Piecewise Constant Level Set Method for Interface Problems

Hongwei Li and Xue-Cheng Tai

Abstract. We apply the Piecewise Constant Level Set Method (PCLSM) to interface problems, especially for elliptic inverse and multiphase motion problems. PCLSM allows using one level set function to represent multiple phases, and the interfaces are represented implicitly by the discontinuity of a piecewise constant level set function. The inverse problem is solved using a variational penalization method with total variation regularization of the coefficient, while the multiphase motion problem is solved by an Additive Operator-Splitting (AOS) scheme.

1. Introduction

The traditional level set method of Osher and Sethian[12] plays a great role in dealing with interface problems. By level set method, one doesn't evolve the interfaces, instead one just evolves the level set function. The main advantage of level set approach is that the interfaces are implicitly represented by a level set function, and so complicated topological changes can be dealt naturally and easily. In [7, 6, 5], some variants to the traditional level set method have been proposed. In this work, we are trying to show the applications of the Piecewise Constant Level Set Method (PCLSM) of [7, 5] for some interface problems. For traditional level set methods, one needs to reinitialize the level set function to be a signed distance function during the iterations, and cautions must be taken with respect to the discretization of Heaviside and Dirac functions. The piecewise constant level set method doesn't need to care about these issues [7]. We would like to mention that some related ideas have already been used in the following publications, see [13, 3, 4, 2, 17, 8].

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2. Piecewise Constant Level Set Method Formulations

The essential idea of the PCLSM of [7] is to use a piecewise constant level set function to identify the interfaces separating the subdomains. Assume that we need to partition the domain Ω into subdomains Ω_i, $i = 1, 2, \ldots, n$ and the number of subdomains is a priori known. In order to identify the subdomains, we try to identify a piecewise constant level set function ϕ such that

$$\phi = i, \quad \text{in } \Omega_i, \quad i = 1, 2, \ldots, n. \tag{2.1}$$

Thus, for any given partition $\{\Omega_i\}_{i=1}^n$ of the domain Ω, it corresponds to a unique PCLS function ϕ which takes the values $1, 2, \cdots, n$. Associated with such a level set function ϕ, the characteristic functions of the subdomains are given as

$$\psi_i = \frac{1}{\alpha_i} \prod_{j=1, j\neq i}^{n} (\phi - j), \quad \alpha_i = \prod_{k=1, k\neq i}^{n} (i - k). \tag{2.2}$$

If ϕ is given as in (2.1), then we have $\psi_i(x) = 1$ for $x \in \Omega_i$, and $\psi_i(x) = 0$ elsewhere. We can use the characteristic functions to extract geometrical information for the subdomains and the interfaces between the subdomains. For example,

$$\text{Length}(\partial\Omega_i) = \int_\Omega |\nabla \psi_i| dx, \quad \text{Area}(\Omega_i) = \int_\Omega \psi_i dx. \tag{2.3}$$

Define

$$K(\phi) = (\phi - 1)(\phi - 2) \cdots (\phi - n) = \prod_{i=1}^{n} (\phi - i). \tag{2.4}$$

At every point in Ω, the level set function ϕ should satisfy

$$K(\phi) = 0. \tag{2.5}$$

This level set idea has been used for image segmentation in [7, 14, 15] and inverse problems involving shape identification in [16]. Fast algorithms have been also developed for this method for image segmentation in [14, 15].

When one really wants to numerically compute the length or area terms in (2.3), caution must be taken. Because ψ_i is not continuous along the interfaces, the commonly used forward and backward difference scheme may not approximate those terms consistently. However it is still possible to find a way to discretize the length and area terms. In our numerical implementations, for example, we found that the central finite difference scheme can approximate the length term well. However, the central finite difference scheme may suffer some stability problems and produce oscillations in the numerical solution. In our simulation, we have used some special treatments for the nodes close to the boundary.

