

# Texture Separation $BV - G$ and $BV - L^1$ Models

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## Abstract

In this work we compare two models that were proposed to improve the well-known Rudin-Osher-Fatemi model [37]. The first one, proposed by Y. Meyer [29], is the  $BV - G$  model and the second one is the  $BV - L^1$  model. We state the similitudes between both models. In particular we prove a characterization theorem for optimal decompositions for the  $BV - L^1$  model. We then compare these models in the particular case of radial functions.

keywords: function of bounded variations,  $BV - L^1$  decomposition, textures, oscillating patterns, Rudin-Osher-Fatemi model, stability.

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# 1 Introduction

In [37], L. Rudin, S.J Osher and E. Fatemi proposed an algorithm for removing noise from images. Given an observed intensity function  $f$  they reconstructed the clean image  $u$  assuming  $f = u + \eta$ , where  $\eta$  is an additive noise. They proposed to minimize the following functional:

$$J(u) = \int_{\Omega} |\nabla u| + \lambda \int_{\Omega} (f - u)^2 \quad (1)$$

for a certain tuning parameter  $\lambda > 0$ . The set  $\Omega$  is a domain of  $\mathbb{R}^n$  and the term  $\int_{\Omega} |\nabla u|$  denotes the total variation of  $u$ , assuming  $u$  is of bounded variation :  $u \in BV(\Omega)$ . Problem (1) is called the Rudin-Osher-Fatemi model (ROF).

Recently Y. Meyer [29] interpreted the ROF model as a texture separation model. More precisely, an image  $f$  is a sum  $u + v$  between a sketch  $u$  and a second term  $v$  which takes care of the textured components and of some additive noise. The objects which are contained in  $f$  belong to the sketch  $u$ . These objects are assuming to be delimited by contours with finite lengths and  $u$  is a geometric-type image. It is then natural to assume that  $u$  is a function of bounded variation. Y. Meyer pointed out the crucial role played by a certain functional Banach space, called *the space of texture* and denoted  $G$ , and its norm, the G-norm denoted  $\|\cdot\|_*$ , in the study of the ROF model. This norm is, in some sense, the dual norm of  $BV$ . It is adapted to characterize oscillating patterns: *the G-norm of an oscillating pattern vanishes when the frequency tends to infinity* [29] [26].

The ROF model suffers from a severe drawback. Even if  $f$  represents a smooth regular set, the texture component  $v$  is, in general, not identically null. The mapping that, given an image  $f$ , associates the  $u$  component is not a projection whereas the one that associates the textured component  $v$  is a projection. The ROF optimal decomposition (unique) is then characterized as follows [29]:

**Theorem 1.1** *Let  $f$  belongs to  $L^2$  and  $(u_0, v_0) \in BV \times L^2$ .*

*If  $\|f\|_* \leq \frac{1}{2\lambda}$  then the image  $f$  is seen as a texture;  $f = 0 + f$  is the ROF decomposition.*

*The following assertions are equivalent: (a)  $\|f\|_* > \frac{1}{2\lambda}$  and  $f = u_0 + v_0$  is the ROF decomposition of  $f$  ; (b)  $f = u_0 + v_0$ ,  $\|v_0\|_* = \frac{1}{2\lambda}$  and*

$$\int u_0 v_0 = \|u_0\|_{BV} \|v_0\|_* . \quad (2)$$

If (2) holds, the pair  $(u_0, v_0)$  is said to be extremal. Y. Meyer and A. Haddad [Theorem 4.2.1 [26]] proved that replacing the  $BV$  space by other functional Banach space (for instance Besov spaces  $\dot{B}_1^{1,1}$  or  $\dot{B}_1^{1,\infty}$ ) does not overcome the drawbacks. The proof comes from rewriting the minimizing problem as a dual problem using a minimax theorem (see Y. Ekeland, R. Temam [21]). The same characterization Theorem 1.1 holds. Y. Meyer suggested to replace the fidelity term by using the G-norm, weaker than the  $L^2$  norm. This model is called the  $BV - G$  model. It consists in minimizing the following (not strictly) convex functional:

$$J(u) = \|u\|_{BV} + \lambda \|v\|_* . \quad (3)$$

This new model was studied by Y. Meyer and A. Haddad in [26][27] and first implemented by S. Osher and L. Vese [39] [40] then by J.F. Aujol and al [10] [8] [11]. In both cases, the authors added an additive term to  $J(u)$  to retrieve uniqueness. In this article we focus on the  $BV - L^1$  model: we consider the following functional

$$J(u) = \|u\|_{BV} + \lambda \|v\|_1 \quad (4)$$

where  $f = u + v \in L^1 \cap L^2$ . This model was studied by Alliney [1] [2], Nikolova [31] [32] [33], Chan & Esedoğlu [18] and Osher et al [34]. For instance, Chan & Esedoğlu prove that the minimization problem has almost a unique solution (with respect to the Lebesgue measure). They also prove that when  $f$  is a characteristic function of a bounded domain  $\Omega$  then there exists a minimizer  $u$  which is also a characteristic function of a (possibly different) domain  $D$ . Moreover if  $\Omega$  is convex then  $D$  is contained in it. In [34], S. Osher et al consider an approached version of the problem. They work in a compact set  $\Omega$ . They replace  $J$  by  $J_\epsilon = \|u\|_{BV} + \lambda \int_\Omega \sqrt{(f - u)^2 + \epsilon}$ . The related minimization problem has a unique solution  $u_\epsilon$ . They denote  $v_\epsilon = f - u_\epsilon$ . Then they prove the following points:

- (i) If  $\|sign_\epsilon(f)\|_* < \frac{1}{\lambda}$  then  $u_\epsilon = 0$  is optimal, where  $sign_\epsilon(g) = \frac{g}{\sqrt{g^2 + \epsilon}}$  for any function  $g$ .
- (ii) If  $\|sign_\epsilon(f)\|_* > \frac{1}{\lambda}$  then the optimal solution satisfies:
  - (a)  $\|sign_\epsilon(v_\epsilon)\|_* = \frac{1}{\lambda}$  and
  - (b)  $\lambda \int u_\epsilon sign_\epsilon(v_\epsilon) = \|u_\epsilon\|_{BV}$ .

