solution. This equation involves a Heaviside function $H$ that is dependent on the solution ($H(u) = 1$ if $u \geq 0$, $H = 0$ otherwise), and therefore, established analysis for conventional Hamilton-Jacobi equations does not applied directly. We derive an upwind discretization of the nonlinear Hamilton-Jacobi equation (4.2). We shall see that $H$ being upper semi-continuous plays a crucial factor in both the analytical properties of the solution and the numerical discretization. Comparing with the eikonal approach mentioned above, which has to handle discontinuity along some surfaces over the numerical grid, we treat the discontinuities in the evaluations of parts of the equations and thus avoid the possible lost of accuracy due to grid resolution. We show convergence of the resulting numerical solutions in the one dimensional case. Finally, we consider some generalizations that include visibility computation under curved ray paths and a new efficient visibility algorithm for environments in which occluders can be described by the graph of a function. The corresponding differential equations and discretizations are derived. Numerical examples in one and two dimensions, including numerical convergence tests both these new algorithms are presented.

## 3. PROPERTIES OF $\psi$

We assume that the wave speed is constant in the whole domain so that rays are straight lines. We first prove that $\psi$ defined in (2.1) satisfies a dynamic programming principle, and is the solution to an integral equation as well as a Hamiton-Jacobi equation.

We begin be some necessary lemmas and definitions. We denote the left and right partial derivatives of a differentiable function$v:\mathbb{R}^d\mapsto\mathbb{R}$ by

$$\partial^{\pm}_{x_j}v(\mathbf{x})=\lim_{h\to 0\pm}\frac{v(x_1,\cdots,x_j+h,x_{j+1},\cdots,x_d)-v(\mathbf{x})}{h},$$

where $\mathbf{x}=(x_1,x_2,\cdots,x_d)$ is a vector in $\mathbb{R}^d$. Correspondingly, we define the upwind gradient of a function $v$ related to the unit vector $\mathbf{r}=(r_1,r_2,\cdots,r_d)$ as

$$\nabla^r v=\begin{pmatrix} sgn(-r_1)\partial^+_{x_1}v+sgn(r_1)\partial^-_{x_1}v \\ sgn(-r_2)\partial^+_{x_2}v+sgn(r_2)\partial^-_{x_2}v \\ \vdots \\ sgn(-r_d)\partial^+_{x_d}v+sgn(r_d)\partial^-_{x_d}v \end{pmatrix}.$$

Also, we define one-sided directional derivative of $v$ in the direction $\mathbf{r}$ as

$$v_{r+}(\mathbf{x})=\lim_{h\to 0+}\frac{v(\mathbf{x}+h\mathbf{r})-v(\mathbf{x})}{h},$$

$$v_{r-}(\mathbf{x})=\lim_{h\to 0+}\frac{v(\mathbf{x})-v(\mathbf{x}-h\mathbf{r})}{h}.$$

Of course when $v$ is differentiable in the direction of $\mathbf{r}$, each respective one-sided notion above is equivalent to its counter part; i.e.

$$v_{r+}(\mathbf{x})=v_{r-}(\mathbf{x}),$$

and we will use $v_{\mathbf{r}}$ and $\nabla v$ to denote, respectively, the directional derivative and the gradient of $v$.

**Lemma 3.1.** *Let $v$ be a function in $C^1(\mathbb{R}^d)$. Given $\mathbf{r}=(r_1,\cdots,r_d)\in\mathbb{R}^d$.*

$$v_{r-}(\mathbf{x})=\lim_{h\to 0+}\frac{v(\mathbf{x})-v(\mathbf{x}-h\mathbf{r})}{h}\le 0 \implies \nabla^r v\cdot\mathbf{r}\le 0.$$

*Proof.* By manipulation,

$$\begin{aligned} v(\mathbf{x})-v(\mathbf{x}-h\mathbf{r}) &= v(\mathbf{x})-v(x_1-hr_1,x_2,\cdots,x_d) \\ &\quad +v(x_1-hr_1,x_2,\cdots,x_d)-v(x_1-hr_1,x_2-hr_2,\cdots,x_d) \\ &\quad +v(x_1-hr_1,x_2-hr_2,\cdots,x_d)-\cdots \\ &\quad +v(x_1-hr_1,\cdots,x_d)-v(x_1-hr_1,\cdots,x_d-hr_d). \end{aligned}$$

By continuity, taking $h\to 0+$, we have the inequality. □

The following theorems show that the solution has a dynamic programming principle.

**Theorem 3.2.** *For any point $\mathbf{y}$ bounded between $\mathbf{x}_0$ and $\mathbf{x}$,*

$$\psi(\mathbf{x};\mathbf{x}_0)=\min(\psi(\mathbf{x};\mathbf{x}_0),\min_{t\in[0,1]}\phi(\mathbf{y}+t(\mathbf{x}-\mathbf{y}))). \tag{3.1}$$

*Proof.* Since $\psi(\mathbf{x};\mathbf{x}_o) = \min_{t\in[0,1]} \phi(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0))$, we have

$$\psi(\mathbf{x};\mathbf{x}_o) = \min\left(\min_{t\in[0,t*]} \phi(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0)),\ \min_{t\in[t*,1]} \phi(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0))\right)$$

for any $t^*$ in [0,1]. Note that when $t^* = 0$ or 1, we have the original definition of $\psi$, (2.1). Denote $\mathbf{y} = \mathbf{x}_0 + t^*(\mathbf{x}-\mathbf{x}_0)$, $t_1 = t/t*$, and $t_2 = (t-t^*)/(1-t*)$. We have

$$\psi(\mathbf{x};\mathbf{x}_o) = \min\left(\min_{t_1\in[0,1]} \phi(\mathbf{x}_0 + t_1(\mathbf{y}-\mathbf{x}_0)),\ \min_{t_2\in[0,1]} \phi(\mathbf{y} + t_2(\mathbf{x}-\mathbf{y}))\right).$$

Thus

$$\psi(\mathbf{x};\mathbf{x}_0) = \min(\psi(\mathbf{y};\mathbf{x}_0), \min_{t\in[0,1]} \phi(\mathbf{y}+t(\mathbf{x}-\mathbf{y}))).$$

□

**Lemma 3.3.** *Let $\phi : \mathbb{R}^d \mapsto \mathbb{R}$ be a bounded and Lipschitz continuous function, and $\psi$ be defined as in (2.1). Let $\mathbf{r}$ be the vector field defined by $(\mathbf{x}-\mathbf{x}_0)/|\mathbf{x}-\mathbf{x}_0|$. If $\psi(\mathbf{x}) = \phi(\mathbf{x})$ at some point $\mathbf{x} \neq \mathbf{x}_0$ then*

$$\psi_{r-} \geq \phi_{r-},$$

$$\psi_{r+} \leq \phi_{r+},$$

*Proof.* By definition of $\psi(\mathbf{y}) \leq \phi(\mathbf{y})$ for all $\mathbf{y} \in \mathbb{R}^d$. So

$$\psi(\mathbf{x}) - \psi(\mathbf{x}-h\mathbf{r}) = \phi(\mathbf{x}) - \psi(\mathbf{x}-h\mathbf{r}) \geq \phi(\mathbf{x}) - \phi(\mathbf{x}-h\mathbf{r}).$$

Similarly,

$$\psi(\mathbf{x}+h\mathbf{r}) - \psi(\mathbf{x}) = \psi(\mathbf{x}+h\mathbf{r}) - \phi(\mathbf{x}) \leq \phi(\mathbf{x}+h\mathbf{r}) - \phi(\mathbf{x}).$$

Taking the limit $h \to 0+$, we obtain the stated inequalities. □

**Lemma 3.4.** *The following are true:*

(1) If $\phi_{r+}(\mathbf{x}) > 0$ then $\psi_{r+}(\mathbf{x}) = 0$;
(2) If $\phi_{r-}(\mathbf{x}) > 0$ then $\psi_{r-}(\mathbf{x}) = 0$;
(3) if $\psi(\mathbf{x}) < \phi(\mathbf{x})$, then $\psi_{r+}(\mathbf{x}) = \psi_{r-}(\mathbf{x}) = 0$.

