Local and Global Existence for an Aggregation Equation

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Abstract

The purpose of this work is to develop a satisfactory existence theory for a general class of aggregation equations. An aggregation equation is a non-linear, non-local partial differential equation that is a regularization of a backward diffusion process. The non-locality arises via convolution with a potential. Depending on how regular the potential is, we prove either local or global existence for the solutions. Aggregation equations have been used recently to model the dynamics of populations in which the individuals attract each other [18, 17, 13, 12, 9, 2].

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1 Introduction

We investigate the non-linear, non-local partial differential equation

$$u_t = -\text{div}\,[u\,(u * \nabla G)] \qquad \text{in } [0,T] \times \mathbb{R}^n \tag{1.1}$$

subject to the initial condition

$$u(0,x) = f(x) \qquad \text{for all } x \in \mathbb{R}^n\ . \tag{1.2}$$

The asterisk denotes spatial convolution and the function $G : \mathbb{R}^n \to \mathbb{R}$ is a given potential. The aim of this paper is to develop a satisfactory existence theory for equation (1.1). In particular, we want to understand the relationship between its well-posedness and the regularity of G. For example, we will see that for $G(x) = e^{-|x|}$ we have local existence whereas for $G(x) = e^{-|x|^2}$ we have global existence. More generally, we will identify classes of potentials for which we have local and global existence.

Several recent papers address similar questions. Bodnar and Velazquez [3] prove existence and uniqueness in one space dimension for C^1 initial data. Burger and Di Francesco [5] prove existence and uniqueness in one space dimension when the potential G is C^2. By contrast, we prove existence and uniqueness in n space dimensions for rough initial data and singular potentials. We derive a priori estimates in H^k and $H^k \cap W^{1,1}$ and use functional analytic methods (weak-star compactness). An analogue of one our estimates, (4.14), appears in Topaz and Bertozzi [17].

Equation (1.1) is derived from simple and intuitive considerations. Consider m particles evolving in $\mathbb{R}^n$ whose dynamics are described by the $n \times m$ system of ordinary differential equations:

$$\dot{X}_i = \sum_{k \neq i} \nabla G(X_i - X_k), \qquad i = 1, \ldots, m \tag{1.3}$$

where $X_i(t)$ is the position of the i^{th} particle at time t. If we choose, for example, $G(x) = e^{-|x|}$ then the m particles attract each other. The closer two particles are, more strongly they attract. Assume now that, instead of a finite number of particles, we have a density $u(x,t)$ of particles. Then, according to (1.3), at time t, a particle located at x should have a velocity

$$\vec{v}(x,t) = \int_{\mathbb{R}^3} u(y,t)\nabla G(x-y)dy = (u * \nabla G)(x,t).$$

This leads to the conservation law $u_t = -\text{div}\,(u\vec{v}) = -\text{div}\,[u\,(u * \nabla G)]$ which is (1.1). We refer to (1.1) as the aggregation equation.

In a recent paper [18] , Topaz, Bertozzi and Lewis used the equation

$$u_t = -\text{div}\,[u\,(u * \nabla G)] + \text{div}\left(u^2\nabla u\right) \tag{1.4}$$

to model the dynamics of a population where individuals experience long range social attraction and short range dispersal. From biological considerations they choose $G(x) = e^{-|x|}$. We will call (1.4) an aggregation-diffusion equation, since it includes both a nonlocal aggregation term and a local diffusion term. The local diffusion has a regularizing effect on the equation and therefore well-posedness is not an issue. Other recent papers address similar biological applications; see for example [17, 13, 12, 9, 2].

If we choose $G(x) = \delta(x)$ then (1.1) becomes the inverse porous media equation

$$u_t = -\text{div}\,(u\nabla u) \tag{1.5}$$

which is ill-posed as is any backward diffusion process. The problem of inverting a diffusion process is well known in image processing, since, if it were possible, it would provide a very valuable deblurring and edge sharpening algorithm (see [14, 1]). The aggregation equation (1.1) can be seen as a regularization of (1.5) and we think it will find applications in image processing.

The idea of regularizing ill-posed problem by introducing convolution with a smooth potential is not new. The Perona-Malik equation $u_t = \text{div}\,(g\,(|\nabla u|)\,\nabla u)$ is used in image processing to smooth pictures away from edges. The idea is to choose the function g such that the diffusion coefficient $g\,(|\nabla u|)$ is small when $|\nabla u|$ is large. Therefore little diffusion will occur along the edges of the picture. Unfortunately this equation is ill-posed. Catte et al. proposed in [6] the following regularization: $u_t = \text{div}\,(g\,(|u * \nabla G|)\,\nabla u)$ with $G(x) = e^{-|x|^2}$. They proved that this equation is well-posed, and it is now widely regarded as the "correct" version of the Perona-Malik equation.