They finally prove a convergence theorem when  $\epsilon$  goes to 0, assuming the  $BV - L^1$  problem has a unique solution  $u$ :

$$\lim_{\epsilon \rightarrow 0} \|u_\epsilon - u\|_1 = 0. \quad (5)$$

This work aims at precisising points (a) and (b) when instead of considering the perturbed functional  $J_\epsilon$  we consider  $J$ . We then show how to find explicit solutions in the particular case of radial functions. Finally, we compare (for these functions) the result obtained for the  $BV - L^1$  decomposition to the one obtained using the  $BV - G$  decomposition.

## 2 Background

From now on we fix the dimension to 2 and choose  $\Omega = \mathbb{R}^2$ . We denote  $\mathcal{S}$  as the Schwartz class in 2-dimension. Following D. Mumford and B. Gidas [30], we consider a natural image as a generalized function. We now define the space  $BV$ . We will not assume the condition  $f \in L^1$  that some authors impose.

**Definition 2.1** *A distribution  $f$  belongs to  $BV$  if the distributional gradient  $\nabla f$  is a (vector valued) bounded Borel measure.*

Then it is proved that  $f = g + c$  where  $c$  is a constant and  $g \in L^2$ . The space  $BV$  can be considered as a subspace of  $L^2$ . Moreover, the isoperimetric inequality [22] implies

$$\|g\|_2 \leq \frac{1}{2\sqrt{\pi}} \|f\|_{BV}. \quad (6)$$

A specific  $BV$  norm is crucially needed when using the ROF model, the  $BV - G$  model or the  $BV - L^1$  model. All models amount to minimizing a functional which contains a  $BV$  norm. We will impose this norm to be isotropic with the same homogeneity as the  $L^2$  norm. Let us begin by the simple case where  $\nabla f$  belongs to  $L^1$ . Then the  $BV$  norm of  $f$  will be defined as  $\|f\|_{BV} = \int |\nabla f(x)| dx$ . We then define the space  $\mathcal{BV}$  as follow:

**Definition 2.2**  $\mathcal{BV} = \{f \in BV \mid \nabla f \in L^1\}$ .

This space will be useful in what follows. Now, consider the general case where  $\nabla f$  is a Borel measure. Our simply minded approach does not work but pave the way to the following definition. We define  $\mu_j = \partial_j f$ , for  $j = 1, 2$ , and the Borel measure  $\sigma = |\mu_1| + |\mu_2|$ . By the Radon-Nikodym theorem we have  $\mu_j = \theta_j(x)\sigma$ ,  $j = 1, 2$ , where  $\theta_j(x)$  are Borel functions with values in  $[-1, 1]$ . Finally, the Borel measure  $|\nabla f|$  is defined by

$$|\nabla f| = \sqrt{\theta_1^2 + \theta_2^2} \sigma. \quad (7)$$

We can conclude:

**Definition 2.3** *The  $BV$  norm of  $f$  is the total mass of the Borel measure  $|\nabla f|$ .*

With an obvious abuse of language, we write  $\|f\|_{BV} = \int |\nabla f| dx$ .

The indicator function  $\chi_E$  of a domain  $E$  with smooth boundaries belongs to the space  $BV$  if and only if its boundary  $\partial E$  has a finite length and the  $BV$  norm of  $\chi_E$  is the length of the boundary. In order to treat the general case, De Giorgi defined the reduced boundary  $\partial^* E$  of a measurable set  $E$  and proved that the  $BV$  norm of  $\chi_E$  is the 1-dimensional Hausdorff measure of its reduced boundary.

For defining the reduced boundary, let us denote  $B(x, r)$  as the ball centered at  $x$  with radius  $r$ . We then follow De Giorgi [20]:

**Definition 2.4** *The reduced boundary  $\partial^* E$  of  $E$  is the set of points  $x$  belonging to the closed support of  $\mu = \nabla \chi_E$  such that the following limit exists*

$$\lim_{r \rightarrow 0} \frac{\mu(B(x, r))}{|\mu|(B(x, r))} = \nu(x) \quad (8)$$

**Theorem 2.1** *An indicator function  $\chi_E$  belongs to  $BV$  if and only if  $\partial^* E$  has a finite 1-dimensional Hausdorff measure:*

$$\|\chi_E\|_{BV} = \mathcal{H}^1(\partial^* E) \quad (9)$$

We now recall some definitions and properties of the space  $G$  and its norm  $\|\cdot\|_*$ . For further details and proofs of results about the space  $G$  and the dual norm, the reader is referred to [29] [26].

**Definition 2.5** *The space of texture  $G$  is defined as the dual of  $\mathcal{BV}$ ,  $G = \mathcal{BV}^*$ .*

Remember that  $\mathcal{BV}$  is the set of all functions such that  $\nabla f$  belongs to  $L^1$ . This space coincides with the closure of the Schwartz class in  $BV$  [29]. Thus  $G$  is a functional Banach space:  $\mathcal{S} \subset G \subset \mathcal{S}'$ . The space  $L^2$  is embedded in  $G$ . Notice that the dual of  $BV$  is not a functional Banach space. Indeed  $BV \neq \mathcal{BV}$  since  $\chi_Q$ ,  $Q = [0, 1]^2$ , belongs to  $BV$  but not to  $\mathcal{BV}$ . There exists a continuous linear form  $\varphi$  on  $BV$  which vanishes on  $\mathcal{S}$  and equals 1 on the function  $\chi_Q$ . This continuous linear form  $\varphi$  is not a distribution  $S$  such that  $\varphi(f) = \langle S, f \rangle$  otherwise  $S$  would be identically null.

The space  $G$  is endowed with the dual norm  $\|\cdot\|_*$ . This norm is also isotropic since the  $BV$  norm is isotropic. The isoperimetric inequality and duality yield the following estimates:

$$\|f\|_* \leq \frac{1}{2\sqrt{\pi}} \|f\|_2 \leq \frac{1}{4\pi} \|f\|_{BV}. \quad (10)$$

Another useful inequality is

**Lemma 2.1** *If  $f$  belongs to  $BV$  and  $g$  to  $L^2$  then*

$$\left| \int f(x)g(x)dx \right| \leq \|f\|_{BV} \|g\|_*. \quad (11)$$

The dual norm can be estimated by duality or by the following lemma.