*Proof.* We prove 1 and 2 together. If $\phi_{r\pm}(\mathbf{x}) > 0$, then for sufficiently small $0 < h \leq h_0$,$\pm\phi(\mathbf{x}\pm h\mathbf{r}) > \pm\phi(\mathbf{x})$, and consequently, by definition 2.1, $\psi(\mathbf{x}) = \psi(\mathbf{x}\pm h\mathbf{r})$. Hence $\psi_{r\pm}(\mathbf{x}) = 0$.

For the case $\psi(\mathbf{x}) < \phi(\mathbf{x})$, let $\psi(\mathbf{x}) < \phi(\mathbf{x}) - \delta$ for some $\delta > 0$. By continuity of $\phi$, there exists $h_\delta > 0$ such that $|\phi(\mathbf{x}+h\mathbf{r}) - \phi(\mathbf{x})| < \delta$ for all $0 \leq |h| < h_\delta$. I.e.$\psi(\mathbf{x}) < \phi(\mathbf{x}+h\mathbf{r})$ for all $0 \leq |h| < h_\delta$. By (3.2), $\psi(\mathbf{x}+h\mathbf{r}) = \psi(\mathbf{x})$ and $\psi_{r\pm}(\mathbf{x}) = 0$. □

**Lemma 3.5.** *If $\phi_{r+}(\mathbf{x}) < 0$, and $\psi(\mathbf{x}) = \phi(\mathbf{x})$, then $\psi_{r+}(\mathbf{x}) = \phi_{r+}(\mathbf{x})$.*

*Proof.* If $\phi_{r+}(\mathbf{x}) < 0$, then there exist a real number $h_0$ s.t. $\phi(\mathbf{x}+h\mathbf{r}) < \phi(\mathbf{x})$ for $0 < h \leq h_0$. So $\min_{y\in[\mathbf{x},\mathbf{x}+h_0\mathbf{r}]} \phi(y) = \phi(\mathbf{x}+h_0\mathbf{r})$. By (3.2), $\psi(\mathbf{x}+h\mathbf{r}) = \psi(\mathbf{x}) = \phi(\mathbf{x})$, and

$$\psi_{r+}(\mathbf{x}) = \lim_{h\to 0^+} \frac{\psi(\mathbf{x}+h\mathbf{r}) - \psi(\mathbf{x})}{h} = \lim_{h\to 0^+} \frac{\phi(\mathbf{x}+h\mathbf{r}) - \phi(\mathbf{x})}{h} = \phi_{r+}(\mathbf{x}).$$

□

**Lemma 3.6.** *Let $v \in C^\infty(\mathbb{R})$ and $u$ be a Lipschitz continuous function on $\mathbb{R}$. If $\mathbf{x}$ be a local maximum of $u - v$, then*

$$u_{r+}(\mathbf{x}) \le v_{\mathbf{r}}(\mathbf{x}) \le u_{r-}(\mathbf{x}).$$

*Conversely, if $\mathbf{x}$ is a local minimum of $u - v$,*

$$u_{r-}(\mathbf{x}) \le v_{\mathbf{r}}(\mathbf{x}) \le u_{r+}(\mathbf{x}).$$

*Proof.* Since $u$ is Lipschitz continuous, $u$ is differentiable almost everywhere and $u_{r\pm}$ exist. Assume $\mathbf{x}$ is a local maximum of $u - v$, then $\exists h_0$ such that for $|h| < h_0$,

$$(u-v)|_{\mathbf{x}+h\mathbf{r}} - (u-v)|_{\mathbf{x}} \le 0 \implies \begin{cases} u_{r+}(\mathbf{x}) - v_{\mathbf{r}}(\mathbf{x}) \le 0, \\ u_{r-}(\mathbf{x}) - v_{\mathbf{r}}(\mathbf{x}) \ge 0. \end{cases}$$

Hence $u_{r+}(\mathbf{x}) \le v_{\mathbf{r}}(\mathbf{x}) \le u_{r-}(\mathbf{x})$. The inequalities at a local minimum of $u - v$ can be obtained in a similar fashion. □

3.1. **Integral Formulation.** We use $H(x)$ to denote the upper semi-continuous Heaviside function:

$$H(x) = \begin{cases} 1, & x \ge 0, \\ 0 & x < 0. \end{cases}$$

For simplicity of exposition, we shall use $z^-$ to denote $\min(z, 0)$.

**Theorem 3.7.** *Given $\phi \in C^1(\mathbb{R}^d)$ such that $\phi$ has finite number of extrema in any bounded interval, $\psi(\mathbf{x};\mathbf{x}_0) = \min_{t\in[0,1]} \phi(\mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0))$ satisfies the integral equation*

$$u(\mathbf{x};\mathbf{x}_0) = \phi(\mathbf{x}_0;\mathbf{x}_0) + \int_0^1 H\left(u(\mathbf{x}_t) - \phi(\mathbf{x_t})\right)\left(\phi_r(\mathbf{x_t})\right)^- |\mathbf{x} - \mathbf{x}_0| dt, \tag{3.2}$$

*where $\mathbf{x}_t = \mathbf{x}_0 + t(\mathbf{x} - \mathbf{x}_0)$.*

*Proof.* Denote Since $\psi(\mathbf{x}_t) - \phi(\mathbf{x}_t)$ is Lipschitz in $t$, we can construct $[0,1] = \bigcup_{k=0}^M \bar{I_k}$, where $I_k = (t_k, t_{k+1})$,$0 = t_0 < t_1 < \cdots < t_M = 1$ such that $\psi - \phi$ is strictly negative in $I_j$ and zero in $I_{j\pm1}$ (since $\psi \le \phi$ by definition). By continuity, $\psi - \phi \equiv 0$ in $\bar{I}_{j\pm1}$. Without loss of generality, we assume that $\psi - \phi \equiv 0$ in $I_0$. (In this case, $\phi \in C^1$ implies $\psi \in C^1$ in $I_0$.) Otherwise, we can always choose an empty set as $I_0$. We then evaluate the integral in (3.2) for $t_0 \le t \le t_1$, since $\phi - \psi \equiv 0$ in $I_0$, and $\psi$ is always non-increasing ( $\psi_r(\mathbf{x}_t) \le 0$),

$$\begin{aligned} \int_0^t H\left(\psi(\mathbf{x}_\tau) - \phi(\mathbf{x}_\tau)\right)\left(\phi_{r-}(\mathbf{x}_\tau)\right)^- |\mathbf{x}_t - \mathbf{x}_0| d\tau &= \int_0^t \left(\phi_{r-}(\mathbf{x}_\tau)\right)^- |\mathbf{x}_t - \mathbf{x}_0| d\tau \\ = \int_0^t \min\left(\phi_r(\mathbf{x}_\tau), 0\right) |\mathbf{x}_t - \mathbf{x}_0| d\tau &= \int_0^t \min\left(\psi_r(\mathbf{x}_\tau), 0\right) |\mathbf{x}_t - \mathbf{x}_0| d\tau = \psi(\mathbf{x}_t) - \psi(\mathbf{x}_0). \end{aligned}$$

Therefore,

$$\psi(\mathbf{x}_t) = \phi(\mathbf{x}_0) + \int_0^t H\left(\psi(\mathbf{x}_\tau) - \phi(\mathbf{x}_\tau)\right)\left(\phi_r(\mathbf{x}_\tau)\right)^- |\mathbf{x}_\tau - \mathbf{x}_0| d\tau, \ \mathbf{x}_t \in I_0.$$

