Large Time Behavior of Nonlocal Aggregation Models with Nonlinear Diffusion

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Abstract

The aim of this paper is to establish rigorous results on the large time behavior of nonlocal models for aggregation, including the possible presence of nonlinear diffusion terms modeling local repulsions. We show that, as expected from the practical motivation as well as from numerical simulations, one obtains concentrated densities (Dirac δ distributions) as stationary solutions and large time limits in the absence of diffusion. In addition, we provide a comparison for aggregation kernels with infinite respectively finite support. In the first case, there is a unique stationary solution corresponding to concentration at the center of mass, and all solutions of the evolution problem converge to the stationary solution for large time. The speed of convergence in this case is just determined by the behavior of the aggregation kernels at zero, yielding either algebraic or exponential decay or even finite time extinction. For kernels with finite support, we show that an infinite number of stationary solutions exist, and solutions of the evolution problem converge only in a set-valued sense to the set of stationary solutions, which we characterize in detail.

Moreover, we also consider the behavior in the presence of nonlinear diffusion terms, the most interesting case being the one of small diffusion coefficients. Via the implicit function theorem we give a quite general proof of a rather natural assertion for such models, namely that there exist stationary solutions that have the form of a local peak around the center of mass. Our approach even yields the order of the size of the support in terms of the diffusion coefficients.

All these results are obtained via a metric gradient flow formulation using the Wasserstein metric for probability measures, and are carried out in the case of a single spatial dimension for convenience.

Keywords: Nonlinear diffusion, nonlocal PDEs, biological aggregation, stationary solutions, asymptotic behavior, Wasserstein metric.

1 Introduction

Nonlocal models for aggregation phenomena recently received growing attention in particular in biological applications. Celebrated examples are the Keller-Segel model for chemotaxis in all its variants (cf. [17, 12, 15, 16]) and models for swarming of populations (cf. [3, 19, 20, 21, 24, 25]). The mathematical analysis of such models is a challenging topic, in

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particular if the equation includes nonlinear diffusion terms modelling local repulsion effects (cf. [14, 21]). Some progress in this context has been made recently (cf. [2, 4, 5, 22]) using either methods of characteristics that are restricted to regular situations or entropy solution techniques that have difficulties to deal with the nonlinearities. A more general framework without these shortcomings of the present approach still seems to be missing and this paper is an attempt in this direction. As has been recently pointed out in [7, 8, 1], models of this form can be formulated as metric gradient flows of certain entropy functionals on a space of probability measures equipped with the Wasserstein metric. However, detecting the long time asymptotics of the models in [7] requires a convexity assumption (the so called displacement convexity, or convexity along geodesics) on the interaction energy functional which does not apply to our case.

In order to avoid technicalities we restrict the analysis to the case of one spatial dimension, where the Wasserstein metric can be computed in a reasonably simple way in terms of pseudo inverses of cumulative distributions (see Section 2). Also, the gradient flow structure is much simpler in this case and the notion of convexity along geodesics turns out to be equivalent to classical convexity at the level of the pseudo–inverse equation.

The models we consider are nonlocal parabolic evolution equations of the form

$$\partial_t \rho = \partial_x \left(\rho \partial_x [a(\rho) - G * \rho + V] \right) \qquad \text{in } \mathbb{R} \times \mathbb{R}^+, \tag{1.1}$$

with initial condition

$$\rho(.,0) = \rho_0 \qquad \text{in } \mathbb{R}. \tag{1.2}$$

In (1.1), $a : \mathbb{R}^+ \to \mathbb{R}$ is a nonlinear diffusion term, $G * \rho$ denotes the convolution

$$(G * \rho)(x) = \int_{\mathbb{R}} G(x-y)\rho(y)\ dy \tag{1.3}$$

with a scalar function $G : \mathbb{R} \to \mathbb{R}$, and $V : \mathbb{R} \to \mathbb{R}$ is an external potential. The nonlinear diffusion term $a(\rho)$ models local repulsions between particles, while the convolution term models wide-range attraction (and possibly also repulsion). Throughout this paper we shall require the follow basic structural conditions on equation (1.1):

(SD) Either a is strictly increasing such that the function $\rho \mapsto \rho a'(\rho)$ is integrable near $\rho = 0$, or $a \equiv 0$.

(EP) $V \in C^1(\mathbb{R})$, $V(-x) = V(x)$.

(IK) $G \in L^\infty(\mathbb{R})$, $G' \in L^\infty(\mathbb{R})$ and $G(-x) = G(x)$ for all $x \in \mathbb{R}$.

Assumption (SD) is usually referred to as *slow diffusion*, because it implies finite speed of propagation of the support (see section 2.3). We stress here that the main difference between our models and those in [7] is that we don't require global convexity of $-G$, due to biological motivations (see e.g. [4]). In the following we always assume that ρ_0 is scaled such that $\int_{\mathbb{R}} \rho_0\ dx = 1$. It is straight-forward to see from (1.1) that this property is conserved in time, i.e.,

$$\int_{\mathbb{R}} \rho(x,t)\ dx = 1, \qquad \forall\ x \in \mathbb{R}, t \in \mathbb{R}^+. \tag{1.4}$$

Moreover, from the interpretation of ρ as a density it is natural to look for nonnegative solutions ρ, and hence, for all $t \in \mathbb{R}^+$, $\rho(.,t)$ can be interpreted as a probability density with associated measure μ_ρ. We are mainly interested in the study of qualitative behavior of L^1 solutions to equation (1.1) and, in particular, in the asymptotic behavior for large times.

The paper is organized as follows: in Section 2 we recall the gradient flow formulation of the model in the Wasserstein space of probability measure. In Section 3 we shall prove existence of stationary solutions and their characterization in the purely aggregative case, i. e. when $a \equiv 0$. In this case we also provide existence of self–similar solutions and asymptotic stability of the stationary solutions in Wasserstein metrics. Some of the results in this section extend certain results about granular media models contained in [18] (in [18] $-G$ is required to be convex and homogeneous). In section 4 we prove existence of compactly supported stationary solutions in case $a(\rho) = \epsilon\rho^2$ for a small enough $\epsilon > 0$.

The major results of this paper concern the structure of stationary solutions and the large-time behavior of solutions of the evolution equation (1.1). In the purely aggregative case, i.e. for $a \equiv 0$ we obtain:

- For kernels with infinite support, there is a unique stationary solution, which is a Dirac δ distribution. For kernels of finite support, there is an infinite number of stationary solutions, which are all certain linear combinations of Dirac δ distributions.
- The solutions of the evolution equation converge weakly to stationary solutions in a set-valued sense.
- For initial values sufficiently close to stationary solutions, we even provide a rate of convergence to the equilibrium depending on the behavior of G' at zero.

In the case of diffusion and aggregation, i.e. a different from zero, we focus on the analysis of stationary solutions and provide a formal asymptotic analysis of the stationary solutions for sufficiently small diffusion. A rigorous analysis is carried out in the particularly interesting case $a(p) = \epsilon p^2$ with the following results:

- For ϵ sufficiently small, there exists a stationary solution with support having a diameter of order $\epsilon^{1/3}$, i.e., one really obtains the expected peak solutions.
- For ϵ sufficiently large (compared to the L^1-norm of the kernel G), there are no stationary solutions of a similar kind.

2 Preliminaries

2.1 Wasserstein Metric, Distribution, and Pseudo-Inverse

In the following we review the basic properties of the p–*Wasserstein metrics* on the space $\mathcal{P}$ of probability measures on $\mathbb{R}^d$ (for more details see for instance the book of Villani [28]). For $p > 1$ we introduce the notation

$$\mathcal{P}_p := \left\{ \mu \in \mathcal{P} \ : \ \int_{\mathbb{R}^d} |x|^p d\mu(x) < +\infty \right\}.$$

For $\mu_1, \mu_2 \in \mathcal{P}_p$, the p–Wasserstein distance between μ_1 and μ_2 is defined by

$$W_p(\mu_1, \mu_2)^p = \inf \left\{ \int\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y), \ \pi \in \Pi(\mathbb{R}^d \times \mathbb{R}^d) \right\},$$

where $\Pi(\mathbb{R}^d \times \mathbb{R}^d)$ is the space of all measures π on the product space $\mathbb{R}^d \times \mathbb{R}^d$ having μ_1 and μ_2 as marginal measures. The product measures $d\pi$ in the above definition are usually referred to as *transference plans*. Under further regularity assumptions on one of the two measures (namely, μ_1 gives no mass to sets of Hausdorff dimension less than $d-1$), the aforementioned minimization problem admits a unique *optimal* transference plan of the form $d\pi(x,y) = d\mu_1(x)\delta[y = T(x)]$ (i.e. Π is concentrated on the graph of a map $T : \mathbb{R} \to \mathbb{R}$), where T satisfies the *push–forward* condition $T_\sharp\mu_1 = \mu_2$ which reads

$$\int \psi \circ T\mu_1 = \int \psi d\mu_2, \qquad \text{for all} \quad \psi \in L^1(d\mu_2). \tag{2.1}$$

In one space dimension, the optimal map T is the same for all $p > 1$ and can be expressed in terms of the cumulative distribution functions of μ_1 and μ_2. This yields to a simplification in the expression of the p–Wasserstein distances. More precisely, let $R_i : \mathbb{R} \to [0,1]$, $i = 1,2$, be defined as the distribution function

$$R_i(x) = \int_{-\infty}^{x} d\mu_i(\xi), \quad i = 1,2. \tag{2.2}$$

The *pseudo-inverse* function of R_i, defined on the interval $[0,1]$, is given by

$$u_i(z) := R_i^{-1}(z) = \inf\left\{x \in \mathbb{R} \mid R_i(x) > z\right\}, \quad i = 1,2. \tag{2.3}$$