**Lemma 2.2** *A distribution  $f$  belongs to  $G$  if and only if there exists  $g = (g_1, g_2) \in (L^\infty)^2$  such that  $f = \text{div } g$ . Then*

$$\|f\|_* = \inf\{\|g\|_\infty = \|(g_1^2 + g_2^2)^{1/2}\|_\infty \mid f = \text{div } g\}. \quad (12)$$

In many cases, we cannot calculate the  $G$  norm but only have an estimate. When the function is radial we have an explicit formula:

**Theorem 2.2** *Consider a radial function  $f \in L^2$ , also noted  $f(r) \in L^2(rdr)$ ,  $r \in \mathbb{R}^+$ . Then,*

$$\|f\|_* = \left\| \frac{1}{r} \int_0^r s f(s) ds \right\|_\infty. \quad (13)$$

In particular the  $G$  norm of a disc of radius  $R$  is  $\frac{R}{2}$ .

Finally, to justify briefly that the  $G$  norm is adapted to textures we recall two results. The complete proofs can be found in [26].

**Proposition 2.1** *If  $\mu$  belongs to  $L^\infty$  and  $\int_{Q+k} \mu(x) dx = 0$  for all  $k \in \mathbb{Z}^2$  and  $Q = [0, 1]^2$ , then  $\mu$  belongs to  $G$  and by duality,  $\|\mu\|_* \leq C \|\mu\|_\infty$ .*

For instance, if  $\mu$  is an  $\alpha$ -periodic pattern in variable  $x_1$ , such that  $\int_0^\alpha \mu(t, x_2) dt = 0$ , identically in  $x_2$ , then  $\|\mu\|_* \propto \alpha$ . Thus the  $G$  norm vanishes when the frequency  $\frac{1}{\alpha}$  tends to infinity.

We now study another example of texture where the pattern is still periodic but located in space.

**Proposition 2.2** *Let  $m \in L^\infty$  such that  $|\nabla m|$  is a Guy David measure. Let  $\mu \in L^\infty$  such that  $\int_{Q_{+k}} \mu = 0$  for all  $k \in \mathbb{Z}$ . Then there exists a constant  $C(m)$  such that for all  $\alpha > 0$ ,*

$$\|m(x)\mu(\omega x)\|_* \leq \frac{C}{\omega}. \quad (14)$$

A Guy David measure is a positive Radon measure such that the mass of any disc is uniformly bounded by the radius. If we replace  $m$  by any function  $f \in L^2$ , is  $\|f(x)\mu(\omega x)\|_*$  of order  $\omega^{-1}$ ? A weaker result is given by the following proposition:

**Proposition 2.3** *Let  $f$  belongs to  $L^2$  and  $\mu$  belongs to  $L^\infty$  such that  $\int_{Q_{+k}} \mu(x)dx = 0$ , for all  $k \in \mathbb{Z}^2$ . Then  $\|f(x)\mu(\omega x)\|_*$  vanishes when  $\omega$  tends to infinity.*

The last three propositions prove that the G norm is “small” when the frequency is high. This justifies the use of this norm to characterize oscillating patterns.

### 3 A review of the $BV - G$ model

In this section we recall some properties concerning the  $BV - G$  model, i.e. minimizing the functional  $J(u) = \|u\|_{BV} + \lambda \|f - u\|_*$ ,  $f \in L^2$ . All the details can be found in [26] [27]. The main results about the  $BV - G$  model is that any optimal decomposition (not unique) is invariant and stable about “simple functions”:

**Definition 3.1 (Simple function)** *A function  $f \in BV$  is called simple if there exists a non identically null function  $g \in BV$  such that  $\int fg = \|f\|_{BV} \|g\|_*$ .*

Roughly speaking, a simple function is the  $u$ -component of a ROF decomposition of a  $BV$ -function for a certain parameter  $\lambda > 0$ . The main theorem is the following one:

#### Theorem 3.1 Invariance and stability

a) *If  $\lambda < 4\pi$  then any function of  $L^2$  has a unique and trivial optimal decomposition  $f = 0 + f$ . The image is considered as a texture.*

b) *Let  $f \in BV$  be a simple function. There exists a rank  $\lambda_0 > 0$  such that for any  $\lambda > \lambda_0$ , the algorithm has a unique solution given by  $\bar{u} = f$ ,  $\bar{v} = 0$ .*

c) *For  $f = g + h$  where  $g$  is simple and  $\|h\|_* \leq \epsilon$ , there exists a rank  $\lambda_0 > 0$ , independent of  $g$ , such that for any  $\lambda > \lambda_0$ , the algorithm yields an optimal decomposition  $f = \bar{u} + \bar{v}$  where  $\bar{u}$  is close to  $g$  in  $L^2$ . More precisely*

$$\|\bar{u} - g\|_2 \leq C(g, \lambda)\epsilon^{1/2}, \quad \|\bar{u} - g\|_* \leq \frac{2\lambda\epsilon}{\lambda - \lambda_0}. \quad (15)$$

The constants  $\lambda_0$  and  $C(g, \lambda)$  are explicit. The exponent  $\frac{1}{2}$  is optimal. This theorem says that any simple function is regarded by the algorithm either as a texture if the tuning parameter is small enough (parameter larger than  $4\pi$ , depending on the function) or as an object. This never occurs when using the ROF algorithm since the texture component is, in general, never canceled. This theorem also says that the object components of two images that only differ from an oscillating pattern are close in the  $L^2$  sense since oscillating patterns have a “small” G-norm.

The lack of uniqueness is emphasized by considering extremal functions :

**Definition 3.2 (extremal function)** *A function  $f \in BV$  is said extremal if  $\|f\|_2^2 = \|f\|_{BV} \|f\|_*$ .*

Extremal functions are a particular case of simple functions where  $f = g$  in definition 3.1. They are also considered either as objects or as texture components. But

**Proposition 3.1** *Let  $f$  be an extremal function. Then it exists a certain parameter  $\lambda_0$  such that all decompositions  $\alpha f + (1 - \alpha)f$ ,  $\alpha \in [0, 1]$ , are optimal.*

It suffices to consider  $\lambda_0 = \frac{\|f\|_{BV}}{\|f\|_*}$ . An example of extremal function is a characteristic function of a disc. Other examples of extremal functions were given by G. Bellettini, V. Caselles and M. Novaga in [14]. They were investigating solutions of  $\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = u$  among characteristic functions. Before stating their result, we need few definitions.