For $t_1 < t < t_2$, $\phi - \psi > 0$,

$$\begin{aligned} \int_0^t H\left(\psi(\mathbf{x}_\tau) - \phi(\mathbf{x}_\tau)\right)\left(\phi_r(\mathbf{x}_\tau)\right)^- |\mathbf{x}_\tau - \mathbf{x}_0| d\tau &= \int_0^{t_1} + \int_{t_1}^t H\left(\psi(\mathbf{x}_\tau) - \phi(\mathbf{x}_\tau)\right)\left(\phi_r(\mathbf{x}_\tau)\right)^- |\mathbf{x}_\tau - \mathbf{x}_0| d\tau \\ &= \psi(\mathbf{x}_{t_1}) - \psi(\mathbf{x}_0) + \int_{t_1}^t 0 \cdot \left(\phi_r(\mathbf{x}_\tau)\right)^- |\mathbf{x}_\tau - \mathbf{x}_0| d\tau \\ &= \psi(\mathbf{x}_{t_1}) - \phi(\mathbf{x}_0) \end{aligned}$$

Now we want to prove $\psi(\mathbf{x}_t) = \psi(\mathbf{x}_{t_1})$ for $\mathbf{x}_t \in I_1$. Due to the continuity of $\phi$, $\psi(\mathbf{x}_t) = \phi(\mathbf{x}_{t_*})$ for some $t_* \in [t_1, t]$. Suppose that $t_* > t_1$. By definition, $\psi(\mathbf{x}_t) \leq \psi(\mathbf{x}_{t_*})$. But $\psi(\mathbf{x}_t) = \phi(\mathbf{x}_{t_*}) > \psi(\mathbf{x}_{t_*})$, therefore, $t_*$ must be $t_1$ by contradiction. Then $\psi(\mathbf{x}_t) = \phi(\mathbf{x}_{t_1}) = \psi(\mathbf{x}_{t_1})$.

Thus $\psi$ satisfies (3.2) in $\bar{I}_1 \bigcup I_2$. We may continue this calculation inductively, using $\phi(\mathbf{x}_{t_{2k}})$ as new initial value, and the claim is proved. $\square$

**Lemma 3.8.** *Assume that $\phi \in C^1$ is strictly decreasing in the interval $I_0 = [\mathbf{x}_0, \mathbf{x}_1]$. If $u$ satisfies Equation (3.2) and $u(\mathbf{x}_0) = \phi(\mathbf{x}_0)$, then $u(\mathbf{x}) = \phi(\mathbf{x})$ for all $\mathbf{x} \in I_0$.*

*Proof.* If $u$ satisfies (3.2), then $u$ is continuous and monotonically decreasing in $I_0$. Since $u(\mathbf{x}_0) = \phi(\mathbf{x}_0)$ , $\phi_r(\mathbf{x}_0) < 0$, and $H(u-\phi) \leq 1$, we have

$$\begin{aligned} \phi(\mathbf{y}) &= \phi(\mathbf{x}_0) + \int_0^{\mathbf{1}} 1 \cdot \phi_r(\mathbf{x}_0 + t(\mathbf{y}-\mathbf{x}_0))|\mathbf{y}-\mathbf{x}_0| dt \\ &\leq u(\mathbf{x}_0) + \int_0^1 H(u-\phi)\min(\phi_r, 0)|\mathbf{y}-\mathbf{x}_0| dt = u(\mathbf{y}). \end{aligned}$$

Assume that $u(\mathbf{y}) > \phi(\mathbf{y})$ for some $\mathbf{y} \in I_0$. Define $t_* = \inf\{t \in [0,1] : u(\mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0)) > \phi(\mathbf{x}_0 + t(\mathbf{x_1} - \mathbf{x}_0))\}$. This implies $u(\mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0)) = \phi(\mathbf{x}_0 + t_*(\mathbf{x}_1 - \mathbf{x}_0))$ for $0 \leq t \leq t_*$. Let $\mathbf{y}_l = \mathbf{x}_0 + t_*(\mathbf{x}_1 - \mathbf{x}_0)$, and $\mathbf{y} = \mathbf{x}_0 + t^{'}(\mathbf{x}_1 - \mathbf{x}_0)$. Since $\phi_r < 0$ in $I_0$,$\min(\phi_r, 0) = \phi_r$, and

$$\begin{aligned} \phi(\mathbf{y}) &= \phi(\mathbf{y}_l) + \int_{t_*}^{t^{'}} 1 \cdot \phi_r(\mathbf{x}_0 + t(\mathbf{y}-\mathbf{x}_0))|\mathbf{y}-\mathbf{x}_0| dt \\ &= u(\mathbf{y}_l) + \int_{t_*}^{t^{'}} H(u-\phi)\min(\phi_r, 0)|\mathbf{y}-\mathbf{x}_0| dt \\ &= \phi(\mathbf{x}_0) + \int_0^{t_*} + \int_{t_*}^{t^{'}} H(u-\phi)\min(\phi_r, 0)|\mathbf{y}-\mathbf{x}_0| dt = u(\mathbf{y}). \end{aligned}$$

We have a contradiction. Together with the previous inequality, $u = \phi$ in $I_0$. $\square$

**Theorem 3.9.** *Given $\phi \in C^1(\mathbb{R}^d)$ such that $\phi$ is finite number of extrema in any bounded interval. If $u_1$ and $u_2$ be two solutions of Equation (3.2), then $u_1(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0)) = u_2(\mathbf{x}_0 + t(\mathbf{x}-\mathbf{x_0}))$ for $0 \leq t < 1$ .*

*Proof.* Define $\mathbf{x}_t = \mathbf{x}_0 + t(\mathbf{x}-\mathbf{x}_0)$. We partition $[0,1] = \bigcup_{k=0}^{x_k} \bar{I}_k$, where $I_k = (t_k, t_{k+1})$, such that $\phi(\mathbf{x_t})$ is strictly decreasing for $t \in I_{2j}$ and strictly increasing for $t \in \underline{I}_{2j}$+1, $j = 0,1,2,\cdots$. Without loss of generality, we assume that $\phi$ is strictly decreasing in $\bar{I}_0$. Lemma 3.8 shows that $u_1 = u_2 = \phi$ in $\bar{I}_0$. Since $\phi$ is strictly increasing in $I_1$, if $u_1(\mathbf{x}) = u_2(\mathbf{x})$, then for (3.2) implies that $u_1(\mathbf{x}) = u_2(\mathbf{x})$ all $\mathbf{x} \in \bar{I}_1$.

Now, consider $u_1$ and $u_2$ at $x_{t_3}$. If $\phi(\mathbf{x}_{t_3}) \leq \phi(\mathbf{x}_{t_1}) = u_1(\mathbf{x}_{t_2}) = u_2(\mathbf{x}_{t_2})$, then by continuity, the hypothesis on the extrema, the monotone decrease of $\phi$ in $I_2$, there is a unique point $\mathbf{x}_{t_*}$, such that $\mathbf{x}_{t_2} \leq \mathbf{x} \leq \mathbf{x}_{t_3}$ and $\phi(\mathbf{x}_{t_*}) = \phi(\mathbf{x}_{t_1})$. Moreover, $\phi(\mathbf{x}_t) \leq \phi(\mathbf{x}_*)$ for $t_2 \leq t \leq t_*$. So by (3.2), $u_1(\mathbf{x}_t) = u_1(\mathbf{x}_{t_1})$ for $t_2 \leq t \leq t_*$, and $u_1(\mathbf{x}_t) = \phi(\mathbf{x}_t)$ for $t_* \leq t < t_{t_3}$. Clearly, for this case, $u_2 \equiv u_1$ in $I_2$.

If $\phi(\mathbf{x}_{t_3}) > \phi(\mathbf{x}_{t_1}) = u_1(\mathbf{x}_{t_2}) = u_2(\mathbf{x}_{t_2})$, then the right hand side of (3.2) is 0 and $u_1 \equiv u_2$ in $I_2$.

Proceed iteratively, using similar arguments, we show that $u_1 \equiv u_2$. $\square$

This integral equation implies that $u$ decreases by the rate $d\phi(\mathbf{x})/d\mathbf{r}$, $\mathbf{r} = (\mathbf{x}-\mathbf{x}_0)/|\mathbf{x}-\mathbf{x}_0|$, if it is negative and $u \leq \phi$.