The choice of the potential G can significantly influence the way in which aggregation occurs. Thus it is important to identify the class of potentials for which the aggregation equation is well-posed.

In section 2 we define the class of acceptable potentials and we prove that if G is acceptable then we have local existence (Theorem 1). The proof uses standard functional analytic arguments together with the a priori estimate

$$\frac{d}{dt}\|u(t)\|^2_{H^k} \le C\|u(t)\|^3_{H^k}. \tag{1.6}$$

In section 3, we prove that the weak H^k solution constructed in Theorem 1 also conserves mass and preserves positivity (Theorem 2). The key ingredient of the proof is a careful a priori estimate of the $W^{1,1}$-norm (3.19). In the last section, section 4, we show that if G is acceptable and belongs to $W^{k+2,\infty}$ then the a priori estimate (1.6) becomes

$$\frac{d}{dt}\|u(t)\|^2_{H^k} \le C\|u(t)\|_{L^1}\|u(t)\|^2_{H^k}.$$

This a priori estimate is actually linear since (1.1) conserves mass. This allow us to prove global existence (Theorem 3).

2 Local Existence for Acceptable Potential

2.1 Definitions and Setting of the Problem

Define p^* by

$$\frac{1}{p^*} = \frac{1}{2} + \frac{1}{n},$$

and the intervals $\mathcal{I}_n$ by

$$\mathcal{I}_1 = [1,2], \qquad \mathcal{I}_2 = (1,2], \qquad \mathcal{I}_n = [p^*, 2] \quad \text{if } n \geq 3.$$

Definition 2.1. *(i) We say that the potential $G : \mathbb{R}^n \mapsto \mathbb{R}$, $n \geq 2$, is acceptable if $\nabla G \in L^2(\mathbb{R}^n)$ and $\Delta G \in L^p(\mathbb{R}^n)$ for some $p \in \mathcal{I}_n$.*

(ii) We say that the potential $G : \mathbb{R} \mapsto \mathbb{R}$ is acceptable if $G' \in L^2(\mathbb{R})$ and if there exists a function $g \in L^p(\mathbb{R})$ for some $p \in \mathcal{I}_1$ and a constant $c \in \mathbb{R}$ such that $G'' = g + c\delta$ where δ is the Dirac delta function.

One can check that the potentials

$$G_1(x) = e^{-|x|} \qquad \textit{and} \qquad G_2(x) = (1+|x|)^{-\alpha}, \quad \alpha > n/2 - 1.$$

are acceptable in all dimensions. These type of mildly singular potentials are commonly used in application ([18], [9]) and it is therefore important to choose the class of acceptable potentials large enough to include them. Note, that, in dimension $n = 2$ (the case most important for image processing), ΔG_1 and ΔG_2 are not in $L^2(\mathbb{R}^n)$ because of their singularity of order $|x|^{-1}$ at the origin. This is the reason for allowing the laplacian of an acceptable potential to be in some L^p-space with $p < 2$. In dimension $n = 1$, G_1'' and G_2'' each involve a delta function and so we must allow for this in definition (2.1)-(ii). The intervals $\mathcal{I}_n$ have been chosen so that we can use the Gagliardo-Nirenberg-Sobolev inequalities.

The goal of this section is to prove the following theorem:

Theorem 1 (Local Existence)**.** *Assume that G and f satisfy:*

$$G \textit{ is acceptable}, \tag{2.1}$$

$$f \in H^k(\mathbb{R}^n) \quad \textit{for some } k \geq 1. \tag{2.2}$$

If $T > 0$ is small enough, then there exist a unique function u satisfying

$$u \in L^\infty\left(0,T; H^k(\mathbb{R}^n)\right), \tag{2.3}$$

$$u_t \in L^\infty\left(0,T; H^{k-1}(\mathbb{R}^n)\right), \tag{2.4}$$

$$u_t + \mathit{div}[u\,(u * \nabla G)] = 0 \quad \textit{in } (0,T) \times \mathbb{R}^n, \tag{2.5}$$

$$u(0) = f. \tag{2.6}$$

Let's make precise the sense in which (2.5) and (2.6) hold. If G and u satisfy (2.1) and (2.3) then it can be checked that

$$\operatorname{div}[u\,(u * \nabla G)] \in L^\infty\left(0,T; H^{k-1}(\mathbb{R}^n)\right). \tag{2.7}$$

Therefore (2.5) has a sense as an equality in $L^\infty\left(0,T; H^{k-1}\right)$. From (2.3) and (2.4) it follows that u is continuous from $[0,T] \to H^{k-1}(\mathbb{R}^n)$ (see for example [10] Lemma 1.2, page 7). Therefore (2.6) makes sense.