3. PCLSM for elliptic inverse problem

We try to use PCLSM for an inverse problem. Consider the partial differential equation:

$$-\nabla \cdot (q(x)\nabla u) = f,\ x \in \Omega \subset R^2, \quad u(x) = 0,\ x \in \partial\Omega. \tag{3.1}$$

Suppose we have some observations of the solution u, and we want to recover the coefficient $q(x)$ by using the observations. In [1], the standard level set method has been applied to elliptic inverse problems.

Due to the ill-posedness of the problem, output-least-squares method is often used for recovering $q(x)$. Assume that $u_d \in L^2(\Omega)$ is an observation for u, and let K be the set of admissible coefficients

$$K = \{q(x) \mid q(x) \in L^\infty(\Omega) \cap TV(\Omega), \quad 0 < \underline{q}(x) \leqslant q(x) \leqslant \bar{q}(x) < \infty\}. \tag{3.2}$$

with $\underline{q}(x)$ and $\bar{q}(x)$ known a priori, and $TV(\Omega)$ denotes the space of function of bounded total variation. We solve the following minimization problem for the output-least-squares method to find the parameter $q(x)$.

$$\min_{q\in K} F(q), \quad F = \int_\Omega \frac{1}{2}|u(q) - u_d|^2 dx + \beta R(q), \tag{3.3}$$

above $R(q) = \int_\Omega |\nabla q| dx$ is the total variation norm of q, β is the regularization parameter and $u(q)$ is the solution of (3.1) with a given q. We assume that $q(x)$ is piecewise constant and represent $q(x)$ by piecewise constant level set function

$$q(x) = \sum_{i=1}^{n} c_i \psi_i(x). \tag{3.4}$$

Incorporating it into (3.3), and letting $G(c_i, q) = F(q(c_i, \phi))$, we then need to solve

$$\min_{\substack{c_i,\phi \\ K(\phi)=0}} G(c_i, \phi), \quad G(c_i, \phi) = \int_\Omega \frac{1}{2}|u(q(c_i, \phi) - u_d|^2 + \beta R(q). \tag{3.5}$$

To deal with the constraint, we use the common penalization method

$$\min_{c_i,\phi} L, \quad L = G + \frac{1}{2\mu} W = \int_\Omega \frac{1}{2}|u - u_d|^2 + \beta R(q) + \frac{1}{2\mu}\int_\Omega K^2(\phi) dx. \tag{3.6}$$

The following algorithm is used to solve (3.6).

Algorithm 1. Choose initial values for ϕ^0 and $\bar{c}^0$. For $k = 1, 2, \ldots$, do

1. Find $\bar{c}^{k+1} = \{c_i^{k+1}\}_{i=1}^n$, such that

$$\bar{c}^{k+1} = arg \min_{\bar{c}} L(\bar{c}^k, \phi^k). \tag{3.7}$$

2. Find ϕ^{k+1} such that

$$\phi^{k+1} = arg \min_{\phi} L(\bar{c}^{k+1}, \phi^k). \tag{3.8}$$

3. Check convergence, if converged, stop; else goto 1.

Above $arg\min L(\cdot)$ denotes the minimizer of $L(\cdot)$. In order to find a minimizer of $L(\cdot)$ with respect to c_i, $i = 1, 2, \ldots, n$, we use a gradient based method with line search. Usually, we update c_i after each 5 - 10 outer iterations. The most difficult part of the above algorithm for our model problem is the second step – minimizing ϕ, so we concentrate on the minimizing of ϕ. At the minimizer, we should have

$$\frac{\partial L}{\partial \phi} = \frac{\partial G}{\partial \phi} + \frac{1}{2\mu} W'(\phi) = 0. \tag{3.9}$$