3.2. **Differential formulation.** The corresponding differential equation in $\mathbb{R}^d$ is as following

$$\nabla_{\mathbf{x}} u(\mathbf{x};\mathbf{x}_0)\cdot\mathbf{r}(\mathbf{x}) = H\left(u(\mathbf{x};\mathbf{x}_0)-\phi(\mathbf{x})\right)\min(\nabla\phi\cdot\mathbf{r},0). \tag{3.3}$$

This equation described how the directional derivative should change so that $u$ is decreasing in the ray direction whenever $\phi$ is decreasing in that direction and the value of u is no less than $\phi$.

We note that the usual viscosity solution theory[15, 9, 5, 8] for first order nonlinear equations, because of the discontinuous dependence on the solution $u$. The Viscosity solution theory of Hamilton-Jacobi equations with discontinuous coefficients was developed by H. Ishii [12], and can also be found in for [4, 5]. We repeat the definition here for our exposition.

**Definition 3.10.** Let $\phi$ be a Lipschitz continuous function in $\mathbb{R}^d$ and $\phi(\mathbf{x}_0)>0$. We consider the Dirichlet boundary value problem

$$\begin{cases} F(\mathbf{x},u,\nabla u)=0 \quad \text{in}\,\mathbb{R}^d\setminus\{\mathbf{x}_0\},\\ u(\mathbf{x}_0)=\phi(\mathbf{x}_0), \end{cases} \tag{3.4}$$

where

$$F(\mathbf{x},u,\mathbf{p}) \quad = \mathbf{p}\cdot\mathbf{r}(\mathbf{x}) - \quad H(u(\mathbf{x})-\phi)\min(\nabla\phi\cdot\mathbf{r},0)$$

is piecewise continuous in $\mathbf{x}$, and $\mathbf{r}(\mathbf{x})$ is smooth in $\mathbb{R}^d\setminus\{\mathbf{x}_0\}$. Let $v\in C^\infty$ be a test function.

(1) $u$ is a viscosity super-solution if $F^*(\mathbf{x},u(\mathbf{x}),\nabla v(\mathbf{x}))\geq 0$ at local minima of $u-v$;
(2) $u$ is a viscosity sub-solution if $F_*(\mathbf{x},u(\mathbf{x}),\nabla v(\mathbf{x}))\leq 0$ at local maxima of $u-v$;
(3) $u$ is a viscosity solution if it is both a viscosity sub-solution and super-solution.

Here, $F_*$ and $F^*$ are respectively the lower and upper semi-continuous envelope of $F$ with respect to its second argument; i.e.

$$F_*(\mathbf{x},u,\mathbf{p}) = \liminf_{y\to u} F(\mathbf{x},y,\mathbf{p})$$

and

$$F^*(\mathbf{x},u,\mathbf{p}) = \limsup_{y\to u} F(\mathbf{x},y,\mathbf{p}).$$

We shall prove that in the one dimensional setting, $u(\mathbf{x})$, defined in (2.1), is the viscosity sup- and sub-solution of (3.2).

**Theorem 3.11.** *Consider the problem:*

$$\begin{cases} F(\mathbf{x},u,\nabla u)=\nabla u\cdot\mathbf{r}-H(u-\phi)\min(\nabla\phi\cdot\mathbf{r},0)=0 \quad \mathbb{R}^d\setminus\{\mathbf{x}_0\}\\ u(\mathbf{x}_0)=\phi(\mathbf{x}_0) \end{cases} \tag{3.5}$$

$\psi(\mathbf{x};\mathbf{x}_o):=\min_{t\in[0,1]}\phi\left(\mathbf{x}_0+t(\mathbf{x}-\mathbf{x}_0)\right)$ *is the viscosity sup- and sub-solution of (3.2) under Definition 3.10. Here* $\mathbf{r}(\mathbf{x})=(\mathbf{x}-\mathbf{x}_0)/|\mathbf{x}-\mathbf{x}_0|$.

*Proof.* Let $v$ be an arbitrary function in $C^\infty(\mathbb{R})$. Let $\mathbf{x}$ be a local extremum of $\psi-v$. By definition, $\psi\leq\phi$ for all $\mathbf{x}\in\mathbb{R}^d$. We break the evaluation of $F$ down to the cases $\psi(\mathbf{x})=\phi(\mathbf{x})$ and $\psi(\mathbf{x})<\phi(\mathbf{x})$.

(1) $\psi(\mathbf{x})=\phi(\mathbf{x})$: We first observe that $\phi_r(\mathbf{x})=\nabla\phi(\mathbf{x})\cdot\mathbf{r}(\mathbf{x})\leq 0$, otherwise, for sufficiently small $h>0$, $\phi(\mathbf{x}-h\mathbf{r}(\mathbf{x}))<\phi(\mathbf{x})$. Since $\psi(\mathbf{x}-h\mathbf{r}(\mathbf{x}))\leq\phi(\mathbf{x}-h\mathbf{r}(\mathbf{x}))$, we have

$\psi(\mathbf{x}-h\mathbf{r}(\mathbf{x})) < \psi(\mathbf{x})$ which contradicts the definition of $\psi$. If $\psi - v$ reaches a local minimum at $\mathbf{x}$, $F^*(\mathbf{x},\psi(\mathbf{x}),\nabla v(\mathbf{x})) = v_r(\mathbf{x}) - \phi_r(\mathbf{x})$. By Lemma 3.6, $v_r(\mathbf{x}) \geq \psi_{r-}(\mathbf{x})$, while by Lemma 3.3, $\psi_{r-}(\mathbf{x}) \geq \phi_{r-}(\mathbf{x})$. So

$$F^*(\mathbf{x},\psi(\mathbf{x}),\nabla v(\mathbf{x})) = v_r(\mathbf{x}) - \phi_r(\mathbf{x}) \geq \psi_{r-}(\mathbf{x}) - \phi_{r-}(\mathbf{x}) \geq 0.$$

If $\psi - v$ reaches a local maximum at $\mathbf{x}$, then by Lemma 3.3,

$$\psi_{r+}(\mathbf{x}) \leq v_r(\mathbf{x}) \leq \psi_{r-}(\mathbf{x}) \leq 0,$$

we have $F_*(\mathbf{x},\psi,\nabla(\mathbf{x})) = v_r(\mathbf{x}) \leq 0$.

(2) $\psi(\mathbf{x}) < \phi(\mathbf{x})$: $F(\mathbf{x},\psi(\mathbf{x}),v_r(\mathbf{x})) = v_r(\mathbf{x})$. From Lemma 3.6, we have

$$\min(\psi_{r+}(\mathbf{x}),\psi_{r-}(\mathbf{x})) \leq v_r(\mathbf{x}) \leq \max(\psi_{r+}(\mathbf{x}),\psi_{r-}(\mathbf{x})).$$

However, Lemma 3.4 implies $\psi_{r\pm}(\mathbf{x}) = 0$ and consequently, $F(\mathbf{x},\psi(\mathbf{x}),v_r(\mathbf{x})) = v_r(\mathbf{x}) = 0$ at extremum. Therefore, $\psi$ is automatically both a sub- and super-solution. □

## 4. GENERALIZATIONS AND DISCRETIZATIONS

4.1. **Visibility of a single observer in inhomogeneous ray fields.** Formula (2.1) can be generalized to visibility in nonhomogeneous ray fields, in which rays are not straight lines anymore. Consider a medium with a smooth nonconstant index of refraction $\eta(x)$. In the setting of geometrical optics [14], the solution of the eikonal equation

$$|\nabla S| = \eta(\mathbf{x}),\ S(\mathbf{x}_o) = 0$$

determines the velocity of rays emanating from $\mathbf{x}_o$. Thus, a ray passing through a point $y$ is the integral curve of the field $\mathbf{r}(\mathbf{x}) = \nabla S(\mathbf{x})/|\nabla S(\mathbf{x})|$ connecting $\mathbf{x}_o$ to $\mathbf{y}$. Denote the segment of this ray between $\mathbf{x}_o$ and $\mathbf{y}$ by $\mathcal{L}(\mathbf{x}_o,\mathbf{y})$. We define the visibility of $\mathbf{x}_o$ as

$$\psi(\mathbf{y};\mathbf{x}_o) = \min_{\mathbf{z}\in\mathcal{L}(\mathbf{x}_o,\mathbf{y})} \phi(\mathbf{z}). \tag{4.1}$$

This is a generalization of Formula (2.1) and defines the visibility function as the minimum value of $\phi$ along each ray $\mathcal{L}(\mathbf{x}_o,\mathbf{y})$.