Under these notations, the p–Wasserstein metric between μ_1 and μ_2 can be expressed by

$$W_p(\mu_1,\mu_2) = \|u_1 - u_2\|_{L^p([0,1])}.$$

The previous relation can be heuristically proven as follows, at least in the case μ_1 and μ_2 do not concentrate on points. Since the optimal map T has to be monotone (in order to achieve the minimum rearrangement of the mass), we can take $\psi(y) = \chi_{(-\infty,x]}(y)$ in the push–forward condition (2.1) and we obtain

$$\int_{-\infty}^{T^{-1}(x)} d\mu_1(y) = \int_{-\infty}^{x} d\mu_2(y),$$

which implies $T^{-1} = R_1^{-1} \circ R_2$. Hence, we can use this expression in the definition of the Wasserstein distance

$$\begin{aligned} W_p(\mu_1,\mu_2)^p &= \int_{\mathbb{R}} |x - T(x)|^p d\mu_1(x)dx = \int_0^1 |R_1^{-1}(z) - T(R_1^{-1}(z))|^p dz \\ &= \int_0^1 |R_1^{-1}(z) - R_2^{-1}(z)|^p dz = \int_0^1 |u_1(z) - u_2(z)|^p dz. \end{aligned} \tag{2.4}$$

The simplified expression (2.4) suggests writing down the explicit time evolution equation for the pseudo inverse u. In this way, one can estimate the p–Wasserstein distances between two solutions to our equation (1.1) in terms of direct L^p estimate of the difference between the two corresponding pseudo–inverses. Hence, let $\rho(t)$ be a (eventually measure valued) solution to (1.1) and let $R(t)$ be its cumulative distribution function, then the pseudo–inverse u of R satisfies the following partial differential equation

$$\partial_t u(z,t) = -\partial_z\left(b(\partial_z u(z,t))\right) + \int_0^1 G'(u(z,t) - u(\zeta,t))\, d\zeta - V'(u(z,t)) \qquad \text{in } [0,1]\times\mathbb{R}^+, \tag{2.5}$$

where

$$b(p) := \int_0^{1/p} za'(z)dz.$$

Throughout this paper we shall often identify a density ρ with its corresponding pseudo–inverse u and viceversa.

2.2 Gradient Flow Formulation

In the following we recall the gradient flow formulation in terms of the Wasserstein metric, which turns out to be a simple gradient flow for the pseudo-inverse in spatial dimension one. In particular, in this case one does not need to introduce the notion of displacement convexity, which is equivalent to classical convexity at the level of the pseudo–inverse variable u. For the theory in more than one space dimension, see [7].

We start by writing the equation for the pseudo-inverse $u : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$, which is given by (2.5) with $b(p) := -p\ a(\frac{1}{p})$. We shall see below that (2.5) can be written as a standard gradient flow of the form

$$\partial_t u = -E'[u], \tag{2.6}$$

for a suitable energy functional. By transforming back to the density ρ, this allows to formulate the original model as a so-called *metric gradient flow* (cf. [1]).

Before introducing the gradient flow formulation we take a closer look on conserved quantities of (1.1) and (2.5), respectively. As we have noticed above, the total mass $\int_{\mathbb{R}} \rho\ dx$ is conserved during the evolution, which was the basis of introducing the distribution function and its pseudo-inverse. Consequently there is no equivalent conservation property of the model (2.5), it is somehow hidden in the fact that one can always consider u as a function on the interval $[0,1]$. In absence of an external potential V, the solutions also conserve the center of mass, given by the first moment of the density (if it exists)

$$CM := \int_{\mathbb{R}} \rho x\ dx = \int_{\mathbb{R}} x\ d\mu_\rho(x), \tag{2.7}$$

which can be seen as follows: taking the test function x for (1.1) we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \rho(x,t) x\ dx &= \int_{\mathbb{R}} \partial_t \rho(x,t) x\ dx \\ &= -\int_{\mathbb{R}} \rho \partial_x [a(\rho) - G * \rho]\ dx \end{aligned}$$

Now $\rho \partial_x a(\rho) = \partial_x \tilde{a}(\rho)$ for a function $\tilde{a}$ with derivative $\tilde{a}'(p) = pa'(p)$, and thus, the first term integrates to zero. The second term is

$$\int_{\mathbb{R}} \rho G' * \rho\ dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \rho(x) G'(x-y) \rho(y)\ dy\ dx$$

and from the symmetry of G we deduce

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \rho(x) G'(x-y) \rho(y)\ dy\ dx = -\int_{\mathbb{R}} \int_{\mathbb{R}} \rho(y) G'(y-x) \rho(x)\ dy\ dx = -\int_{\mathbb{R}} \rho(G' * \rho)\ dy,$$

and hence, the second term vanishes, too. In terms of the pseudo-inverse, the center of mass can be rewritten as

$$CM = \int_0^1 u\ dz, \tag{2.8}$$

which means that the mean value of u is conserved in time if there is no external potential.

2.2.1 Energy Functional

In the following we introduce an energy functional E such that $-E'[u]$ coincides with the right-hand side of (2.5). Let $B : \mathbb{R} \to \mathbb{R}$ be the primitive of b, i.e., $B' = b$. Then we define

$$E[u] := \int_0^1 \left[B\left(\partial_z u(z)\right) + V(u(z))\right]\, dz - \frac{1}{2}\int_0^1 \int_0^1 G(u(z) - u(\zeta))\, d\zeta\, dz \tag{2.9}$$

Note that E is well-defined at least for $u \in C^1([0,1])$ such that $B(\partial_z u) < \infty$ in $[0,1]$. On this set, the directional derivative of E is well defined and given by

$$\begin{aligned} E'[u]v &= \int_0^1 \left[b\left(\partial_z u(z)\right)\partial_z v(z) + V'(u(z))v(z)\right]\, dz + \\ &\quad -\frac{1}{2}\int_0^1 \int_0^1 G'(u(z) - u(\zeta))(v(z) - v(\zeta))\, d\zeta\, dz \end{aligned} \tag{2.10}$$

Exploiting the symmetry of G (and the consequent anti-symmetry of G') we obtain

$$-\frac{1}{2}\int_0^1 \int_{\mathbb{R}} G'(u(z) - u(\zeta))(v(z) - v(\zeta))\, d\zeta\, dz = -\int_0^1 \int_0^1 G'(u(z) - u(\zeta))\, d\zeta\, v(z)\, dz$$

After integration by parts, we can identify the derivative of E with

$$E'[u] = -\partial_z\left(b(\partial_z u(z))\right) + V'(u(z)) - \int_0^1 G'(u(z) - u(\zeta))\, d\zeta,$$

i.e., (2.5) can indeed be formulated as a gradient flow (2.6).

Before we proceed to an analysis of the energy functional, we discuss the interpretation of the gradient flow structure for the original formulation, which is also needed for an extension to multiple dimensions. Note that if a density ρ exists, then the energy functional can be rewritten as a functional of ρ and even generalized to multiple dimensions, namely as

$$\mathcal{E}[\rho] := \int_{\mathbb{R}^d} \left[A(\rho) + \rho V - \frac{1}{2}\rho(G * \rho)\right] dx. \tag{2.11}$$

with $A' = a$. The derivative is then given as

$$\mathcal{E}'[\rho] = a(\rho) + V - (G * \rho), \tag{2.12}$$

and consequently, (1.1) can be reformulated as the generalized gradient flow structure

$$\partial_t \rho = \operatorname{div}\left(\rho \nabla(\mathcal{E}'[\rho])\right). \tag{2.13}$$

2.2.2 Convexity Properties

In general, one cannot expect the energy functional E to be convex, in particular due to the convolution term. However, we can verify a generalized property, namely λ-convexity (see also [1])

$$E[\alpha u + (1-\alpha)v] \le \alpha E[u] + (1-\alpha)E[v] - \frac{\lambda}{2}\alpha(1-\alpha)\|u - v\|_2^2 \qquad \forall\, u, v \in L^2([0,1]), \tag{2.14}$$

which is fundamental property for existence and uniqueness of the gradient flow.

For the analysis we split the energy functional in the form $E = E_1 + E_2$ with

$$E_1[u] := \int_{\mathbb{R}} B\left(\partial_z u(z)\right)\, dz \tag{2.15}$$

and

$$E_2[u] := \int_{\mathbb{R}} V(u(z))\, dz - \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} G(u(z) - u(\zeta))\, d\zeta\, dz. \tag{2.16}$$

Due to the convexity of B we immediately obtain that E_1 is a convex functional, and therefore we concentrate our attention on the properties of E_2.

Lemma 2.1. *Let G and V be twice continuously differentiable with bounded derivatives. Then the functional E_2 is twice continuously Frechet-differentiable on $L^2([0,1])$ and there exists a constant C such that*

$$E_2''[u](\varphi,\varphi) \geq -C\|\varphi\|^2 \tag{2.17}$$

Proof. We can compute

$$E_2'[u]\varphi = \int_{\mathbb{R}} V'(u(z))\varphi(z)\, dz - \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} G'(u(z) - u(\zeta))(\varphi(z) - \varphi(\zeta))\, d\zeta\, dz$$

and

$$\begin{aligned} E_2''[u](\varphi,\psi) \;&=\; \int_{\mathbb{R}} V''(u(z))\varphi(z)\psi(z)\, dz - \\ &\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} G''(u(z) - u(\zeta))(\varphi(z) - \varphi(\zeta))(\psi(z) - \psi(\zeta))\, d\zeta\, dz. \end{aligned}$$

By standard estimates one can verify that E_2' and E_2'' are Frechet-derivatives, and one obtains (2.17) with $C = \|V''\|_\infty + 2\|G''\|_\infty$. □

As a consequence of (2.17) we have (with the notation $w = \alpha u + (1-\alpha)v$)

$$\begin{aligned} E_2[v] \;&=\; E_2[w] + \alpha E_2'[w](v-u) + \frac{\alpha^2}{2}\int_0^1 E_2''[v + \sigma\alpha(v-u)](v-u, v-u)\, d\sigma \\ &\geq\; E_2[w] + \alpha E_2'[w](v-u) - \frac{C\alpha^2}{2}\|v-u\|_2^2, \end{aligned}$$

and by analogous reasoning we obtain

$$E_2[u] \;\geq\; E_2[w] + (1-\alpha)\alpha E_2'[w](u-v) - \frac{C(1-\alpha)^2}{2}\|v-u\|_2^2.$$

Taking a convex combination of the last two inequalities we obtain

$$E_2[w] \leq \alpha E_2[u] + (1-\alpha)E_2[v] + C(1-\alpha)^2\|v-u\|_2^2.$$

Hence, adding the convex functional E_1, which satisfies

$$E_1[w] \leq \alpha E_1[u] + (1-\alpha)E_1[v],$$

we obtain (2.14) with $\lambda = -2C$.