Let  $\Omega$  be a subset of  $\mathbb{R}^2$  such that  $\chi_\Omega \in BV$ . We denote  $\lambda_\Omega = \frac{\|\chi_\Omega\|_{BV}}{|\Omega|}$ . We say that  $\partial\Omega$  is of class  $\mathcal{C}^{1,1}$  if, to a change of coordinates system,  $\partial\Omega$  is locally in each point the graph of a function  $f$  of class  $\mathcal{C}^1$  such that  $\frac{d}{dx_1}f$  is lipschitzian and continuous; Moreover  $\Omega$  is locally the epigraph of  $f$ . If  $\partial\Omega \in \mathcal{C}^{1,1}$  then we denote  $\kappa_{\partial\Omega}$  as the curvature of  $\partial\Omega$ . The curvature is defined  $\mathcal{H}^1$ -almost-everywhere. Here is the result obtained in [13], rewritten in terms of extremal functions.

**Theorem 3.2** *Let  $\Omega$  be a rectifiable connected set. The following properties are equivalent: (a) The function  $u = \chi_\Omega$  is extremal ; (b)  $\Omega$  is convex,  $\partial\Omega \in \mathcal{C}^{1,1}$  and  $\operatorname{ess\,sup}_{\partial\Omega} \kappa_{\partial\Omega} \leq \lambda_\Omega$ .*

However proposition 3.1 is not true in general for simple functions. In fact, proposition 3.1 is a characterization of extremal functions.

**Theorem 3.3** *Let  $f \in BV$  and assume that it exists  $\alpha \notin \{0, 1\}$  and  $\lambda > 0$  such that  $\bar{u} = \alpha f$  is an optimal solution for the BV – G model. Then  $f$  is extremal.*

Theorem 3.3 follows from a nice characterization of optimal decompositions established by S. Kindermann and S. Osher [Theorem 3.4 [28]]. They proved the following theorem, after reformulating the minimizing problem as a dual problem:

**Theorem 3.4** *Let  $f = u + v$  be a decomposition in  $BV + G$  of  $f \in L^2$ . Let  $\lambda > 0$ . Then  $f = u + v$  is optimal if and only if it exists a function  $g \in BV$  such that :*

$$\lambda \int ug = \|u\|_{BV} \quad (16)$$

$$\int gv = \|v\|_* \quad (17)$$

$$\|g\|_{BV} \leq 1 \quad (18)$$

$$\|g\|_* \leq \frac{1}{\lambda}. \quad (19)$$

In particular if  $\|g\|_{BV} < 1$  then necessarily  $v = 0$  and  $u = f$  and if  $\|g\|_* < \frac{1}{\lambda}$  then necessarily  $v = f$  and  $u = 0$ . Roughly speaking, a decomposition  $f = u + v$  is optimal if and only if it exists a function  $g$  such that  $\|g\|_{BV} \leq 1$ ,  $\|g\|_* \leq \lambda^{-1}$  and the pairs  $(u, g)$  and  $(g, v)$  are extremal.

## 4 The $BV - L^1$ model

The goal of this section is to give a characterization of optimal decompositions of a function  $f \in L^1 \cap L^2$  for the  $BV - L^1$  algorithm. We minimize the functional  $J(u) = \|u\|_{BV} + \lambda \|f - u\|_1$ . An optimal decomposition of  $f$  is  $f = \bar{u} + \bar{v}$ , where  $\bar{u} \in BV$  and  $\bar{v} \in L^1$ , such that  $J(\bar{u}) \leq J(u)$ ,  $\forall u \in BV$  (if  $f - u \notin L^1$  then  $J(u) = +\infty$ ). Following Y. Meyer, A. Haddad [26] and S. Kindermann S. Osher [28], we reformulate the minimizing problem as a dual problem. We then use a minimax theorem to get the characterization. This characterization allows us to find explicitly optimal decompositions for certain functions.

To rewrite the problem as a minimax problem, we replace the  $L^1$  norm by a supremum. Indeed, we have

$$\|v\|_1 = \sup_{\|g\|_\infty \leq 1} \int vg. \quad (20)$$

Then the minimization problem  $\inf_{u \in BV} J(u)$  becomes

$$\inf_{u \in BV} \sup_{\|g\|_\infty \leq 1} \|u\|_{BV} + \lambda \int (f - u)g. \quad (21)$$

We now state a version of the minimax theorem. It will allow us to swap the infimum and the supremum.

Let  $E$  be a compact and convex set. Let  $F$  be a convex set which does not need to be given a topological structure. We consider a functional  $V : E \times F \rightarrow \mathbb{R}$ . We define  $P : E \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $Q : F \rightarrow \{-\infty\} \cup \mathbb{R}$  by

$$P(u) = \sup \{V(u, g); g \in F\} \quad (22)$$



and similarly

$$Q(g) = \inf \{V(u, g); u \in E\}. \quad (23)$$

We obviously have  $Q(g) \leq V(u, g) \leq P(u)$  which implies  $\sup \{Q(g); g \in F\} := \beta \leq \alpha := \inf \{P(u); u \in E\}$ . The minimax theorem yields  $\alpha = \beta$  under the following assumptions

**Theorem 4.1** *Let us assume that (a)  $u \rightarrow V(u, g)$  is convex and lower semi-continuous on  $E$  for every  $g \in F$ . Let us also assume that (b)  $g \rightarrow V(u, g)$  is concave on  $F$  for every  $u \in E$ . Then there exists an element  $\bar{u} \in E$  such that*

$$P(\bar{u}) = \alpha = \beta. \quad (24)$$

Moreover, if (a) and (b) hold and if (c)  $F$  is compact then there exists a saddle point  $(\bar{u}, \bar{g}) \in E \times F$ , i.e. for all  $u \in E$  and  $g \in F$  we have

$$V(\bar{u}, g) \leq V(\bar{u}, \bar{g}) \leq V(u, \bar{g}). \quad (25)$$

We now show how to apply Theorem 4.1 to the  $BV - L^1$  problem. We naturally define the functional

$$V(u, g) = \|u\|_{BV} + \lambda \int (f - u)g. \quad (26)$$

Any optimal decomposition should satisfy  $\|\bar{u}\|_{BV} \leq \lambda \|f\|_1$  and  $\|\bar{u}\|_1 = \|f - \bar{v}\|_1 \leq 2\|f\|_1$  since  $f \in L^1 \cap L^2$  and  $J(\bar{u}) \leq J(0)$ . Let us choose  $E = \{u \in BV \cap L^1 \mid \|u\|_{BV} \leq R_1, \|u\|_1 \leq R_2\}$  where  $R_1 \geq \lambda \|f\|_1$  and  $R_2 \geq 2\|f\|_1$  are fixed. The set  $E$  is then convex. We equip  $E$  with the weak\*  $\sigma(L^2, L^2)$  topology. We need to verify that  $E$  is compact in order to apply Theorem 4.1.