For general $n$ dimensions, we have

$$\nabla^r u \cdot \mathbf{r} = H(u-\phi)\min(\phi_r,0) \tag{4.2}$$

where $\mathbf{r}(\mathbf{x}) = \nabla S/|\nabla S|$. We may factor out $|\nabla S|$ from both sides of the equation without effecting he solution. So in the following, we shall use $\mathbf{r}(\mathbf{x}) = \nabla S(\mathbf{x})$ instead.

4.2. **Numerical algorithms.** We discretize (4.2) with the standard upwinding finite differencing:

$$\sum_{\nu=1}^{d} r_\nu^+ D_{x_\nu}^- u + r_\nu^- D_{x_\nu}^+ u = H(u-\phi)\min(\sum_{\nu=1}^{d} r_\nu^+ D_{x_\nu}^- \phi + r_\nu^- D_{x_\nu}^+ \phi, 0). \tag{4.3}$$

We first show that this discretization is equivalent to the algorithm proposed in [22]. We then prove the convergence of this algorithm in one dimension. We shall use the integral formula of the analytical solution (3.2).

It suffices to show the equivalence in two space dimensions with $r = (r_1 > 0, r_2 > 0)$. Formula (4.3) reduces to

$$r_1 \frac{u_{i,j} - u_{i-1,j}}{h} + r_2 \frac{u_{i,j} - u_{i,j-1}}{h} = H(u_{i,j} - \phi_{i,j})(r_1 \frac{\phi_{i,j} - \phi_{i-1,j}}{h} + r_2 \frac{\phi_{i,j} - \phi_{i,j-1}}{h})^-, \tag{4.4}$$

and
(4.5)
$$u_{i,j} = \frac{1}{r_1+r_2}\left(r_1 u_{i-1,j} + r_2 u_{i,j-1} + hH(u_{i,j}-\phi_{i,j})(r_1\frac{\phi_{i,j}-\phi_{i-1,j}}{h} + r_2\frac{\phi_{i,j}-\phi_{i,j-1}}{h})^-\right).$$

In solving the above equation (4.5) for $u_{i,j}$, we may assume that $u_{i-1,j}$ and $u_{i,j-1}$ are identical to the analytical solution, and therefore we have $u_{i-1,j} \le \phi_{i-1,j}$ and $u_{i,j-1} \le \phi_{i,j-1}$.

Apparently, the solution of $u_{i,j}$ depends on the possible evaluations of $H(u_{i,j}-\phi_{i,j})$. If $H(u_{i,j}-\phi_{i,j}) = 0$, i.e. $\phi_{i,j} > u_{i,j}$, we have

$$u_{i,j} = \frac{1}{r_1+r_2}\left(r_1 u_{i-1,j} + r_2 u_{i,j-1}\right), \tag{4.6}$$

which is identical to the solution of the upwind discretization of $\nabla u \cdot \mathbf{r} = 0$. If (4.6) yields $u_{i,j} < \phi_{i,j}$, it is a solution to (4.5). However, if the value $u_{i,j}$ computed above does not satisfy the hypothesis that $\phi_{i,j} > u_{i,j}$, then we conclude that the case $H(u_{i,j}-\phi_{i,j}) = 0$ is not possible, and we should solve (4.5) under the hypothesis that $H(u_{i,j}-\phi_{i,j}) = 1$. In this case, $\phi_{i,j} \le u_{i,j}$, we have

$$u_{i,j} = \frac{1}{r_1+r_2}\left(r_1 u_{i-1,j} + r_2 u_{i,j-1} + h(r_1\frac{\phi_{i,j}-\phi_{i-1,j}}{h} + r_2\frac{\phi_{i,j}-\phi_{i,j-1}}{h})^-\right). \tag{4.7}$$

Consequently,

$$\begin{aligned} u_{i,j} &\le \frac{1}{r_1+r_2}\left(r_1 u_{i-1,j} + r_2 u_{i,j-1} + r_1(\phi_{i,j}-\phi_{i-1,j}) + r_2(\phi_{i,j}-\phi_{i,j-1})\right) \\ &\le \phi_{i,j}, \end{aligned}$$

and the only possibility is $u_{i,j} = \phi_{i,j}$. Thus, we see that the solution of (4.5) can be constructed by performing the two steps as proposed in [22]:

(1) Solve $\nabla u \cdot \mathbf{r} = 0$ at $\mathbf{x}_{i,j}$ by upwinding, and set the solution to $u^{\text{tmp}}_{i,j}$. In the settings above,

$$u^{\text{tmp}}_{i,j} = \frac{1}{r_1+r_2}\left(r_1 u_{i-1,j} + r_2 u_{i,j-1}\right). \tag{4.8}$$

(2) Update:

$$u_{i,j} = \min(u^{\text{tmp}}_{i,j}, \phi_{i,j}). \tag{4.9}$$

This discretization can be easily generalized all possible directions, $\mathbf{r} \in S^1$. Algorithm 1 suggest one possible numerical algorithm solving the discetized system. it is based on a sweeping algorithm which can be interpreted as a version of Gauss-Seidel method combined with a predefined set of grid node ordering so that characteristics are better approximated. For more discussion on fast sweeping algorithms, we refer the readers to[13, 23, 25]. It is also straighforward to apply a fast marching algorithm [24][21] for this discretization. We remark that further extension to higher dimensions is rather straight forward.

**Theorem 4.1.** *The discretization is monotone and consistent.*

*Proof.* The discretization if also clearly consistent, since all the derivatives in (4.4) are approximated by standard one sided finite differences. By (4.8)-(4.9), $u_{i,j} = u_{i,j}(a,b)$ is a function of $a = u_{i-\text{sign}(r_1),j}$ and $b = u_{i,j-\text{sign}(r_2)}$:

$$u_{i,j}(a,b) = \min(\frac{1}{|r_1|+|r_2|}\left(|r_1|a+|r_2|b\right), \phi_{i,j}),$$

and clearly, it is a non-decreasing function of $a$ and $b$. □

The monotonicity together with (4.9) implies that we also have a decreasing sequence of approximations when we refine the grid dyadically:

**Corollary 4.2.** *Consider the one dimensional domain $[0,L]$ in which $\mathbf{x}_0 = 0$ and $\mathbf{r}(\mathbf{x}) = 1$. Let $u^h$ denote the solution of Algorithm 1 on the mesh $x_j = jh$. If $u_0^{2h} \geq u_0^h$ , then $u_j^{2h} \geq u_{2j}^h$ for $j = 1,2,\cdots$.*

*Proof.* Under the given hypotheses, Steps (4.8)-(4.9) yield the solution $u_{2j}^h = \min(u_{2j-1}^h, \phi_{2j})$. By induction,

$$\begin{aligned}
u_{2j}^h &= \min(u_{2j-1}^h, \phi_{2j}) = \min(\min(u_{2j-2}^h, \phi_{2j-1}), \phi_{2j}) = \min(u_{2j-2}^h, \phi_{2j-1}, \phi_j) = \cdots \\
&= \min(u_0^h, \{\phi_k\}_{k=0}^{2j}) \\
&= \min(u_0^h, \{\phi_{2i}\}_{i=0}^{j}, \{\phi_{2i+1}\}_{i=0}^{2j-1}) \\
&\leq \min(\min(u_0^{2h}, \{\phi_{2i}\}_{i=0}^{j}), \{\phi_{2i+1}\}_{i=0}^{2j-1}) \\
&\leq \min(u_j^{2h}, \{\phi_{2i+1}\}_{i=0}^{2j-1}).
\end{aligned}$$