Note that from the λ-convexity (2.14) we can immediately deduce existence and uniqueness of solutions of the gradient flow (2.6) by specialization of general results in [1]. However, we will investigate the correct formulation of weak solutions for the gradient flow and the ideas of existence and uniqueness proofs below, since they yield further insight into the structure of solutions that is useful also for the analysis below.

2.2.3 Existence and Uniqueness of Solutions

Now we turn our discussion to the existence and uniqueness of solutions of the gradient flow (2.5), which, due to the correspondence via the pseudo-inverse, also yields existence and uniqueness of solutions of (1.1) (in an appropriately chosen weak solution context). We mention that existence and uniqueness for the density equation (1.1) has been already considered before: In the absence of a nonlinear diffusion term, this problem has been solved via a method of characteristics (cf. [2]). For quadratic diffusion terms the existence and uniqueness could be verified in terms of entropy solutions (cf. [4]), and one observes that the proofs extend to a larger class of nonlinear diffusions. However, all these approaches are very involved and need very strong assumptions on the data (in particular for proving uniqueness). Using the gradient flow formulation we derived above together with the λ-convexity of the energy functional, the existence and uniqueness for equation (2.5) is straight-forward. The only delicate point is to keep the connection with the original Eulerian formulation, which is guaranteed only for monotone solutions of (2.5).

A weak solution $u \in C(0,T;L^2([0,1]))$ of the gradient flow (2.5) with initial value u_0 is characterized by (cf. [1])

$$u|_{t=0} = u_0 \tag{2.18}$$

$$\langle \partial_t u, g\rangle = \frac{1}{2}\|g\|_{L^2}^2 + \frac{1}{2}\|\partial_t u\|_{L^2}^2 \qquad L^1\text{–a.e. in } (0,T), \tag{2.19}$$

$$g = h + E_2'[u] \qquad L^2\text{–a.e. in } (0,T), \tag{2.20}$$

$$\langle h, \varphi - u\rangle \le E_1[\varphi] - E_1[u] \qquad L^1\text{–a.e. in } (0,T), \forall\ \varphi \in C(0,T;L^2[0,1])). \tag{2.21}$$

Note that here we have used the special properties of the energy E, which can be split into the convex part E_1 and a continuously differentiable part E_2, so that upper gradients of E are obtained as the sum of a subgradient of E_1 and the Frechet-derivative of E_2.

Theorem 2.2 (Existence and Uniqueness). *Let F and V be twice continuously differentiable with bounded derivatives, and let $u_0 \in L^2([0,1])$ be monotone. Then there exists a unique monotone solution $u \in C(0,T;L^2([0,1]))$ of (2.5) with initial value u_0.*

Proof. The existence and uniqueness of a solution defined as above follows from the general results in [1]. Having established the existence, we can compute

$$W(z,t) := \int_0^1 G''(u(z,t) - u(\zeta,t))\ d\zeta - V''(u(z,t))$$

and show that $\partial_z u$ is a measure-valued solution of the transport equation

$$\partial_t v = -\partial_{zz}\left(b(v)\right) + Wv \qquad \text{in } [0,1]\times\mathbb{R}^+.$$

Since this first-order equation preserves positivity, we conclude that u is monotone for all $t > 0$ if u_0 is monotone. □

We finally mention that Theorem 2.2 also covers the case without diffusion, where the original equation (1.1) could have measure-valued solutions. Note that a density for the original equation only exists if $\partial_z u$ exists and is strictly positive in the whole interval $[0,1]$.

2.3 Finite Speed of Propagation

The structural condition (SD) on the nonlinear diffusion term in (1.1) implies a *slow* propagation of the support of the solution, in the same fashion as in the porous medium equation (see [26, 27]). Following the ideas in [6, 5], one can prove that the supports of the solutions propagate with finite speed by using an interpretation of the limit as $p \to +\infty$ of the p–Wasserstein distances. Let us first introduce the ∞–Wasserstein distance (see also [9])

$$W_\infty(\mu_1, \mu_2) := \lim_{p \to +\infty} W_p(\mu_1, \mu_2).$$

From (2.4) it is clear that

$$W_\infty(\mu_1, \mu_2) = \|u_1 - u_2\|_{L^\infty([0,1])}.$$

Moreover, the following estimate on the speed of propagation of the support can be easily proven,

$$\max\{|\inf(\mathrm{supp}\mu_1) - \inf(\mathrm{supp}\mu_2)|, |\sup(\mathrm{supp}\mu_1) - \sup(\mathrm{supp}\mu_2)|\} \le W_\infty(\mu_1, \mu_2). \tag{2.22}$$

We can state the following result, the proof of which is a trivial generalization of a similar theorem in [5]. Therefore it will be omitted.

Theorem 2.3. *Let ρ be a solution to (1.1) having a compactly supported initial datum. Then, the support of $\rho(t)$ at any time $t > 0$ is compact.*

3 Pure Aggregation Models

In the following we specialize to the case of a pure aggregation model, i.e., $a \equiv 0$. We shall require G to satisfy (IK) above and the additional hypotheses

$$G \ge 0, \quad G \text{ has a unique maximum } \ g_0 = G(0), \quad G'(x) < 0 \text{ for } x \in \mathrm{supp}G \cap [0, +\infty). \tag{3.1}$$

We shall consider several assumptions on the potential V according to different cases. The continuity equation for the density in this case reads

$$\partial_t \rho + \partial_x \left(\rho \partial_x [G * \rho - V]\right) = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}^+, \tag{3.2}$$

and the equation for the pseudo-inverse becomes

$$\partial_t u(z,t) = \int_0^1 G'(u(z,t) - u(\zeta,t))\, d\zeta - V'(u(z,t)) \qquad \text{in } [0,1] \times \mathbb{R}^+, \tag{3.3}$$

In this case the above analysis allows for measure-valued solutions ρ of (3.2) due to the absence of a diffusion term, e.g. Dirac δ-distributions. In particular one expects a concentration to such measures in the long time limit, and we will show below that this is indeed true under reasonable assumptions.

For the analysis below we define the Radon measure δ_γ via

$$\langle \delta_\gamma, \varphi \rangle := \varphi(\gamma), \qquad \forall\ \varphi \in C(\mathbb{R}). \tag{3.4}$$

In spatial dimension one the distribution function corresponding to δ_γ is given by

$$R_\gamma := \begin{cases} 0 & \text{if } x \leq \gamma \\ 1 & \text{if } x < \gamma. \end{cases}$$

The corresponding pseudo-inverse function is then the constant function $v \equiv \gamma$. Hence, asymptotic concentration of the density ρ to a Dirac-δ is equivalent to convergence of the pseudo-inverse to a constant state. We conclude this subsection by recalling the entropy dissipation identity for equation (3.2), given by

$$\int_{\mathbb{R}} (V - G * \rho(t)) d\rho(t)(x) + \int_0^t \int_{\mathbb{R}} |\partial_z (V - G * \rho(\tau))|^2 d\rho(\tau)(x) d\tau = \int_{\mathbb{R}} (V - G * \rho(0)) d\rho(0)(x). \tag{3.5}$$

3.1 Stationary Solutions

We first investigate possible stationary states. As noticed above, a Dirac-δ is a good candidate, and in the following we verify that such a stationary solution indeed exists (with location determined by the potential):

Proposition 3.1 (Existence of Stationary States). *Let $\gamma \in \mathbb{R}$ be such that $V'(\gamma) = 0$. Then $v \equiv \gamma$ (respectively $\rho = \delta_\gamma$) is a stationary solution of* (3.3) *(respectively* (3.2)*).*

Proof. By inserting $v \equiv \gamma$ into the right-hand side of (3.3) we obtain

$$\int_0^1 G'(v(z) - v(\zeta))\, d\zeta - V'(v(z)) = \int_0^1 G'(\gamma - \gamma)\, d\zeta - V'(\gamma) = G'(0) - V'(\gamma)$$

and since G has a maximum at 0 and V a minimum at γ, the last term vanishes. Hence, v is a stationary solution. □

Remark 3.2. Note that if V has a unique stationary point, then Proposition 3.1 describes only a single stationary solution, while in the case $V \equiv 0$ each constant function v (and therefore each Dirac δ) is a stationary state. On the other hand, the center of mass is conserved by the evolution in the latter case, and therefore the only reasonable stationary state to consider is the one with the same center of mass as the initial value, since no other constant function v could be a reasonable limit.

Under certain extra assumptions on G and V it is possible to construct nontrivial stationary solutions to (3.2) which are linear combinations of delta distributions. We remark that the following result is a generalization of a similar existence result proven in [18], where the authors prove existence of nontrivial singular equilibria for equation (3.2) in the special case $G(x) = |x|^\gamma$, $1 < \gamma \leq 4$ and $V(x) = -|x|^2/2$.

Proposition 3.3 (Nontrivial equilibria). *Suppose that G satisfies $G'(0) = 0$ and suppose there exists $x_0 \in (0, +\infty)$ such that*

$$G'(2x_0) = 2V'(x_0). \tag{3.6}$$

Then, the measure

$$\rho^\infty := \frac{1}{2}\left[\delta_{-x_0} + \delta_{x_0}\right]$$

is a (distributional) stationary solution to (3.2).