To solve (3.9), we can instead solve the following evolution differential equation to steady state $\phi_t + \dfrac{\partial L}{\partial \phi} = 0$. According to the operator splitting scheme [9, 10], we can solve this equation in the following way: For $l = 1, 2, \ldots$, until convergence, do

$$\frac{\phi^{l+1/2} - \phi^l}{\tau} + \frac{\partial G}{\partial \phi}(\bar{c}^l, \phi^{l+1/2}) = 0, \tag{3.10}$$

$$\frac{\phi^{l+1} - \phi^{l+1/2}}{\tau} + \frac{1}{2\mu} W'(\phi^{l+1}) = 0. \tag{3.11}$$

where τ is a pseudo time-step. Notice that (3.11) is trying to solve

$$\phi - \phi^{l+1/2} + \frac{\tau}{\mu} K(\phi) K'(\phi) = 0. \tag{3.12}$$

where τ/μ is a parameter that should be chosen properly. Notice that (3.12) is a polynomial of ϕ and it has $2n - 1$ roots. We will use Newton method to solve it. We apply one technique to guarantee that (3.12) only has one real root and all the other roots are complex. Choosing τ/μ such that

$$\tau/\mu (K(x)K''(x) + (K'(x))^2) + 1 > 0 \ \forall x,$$

then we can guarantee that (3.12) only has one real root. Notice that $K(\phi)$ is a polynomial, It's easy to compute the bounds for τ/μ from the above inequality. For example, we need $\tau/\mu \leq 2$ for 2-phase problems and $\tau/\mu \leq 0.71$ for 3-phase problems in order to satisfy the above inequality.

The two minimization problems in Algorithm 1 are never solved exactly. A fixed number of gradient iterations is used for solving (3.7). A fixed number of iterations of (3.10)-(3.11) is used for solving (3.8) for each outer iteration in Algorithm 1.

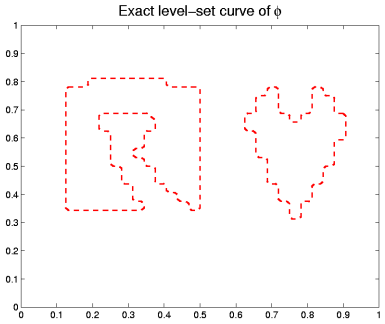
3.1. Numerical experiments for elliptic inverse problem

We take the examples in [1] to testify the efficiency of our **Algorithm 1**. Let $\Omega = (0,1) \times (0,1)$, $f = 20\pi^2 \sin(\pi x)\sin(\pi y)$. Let $u^\star$ be the exact finite element solution for the exact q and σ be the noise level. We get the observed solution $u_d = u^\star + \sigma ||u^\star||_{L^2}/||R_d||_{L^2} R_d$. Where R_d is a finite element function with nodal values being uniform random numbers between $[-1, 1]$ with zero mean.

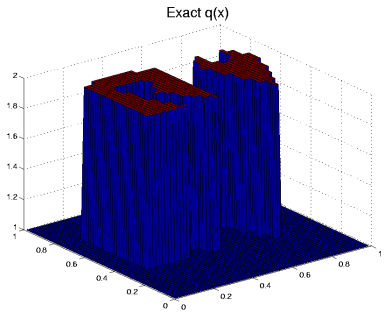
The domain Ω consists of a rectangular mesh with uniform mesh size $h = 1/64$ for both x and y directions. In all the figures, the dotted lines in the background

show the true level set curves and the dashed lines are the computed level set curves.

In this example, the exact coefficient $q(x)$ is given in Fig.1, i.e. $q(x) = 2$ inside the two closed curves and $q(x) = 1$ outside the curves. See Fig. 2 for the numerical results. We see that only 300 iterations are needed to recover $q(x)$ rather accurately.



(a) exact level set curve



(b) exact q(x)

FIGURE 1. The exact $q(x)$ and the location of the discontinuity

4. PCLSM for multiphase motion problem

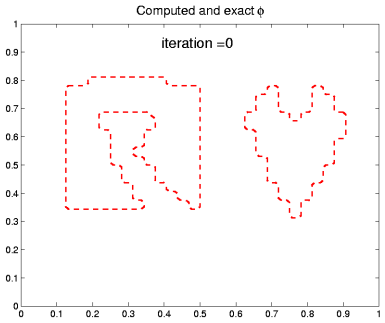
Usually, the multiphase motion problem involves curves meeting at a point with prescribed angles. Each interface Γ_{ij}, separates regions Ω_i and Ω_j and moves with a normal velocity