□

**Theorem 4.3.** *Consider the one-dimensional case with $r(x) = (x-x_0)/|x-x_0| = 1$ in the domain $[x_0, x_0+L]$. Let $u_i^h$ denote the numerical solution constructed by (4.4) at $x_i = x_0 + ih$, $i = 0,\cdots,N$, and let $u(x_i)$ denote the analytical solution defined in 2.1. Assume that $u_0^h = u(x_0)$. We have*

$$0 \leq E_1 := \sum_{i=0}^{N}(u_i^h - u(x_i))h \leq C_1 h, \tag{4.10}$$

*and*

$$0 \leq E_\infty := \max_{0\leq i\leq N} u_i^h - u(x_i) \leq C_1 h, \tag{4.11}$$

*Where $C_1$ is a Lipschitz constant of $\phi$ in $[x_0, x_0+L]$.*

*Proof.* In one dimension,

$$u_{j+1}^h = \min(u_j^h, \phi_{j+1}) = \min(u_{j-1}^h, \phi_j, \phi_{j+1}) = \cdots = \min(u_0^h, \phi_1, \phi_2, \cdots, \phi_{j+1}) = \min_{0\leq k\leq j+1} \phi_k.$$

$$u(x_{j+1}) = \min(u(x_j), \min_{x_j\leq x\leq x_{j+1}} \phi(x)) = \cdots = \min_{0\leq k\leq j}(\phi_0, \min_{x_k\leq x\leq x_{k+1}} \phi(x)).$$

So $u_{j+1}^h \geq u(x_{j+1})$. Since $|\phi(x) - \phi(x_{k+1})| \leq C_1|x - x_{k+1}| \leq Ch$ for all $x \in [x_k, x_{k+1}]$.

$$\min_{x_k\leq x\leq x_{k+1}} \phi(x) \geq \phi(x_{k+1}) - C_1 h.$$

$$\Longrightarrow u(x_{j+1}) = \min_{0\leq k\leq j}(\phi_0, \min_{x_k\leq x\leq x_{k+1}} \phi(x)) \leq \min_{0\leq k\leq j+1}(\phi(x_j) - C_1 h) = u_{j+1}^h - C_1 h.$$

$$\Longrightarrow 0 \leq u_{j+1}^h - u(x_{j+1}) \leq C_1 h.$$

The two inequalities follow. □

**Algorithm 1** Sweeping Algorithm for solving Equation (4.2)

Sweeping Algorithm: $\phi_{i,j}$ is given on the domain. We initialize the unknown $u_{i,j}$ to be $\infty$ except $u_{i,j} = \phi_{i,j}$ at the observer.

Do the following steps while $|u^{(n+1)} - u^{(n)}| > \delta$:($\delta > 0$ is the given tolerance)

Sweeping process: A compact way of writing this sweeping iterations in C/C++ is:

```
for (s1 = -1; s1<=1; s1+=2)
for (s2 = -1; s2<=1; s2+=2)
for (i=(s1<0?nx:0);(s1<0?i>=0:i<=nx);i+=s1 )
for (j=(s2<0?ny:0);(s2<0?j>=0:j<=nx);j+=s2 )
```

calculate $r_1$ and $r_2$

$u^{\text{tmp}}_{i,j} = \frac{1}{|r_1|+|r_2|}\left(|r_1|u_{i-\text{sign}(r_1),j} + |r_2|u_{i,j-\text{sign}(r_2)}\right)$

$u_{i,j} = \min(u^{\text{tmp}}_{i,j}, \phi_{i,j})$

4.3. **Visibility in a graph environment.** Assume that we are given a function $f : \Omega \subset \mathbb{R}^d \mapsto \mathbb{R}$. The graph of $f$ over $\Omega$ describes the occluder. Assume further that the observer is always on or above the graph; i.e. in the full space $\Omega \times \mathbb{R}$, the observer location is always $(\mathbf{x}_0, f_0)$ and $f_0 \geq f(\mathbf{x}_0)$.

With these assumptions, we construct a function $g : \Omega \mapsto \mathbb{R}$ such that a point $(\mathbf{x}, z)$ in space is occluded from the observer if $z < g(\mathbf{x})$. Furthermore, $g(\mathbf{x}) = f(\mathbf{x})$ where the graph of $f$ is visible from the observer.

We want to generalize our previous visibility algorithm so that it constructs such a function. Starting from the vantage point location $x_0$, following each ray to the boundary of $\Omega$, we need to determine how $g$ changes according to $f$.

This can be done by defining $u(\mathbf{x}, z) = z - g(\mathbf{x})$ and $\phi = z - f(\mathbf{x})$ in (3.3). We have

$$(-\nabla g, 1) \cdot \mathbf{r} = H(f - g)\left((-\nabla f, 1) \cdot \mathbf{r}\right)^-$$

where

$$\mathbf{r}(\mathbf{x}) \quad = \quad \begin{pmatrix} \mathbf{x} - \mathbf{x}_0 \\ g(\mathbf{x}) - f_0 \end{pmatrix}.$$

Thus

$$\nabla g \cdot (\mathbf{x} - \mathbf{x}_0) = -H(f - g)\left(-\nabla f \cdot (\mathbf{x} - \mathbf{x}_0) + g(\mathbf{x}) - f_0\right)^- + g(\mathbf{x}) - f_0.$$

Let $\tilde{\mathbf{r}}$ denote the direction $(\mathbf{x} - \mathbf{x}_0)$. The equation becomes

$$\nabla g \cdot \tilde{\mathbf{r}} = -H(f - g)\left(-\nabla f \cdot \tilde{\mathbf{r}} + g(\mathbf{x}) - f_0\right)^- + g(\mathbf{x}) - f_0. \tag{4.12}$$

This gives

$$\nabla g \cdot \tilde{\mathbf{r}} = \begin{cases} g(\mathbf{x}) - f_0, & -\nabla f \cdot \tilde{\mathbf{r}} + g(\mathbf{x}) - f_0 > 0 \text{ or } g(\mathbf{x}) > f(\mathbf{x}) \\ \nabla f \cdot \tilde{\mathbf{r}}, & \text{otherwise.} \end{cases}$$

with the boundary condition $g(\mathbf{x}_0) = f(\mathbf{x}_0)$.

We now derive an upwind discretization of Equation (4.12). For simplicity of derivation, we only consider the discretization for $d = 2$ with $\tilde{\mathbf{r}} = (r_1 > 0, r_2 > 0)$. The upwind discretization thus takes the form:

(4.13)
$$\tilde{r}_1\frac{g_{i,j}-g_{i-1,j}}{h}+\tilde{r}_2\frac{g_{i,j}-g_{i,j-1}}{h}=-H(f_{i,j}-g_{i,j})\min(\nabla^{\tilde{\mathbf{r}}}f(\mathbf{x}_{i,j})\cdot\tilde{\mathbf{r}}+g_{i,j}-f_0,0)+g_{i,j}-f_0.$$

We first consider the different possible evaluations of this nonlinear equation and then summarize to get a compact algorithm. We shall use the superscript 'tmp1' and 'tmp2' in $g_{i,j}^{tmp1}$ and $g_{i,j}^{tmp2}$to denote that the tentative values obtained by the possible reduction of the nonlinear discretization.