Proof. In order to proof the above assertion we have to check that

$$0 = \int_{\mathbb{R}} \phi'(x) \left[G' * \rho^\infty(x) - V'(x)\right] d\rho^\infty(x), \tag{3.7}$$

for any test function ϕ. By definition of ρ^∞, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \phi'(x) \left[G' * \rho^\infty(x) - V'(x)\right] d\rho^\infty(x) \\
&= \frac{1}{2}\int_{\mathbb{R}} \phi'(x) \left[G'(x-x_0) + G'(x+x_0)\right] d\rho^\infty(x) - \frac{1}{2}\left[\phi'(-x_0)V'(-x_0) + \phi'(x_0)V'(x_0)\right] \\
&= \phi'(-x_0)\left[\frac{1}{4}\left(G'(-2x_0) + G'(0)\right) - V'(-x_0)\right] + \phi'(x_0)\left[\frac{1}{4}\left(G'(2x_0) + G'(0)\right) - V'(x_0)\right]
\end{aligned}$$

and the above expression vanishes because of the symmetry of G and V and of the condition $G'(0) = 0$ and in view of (3.6). □

Remark 3.4. We remark that the results stated in Propositions 3.1 and 3.3 require neither any convexity assumption on V nor the fact that the stationary point 0 is the unique maximizer for G. If the latter is satisfied, then it is clear from (3.6) that the nontrivial singular states found in Proposition 3.3 cannot exist when $V(x)$ is increasing for positive x, which is the case when e. g. V is uniformly convex. This fact suggests that such nontrivial states exist when the external potential V produces a repulsive drift, which is unlikely in the applications. On the other hand, a nonlocal interaction kernel with changing sign in its second derivative still allows the existence of such stationary states even when V is uniformly convex (provided that (3.6) is satisfied). An interesting situation occurs when $V \equiv 0$ and G has compact support. In this case, there exist infinite stationary states of the previous form, corresponding to a point x_0 such that

$$x_0 > \frac{\eta}{2}, \quad \eta = \sup_{x \in \mathrm{supp}(G)} |x|.$$

As pointed out at the end of the previous remark, stationary solutions different from the trivial one (i. e. the Dirac δ centered at the center of mass) may exists in case $V \equiv 0$ and G compactly supported. In what follows we show that the assumption of compact support for G is necessary in order to detect non trivial stationary states if G has only one stationary point, whereas infinitely many stationary states can be produced when G has compact support.

Theorem 3.5 (Uniqueness of Stationary States for Infinite Range). *Let $G \in L^1(\mathbb{R})$ be such that* $\mathrm{supp}G = \mathbb{R}$ *and $G'(p) > 0$ for $p < 0$ (in addition to standard properties of G assumed above), and let $V \equiv 0$. Then each stationary solution of* (3.3) *is of the form $v \equiv \gamma$ for $\gamma \in \mathbb{R}$.*

Proof. Let $q = \sup_{z\in[0,1]} v(z)$, and first assume $q < \infty$. The function v is monotonely non-decreasing, and thus, for each $\epsilon > 0$ we can find $\delta > 0$ such that

$$v(z) > q - \epsilon. \qquad \text{if } z > 1 - \delta.$$

Hence,

$$0 = \int_0^1 G'(v(z) - v(\zeta))\, d\zeta \le \int_0^{1-\delta} G'(v(z) - v(\zeta))\, d\zeta + \int_{1-\delta}^1 G'(v(z) - v(\zeta))\, d\zeta$$

As $\epsilon \to 0$, we obtain the limiting inequality

$$0 \le \int_0^1 G'(q - v(\zeta))\, d\zeta, \tag{3.8}$$

but since $G'(q - v(\zeta)) > 0$ for $v(\zeta) > q$, then (3.8) can only be true for $v \equiv q$.

If $q = \infty$, then there exists $r \in \mathbb{R}$ such that the original measure μ satisfies $G' * \mu = 0$ in (r, ∞). Consequently, $G * \mu$ is constant in (r, ∞). On the other hand, since $G \in L^1(\mathbb{R})$ and μ is a probability measure, we obtain that $G * \mu \in L^1(\mathbb{R})$ and therefore the constant value in (r, ∞) can only be zero. Hence, $G * \mu = 0$ in (r, ∞), which is only possible for $\mu \equiv 0$ due to the positivity of G, but this contradicts $\mu(\mathbb{R}) = 1$. □

If on the other hand the interaction range is finite (i.e., the support of the kernel is compact) and there is no external potential, then we can immediately construct an infinite number of different stationary states:

Theorem 3.6 (Infinite number of Stationary States for Finite Range). *Let G be such that* $\text{supp}G = [-\eta, \eta]$ *and let $V \equiv 0$. Then, for each $N \in \mathbb{N}$ and each $(\gamma_j)_{j=1,\dots,N} \in \mathbb{R}^N$ such that $\gamma_j + \eta < \gamma_{j+1}$, the function*

$$v^N(z) := \begin{cases} \gamma_j & \text{if } \frac{j-1}{N} \le z < \frac{j}{N} \\ \gamma_N & \text{if } z = 1 \end{cases} \tag{3.9}$$

is a stationary solution of (3.3).

Proof. Let $\frac{k-1}{N} \le z < \frac{k}{N}$. Then,

$$\int_0^1 G'(v(z) - v(\zeta))\, d\zeta = \frac{1}{N} \sum_{j=1}^{N} G'(\gamma_k - \gamma_j)$$

and $G'(\gamma_k - \gamma_j) = 0$ either since $G'(0) = 0$ or since $|\gamma_k - \gamma_j| > \eta$. □

To conclude this subsection, we prove that we can characterize the compactly supported stationary states also in case G has a finite range. More precisely, in terms of the pseudo inverse variable, a function v is a stationary solution to (3.3) with $V \equiv 0$ if and only if v is piecewise constant.

Theorem 3.7 (Characterization of steady states with finite range). *Let G be such that (3.1) is satisfied and such that* $\text{supp}G = [-\eta, \eta]$*, and let $V \equiv 0$. Then, all bounded stationary solutions to (3.3) are of the form (3.9) with $|\gamma_i - \gamma_j| > \eta$ for all i, j.*

Proof. The proof uses the same strategy of the proof of Theorem 3.5. Let once again $q = \sup_{z\in[0,1]} v(z)$. With similar arguments as in Theorem 3.5, we deduce

$$0 \le \int_0^1 G'(q - v(\zeta))\, d\zeta.$$

Now, $G'(q - v(\zeta)) \ge 0$ may occur either for $v(\zeta) \ge q$ or for $v(\zeta) \le q - \eta$. Let

$$\bar{\zeta} = \sup\{\zeta \in [0, 1) \mid v(\zeta) \le q - \eta\}.$$

Then, we have

$$\int_{\bar{\zeta}}^{1} G'(q - v(\zeta))\, d\zeta = \int_0^1 G'(q - v(\zeta))\, d\zeta - \int_0^{\bar{\zeta}} G'(q - v(\zeta))\, d\zeta \geq 0.$$

Now, since $G'(q - v(\zeta)) \leq 0$ for $\zeta \in [\bar{\zeta}, 1]$, we can deduce that $v(\zeta) \equiv q$ over the interval $[\bar{\zeta}, 1]$. We can now repeat the same argument on the interval $[0, \bar{\zeta}]$ in order to prove that v is constant on an interval $[\tilde{\zeta}, \bar{\zeta}]$. The proof can be completed by iteration, where the finiteness of the number of steps is guaranteed by the condition $q - v(\bar{\zeta}) \geq \eta$ and by the fact that v is bounded. The last assertion of the Theorem is a consequence of Theorem 3.8. □

3.2 Self–Similar Solutions

In this subsection we shall analyze the existence of so called self–similar solutions of the form

$$\rho(x,t) = \frac{1}{2}\delta_{-y(t)} + \frac{1}{2}\delta_{y(t)} \tag{3.10}$$

of the equation (3.2) under the standard assumption that $V' \geq 0$ on $[0, +\infty)$ and G satisfying (3.1). In terms of the pseudo inverse equation (3.3), a solution of the form (3.10) can be expressed in terms of a function $t \to y(t)$ solving the ordinary differential equation

$$\dot{y}(t) = \frac{1}{2}G'(2y(t)) - V'(y(t)). \tag{3.11}$$

The equation (3.11) can be easily recovered by substituting

$$u(z,t) = \begin{cases} -y(t) & \text{if } 0 < \frac{1}{2} \\ y(t) & \text{if } \frac{1}{2} \leq 1 \end{cases}$$

in the equation (3.3) and by choosing $z \in (1/2, 1]$, whereas the equation will be automatically satisfied for $z \in [0, 1/2]$. As can be easily seen, the stationary solutions found in Theorem 3.3 are obtained by means of constant solutions to (3.11), since condition (3.6) is nothing but the stationary equation corresponding to (3.11). Suppose now that (3.6) is not satisfied by the initial datum, more precisely suppose that $y(0) = y_0 > 0$ and

$$G'(2y_0) - 2V'(y_0) < 0$$

(we recall that $G'(2y_0) - 2V'(y_0) \leq 0$ for all $y_0 > 0$ because of the assumptions on G and V). Then we can explicitly compute the solution to (3.11) by direct integration to obtain

$$y(t) = F^{-1}(-t), \quad F(x) = \int_{y_0}^{x} \frac{1}{V'(\xi) - \frac{1}{2}G'(2\xi)} d\xi. \tag{3.12}$$

Clearly, the solution $t \to y(t)$ is decreasing and therefore it admits a limit for large time. Let then $\bar{y} := \lim_{t\to+\infty} y(t)$ and suppose that $\bar{y} \neq 0$. Then we can write

$$+\infty = \int_{\bar{y}}^{y_0} \frac{1}{V'(\xi) - \frac{1}{2}G'(2\xi)} d\xi$$

which is a contradiction because the integrand above is bounded away from $\xi = 0$. Therefore $y(t)$ tends to zero for large times. We have thus proven that the equation (3.2) exhibits self–similar solutions of the form (3.10) with $y(t) \to 0$ as $t \to +\infty$. Depending on the behavior of G' and V' at zero, $y(t)$ attains the value zero at finite or infinite time. More precisely, let

$$t^\infty := \sup\{t > 0 \mid y(t) > 0\}.$$

Then, we have

$$t^\infty = \int_0^{y_0} \frac{1}{V'(\xi) - \frac{1}{2}G'(2\xi)} d\xi.$$

We summarize all the previous computations in the following Theorem.

Theorem 3.8. *Suppose $V'' \geq 0$ and G satisfies (3.1). Let*

$$G'(2y_0) - 2V'(y_0) < 0$$

and let $y(t) > 0$ be given by (3.12). Then, the measure

$$\rho(x,t) = \frac{1}{2}\delta_{-y(t)} + \frac{1}{2}\delta_{y(t)}$$

solves the equation (3.2) in weak sense. Moreover, given the (eventually infinite) time

$$t^\infty = \int_0^{y_0} \frac{1}{V'(\xi) - \frac{1}{2}G'(2\xi)} d\xi,$$

$y(t)$ is strictly increasing on $[0, t^\infty)$ and $\lim_{t \to t^\infty} y(t) = 0$.