Consider a sequence  $(u_n)$  of  $E$ . This subsequence is bounded in  $L^2$  since  $\|\cdot\|_2 \leq \|\cdot\|_{BV}$ . We can extract a subsequence that converges in  $L^2$  for the weak\* topology. We still denote this subsequence  $(u_n)$  and the limit  $u \in L^2$ . But  $\|u_n\|_{BV} \leq R_1$ . Then  $u \in BV$  and  $\|u\|_{BV} \leq \liminf \|u_n\|_{BV} \leq R_1$ . Now, this subsequence is also bounded in  $L^1$  by  $R_2$ . The Dunford-Pettis Theorem [Theorem IV.29 [15]] can be applied since the sequence is bounded in  $L^2$ . This theorem allows us to extract a subsequence that weakly\* converges towards  $u_2 \in L^1$ : it means that  $\int u_n g \rightarrow \int u_2 g, \forall g \in L^\infty$ , where  $(u_n)$  still represents the subsequence. Obviously we get  $\|u_2\|_1 \leq R_2$  and  $u_2 = u$ . Thus  $u \in E$ . The set  $E$  is compact. It also follows from this proof that  $u \rightarrow V(u, g)$  is lower semi-continuous for all  $g \in L^\infty$ . We now define the convex set  $F = \{g \in L^\infty \mid \|g\|_\infty \leq 1\}$ . Notice that  $P(u) = J(u)$ .

Theorem 4.1 says that it exists a function  $\bar{u}$  such that  $P(\bar{u}) = \alpha = \beta$ . This means that  $\bar{u}$  minimizes the functional  $J$  and we can swap the infimum and supremum. We then study what is called the dual problem:

$$\sup_{g \in F} \inf_{u \in E} \|u\|_{BV} - \lambda \int ug + \lambda \int fg. \quad (27)$$

To compute  $\inf_{u \in E} \|u\|_{BV} - \lambda \int ug$ , we consider three cases:  
First case: if  $g \in G$  and  $\|g\|_* \leq \frac{1}{\lambda}$ .

Then  $\|u\|_{BV} - \lambda \int ug \geq \|u\|_{BV} (1 - \lambda \|g\|_*) \geq 0$ . Thus the infimum is 0 since  $u = 0$  reaches the lower bound.

Second case: if  $g \in G$  and  $\|g\|_* > \frac{1}{\lambda}$ .

It exists a function  $u$  such that  $\|u\|_{BV} \leq 1$  and  $\epsilon > 0$  such that  $\lambda \int ug \geq (1 + \epsilon)$ . Thus  $\|u\|_{BV} - \lambda \int ug \leq -\epsilon < 0$ . Thus the infimum is strictly negative.

Third case: if  $g \notin G$ .

Then obviously, the infimum is  $-\infty$ .

Finally, the dual problem can be reformulated as follow:

$$\sup_{\substack{\|\bar{g}\|_\infty \leq 1 \\ \|\bar{g}\|_* \leq \lambda^{-1}}} \lambda \int f\bar{g}. \quad (28)$$

By considering a maximizing sequence, we can easily prove that the supremum is in fact a maximum: it exists a function  $\bar{g}$  such that  $\|\bar{g}\|_\infty \leq 1$ ,  $\|\bar{g}\|_* \leq \lambda^{-1}$  and  $\beta = Q(\bar{g}) = \lambda \int f\bar{g}$ .

Let us summarize what we get:  $\alpha = J(\bar{u}) = \beta = Q(\bar{g})$ ,  $\|\bar{g}\|_\infty \leq 1$  and  $\|\bar{g}\|_* \leq \frac{1}{\lambda}$ .

But  $V(\bar{u}, \bar{g}) \leq J(\bar{u}) = \lambda \int f\bar{g}$ . Thus  $\|\bar{u}\|_{BV} \leq \lambda \int \bar{u}\bar{g}$ . The right term is always bounded by  $\|\bar{u}\|_{BV}$  since  $\lambda \|g\|_* \leq 1$ . Then,

$$\|\bar{u}\|_{BV} = \lambda \int \bar{u}\bar{g}. \quad (29)$$

But

$$J(\bar{u}) = \lambda \int \bar{u}\bar{g} + \lambda \int \bar{v}\bar{g} = \|\bar{u}\|_{BV} + \lambda \int \bar{v}\bar{g}.$$

After simplification, it comes:

$$\|\bar{v}\|_1 = \int \bar{v}\bar{g}. \quad (30)$$

It is then obvious to verify that the pair  $(\bar{u}, \bar{g})$  is a saddle point:

$$V(\bar{u}, g) \leq V(\bar{u}, \bar{g}) \leq V(u, \bar{g}) \quad (31)$$

for all  $u \in E$  and  $g \in F$ .

Conversely, let  $(\bar{u}, \bar{g})$  be a saddle point. Necessarily  $\alpha = \beta = V(\bar{u}, \bar{g})$ . We have  $V(\bar{u}, g) \leq V(\bar{u}, \bar{g})$ . Thus  $\int \bar{v}g \leq \int \bar{v}\bar{g}$ ,  $\forall g \in F$ , i.e  $\|\bar{v}\|_1 \leq \int \bar{v}\bar{g}$ . But  $\|\bar{g}\|_\infty \leq 1$ , thus  $\int \bar{v}\bar{g} = \|\bar{v}\|_1$  and  $V(\bar{u}, \bar{g}) = J(\bar{u}) = \alpha$ . Thus  $\bar{u}$  is a minimizer of  $J$ . From the definition of a saddle point, it is clear that  $\bar{g}$  is such that  $\beta = \lambda \int f\bar{g}$  and  $\|\bar{g}\|_* \leq \frac{1}{\lambda}$ . From  $V(\bar{u}, \bar{g}) = \beta$  we get  $\|\bar{u}\|_{BV} = \lambda \int \bar{u}\bar{g}$ .