In the first case, $-\nabla f\cdot\tilde{\mathbf{r}}+g(\mathbf{x})-f_0>0$ or $g(\mathbf{x})>f(\mathbf{x})$, the equation is discretized by upwind differencing, leading to

$$(r_1+r_2-h)\,g_{i,j}^{tmp1}=r_1g_{i-1,j}+r_2g_{i,j-1}-f_0h.$$

If $r_1+r_2\neq h$, we then have

$$g_{i,j}^{\mathrm{tmp1}}=\left(r_1g_{i-1,j}+r_2g_{i,j-1}-f_0h\right)/\left(r_1+r_2-h\right).$$

The degenerate case of $r_1+r_2-h=0$ corresponds to $\mathbf{x}_{i,j}$ being $h$ distance from $\mathbf{x}_0$, and it means that the first case cannot happen on these grid points. If the computed value of $g_{i,j}^{tmp1}$ does not satisfy $-\nabla f\cdot\tilde{\mathbf{r}}+g(\mathbf{x})-f_0>0$ or $g(\mathbf{x})>f(\mathbf{x})$, we then abandon the first case and consider the second case: $r_1\frac{g_{i,j}-g_{i-1,j}}{h}+r_2\frac{g_{i,j}-g_{i,j-1}}{h}=\nabla^{\tilde{\mathbf{r}}}f(\mathbf{x}_{i,j})\cdot\tilde{\mathbf{r}}$. Thus

$$g_{i,j}^{tmp2}=\left(r_1g_{i-1,j}+r_2g_{i,j-1}+h\nabla^{\tilde{\mathbf{r}}}f_{i,j}\cdot\tilde{\mathbf{r}}_{i,j}\right)/(r_1+r_2).$$

Here the gradient $\nabla^{\tilde{r}}f_{i,j}$ is approximated by the corresponding upwind differencing. In particular, if $r_1+r_2=h$ at he grid point $x_{i,j}$, then the vantage point is located at either $x_{i-1,j}$, or $x_{i,j-1}$ (assuming that the vantage point always lies on the grid and $r_1,r_2>0$). Let's assume that $\mathbf{x}_0=x_{i-1,j}$ and $f_0=f_{i-1,j}$, then $\tilde{\mathbf{r}}_{i,j}=(h,0)$ and the upwind discretization of $\nabla^{\tilde{r}}f_{i,j}\cdot\tilde{\mathbf{r}}_{i,j}$ corresponds to

$$\frac{f_{i,j}-f_{i-1,j}}{h}r_1=f_{i,j}-f_{i-1,j}.$$

Since $g(\mathbf{x}_0)=f_0$, i.e. $g_{i,j-1}=f_0$, $g_{i,j}^{tmp2}=g_{i-1,j}+(f_{i,j}-f_{i-1,j})=f_{i,j}$.

In the discretization for $d=2$ with $\tilde{\mathbf{r}}=(r_1,r_2),r_1>0,r_2>0$, we have the algorithm.
If $r_1+r_2\neq h$,

(1) Solve

$$g_{i,j}^{\mathrm{tmp1}}=\left(r_1g_{i-1,j}+r_2g_{i,j-1}-f_0h\right)/\left(r_1+r_2-h\right).$$

$$g_{i,j}^{\mathrm{tmp2}}=\left(r_1g_{i-1,j}+r_2g_{i,j-1}+h\nabla^{\tilde{\mathbf{r}}}f_{i,j}\cdot\tilde{\mathbf{r}}_{i,j}\right)/\left(r_1+r_2-h\right).$$

(2) Set

$$g_{i,j}=\begin{cases}g_{i,j}^{tmp1}, & g_{i,j}^{tmp1}>f_{i,j}\text{ or }-\nabla^{\tilde{\mathbf{r}}}f\cdot(\mathbf{x}-\mathbf{x}_0)+g_{i,j}^{tmp1}-f_0>0;\\ g_{i,j}^{tmp2}, & \text{otherwise.}\end{cases}$$

If $r_1+r_2=h$, $g_{i,j}=f_{i,j}$.

**Algorithm 2** Sweeping Algorithm for graph environment.

Sweeping Algorithm: $f_{i,j}$ is given on the domain. We initialize the unknown $g_{i,j}$ to be $\infty$ except $g_{i,j} = f_{i,j}$ at the observer.

Do the following steps while $|g^{(n+1)} - g^{(n)}| > \delta$:($\delta > 0$ is the given tolerance)

Sweeping process: A compact way of writing this sweeping iterations in C/C++ is:

```
for (s1 = -1; s1<=1; s1+=2)
for (s2 = -1; s2<=1; s2+=2)
for (i=(s1<0?nx:0);(s1<0?i>=0:i<=nx);i+=s1)
for (j=(s2<0?ny:0);(s2<0?j>=0:j<=nx);j+=s2 )
```

$signx = sign(x_i - x_0)$

$signy = sign(y_j - y_0)$

$r_1 = x_i - x_0$

$r_2 = y_j - y_0$

If $r_1 + r_2 \neq h$,

$g^{\text{tmp1}}_{i,j} = \left(|r_1| g_{i-signx,j} + |r_2| g_{i,j-signy} - f_0 h\right) / \left(|r_1| + |r_2| - h\right).$

$g^{\text{tmp2}}_{i,j} = \left(|r_1| g_{i-signx,j} + |r_2| g_{i,j-signy} - h \nabla^{\tilde{\mathbf{r}}} f_{i,j} \cdot \tilde{\mathbf{r}}_{i,j}\right) / \left(|r_1| + |r_2|\right).$

If $g^{tmp1}_{i,j} > f_{i,j}$ or $-\nabla^{\tilde{\mathbf{r}}} f \cdot (\mathbf{x} - \mathbf{x}_0) + g^{tmp1}_{i,j} - f_0 > 0$

$g_{i,j} = g^{tmp1}_{i,j}$

Else

$g_{i,j} = g^{tmp2}_{i,j}$

Else

$g_{i,j} = f_{i,j}.$

4.4. **Numerical results.** In Table 1, we present a numerical convergence study of the algorithm. A disc of radius 0.5 is placed at the origin and the observer is placed at $(-1,-1)$. In this example, the light speed is constant outside of the obstacle.

Table 2 shows a numerical convergence study of the variable wave speed case. For simplicity, we work on the complex plane and denote a point $(x, y) \in \mathbb{R}^2$ by its equivalent in $z = x + iy$ in the complex plane. We set up the ray field $r(z) = (i/a - 1/b)z$, $a = 1.5$, $b = 0.75$, and place a circular obstacle $\phi(z) = |z - (1+i)/4|^2 - 0.25^2$ in the computational domain $\{Z :\in \mathbb{C} : -1 \leq \mathrm{Re}(z) \leq 1 \text{ and } -1 \leq \mathrm{Im}(z) \leq 1\}$. Hence, the analytical solution is

$$\psi(z_0) = \min_{t \in [0,\infty)} \phi(e^{(-1/b + i/a)t} z_0).$$

Next, we test the numerical convergence of the visibility algorithm for a graph environment that is introduced in Section 4.3. In our test case, the occluders/terrain is described by the graph of a radial symmetric function $f(r)$, $r = \sqrt{x^2 + y^2}$ for $x, y \in [-8, 8]$:

$$f(r) = -\frac{1}{100} r^4 + \frac{7}{50} r^3 - \frac{127}{200} r^2 + \frac{243}{200} r + \frac{1639}{1600}$$

with vantage point location $\mathbf{x}_0 = (0, 1.6)$. We tabulate the absolute errors for the in $l_1$- and $l_\infty$-norms in Table 3. Notice that the domain for this computation is 64 times larger than the

TABLE 1. Numerical convergence study for the case $r(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)$.

| | $h = 1/50$ | $1/100$ | $1/200$ | $1/400$ | $1/800$ |
|---|---|---|---|---|---|
| $\|\|\cdot\|\|_1$ | 0.0165330 | 0.0079895 | 0.0039365 | 0.0019556 | 0.0009744 |
| $\|\|\cdot\|\|_\infty$ | 0.0202971 | 0.0100736 | 0.0050185 | 0.0025065 | 0.0012512 |

TABLE 2. Numerical convergence study for the case $r(z) = (i/a - 1/b)z$, with $a = 1.5$ and $b = 0.75$.

| | $h = 1/50$ | $1/100$ | $1/200$ | 1/400 |
|---|---|---|---|---|
| $\|\|\cdot\|\|_1$ | 1.1327e-4 | 4.2022e-5 | 1.8178e-5 | 8.6035e-6 |
| $\|\|\cdot\|\|_\infty$ | 2.1822e-3 | 1.0826e-3 | 5.3991e-4 | 2.7083e-4 |

TABLE 3. Numerical convergence study for the case $\mathbf{r}(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)$.

| | $h = 1/10$ | $1/20$ | $1/40$ | $1/80$ |
|---|---|---|---|---|
| $\|\|\cdot\|\|_1$ | 9.2951e-1 | 4.4469e-1 | 2.1080e-1 | 1.0426e-1 |
| $\|\|\cdot\|\|_\infty$ | 1.8051e-2 | 8.1644e-3 | 4.7167e-3 | 2.4812e-3 |

previous numerical examples, and the effect of accumulating error over this larger domain is reflected in the computed $l_1$ errors in the table.