$$v_{ij} = f_{ij}\kappa_{ij} + (e_i - e_j). \tag{4.1}$$

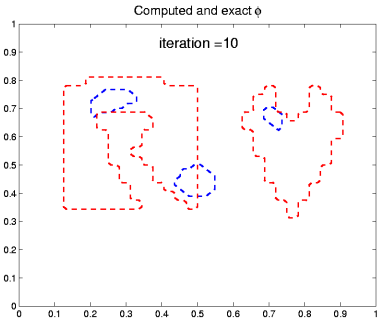
where κ_{ij} is the local curvature, f_{ij} is the constant surface tension of Γ_{ij}, and e_i corresponds to the bulk energy. This model problem can be obtained by associating an energy functional E to the motion, which involves the length of each interface and the area of each subregion, i.e.

$$\begin{aligned} E &= E_1 + E_2 \\ E_1 &= \sum_{1 \leqslant i < j \leqslant n} f_{ij} Length(\Gamma_{ij}) \\ E_2 &= \sum_{1 \leqslant i \leqslant n} e_i Area(\Omega_i). \end{aligned} \tag{4.2}$$

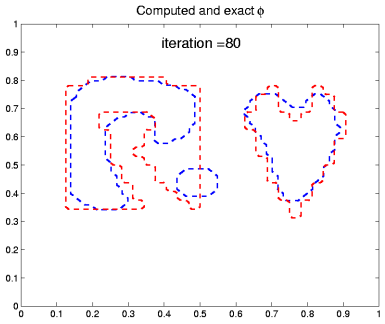
By minimizing this energy functional, the internal interfaces are driven to equilibrium. Our method is especially inspired by [11] and [19].



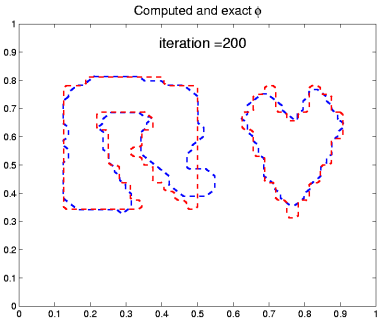
(a) Initial



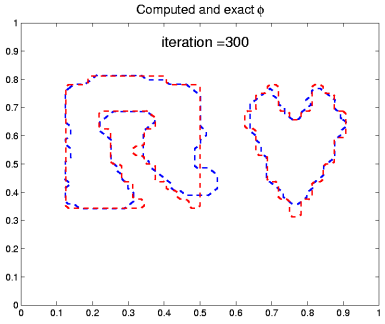
(b) 10 iterations



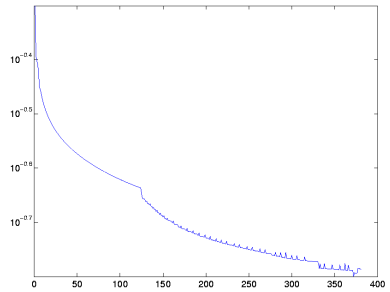
(c) 80 iterations



(d) 200 iterations



(e) 300 iterations



(f) The error $||q - q^k||_{L^2(\Omega)}$ v.s. iteration number

FIGURE 2. The computed solution at different iterations , with $\sigma = 1\%$. Initial $q_i = [1.2,\ 1.8]$, Initial level set function $\phi = 1.5$

In the following, the PCLSM will be used to solve the motion by mean curvature problem. For simplicity, let us consider problem (4.2) with

$$e_i = 0, \quad f_{ij} = 1. \tag{4.3}$$

We want to emphasize that we can apply the PCLSM for general setting for (4.2). Under condition (4.3), the problem (4.2) reduces to the model problem:

$$\min_{\Gamma_{ij}} \sum_{1\leqslant i<j\leqslant n} Length(\Gamma_{ij}). \tag{4.4}$$

There are different ways to find the curves that minimize the above energy functional. Under the condition that Γ_{ij} is the interface between Ω_i and Ω_j and $\{\Omega_i\}_{i=1}^n$ are represented by (2.1), we see that

$$\sum_{i=1}^{n}\int_{\Omega}|\nabla\psi_i|dx = 2\sum_{1\leqslant i<j\leqslant n} Length(\Gamma_{ij}).$$