To summarize, we proved that if  $(\bar{u}, \bar{g})$  is a saddle point, then  $\bar{u}$  minimizes  $J$ ,  $\bar{g}$  maximizes (28) and relations (29) (30) hold. We then get the following theorem:

**Theorem 4.2** *A function  $\bar{u}$  is a minimizer of  $J$  if and only if it exists a function  $\bar{g}$  such that*

$$\|\bar{u}\|_{BV} = \lambda \int \bar{u}\bar{g} \quad (32)$$

$$\|\bar{v}\|_1 = \int \bar{v}\bar{g} \quad (33)$$

$$\|\bar{g}\|_\infty \leq 1 \quad (34)$$

$$\|\bar{g}\|_* \leq \frac{1}{\lambda} \quad (35)$$

In particular, if  $\|\bar{g}\|_\infty < 1$  then  $\bar{v} = 0$  and  $\bar{u} = f$ . If  $\|\bar{g}\|_* < \frac{1}{\lambda}$  then  $\bar{u} = 0$  and  $\bar{v} = f$ . Otherwise we have  $\|\bar{g}\|_\infty = 1$  and  $\|\bar{g}\|_* = \frac{1}{\lambda}$ . Roughly speaking, from  $\int \bar{v}\bar{g} = \|\bar{v}\|_1$  we get  $\bar{g} = \text{sign}(\bar{v})$  and then

$$\|\text{sign}(\bar{v})\|_* = \frac{1}{\lambda}. \quad (36)$$

It is then trivial to prove the following theorem

**Theorem 4.3** *If  $\text{sign}(f) \in G$  and  $\|\text{sign}(f)\|_* < \frac{1}{\lambda}$  then 0 is the only minimizer of  $J$ . If  $f > 0$  (or  $f < 0$ ) then 0 is never a minimizer.*

Now, assume  $f$  to be positive with bounded support  $\Omega$ . Then  $\|\chi_\Omega\|_* < \frac{1}{\lambda}$  implies  $u = 0$  is a minimizer. For instance if  $\Omega$  is a rectifiable connected set that satisfies condition (b) of Theorem 3.2 then  $\|\chi_\Omega\|_* = \frac{P(\Omega)}{|\Omega|}$  and

$$\lambda < \frac{|\Omega|}{P(\Omega)} \Rightarrow 0 \text{ is the only minimizer of } J. \quad (37)$$

We now generalize this result to simple functions. The definition of simple function is slightly different for the  $BV - L^1$  model:

**Definition 4.1** *A function  $f \in BV \cap L^1$  is simple if it exists a function  $h \in G \cap L^\infty$  such that  $\|h\|_* = 1$  and  $\int fh = \|f\|_{BV}$ .*

Let  $g = \frac{1}{\lambda}h$ . Then conditions (32) (33) (35) are satisfied for  $\bar{u} = f$  and  $\bar{v} = 0$ . It remains to satisfy condition (34) to conclude that  $\bar{u} = f$  is a minimizer. This occurs when  $\|h\|_\infty \leq \lambda$ . More precisely, we get

**Proposition 4.1** *When  $\lambda > \frac{\|h\|_\infty}{\|h\|_*}$  then  $\bar{u} = f$  is the only minimizer of  $J$ .*

This proposition was independently proved by Chan & Esedeglu (Lemma 1 [18]). In particular, if  $f \in BV \cap L^\infty$  is extremal then

$$\lambda > \frac{\|f\|_\infty}{\|f\|_*} \Rightarrow \bar{u} = f. \quad (38)$$

To conclude this section, we would like to investigate the stability of the algorithm. More precisely, considering a simple function  $g$  and a function  $h \in L^1$ , is any optimal  $u$ -component of  $f = g + h$  (using  $BV - L^1$ ) close to  $g$ ? We need to precise what “close” means. Actually, we were only able to prove a weak result:

**Proposition 4.2** *Let  $f = g + h$  where  $g$  is simple and  $h \in L^1$ . There exists a parameter  $\lambda_0$  such that for any  $\lambda > \lambda_0$ , we have*

$$\|\bar{u} - g\|_1 \leq C(\lambda) \|h\|_1 \quad (39)$$

where  $\bar{u}$  is a minimizer of  $J$ .

To prove this, consider the integral  $I = \left| \int f v_0 \right|$  where  $v_0 \in L^\infty \cap G$  is associated to the simple function  $g$ . Obviously,  $I \geq \|g\|_{BV} - \|h\|_1 \|v_0\|_*$  since  $f = g + h$  and  $g$  is simple. But  $I = \int (\bar{u} + \bar{v}) v_0$ . Then  $I \leq \|\bar{u}\| \|v_0\|_* + \|\bar{v}\|_1 \|v_0\|_\infty$ . If we combine these two relations, we get  $\|g\|_{BV} \leq \|\bar{u}\|_{BV} + \lambda_0(\|\bar{v}\|_1 + \|h\|_1)$  where  $\lambda_0 = \frac{\|v_0\|_\infty}{\|v_0\|_*}$ . Now, since  $\bar{u}$  is optimal, we have  $\|\bar{u}\|_{BV} + \lambda \|\bar{v}\|_1 \leq \|g\|_{BV} + \lambda \|h\|_1$ . All this together gives, assuming  $\lambda > \lambda_0$ ,

$$\|\bar{v}\|_1 \leq \frac{\lambda + \lambda_0}{\lambda - \lambda_0} \|h\|_1. \quad (40)$$

But  $\bar{u} - g = h - \bar{v}$ . Thus,  $\|\bar{u} - g\|_1 \leq \|\bar{v}\|_1 + \|h\|_1$ , i.e.

$$\|\bar{u} - g\|_1 \leq \frac{2\lambda}{\lambda - \lambda_0} \|h\|_1. \quad (41)$$

Let us make a remark. This result is weaker than the one we have for the  $BV - G$  model: the error bound we get for the  $BV - L^1$  model depends on  $\|h\|_1$  which has no reason to be very small for high frequency functions. For instance, let  $\mu \in L^\infty$  be 1-periodic in each variable and  $\int_{[0,1]^2} \mu = 0$ . Consider a bounded connected set  $\Omega$  with rectifiable boundaries. Then  $\|\mu(Nx)\chi_\Omega\|_* \sim \frac{1}{N}$  but  $\|\mu(Nx)\chi_\Omega\|_1 \sim |\Omega|$ : the error bound (39) does not blow up when  $N$  increases.

## 5 Application

In this section we apply Theorem 4.2 in the particular case of positive decreasing radial functions. We explicitly find an optimal decomposition.