In Figure 4.1, we show the contour for a visibility function with five-circle occluder in two dimensions. The observer is located at $(0,0)$. The origin and radius of the circles are $(0.6, 1.0)$, (2.0,1.7), $(-0.6, -0.6)$,$(1.5, -0.5)$,$(2.5, -0.7)$ and 0.3,0.7, 0.15, 0.3,0.2 respectively. We can see that if a circle is totally invisible to observer, it does not affect the visibility function.

Figure 4.2 shows the contour for a visibility function with two different material with reflecting index is 2. The observer is located at (-0.5,-0.75). The origin and radius of the circles are (0.5,0.5), (0,-0.35) and 0.3, 0.25 respectively.

Figure 4.3 shows the zero level set of the visibility function which indicates the boundary of invisible region. in the figure, the shadow boundaries are shown by the blue surfaces. The velocity field is

$$(x - y,\, x + y,\, \sqrt{2}z)$$

which gives the curved rays. The occluders consist of three spheres $(x-1)^2 + (y-1)^2 + (z-1)^2 = (0.3)^2$, $(x-1.8)^2 + (y-1.8)^2 + (z-1.8)^2 = (0.8)^2$ and $(x+0.2)^2 + (y+0.2)^2 + (z+0.,2)^2 = (0.15)^2$. The observer is located at (0,0,0). It takes five iterations for sweeping scheme to reduce the difference of two successive iterations, $\left\|u^n - u^{n-1}\right\|_{l_1}$ to be less than $10^{-12}$.

Figures 4.4,4.5, and 4.6 are three examples of the algorithm for the graph environment. The shadow boundaries are shown as the bue surfaces (curves).

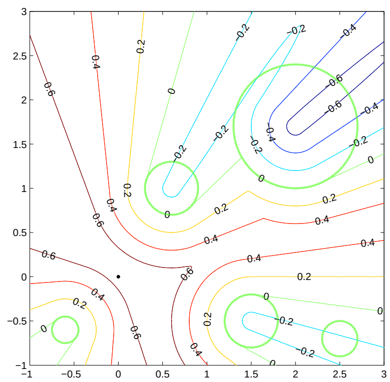


FIGURE 4.1. Contour plot of visibility function with five-circle occluder.

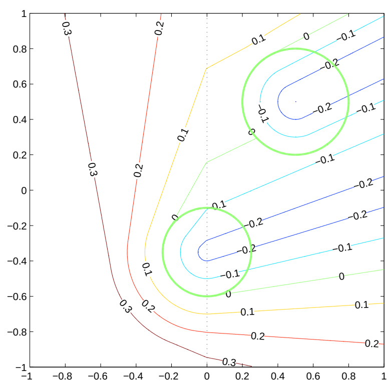


FIGURE 4.2. Contour plot of visibility function with refraction index 2.

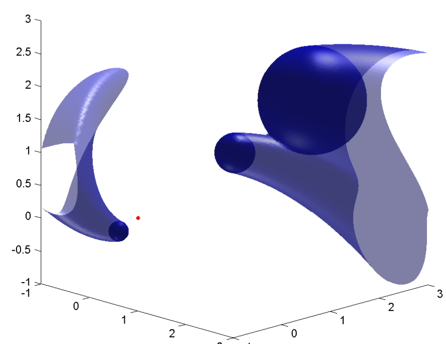


FIGURE 4.3. The observer is located at $(0,0,0)$.$\phi$ is the signed distance function to the three spheres $(x-1)^2+(y-1)^2+(z-1)^2=(0.3)^2$, $(x-1.8)^2+(y-1.8)^2+(z-1.8)^2=(0.8)^2$ and $(x+0.2)^2+(y+0.2)^2+(z+0.,2)^2=(0.15)^2$. The velocity field is $(x-y, x+y, \sqrt{2}z)$

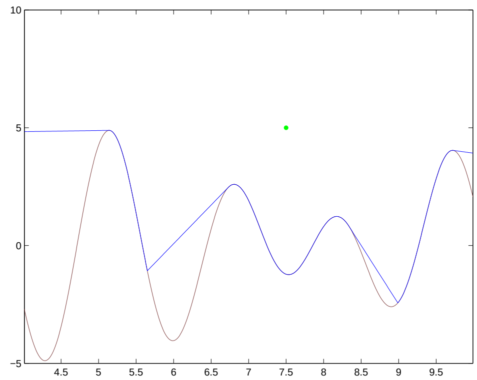


FIGURE 4.4. $f = 3\sin(4x) - 2\cos(3x)$, domain [4 10]x[0 1], dx = 0.01, $x_0 = (0.02, 0.02)$

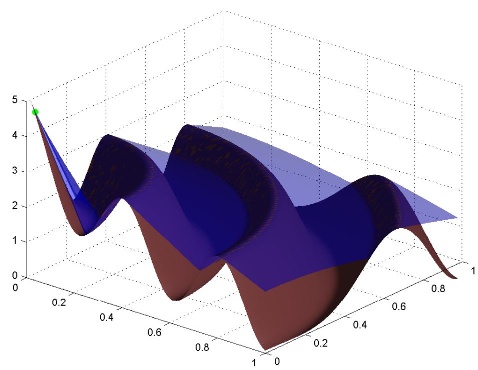


FIGURE 4.5. $f = \cos(5\sqrt{x^2+y^2}\pi) - 4(x^2+y^2)^{\frac{1}{6}} + 5$, domain [0 1]x[0 1], dx = 0.01, $x_0 = (0.02, 0.02)$

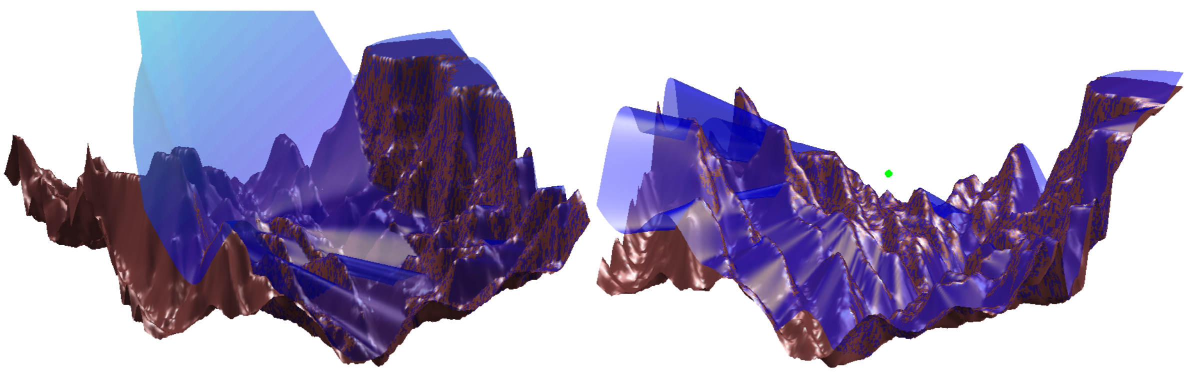

FIGURE 4.6. Shadow boundaries (blue surface) of the visibility from a vantage point (green dot) over a region of the Grand Canyon.

## 5. CONCLUSION

In this paper, we discuss some properties of the visibility function (2.1) and the corresponding discretization (1) that is introduced in the second authors earlier work [22]. We show that this notion of visibility, i.e. formula (2.1), satisfies a dynamic programming principle, is the unique solution to an integral equation and the viscosity solution to a Hamilton-Jacobi equation with discontinuous dependence on the solution. We show further that Algorithm 2.1 can be derived directly by the upwind discretization of the nonlinear equation (4.2). Finally, we consider some generalizations that include visibility computation under curved ray paths and a new efficient visibility algorithm for environments in which that occluders can be described by the graph of a function. Future direction includes the extension of Algorithm 2 to the case of non-constant wave speed, and developing and analyzing multi-resolution algorithms.

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