3.3 Attractors of the Evolution

For long time we expect the convergence of arbitrary solutions to stationary solutions (whose existence is guaranteed from the results above). In a general setup, this is true at least in a weak sense for subsequences as we shall show now:

Theorem 3.9. *Let u be the solution of* (3.3) *and $\mu(t)$ the probability measure corresponding to $u(.,t)$ with initial value $\mu_0 = \mu(0)$. Suppose $\int_{\mathbb{R}} V(x) d\mu_0(x) < +\infty$. Then, there exists a sequence $t_k \to \infty$ and a probability measure μ^∞ such that $\mu(t) \to \mu^\infty$ weakly in $\mathcal{P}(\mathbb{R})$, and each weak limit of a subsequence of $\mu(t)$ is a stationary solution of* (3.2) *(respectively the pseudo-inverse of the distribution function is a stationary solution of* (3.3)*).*

Proof. Due to the energy dissipation relation (3.5) we know that

$$\int_0^\infty \langle \mu(t), |\partial_x(G * \mu(t) - V)|^2 \rangle dt < \infty.$$

Hence, there exists a subsequence such that

$$\langle \mu(t_k), |\partial_x(G * \mu(t_k) - V)|^2 \rangle \to 0.$$

Due to standard compactness properties of probability measures we can extract a convergent subsequence (again denoted by t_k) such that $\mu(t_k)$ converges to some limit μ^∞ weakly in

$\mathcal{P}(\mathbb{R})$. Then by the properties of the convolution, $\partial_x G * \mu(t_k) \to \partial_x G * \mu^\infty$ strongly in $C(\mathbb{R})$ and hence

$$\langle \mu(t_k), |\partial_x(G * \mu(t_k) - V)|^2 \rangle \to \langle \mu^\infty, |\partial_x(G * \mu^\infty - V)|^2 \rangle \quad \text{weakly in } \mathcal{P}(\mathbb{R})$$

and by the uniqueness of the limit $\langle \mu^\infty, |\partial_x(G * \mu^\infty - V)|^2 \rangle = 0$, which implies μ^∞ is a stationary solution. □

For infinite range interactions, the results can be strengthened, since the only stationary solution is a Dirac-δ at the center of mass

Corollary 3.10. *Let the assumptions of Theorem 3.9 be satisfied and let G be as in Theorem 3.5. Then $\mu(t) \to \delta_{CM}$ weakly in $\mathcal{P}(\mathbb{R})$ as $t \to \infty$.*

3.4 Moment Estimates and Convergence Rates to Singular Equilibria

In this section we focus on contraction and stability properties of solutions of (3.2) with respect to the Wasserstein distances. In order to perform this task, we shall deal with L^p estimates in terms of solutions to the pseudo–inverse equation (3.3). First we shall prove that the solutions to (3.3) are stable in L^p. Such a stability turns out to be a strict contraction in case V has a unique minimum at zero, and this implies that the p–moments of the solution ρ to equation (3.2) are decreasing in time. The convolution kernel G will be supposed to satisfy (IK) and (3.1). The estimates performed in this section and in the following one are natural generalizations of the estimates in [18]. Throughout this section, $\rho(t)$ will denote the solution of (3.2) while $u(t)$ will denote the corresponding pseudo–inverse satisfying (3.3). We have the following theorem.

Theorem 3.11. *Suppose that G satisfies (IK) and (3.1). Moreover assume that the potential V satisfies*

$$V'(u) \geq \alpha u, \qquad \forall\ u \geq 0.$$

Let $u(z,t)$ be the solution of (3.3) having initial datum $u_0 \in L^p([0,1])$. Then, for $p \in [1, +\infty]$

$$W_p(\rho(t), \delta_0) = \|u(t)\|_{L^p([0,1])} \leq e^{-\alpha t} \|u_0\|_{L^p([0,1])},$$

for all $t \geq 0$.

Proof. We formally compute the time evolution of the L^p norm of u. The following estimate can be made rigorous by smoothing the function $|\cdot|^p$ in a standard way. In the following we drop the time dependency in the notation for simplicity, and derive

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 |u(z,t)|^p dz = p\int_0^1 |u(z)|^{p-1}\text{sign}(u(z))u_t dz \\
&= p\int_0^1\int_0^1 |u(z)|^{p-1}\text{sign}(u(z))[G'(u(z)-u(\zeta))]dz d\zeta - p\int_0^1 |u(z)|^{p-1}\text{sign}(u(z))V'(u(z))dz \\
&=: I_1 + I_2.
\end{aligned}$$

Due to the antisymmetry of G' and to $G'(z)$ being nonnegative for nonnegative z, using the monotonicity of the function $u \mapsto |u|^{p-1}\mathrm{sign}(u)$, we can compute the term I_1 as follows,

$$\begin{aligned}
I_1 = p \int\int_{z\leq\zeta} |u(z)|^{p-1}\mathrm{sign}(u(z))[G'(u(z)-u(\zeta))]dzd\zeta +\\
p \int\int_{z\geq\zeta} |u(z)|^{p-1}\mathrm{sign}(u(z))[G'(u(z)-u(\zeta))]dzd\zeta\\
\leq p \int\int_{z\leq\zeta} |u(\zeta)|^{p-1}\mathrm{sign}(u(\zeta))[G'(u(z)-u(\zeta))]dzd\zeta +\\
- p \int\int_{z\geq\zeta} |u(z)|^{p-1}\mathrm{sign}(u(z))[G'(u(\zeta)-u(z))]dzd\zeta = 0.
\end{aligned}$$

The assumption on V' implies

$$I_2 \leq -\alpha p \int_0^1 |u(z)|^p dz$$

and the assertion follows with the Gronwall lemma for finite p. The case $p = \infty$ can be obtained by taking the limit $p \to +\infty$. □

The results in Theorem 3.11 provide exponential convergence of $\rho(t)$ in the p–Wasserstein distance towards the Dirac δ centered at zero in case $\alpha > 0$. A similar situation occurs in case $V \equiv 0$, namely

$$\partial_t u = \int_0^1 G'(u(z) - u(\zeta))d\zeta. \tag{3.13}$$

As shown in a previous section, in this case the suitable stationary state $u \equiv \gamma$ is determined by the center of mass, which is invariant under the flow. We shall therefore consider initial data with zero center of mass for simplicity. Again, the behavior of the solutions depends on the range of G being finite of infinite. In case of infinite range, we shall prove that the second moment (or, equivalently the 2–Wasserstein distance to δ_0) of any compactly supported solution converges to zero for large times. The rate of convergence depends on the size of the support of initial data and on the behavior of G' near zero, as it can be seen in the statement of the following theorem. Again, the proof is reminiscent of [18].

Theorem 3.12. *Suppose G satisfies (IK) and (3.1) and suppose $V \equiv 0$. Moreover, suppose that* $\mathrm{supp}(G) = \mathbb{R}$ *and that*

$$\lim_{x\to 0^+} \frac{G'(x)}{x^\alpha} = l < 0, \tag{3.14}$$

for some $\alpha > 0$. Let $u(z,t)$ be the solution to (3.13) having initial datum $u_0 \in L^\infty([0,1])$ (or, equivalently, $\mathrm{supp}(\rho(0))$ *compact). Then, there exists a constant $C > 0$ depending on G and on*

$$\|u_0\|_{L^\infty([0,1])} = \max\{|\inf\mathrm{supp}(\rho(0))|, |\sup\mathrm{supp}(\rho(0))|\}$$

such that,

- $\|u(t)\|_{L^{2k}([0,1])} \leq e^{-Ct}\|u_0\|_{L^{2k}([0,1])}$*, for any positive integer k, if $\alpha = 1$,*
- $\|u(t)\|_{L^2([0,1])} \leq C(1+t)^{-\frac{1}{\alpha-1}}$ *if $\alpha > 1$,*
- *there exists $t^* > 0$ such that $\|u(t)\|_{L^\infty([0,1])} = 0$ for $t \geq t^*$ if $0 < \alpha < 1$.*

Proof. As in the previous theorem, by direct computation of the evolution of the L^{2k} norm of $u(t)$ we have

$$
\begin{aligned}
\frac{d}{dt}\int_0^1 u(z,t)^{2k}dz &= 2k\int_0^1 u(z)^{2k-1}u_t dz = 2k\int_0^1\int_0^1 u(z)^{2k-1}[G'(u(z)-u(\zeta))]dzd\zeta \\
&= 2k\int\int_{z\leq\zeta} u(z)^{2k-1}[G'(u(z)-u(\zeta))]dzd\zeta + 2k\int\int_{z\geq\zeta} u(z)^{2k-1}[G'(u(z)-u(\zeta))]dzd\zeta \\
&= 2k\int\int_{z\geq\zeta}[u(z)^{2k-1}-u(\zeta)^{2k-1}][G'(u(z)-u(\zeta))]dzd\zeta.
\end{aligned}
$$

Now, thanks the result in Theorem 3.11,

$$
|u(z)-u(\zeta)| \leq 2\|u(t)\|_{L^\infty} \leq 2\|u(0)\|_{L^\infty},
$$

and the above inequality, together with hypotheses (3.14) and (3.1), guarantees that

$$
\frac{[G'(u(z)-u(\zeta))]}{[u(z)-u(\zeta)]^\alpha} \leq -L \tag{3.15}
$$

for a certain $L>0$ (L depending on $\|u(0)\|_{L^\infty}$) when $z\geq\zeta$. We observe here that, in view of the hypotheses $G\in L^1$, one has to choose a suitably small constant L in (3.15) when the size of the initial support of ρ (i. e. the initial L^∞–norm of u) is very large. Therefore, at this level of the proof the infinite size of the range of G plays a key role. Due to (3.15) we have

$$
\frac{d}{dt}\int_0^1 u(z,t)^{2k}dz \leq -2kL\int\int_{z\geq\zeta}[u(z)^{2k-1}-u(\zeta)^{2k-1}][u(z)-u(\zeta)]^\alpha dzd\zeta. \tag{3.16}
$$