From now on, we assume  $f$  to be radial. We use the notation  $f(r)$  for radial functions,  $r \in \mathbb{R}^+$ . Then among minimizers of  $J$  there exists one minimizer that is radial. Indeed, if  $\bar{u}$  is a minimizer then  $u \circ r_\theta$ , where  $r_\theta$  is a rotation, is optimal since  $J$  is rotation-invariant. It suffices to consider  $u = \frac{1}{2\pi} \int_0^{2\pi} \bar{u} \circ r_\theta d\theta$ . This function is radial and is still a minimizer of  $J$ . Also a maximizer of (28) can be chosen radial too by a same argument.

We are looking for radial minimizer of  $J$ , let say  $u$ . Theorem 4.2 says that it suffices to find a function  $g$  such that relations (32) (33) (34) and (35) hold. The previous remark allows us to suppose  $g$  radial. From Theorem 2.2 relations (34) (35) become

$$\|g(r)\|_\infty \leq 1 \quad (42)$$

$$\|G(r)\|_\infty \leq \frac{1}{\lambda} \quad (43)$$

where  $G(r) = \frac{1}{r} \int_0^r sg(s)ds$ . Notice that  $G$  is continuous and  $G(r) \rightarrow 0$  when  $r \rightarrow 0$ . Also  $\int u(x)g(x)dx = 2\pi \int_0^\infty ru(r)g(r)dr$ . But  $rg(r) = \frac{d}{dr}rG(r)$  and  $\|u\|_{BV} = 2\pi \int_0^\infty r|u'|dr$ . Relations (32) and (33) become

$$r[|u'(r)| + \lambda G(r)u'(r)] = 0 \quad \text{in measure} \quad (44)$$

$$v(r)g(r) = |v(r)| \quad (45)$$

where  $v = f - u$ .

The first remark we can make is that  $u$  should be constant about 0 since  $|G(r)| < \|G\|_\infty$  about 0 ( $G(0) = 0$  and  $G$  is continuous). Secondly, if  $u$  is no more constant then we should have  $|G(r)| = \frac{1}{\lambda}$ . Thus  $\|G\|_\infty = \frac{1}{\lambda}$  when  $u$  is not a constant function, i.e. when  $u$  is not identically null. Notice that if  $G(r) = \pm \frac{1}{\lambda}$  on a non empty interval, then  $g(r) = \pm \frac{1}{\lambda r}$  on the same interval. If  $v$  is not identically null, (45) implies  $\|g\|_\infty = 1$ .

We now consider a decreasing radial function  $f \in L^1 \cap L^2$  ( $f$  is then non-negative). We search  $g$  such that  $g(r) = 1$  on an interval  $[0, R]$  and  $g(r) = \frac{1}{\lambda r}$  for  $r > R$ . Then the function  $G$  is such that  $G(r) = \frac{r}{2}$  for  $r \leq R$  and  $G(r) = \frac{R}{2}$  for  $r > R$ . But  $\|G\|_\infty = \frac{1}{\lambda}$ . Thus  $R = \frac{2}{\lambda}$ . We also need  $\|g\|_\infty \leq 1$ . This is automatically satisfied since  $\frac{1}{\lambda R} = \frac{1}{2}$ . The function  $g$  satisfies (42) and (43). The fact that  $G$  does not reach its maximum before  $R$  implies that  $u$  is first a constant  $\gamma$  on  $[0, R]$ . If  $r > R$  then  $0 \leq g(r) < 1$ . This implies  $v(r) = 0$  and  $u(r) = f(r)$ . To conclude we need to find the constant  $\gamma$ . We assumed  $g = 1$  on  $[0, R]$  or equivalently  $v \geq 0$ . Thus  $f(R) \geq \gamma$ . But to satisfy the relation (44) we need to assume  $\gamma \geq f(R)$ . Thus  $\gamma = f(R)$  is unique.

**Proposition 5.1** *Let  $f$  be a non-negative decreasing radial function and  $\lambda > 0$ . Let  $R = \frac{2}{\lambda}$ . We define the function  $u$  by*

$$u(r) = \begin{cases} f(R) & r < R \\ f(r) & r > R \end{cases} \quad (46)$$

*Then,  $u$  is a minimizer of  $J$ .*

**Remark 5.1** *If  $f$  is radial whose support is a disc  $D$  of radius  $R_1$  then  $u = 0$  whenever  $R \geq R_1$ , i.e. when  $\lambda \leq \frac{2}{R_1}$ . This remark is coherent with relation (37) since  $\frac{P(D)}{|D|} = \frac{2}{R_1}$ .*

Figure 1 shows the  $BV - L^1$  decomposition performed on the function  $f(r) = e^{-r}$ . We choose a parameter  $\lambda = 4$ . The solution  $u$  is first constant then matches exactly the function  $f$ . We would like to compare this decomposition with the one obtained using the  $BV - G$  algorithm (for  $\lambda > 4\pi$ ). We still consider the function  $f(r) = e^{-r}$ . We know that  $f = u + v$  is an optimal decomposition if a function  $g$  exists such that Theorem 3.4 is satisfied. Again, when everything is radial this theorem can be rewritten as:

**Theorem 5.1** *Let  $f(r) = u(r) + v(r)$  be a decomposition of  $f \in L^2(rdr)$  such that  $u \in BV$  and  $v \in G$ . Let  $\lambda > 0$ . This decomposition is optimal if and only if it exists a radial function  $g \in BV$  such that :*

$$r[|u'| (r) + \lambda u'G(r)] = 0 \text{ in measure} \quad (47)$$

$$r[g'(r)V(r) + \|V\|_\infty] = 0 \text{ in measure} \quad (48)$$

$$\|g\|_{BV} \leq 1 \quad (49)$$

$$\|g\|_* \leq \frac{1}{\lambda}. \quad (50)$$

where  $G(r) = \frac{1}{r} \int_0^r sg(s)ds$  and  $V(r) = \frac{1}{r} \int_0^r sv(s)ds$ . Notice that  $\|G\|_\infty = \|g\|_*$  and  $\|V\|_\infty = \|v\|_*$ .