2.2 A Priori Estimate of the H^k-Norm and Uniqueness

Define q^* by

$$\frac{1}{q^*} = \frac{1}{2} - \frac{1}{n},$$

and the intervals $\mathcal{J}_n$ by

$$\mathcal{J}_1 = [2, +\infty], \qquad \mathcal{J}_2 = [2, +\infty), \qquad \mathcal{J}_n = [2, q^*] \quad \text{if } n \geq 3.$$

and recall the Sobolev embedding

$$H^1(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \qquad \forall q \in \mathcal{J}_n \tag{2.8}$$

with continuous injection. Note that p^* and q^* are conjugate of each other, i.e., $1/p^* + 1/q^* = 1$. Actually the intervals $\mathcal{I}_n$ and $\mathcal{J}_n$ are conjugate of each other in the sense that if p belongs to $\mathcal{I}_n$ then its conjugate q belongs to $\mathcal{J}_n$.

The next two lemmas play a key role in the derivation of the a priori estimate (1.6).

Lemma 2.2. *Suppose* $1 \leq p \leq 2$, *and* q *is the conjugate of* p. *If* $\phi_1 \in L^2(\mathbb{R}^n)$, $\phi_2 \in L^q(\mathbb{R}^n)$, $\phi_3 \in L^2(\mathbb{R}^n)$ *and* $\phi_4 \in L^p(\mathbb{R}^n)$, *then*

$$\left| \int_{\mathbb{R}^n} \phi_1 \, \phi_2 \, (\phi_3 * \phi_4) \, dx \right| \leq \|\phi_1\|_{L^2} \|\phi_2\|_{L^q} \|\phi_3\|_{L^2} \|\phi_4\|_{L^p}.$$

Proof. Using Young's inequality, we get $\|\phi_3 * \phi_4\|_{L^r} \leq \|\phi_3\|_{L^2} \|\phi_4\|_{L^p}$ for r defined by $1/r = 1/2 + 1/p - 1$. Note that the condition $p \leq 2$ ensure that $1/r$ is nonnegative. One can then check that $1/r + 1/q = 1/2$. This allow us to use Hölder's inequality to show that $\|\phi_2 (\phi_3 * \phi_4)\|_{L^2} \leq \|\phi_2\|_{L^q} \|\phi_3 * \phi_4\|_{L^r}$. Then the Schwarz inequality concludes the proof. □

Lemma 2.3. *Suppose* $p \in \mathcal{I}_n$. *If* $\phi_1 \in L^2(\mathbb{R}^n)$, $\phi_2 \in H^1(\mathbb{R}^n)$, $\phi_3 \in L^2(\mathbb{R}^n)$ *and* $\phi_4 \in L^p(\mathbb{R}^n)$, *then*

$$\left| \int_{\mathbb{R}^n} \phi_1 \, \phi_2 \, (\phi_3 * \phi_4) \, dx \right| \leq C \|\phi_1\|_{L^2} \|\phi_2\|_{H^1} \|\phi_3\|_{L^2} \|\phi_4\|_{L^p}$$

The constant $C > 0$ *depends only on* p *and* n.

Proof. Let q be the conjugate of p. Since $p \in \mathcal{I}_n$ then $q \in \mathcal{J}_n$. The inequality therefore follows from the Sobolev embedding (2.8) together with Lemma 2.2. □

Suppose that $u(x,t)$ satisfies the differential equation (2.5). Then we have, at least formally,

$$(D^\alpha u, D^\alpha u_t) = -T^\alpha(u,u,u)$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\mathbb{R}^n)$ and T^α is the trilinear form defined by

$$T^\alpha(u,v,w) = \int_{\mathbb{R}^n} (D^\alpha u) \; D^\alpha \mathrm{div}\left[v \, (w * \nabla G)\right] \, dx. \tag{2.9}$$

Lemma 2.4. *Assume* G *is an acceptable potential.*

(i) If α *is a multiindex of order* $k \geq 1$, *then*

$$|T^\alpha(u,u,w)| \leq C \|u\|_{H^k}^2 \|w\|_{H^k} \qquad \forall u \in H^{k+1}(\mathbb{R}^n) \text{ and } \forall w \in H^k(\mathbb{R}^n).$$

The constant $C > 0$ *depends only on* α, G *and* n.

(ii) If $\alpha = 0$ then

$$\left|T^0(u,u,w)\right| \le C \left\|u\right\|_{L^2}^2 \left\|w\right\|_{H^1} \qquad \forall u, w \in H^1(\mathbb{R}^n),$$
$$\left|T^0(u,v,w)\right| \le C \left\|u\right\|_{L^2} \left\|v\right\|_{H^1} \left\|w\right\|_{L^2} \;\; \forall u, w \in L^2(\mathbb{R}^n) \;\; \text{and } \forall v \in H^1(\mathbb{R}^n).$$

The constant $C > 0$ depends only on G and n.