Thus, If we use our PCLSM for (4.4), then we need to find a function ϕ that solves the following constrained minimization problem:

$$\min F, \quad F=\sum_{i=1}^{n}\int_{\Omega}|\nabla\psi_i|dx, \quad \text{subject to } K(\phi)=0 \text{ and } \phi|_{\partial\Omega}=g. \tag{4.5}$$

Usually, Neumann boundary condition is supposed. However, in this paper, we would like to try Dirichlet boundary conditions, which should produce a constrained motion. By using the same penalization technique and gradient method, we found that the equation we need to solve is

$$\phi_t+\frac{\partial F}{\partial\phi}+\frac{1}{2\mu}W'(\phi)=0. \tag{4.6}$$

Applying the operator-splitting scheme again, we need to solve the following two equations alternatively

$$\phi_t+\frac{\partial F}{\partial\phi}(\phi)=0, \qquad \phi_t+\frac{1}{2\mu}W'(\phi)=0. \tag{4.7}$$

The first equation is trying to minimize the energy functional and the second equation is trying to enforce that the minimizer is taking the values $1, 2, \cdots, n$.

We have tried to solve the first equation by the Additive version of Operator Splitting (AOS) scheme of [9, 10, 18]. Note that

$$\nabla\psi_i=\psi_i'(\phi)\nabla\phi. \tag{4.8}$$

and

$$\frac{\partial F}{\partial\phi}=-\sum_{i=1}^{n}\nabla\cdot\left(\frac{\nabla\psi_i}{|\nabla\psi_i|}\right)\psi_i'=-\sum_{i=1}^{n}\nabla\cdot\left(sign(\psi_i')\frac{\nabla\phi}{|\nabla\phi|}\right)\psi_i'. \tag{4.9}$$

For two dimensional problems, we have

$$\frac{\partial F}{\partial\phi}=-\sum_{i=1}^{n}\psi_i'\left(sign(\psi_i')\frac{\phi_x}{|\nabla\phi|}\right)_x-\sum_{i=1}^{n}\psi_i'\left(sign(\psi_i')\frac{\phi_y}{|\nabla\phi|}\right)_y. \tag{4.10}$$

If we apply the AOS [9, 10] and do some standard linearization, we need to solve

$$\frac{\tilde{\phi}^{k+1/4}-\phi^k}{\tau}-\sum_{i=1}^{n}\psi_i'(\phi^k)\left(sign(\psi_i'(\phi^k))\frac{\tilde{\phi}_x^{k+1/4}}{|\nabla\phi^k|}\right)_x=0, \tag{4.11}$$

$$\frac{\tilde{\phi}^{k+1/2}-\phi^k}{\tau}-\sum_{i=1}^{n}\psi_i'(\phi^k)\left(sign(\psi_i'(\phi^k))\frac{\tilde{\phi}_y^{k+1/2}}{|\nabla\phi^k|}\right)_y=0. \tag{4.12}$$

Then, set

$$\phi^{k+1/2}=\frac{1}{2}(\tilde{\phi}^{k+1/4}+\tilde{\phi}^{k+1/2}). \tag{4.13}$$

When the value of $\phi^{k+1/2}$ is obtained, we solve (3.11) to get ϕ^{k+1}. The two equations (4.11)–(4.12) can be solved efficiently on lines parallel to the x and y-axes.

4.1. Numerical experiments for multiphase motion problem

We take $\Omega=(0,1)\times(0,1)$ and use Dirichlet boundary conditions. And the domain Ω is divided into square elements with uniform mesh size $h=hx=hy=1/64$.

In this example, we test our algorithm on the well-known triple-junction problem which involves three phases. The boundary and initial values are: $\phi^0|_\Omega=1.0$, $g(0,[0,1/2])=g([0,1],0)=1$, $g(0,[1/2,1])=g([0,1],1)=3$, $g(1,[0,1])=2$.