Using the conservation of the first moment, in a similar fashion as in [18], we can prove the following estimate which will be useful in the sequel,

$$
\begin{aligned}
\int_0^1 u(z)^{2k}dz &\leq \int_0^1\int_0^1 [u(z)-u(\zeta)]^2 u(z)^{2k-2}dzd\zeta \\
&\leq \frac{1}{2}\int_0^1\int_0^1 [u(z)-u(\zeta)]^2[u(z)^{2k-2}+u(\zeta)^{2k-2}]dzd\zeta \\
&\leq \frac{1}{2}\int_0^1\int_0^1 [u(z)-u(\zeta)][u(z)^{2k-1}-u(\zeta)^{2k-1}]dzd\zeta,
\end{aligned} \tag{3.17}
$$

We now consider the following three cases corresponding to different ranges of values of α:
Case $\alpha>1$. The inequality (3.16) when $k=1$ implies in this case

$$
\frac{d}{dt}\int_0^1 u(z,t)^2dz \leq -2L\int\int_{z\geq\zeta}[u(z)-u(\zeta)]^{1+\alpha}dzd\zeta = -L\int_0^1\int_0^1[u(z)-u(\zeta)]^{1+\alpha}dzd\zeta.
$$

By Hölder inequality and by conservation of the first moment, we have

$$
\frac{d}{dt}\int_0^1 u(z,t)^2dz \leq -L\int\left[\int_{z\geq\zeta}[u(z)-u(\zeta)]^2dzd\zeta\right]^{\frac{1+\alpha}{2}} = -L2^{\frac{1+\alpha}{2}}\left(\int_0^1 u(z,t)^2dz\right)^{\frac{1+\alpha}{2}}.
$$

Hence, the variation of constants formula applied to the above inequality implies the assertion.

Case $\alpha = 1$. The inequalities (3.16) and (3.17) imply

$$\frac{d}{dt}\int_0^1 u(z,t)^{2k}dz \leq -2kL\int_0^1 u(z,t)^{2k},$$

and the assertion follows with the Gronwall lemma.
Case $0 < \alpha < 1$. The proof in this case can be performed in the same way as in [18, Section 6, Theorem 6.1] starting from the inequalities (3.16) and (3.17). Therefore, we shall skip the further details here. □

Obviously, the above result has to be interpreted in terms of convergence in p–Wasserstein distance of $\rho(t)$ to the Delta measure. In particular, in the case $0 < \alpha < 1$ the support of $\rho(t)$ degenerates in finite time.

Let us now focus on the case of an interaction kernel G having finite support. As in Theorem 3.6 we define

$$\mathrm{supp}G := [-\eta, \eta],$$

for some $\eta > 0$. As shown in Theorem 3.6, in this case we have infinite singular equilibria for equation (3.2) when $V \equiv 0$. The selection of the right stationary solution as a typical asymptotic state depends on the location of the support of the initial datum. In particular, we shall prove that when the initial datum has many connected components *'sufficiently far from each other'* each one having *'sufficiently small support'*, then each connected component behaves independently from the others. In particular, each connected component will converge weakly to a Dirac δ centered at its first moment. This result is summarized in the following theorem.

Theorem 3.13. *Let ρ be the solution to (3.13), with G satisfying (3.14) for some $\alpha > 0$, with a compactly supported probability measure ρ_0 as initial datum. Suppose that* $\mathrm{supp}\rho_0 = \bigcup_{j=1}^n [a_j, b_j]$*, with $a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n$ for a certain fixed integer n. Moreover, suppose that $b_j - a_j < \eta$ for all $j \in \{1, \ldots, n\}$ and that $a_{j+1} - b_j > \eta$ for all $j \in \{1, \ldots, n-1\}$. Let*

$$m_j := \int_{a_j}^{b_j} d\rho_0(x), \qquad \gamma_j := \int_{a_j}^{b_j} x d\rho_0(x), \quad j = 1, \ldots, n.$$

Then,

$$W_p\left(\rho(t), \sum_{j=1}^n m_j \delta_{\gamma_j}\right) \to 0,$$

for all $p \in [2, +\infty]$.

Proof. Step 1. Let us first consider the case of one single connected component, namely $\mathrm{supp}\rho_0 = [a, b]$ with $b - a < \eta$. Then, in terms of the pseudo inverse variable u it can be easily proven that $u(0, t)$ and $u(1, t)$ are nondecreasing and nonincreasing with respect to t respectively. In order to see that, we compute

$$u_t(1, t) = \int G'(u(1) - u(\zeta))d\zeta \leq 0,$$

due to $G'(z) \leq 0$ for $z \geq 0$. The assertion at $z = 0$ can be proven similarly. As a consequence of that, we have

$$u(1, t) - u(0, t) < \eta$$

for all $t \geq 0$. Therefore, we can derive the inequality (3.15) due to assumption (3.1) and repeat the proof of Theorem 3.12 in order to get the desired assertion.

STEP 2. In order to simplify the notation, we shall perform the proof in the case $n = 2$. The general case $n > 2$ does not bring any further significant difficulty to the problem and it can be performed by an analogous argument. Let us then suppose that $\mathrm{supp}\rho_0 = [a, b] \cup [c, d]$, with $b - a < \eta$, $d - c < \eta$, $c - b > \eta$. Let

$$m_1 = \int_a^b d\rho_0(x), \quad m_2 = \int_c^d d\rho_0(x) = 1 - m_1, \quad \gamma_1 = \int_a^b x d\rho_0(x), \quad \gamma_2 = \int_c^d x d\rho_0(x).$$

We claim that the solution can be expressed in terms of the pseudo inverse variable u as follows:

$$u_t = \int_0^{m_1} G'(u(z) - u(\zeta))d\zeta \qquad \text{if } \ 0 \leq z < m_1 \tag{3.18}$$

$$u_t = \int_{m_1}^1 G'(u(z) - u(\zeta))d\zeta \qquad \text{if } \ m_1 \leq z \leq 1. \tag{3.19}$$

More precisely, assuming that u^0 be the pseudo inverse of the primitive of ρ_0, we claim that the solution u to (3.13) with initial datum u^0 is given by $u_1(z)$ for $z < m_1$ and by $u_2(z)$ for $z \geq m_1$, where u_1 and u_2 solve (3.18) and (3.19) respectively, with initial data $u_1^0 = u^0|_{[0,m_1)}$ and $u_2^0 = u^0|_{[m_1,1]}$ respectively. In order to prove such an assertion, we observe that the aforementioned u actually solves the equation (3.13) with initial datum u_0, because it can be easily proven that $\lim_{z \searrow m_1} u(z, t) - \lim_{z \nearrow m_1} u(z, t) > c - b$ for all $t \geq 0$ (because $G'(z) \leq 0$ when $z \geq 0$). Therefore, the assertion follows by uniqueness of the solution.

Finally, since u_1 and u_2 behaves like two solutions to (3.13) (with masses m_1 and $1 - m_1$ respectively) with initial data having one single component in its support, we can apply step one and the proof is complete. □

Remark 3.14. Explicit rates of convergence in the previous Theorem can be derived as in Theorem 3.12, depending on the value of the constant α.

Remark 3.15. Clearly, an open problem is determining the asymptotic behavior when G has finite range $[-\eta, \eta]$ and when the initial support does not satisfy the hypotheses of the previous theorem. For instance, it is not clear whether a solution with initial support exceeding the value η in its diameter will converge to the Dirac δ centered at the center of mass or it will develop many peaks. We can only state here that the size of the support alone is not sufficient to determine the large time asymptotics. To see this, take for instance a compactly supported initial condition ρ^0 with zero center of mass such that $W_2(\rho^0, \delta_0) \leq \eta/2$. We know by Theorem 3.9 that any sequence of times t_k admits a subsequence $t_{k'}$ such that $\rho(t_{k'})$ converges to a stationary solution in the sense of measures. Since the support of ρ is uniformly bounded in time (see Theorem 3.11), standard properties of the probability measures ensure continuity of the second moments and then

$$W_2(\rho(t_{k'}), \rho^\infty) \to 0$$

for a certain stationary solution ρ^∞. Let us suppose that $\rho^\infty \neq \delta_0$ for a certain subsequence $t_{k'}$. Then, due to Theorem 3.7, ρ^∞ is a finite combination of Dirac δ. Therefore it is not difficult to prove that

$$W_2(\rho^\infty, \delta_0) = W_\infty(\rho^\infty, \delta_0) > \eta,$$

thanks once again to the result in Theorem 3.7. Finally, by triangular inequality,

$$W_2(\rho(t_{k'}),\rho^\infty) \geq W_2(\delta_0,\rho^\infty) - W_2(\rho(t_{k'}),\delta_0) > \frac{\eta}{2},$$

which is in contradiction with $W_2(\rho(t_{k'}),\rho^\infty) \to 0$. Therefore ρ^∞ must equal δ_0 and since the limit is the same for all subsequences, we have thus proven

$$W_2(\rho(t),\delta_0) \to 0 \quad \text{as} \quad t \to +\infty.$$

We could produce such a solution without restrictions on the diameter of the support, which can thus be chosen arbitrarily large.

4 Small Diffusion Models

In the following we consider the case of a small nonlinear diffusion. More precisely, we assume $a(\rho) = \epsilon\rho^{m-1}$ for $0 < \epsilon \ll 1$ and investigate the asymptotic $\epsilon \to 0$ in order to achieve existence of stationary solutions. For simplicity we also assume that there is no external potential, i.e., $V \equiv 0$. The interaction kernel will be supposed to satisfy (IK) plus the additional assumptions

- $G \in C^3(I)$ for some neighborhood I of zero.
- G is decreasing on $\mathbb{R}_+$ and G has a unique maximum at zero.
- $G''(0) < 0$.