We consider the non-trivial case where we assume  $\|g\|_{BV} = 1$  and  $\|g\|_* = \frac{1}{\lambda}$ . A basic study proves that  $u$  and  $g$  should both be constant about 0 since the functions  $V(r)$  and  $G(r)$  are both continuous and they reach the value 0 for  $r = 0$ . Now if  $V(r)$  reaches  $\|V\|_\infty$  and remains constant then necessarily  $v(r) = \frac{\|v\|_*}{r}$  and  $u = f - \frac{\|v\|_*}{r}$ . The same reasoning holds for  $g$ . A natural hypothesis is to assume:

- a)  $u(r) = C_1$  and  $g(r) = C_2$  for  $0 < r < A$
- b)  $u(r) = f(r) - \frac{d_1}{r}$  and  $g(r) = \frac{d_2}{r}$  for  $A < r < R$  where  $d_1, d_2$  are both positive constants
- c)  $u(r) = g(r) = 0$  for  $r > R$ .

We then have to determine the constants  $A, R, C_1, C_2, d_1$  and  $d_2$ . The conditions to satisfy are:

- a)  $u$  is continuous
- b)  $u$  is decreasing between  $A$  and  $R$ .
- c)  $\|V\|_\infty = d_1$  and  $\|G\|_\infty = d_1 = \frac{1}{\lambda}$
- d)  $\|g\|_{BV} = 1$ .

This problem can be solved numerically. Figure 2 presents the decomposition we obtained for  $f(r) = e^{-r}$  with  $\lambda = 20$ .

Both algorithms create a step about 0. But the  $BV - L^1$  performs better for that kind of images since the function  $u$  is exactly equal to  $f$  whereas the distance between the  $BV - G$  optimal solution and  $f$  is proportional to  $\frac{1}{r}$ . Notice that the ROF algorithm and the  $BV - G$  model would be more or less identical for that kind of images (Theorem 3.8.3 [26]):

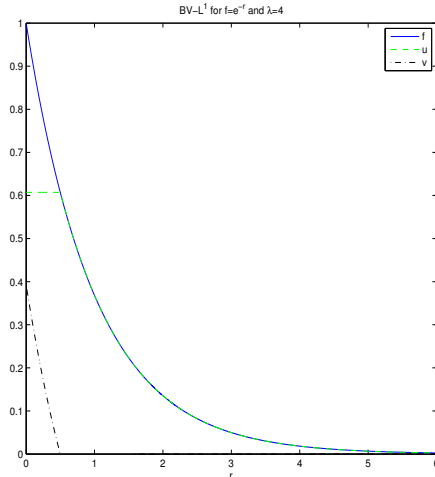


Figure 1:  $BV - L^1$  decomposition of  $f(r) = e^{-r}$  for  $\lambda = 4$

**Theorem 5.2** *Let  $f = f(r)$  be a  $C^2(\mathbb{R}^+)$  radial function such that  $\int_0^\infty f^2(r)rdr < \infty$ . We assume there exists  $t_0 > 0$  such that the function  $rf$  is nondecreasing and concave down on  $[0, t_0]$  and nonincreasing on  $[t_0, +\infty[$ . Then the optimal ROF solution  $u$  (which is unique and radial), given  $\lambda$  such that  $\|f\|_* > (2\lambda)^{-1}$ , satisfies:*

*there exists  $\gamma \geq 0$ ,  $A \geq 0$  and  $R > A$  such that*

- a)  $u(r) = \gamma$  for  $0 \leq r \leq A$*
- b)  $u(r) = f(r) - \frac{M}{r}$  where  $M = (2\lambda)^{-1}$  for  $A \leq r \leq R$*
- c)  $u(r) = 0$  for  $r \geq R$ .*

The ROF and the  $BV - G$  give almost the same solution. The difference is in the computation of the constant  $M$  which is  $\|v\|_*$  for both models. The ROF model imposes  $M = \frac{1}{2\lambda}$  while the  $BV - G$  model does not. The constants  $A$  and  $R$  depend on the model.

## 6 Conclusion

This work emphasized the similitudes between the  $BV - G$  and the  $BV - L^1$  algorithm, in particular the characterization theorems. If we consider the class of simple functions both algorithms were proved to behave in a same manner. However the  $BV - L^1$  algorithm was proved to perform better if one consider smooth radial functions. While the stability of  $BV - G$  algorithm about simple function was proved in [26], the question of stability of  $BV - L^1$  algorithm is still open.

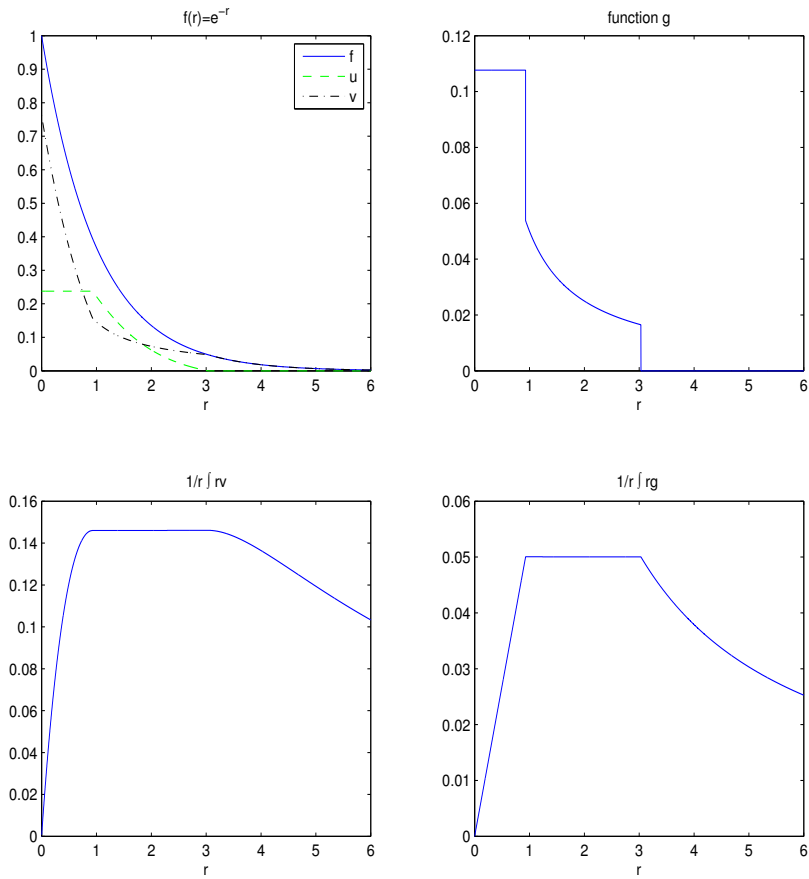


Figure 2: BV-G decomposition of  $f(r) = e^{-r}$  for  $\lambda = 20$



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