For this test problem, the real triple junction point should be at $(1-1/2\sqrt{3},1/2)$ which is approximately $(0.7118,0.5)$. The three interface curves should be straight lines and the three angles around the triple junction point should satisfy the classical angle condition, i.e. $(\frac{2\pi}{3},\frac{2\pi}{3},\frac{2\pi}{3})$. The simulated results are presented in Fig.3. The computed triple junction point is at $(0.69,0.5)$. The algorithm needs only about 2100 iterations to get to a steady state. Due to the use of the AOS scheme, the cost for the computation is rather cheap.

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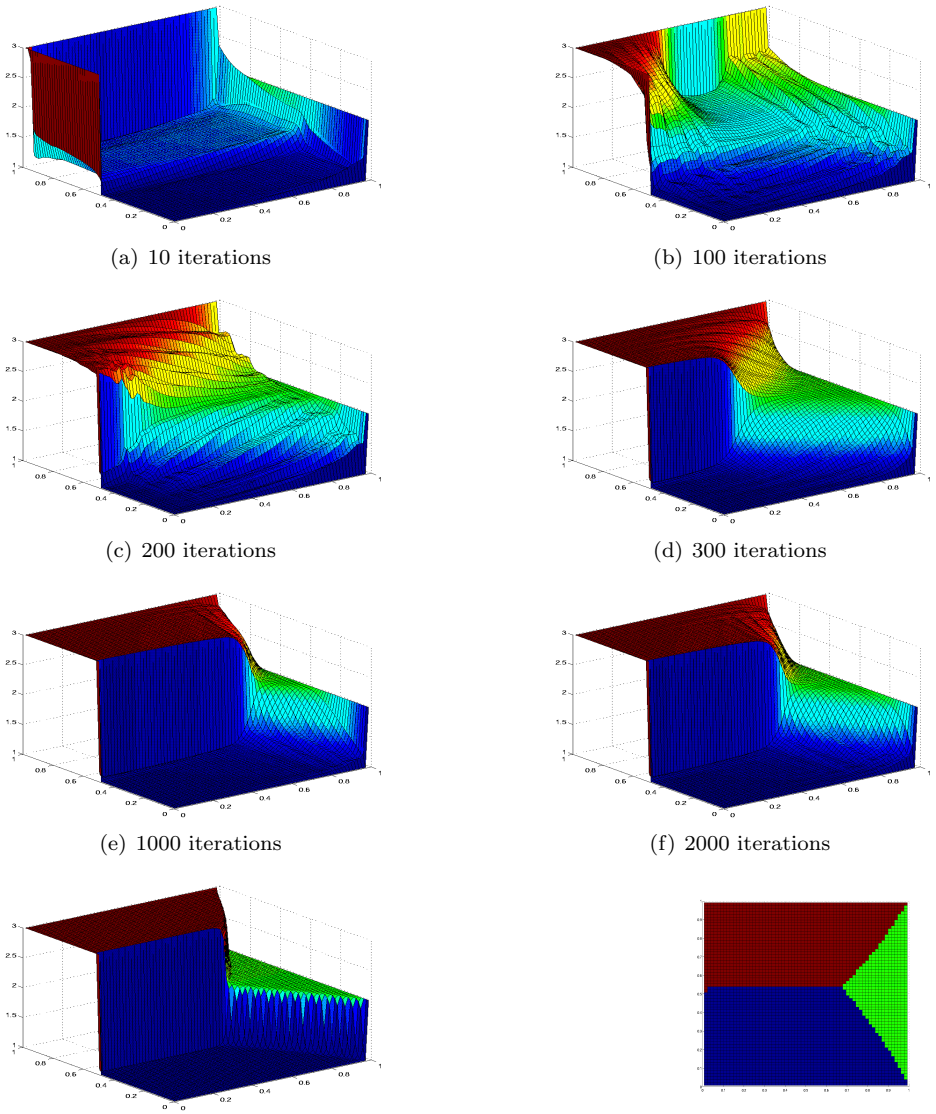

(a) 10 iterations (b) 100 iterations

(c) 200 iterations (d) 300 iterations

(e) 1000 iterations (f) 2000 iterations

(g) 2100 iterations (h) The computed solution

FIGURE 3. Example 2: Computed solution for 3 phases.

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