Equation (2.5) in this case reads

$$\partial_t u(z,t) = -\frac{m-1}{m}\partial_z\left(\frac{\epsilon}{(\partial_z u(z,t))^m}\right) + \int_0^1 G'(u(z,t)-u(\zeta,t))\ d\zeta \qquad \text{in } [0,1]\times\mathbb{R}^+, \quad (4.1)$$

4.1 Formal Asymptotic Expansion

For $\epsilon = 0$ we have studied the stationary states of this equation above, and a possible choice is always the constant state equal to the center of mass CM. Motivated by this limiting solution we study a formal expansion for the pseudo-inverse of the form

$$u = CM + \epsilon^\nu u^1 + o(\epsilon^\nu),$$

with $\int_0^1 u^1\ dz = 0$ (reflecting the conservation of the integral of u). Then we obtain

$$\epsilon^\nu \partial_t u^1(z,t) = -\frac{m-1}{m}\partial_z\left(\frac{\epsilon^{1-m\nu}}{(\partial_z u^1(z,t))^m}\right) + \epsilon^\nu \int_0^1 G''(0)(u^1(z,t)-u^1(\zeta,t))\ d\zeta + o(\epsilon^\nu).$$

which determines the natural choice for the exponent $\nu = \frac{1}{m+1}$. Since the integral term evaluates to

$$\begin{aligned}\int_0^1 G''(0)(u^1(z,t)-u^1(\zeta,t))\ d\zeta &= G''(0)\int_0^1 u^1(z,t)\ d\zeta - G''(0)\int_0^1 u^1(\zeta,t)\ d\zeta \\ &= G''(0)u^1(z,t),\end{aligned}$$

the first-order expansion is determined by the local equation

$$\partial_t u^1 = -\frac{m-1}{m}\partial_z\left(\frac{1}{(\partial_z u^1)^m}\right) + cu^1. \tag{4.2}$$

Since the factor $c := G''(0)$ is nonpositive, the last term adds a decaying mode, thus playing the role of a confining potential in a nonlinear Fokker-Planck equation.

Now we can take a closer look on stationary solutions of (4.2), which solve

$$(m-1)\frac{\partial^2_{zz}v^1}{(\partial_z v^1)^{m+1}} = -\frac{m-1}{m}\partial_z\left(\frac{1}{(\partial_z v^1)^m}\right) = |c|u^1.$$

After multiplying by $\partial_z v^1$, we can integrate with respect to z to obtain

$$-\frac{1}{(\partial_z v^1)^{m-1}} = \frac{|c|}{2}(v^1)^2 - \gamma$$

for $m > 1$, with some integration constant γ. Since $\frac{1}{\partial_z v^1} \to 0$ as $z \to 0$ or $z \to 1$, the solution satisfies

$$\frac{1}{\partial_z v^1} = \left[\left(\gamma - \frac{|c|}{2}(v^1)^2\right)_+\right]^{1/(m-1)},$$

with constant γ determined from the mass.

We can relate v again to stationary solution of (1.1), to leading order we have $\rho = \frac{1}{\epsilon^\nu \partial_z v} \approx \frac{1}{\partial_z v^1}$ and $x \approx CM + \epsilon^\nu v^1$. Hence, a stationary solution is asymptotically determined as

$$\rho^\infty(x) = \epsilon^{-\nu}\left[\left(\gamma - \frac{|c|}{2}\epsilon^{-2\nu}(x - CM)^2\right)_+\right]^{1/(m-1)}$$

Consequently we expect a stationary solution with support of order ϵ^ν. In a similar way we can construct asymptotic expansions around stationary solutions with multiple peaks (as existing in the case of a kernel with compact support). Thus, to leading order it seems that the stationary solutions of (1.1) for small ϵ are of the same structure as the ones for $\epsilon = 0$, but the Dirac δ distributions are changed to finite peaks of height $\epsilon^{-\nu}$.

4.2 Existence of Stationary States with Compact Support

For the particular case $m = 2$ we can make the above reasoning rigorous to some extent, i.e., we can prove the existence of stationary solutions with compact support. We start from the stationary version of (4.1)

$$0 = \frac{1}{2}\partial_z\left(\frac{\epsilon}{(\partial_z u(z))^2}\right) - \int_0^1 G'(u(z) - u(\zeta))\, d\zeta \qquad \text{in } [0,1]. \tag{4.3}$$

Accordingly to the previous formal asymptotic expansion, we make the ansatz

$$u = CM + \delta v, \qquad \delta = \epsilon^{1/3}.$$

We then look for v solving

$$0 = \frac{1}{2}\partial_z\left(\frac{\delta}{(\partial_z v)^2}\right) - \int_0^1 G'(\delta(v(z) - v(\zeta))d\zeta.$$

This equation can be rewritten as

$$-\frac{\delta^2\partial_{zz}v(z)}{(\partial_z v(z))^2} = \delta\int_0^1 G'(\delta(v(z)-v(\zeta)))\,d\zeta\partial_z v(z) = \partial_z\int_0^1 G(\delta(v(z)-v(\zeta)))\,d\zeta,$$

and hence, we can integrate with respect to z to obtain

$$\frac{\delta^2}{\partial_z v(z)} = \int_0^1 G(\delta(v(z)-v(\zeta)))\,d\zeta + \alpha \tag{4.4}$$

for some integration constant α. By substituting $z=1$ into the above identity we get

$$\alpha = -\int_0^1 G(\delta(v(1)-v(\zeta)))\,d\zeta,$$

and the analogous condition is satisfied at $z=0$ if we look for an antisymmetric v, i. e. $v(1-z)=-v(z)$, which implies that the corresponding density is even. We recall that the term $1/v_z$ vanishes at $z=0,1$ if the corresponding density is compactly supported and continuous. Now we can multiply (4.4) by $\delta\partial_z v(z)$ and integrate again to deduce

$$\delta^3 z = \int_0^1 H(\delta(v(z)-v(\zeta)))\,d\zeta + \alpha\delta v(z) + \beta, \tag{4.5}$$

where β is another integration constant and $H'=G$. Without restriction of generality we assume that $H(0)=0$. Since H is an odd kernel, we obtain by integration with respect to z that

$$\frac{\delta^3}{2} = \beta.$$

Equation (4.5) can be rewritten as

$$0 = \frac{1}{2} - z + \delta^{-3}\int_0^1 \left[H(\delta(v(z)-v(\zeta))) - \delta v(z)G(\delta(v(1)-v(\zeta)))\right]d\zeta. \tag{4.6}$$

The expression (4.6) can be viewed as a function equation of the form $F(v,\delta)=0$, where $\delta>0$ is a parameter and where the map F is defined as

$$F(v,\delta) := \begin{cases} \frac{1}{2} - z + \delta^{-3}\int_0^1 \left[H(\delta(v(z)-v(\zeta))) - \delta v(z)G(\delta(v(1)-v(\zeta)))\right]d\zeta & \text{if } \delta\neq 0 \\ \frac{1}{2} - z + \frac{1}{6}G''(0)[v(z)^3 - 3v(z)v(1)^2] & \text{if } \delta = 0. \end{cases}$$

If we assume (so far) that v lives in the functional space

$$\mathcal{X} := \left\{ v\in L^\infty[0,1],\quad v \text{ increasing},\quad \int_0^1 v(z)dz = 0,\quad v(1-z) = -v(z)\right\},$$

we can easily recover the above expression for $\delta=0$ by the following Taylor expansion of the integrand in (4.6). More precisely (recalling $H(0)=0$, $H'=G$, $G'(0)=0$), we have

$$\begin{aligned} &\delta^{-3}\int_0^1 \left[H(\delta(v(z)-v(\zeta))) - \delta v(z)G(\delta(v(1)-v(\zeta)))\right]d\zeta \\ &= \delta^{-3}\int_0^1 \Bigg[\delta G(0)(v(z)-v(\zeta)) + \frac{\delta^3}{6}G''(0)(v(z)-v(\zeta))^3 - G(0)\delta v(z) \\ &\quad -\frac{\delta^3}{2}G''(0)v(z)(v(1)-v(\zeta))^2 + \delta^4 R(\delta,v)\Bigg]\,d\zeta \end{aligned} \tag{4.7}$$

The assumption $v(1-\zeta) = -v(\zeta)$ (which corresponds to $\rho(-x) = \rho(x)$ at the level of the density) and further simple calculations imply that the extension to $\delta = 0$ is identified by the term in the above definition. Moreover, the boundedness of v ensures the remainder R above is uniformly bounded and the extension is therefore continuous. For further use, we observe that $\mathcal{X}$ can be identified with the space

$$\mathcal{X}^* := \{v \in L^\infty[1/2,1], \quad v \text{ increasing}, \quad v(1/2) = 0\},$$

via antisymmetric extension to the interval $[0, 1/2]$.

Our strategy to prove existence of a certain v for small $\delta > 0$ is to use the Implicit Function Theorem (cf. [10, Theorem 15.1]). For $\delta = 0$ the functional equation (4.6) reads

$$\frac{6\left(z-\frac{1}{2}\right)}{G''(0)} = v(z)^3 - 3v(z)v(1)^2. \tag{4.8}$$

It is easily seen that the unique v_0 satisfying (4.8) is the pseudo inverse of the primitive of the density

$$\rho_0(x) = \frac{|G''(0)|}{2}\left(x^2 - |G''(0)|^{-\frac{2}{3}}\right)_+$$

and we have $v_0(1) = -\left(\frac{2G''(0)}{3}\right)^{-1/3}$.

For fixed $\delta \geq 0$ we shall regard the mapping $F(\cdot,\delta)$ as an operator defined on the domain $\mathcal{X}^*_{1/2}$ onto $\mathcal{X}^*_1$, where we have used the notation

$$\mathcal{X}^*_\alpha := \left\{ v \in \mathcal{X}^* \mid \sup_{1/2 \leq z \leq 1} \frac{v(1)-v(z)}{(1-z)^\alpha} < +\infty \right\}$$

$$|||v|||_\alpha := \|v\|_{L^\infty[0,1]} + \sup_{1/2 \leq z \leq 1} \frac{v(1)-v(z)}{(1-z)^\alpha}.$$

It is easy to check that the extension of the map $F(\cdot,\delta) : \mathcal{X}^*_{1/2} \to \mathcal{X}^*_1$ to $\delta = 0$ is still continuous. In order to see that, one can express the remainder in (4.7) by means of mean value formulas and obtain a uniform bound for R in the $|||\cdot|||_1$ norm. We have the following lemmas.

Lemma 4.1. *The solution v_0 to the functional equation $F(v_0, 0) = 0$ belongs to the space $\mathcal{X}^*_{1/2}$ and the ratio*

$$\frac{v_0(1) - v_0(z)}{(1-z)^{1/2}}$$

is strictly positive on $[1/2, 1]$.

Proof. Let $F_0(x) := \int_{-\infty}^x \rho_0(y)dy$. Since F_0 (restricted to its support) is the inverse of v_0, it suffices to prove

$$\lim_{x \to v_0(1)^-} \frac{F_0(x) - 1}{(x - v_0(1))^2} = l > 0.$$

In order to prove that, we use De L'Hospital rule twice to get

$$\lim_{x \to v_0(1)^-} \frac{F_0(x) - 1}{(x - v_0(1))^2} = \lim_{x \to v_0(1)^-} \frac{1}{2}\rho_0'(x) = \frac{|G''(0)|v_0(1)}{2} > 0.$$

□

Lemma 4.2. *For any fixed* $\delta \geq 0$, $F(\cdot, \delta) : \mathcal{X}^*_{1/2} \to \mathcal{X}^*_1$ *is a bounded operator.*

Proof. Let $v \in \mathcal{X}^*_{1/2}$ and let $h(z) := F(v, \delta)$. From the definition of F it is clear that the L^∞ norm of h can be bounded by a constant depending on the L^∞ norm of v. We now compute

$$h(z) - h(1) = 1 - z + \delta^{-3} \int_0^1 [H(\delta(v(z) - v(\zeta))) - H(\delta(v(1) - v(\zeta))) \\ -\delta G(\delta(v(1) - v(\zeta)))v(z) + \delta G(\delta(v(1) - v(\zeta)))v(1)]\, d\zeta.$$

By means of the first order Taylor expansion of H centered at $\delta(v(1) - v(\zeta))$ in the integral above, we obtain

$$h(z) - h(1) = 1 - z + \delta^{-1}(v(z) - v(1))^2 \int_0^1 R(z, \zeta, \delta) d\zeta,$$

for a certain bounded function R. By dividing the above relation by $\zeta - 1$ and thanks to the assumption on v we obtain the desired control of the difference quotient of h at the point $z = 1$. □

We now pass to study the partial (functional) derivative of F with respect to v at the point $(v_0, 0)$. For all $w \in \mathcal{X}^*_{1/2}$ we define

$$g(z) := \frac{\partial F}{\partial v}(v_0, 0)[w](z) = \frac{G''(0)}{2} \left[v(z)^2 w(z) - v_0(1)^2 w(z) - 2v_0(z)v_0(1)w(1)\right].$$

Lemma 4.3 (Boundedness of the partial derivative). *Let* $|||w|||_{1/2} \leq 1$*. Then, there exists a fixed constant* $A > 0$ *such that* $|||g|||_1 \leq A$*.*

Proof. It is clear that $\|g\|_{L^\infty}$ can be uniformly bounded. We then compute

$$g(1) = -G''(0)v_0(1)^2 w(1).$$

We now evaluate the difference $g(z) - g(1)$ as follows,

$$\begin{aligned} g(z) - g(1) &= \frac{G''(0)}{2} \left[w(z)(v_0^2(z) - v_0^2(1)) - 2v_0(1)w(1)(v_0(z) - v_0(1))\right] \\ &= \frac{G''(0)}{2}(v_0(z) - v_0(1)) \left[w(z)(v_0(z) + v_0(1)) - 2v_0(1)w(1)\right] \\ &= \frac{G''(0)}{2}(v_0(z) - v_0(1)) \left[(w(z) - w(1)) (v_0(z) + v_0(1)) + w(1) (v_0(z) - v_0(1))\right]. \end{aligned}$$

By dividing the above relation by $z - 1$ and by using the hypotheses on w and the result in Lemma 4.1, we obtained the desired assertion. □

Finally, we prove that the inverse of $\frac{\partial F}{\partial v}(v_0, 0)$ is also bounded, namely

$$g \in \mathcal{X}^*_1 \mapsto w := \left(\frac{\partial F}{\partial v}(v_0, 0)\right)^{-1} [g] \in \mathcal{X}^*_{1/2}$$
$$w(z) = \left(\frac{2}{G''(0)}\right) \frac{g(z) - \frac{v_0(z)}{v_0(1)} g(1)}{v_0^2(z) - v_0^2(1)}$$

Lemma 4.4 (Boundedness of the inverse). *Let $|||g|||_1 \leq 1$. Then, there exists a fixed constant $A > 0$ such that $|||w|||_{1/2} \leq A$.*

Proof. A simple but tedious calculation gives

$$\begin{aligned} w(z) - w(1) &= \frac{2g(z)v_0^3(1) - 2v_0(z)g(1) + g(1)v_0(1)v_0^2(z) - g(1)v_0^3(1)}{G''(0)v_0^3(1)(v_0^2(z) - v_0^2(1))} \\ &= \frac{2v_0^3(1)(g(z) - g(1)) + g(1)v_0(1)(v_0(z) - v_0(1))^2}{G''(0)v_0^3(1)(v_0^2(z) - v_0^2(1))} \end{aligned}$$

The hypotheses on g near $z = 1$ and the result in Lemma 4.1 ensures $w(z) - w(1)$ is uniformly bounded, and since $g(1)$ can be expressed in terms of $w(1)$ as in the previous lemma, then we obtain a uniform bound for $\|w\|_{L^\infty}$. Moreover, by dividing the above identity by $(1 - z)^{1/2}$ we obtained the desired control of $|||w|||_{1/2}$. □

Using all the previous results, we can prove the following theorem on the existence of stationary solutions:

Theorem 4.5 (Existence of stationary solutions for small ε). *There exists a positive constant $\epsilon_0 > 0$ such that, for all $0 \leq \epsilon < \epsilon_0$, the stationary equation*

$$0 = \partial_x(\rho \partial_x(\epsilon\rho - G * \rho)) \tag{4.9}$$

admits a nonzero, bounded and compactly supported solution ρ. Moreover, the diameter of the support of ρ is of order $\epsilon^{1/3}$.

Proof. The equation (4.9) can be reformulated as

$$\epsilon\rho(x) - G * \rho(x) = \text{const} \quad \text{if } x \notin \text{supp}\rho. \tag{4.10}$$

Let

$$\tilde{\rho}(y) := \rho\left(CM + \epsilon^{1/3}y\right) \quad \tilde{F}(y) := \int_{-\inf(\text{supp}\rho)}^{y} \tilde{\rho}(y')dy'.$$

By evaluating (4.10) at $x = CM + \epsilon^{1/3}y$ and by integrating with respect to y we get

$$\tilde{F}(y) = \epsilon^{-1} \int_{\mathbb{R}} H(\epsilon^{1/3}(y - t))\tilde{\rho}(t)dt + \alpha y + \beta, \tag{4.11}$$

for some constants $\alpha, \beta > 0$, where $H' = G$, $H(0) = 0$. The equation (4.11) holds for all y such that $CM + \epsilon^{1/3}y \in \text{supp}\rho$. The change of variable $\tilde{F}(y) = v$ on (4.11) and a suitable choice of α, β as before lead to the equation (4.6) with $\delta = \epsilon^{1/3}$. By regarding this equation as an operator equation, due to the results in Lemmas 4.1, 4.2, 4.3 and 4.4, we can apply the Implicit Function Theorem (cf. [10, Theorem 15.1]) and deduce existence of a solution to (4.6) for small ϵ. □

4.3 Larger Diffusion

We finally turn to the natural question whether we can obtain such stationary solutions also for larger values of ϵ. From a modeling point of view one would expect that for ϵ exceeding a certain threshold value, the repulsive forces (modeled by the diffusion) will dominate the aggregative forces, so that solutions of the evolution problem will decay. As we shall see in the following, this is indeed true, and the critical value for ϵ scales with the L^1-norm of the kernel G, which exhibits the interplay of diffusion and aggregation.

Let us assume that there exists an antisymmetric stationary solution u of (4.3). We now define $v := u - CM$ and integrate the equation exactly as in the previous section, which yields

$$0 = \epsilon(\frac{1}{2} - z) + \int_0^1 \left[H(v(z) - v(\zeta)) - v(z)G(v(1) - v(\zeta))\right] d\zeta.$$

Now we multiply this relation by $\text{sign}(\frac{1}{2} - z)$ and integrate to obtain

$$\begin{aligned} 0 &= \epsilon \int_0^1 |\frac{1}{2} - z| dz + \int_0^1 \int_0^1 \left[H(v(z) - v(\zeta)) - v(z)G(v(1) - v(\zeta))\right] d\zeta \; \text{sign}(\frac{1}{2} - z) dz \\ &= \frac{\epsilon}{4} + \int_0^1 \int_0^1 H(v(z) - v(\zeta)) \; \text{sign}(\frac{1}{2} - z) d\zeta dz + \int_0^1 |v(z)| dz \int_0^1 G(v(1) - v(\zeta)) d\zeta, \end{aligned}$$

where we have used the antisymmetry of v for the last term. Due to the monotonicity of v we know that $H(v(z) - v(\zeta)) \; \text{sign}(\frac{1}{2} - z) \geq 0$ if $\frac{1}{2} \geq z \geq \zeta$ and $\frac{1}{2} \leq z \leq \zeta$. On the remaining part of $[0,1]^2$ we estimate

$$H(v(z) - v(\zeta)) \; \text{sign}(\frac{1}{2} - z) \geq -\frac{1}{2} \sup_{p \in \mathbb{R}} |H(p)| = -\frac{1}{2} \int_0^\infty G(p) dp = -\frac{1}{4} \int_{\mathbb{R}} G(p) dp.$$

Hence with the above identity, we conclude

$$0 \geq \frac{\epsilon}{4} - \frac{3}{16} \int_{\mathbb{R}} G(p) dp,$$

which implies that $\epsilon \leq \frac{3}{4} \int_{\mathbb{R}} G(p) dp$ is a necessary condition for the existence of an antisymmetric stationary solution.

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