Proof. We first rewrite $T^\alpha(u,v,w)$ as follow

$$T^\alpha(u,v,w) = \int_{\mathbb{R}^n} (D^\alpha u) \; D^\alpha \left[\nabla v \cdot (w * \nabla G) + v \; (\nabla w * \nabla G)\right] \, dx$$
$$= \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} \left(A_\gamma^\alpha(u,v,w) + B_\gamma^\alpha(u,v,w)\right),$$

where the trilinear forms A_γ^α and B_γ^α are defined by

$$A_\gamma^\alpha(u,v,w) = \int_{\mathbb{R}^n} (D^\alpha u) \; (D^\gamma \nabla v) \cdot D^{\alpha-\gamma} (w * \nabla G) \; dx, \tag{2.10}$$
$$B_\gamma^\alpha(u,v,w) = \int_{\mathbb{R}^n} (D^\alpha u) \; \left(D^{\alpha-\gamma} v\right) \; D^\gamma (\nabla w * \nabla G) \; dx. \tag{2.11}$$

The notation $\nabla w * \nabla G$ stands for $\sum_{i=1}^n w_{x_i} * G_{x_i}$.

We now prove (i). Let α be a multiindex of order $k \ge 1$. Using Lemma 2.2 with $p = q = 2$ and the fact that $\nabla G \in L^2$ we find

$$\left|A_\gamma^\alpha(u,u,w)\right| \le \|u\|_{H^k}^2 \|w\|_{H^k} \|\nabla G\|_{L^2} \qquad \text{if } |\gamma| \le k-1, \tag{2.12}$$
$$\left|B_\gamma^\alpha(u,u,w)\right| \le \|u\|_{H^k}^2 \|w\|_{H^k} \|\nabla G\|_{L^2} \qquad \text{if } |\gamma| \le k-1. \tag{2.13}$$

Note that

$$A_\alpha^\alpha(u,u,w) = \int_{\mathbb{R}^n} (D^\alpha u) \; (D^\alpha \nabla u) \cdot (w * \nabla G) \; dx \tag{2.14}$$
$$= -\frac{1}{2} \int_{\mathbb{R}^n} (D^\alpha u)^2 \; (\nabla w * \nabla G) \; dx \tag{2.15}$$

Thus

$$|A_\alpha^\alpha(u,u,w)| \le \frac{1}{2} \|u\|_{H^k}^2 \|w\|_{H^1} \|\nabla G\|_{L^2}. \tag{2.16}$$

We next show that

$$|B_\alpha^\alpha(u,u,w)| \le C \|u\|_{H^k} \|u\|_{H^1} \|w\|_{H^k} \tag{2.17}$$

where the constant $C > 0$ depends only on α, G and n. This estimate, together with (2.12), (2.13) and (2.16), gives (i). For $n \ge 2$ we have

$$B_\alpha^\alpha(u,u,w) = \int_{\mathbb{R}^n} (D^\alpha u) \; (u) \; (D^\alpha w * \Delta G) \; dx$$

where $\Delta G \in L^p$ for some $p \in \mathcal{I}_n$ (by definition of an acceptable potential in dimension $n \ge 2$). We can then use Lemma 2.3 to get

$$|B_\alpha^\alpha(u,u,w)| \le C \|u\|_{H^k} \|u\|_{H^1} \|w\|_{H^k} \|\Delta G\|_{L^p}$$

which gives (2.17) when $n \ge 2$. If $n = 1$, then $B_\alpha^\alpha(u,u,w)$ can be written

$$B_\alpha^\alpha(u,u,w) = \int_{\mathbb{R}^n} (D^\alpha u) \; (u) \; (D^\alpha w * g) \; dx + c \int_{\mathbb{R}^n} (D^\alpha u) \; (u) \; (D^\alpha w) \; dx \tag{2.18}$$

where $g \in L^p$ for some $p \in \mathcal{I}_1$ and $c \in \mathbb{R}$ (see definition 2.1). Using Lemma 2.3 we get

$$\left| \int_{\mathbb{R}^n} (D^\alpha u)\ (u)\ (D^\alpha w * g)\ dx \right| \leq C \left\| u \right\|_{H^k} \left\| u \right\|_{H^1} \left\| w \right\|_{H^k} \left\| g \right\|_{L^p} . \tag{2.19}$$

Using the fact that $\left\| u \right\|_{L^\infty} \leq C \left\| u \right\|_{H^1}$ in dimension $n = 1$, we find

$$\left| \int_{\mathbb{R}^n} (D^\alpha u)\ (u)\ (D^\alpha w)\ dx \right| \leq C \left\| u \right\|_{H^k} \left\| u \right\|_{H^1} \left\| w \right\|_{H^k} . \tag{2.20}$$

Combining (2.18), (2.19) and (2.20) gives (2.17) when $n = 1$.

We now turn to the proof of (ii). Note that $T^0(u, u, v) = A_0^0(u, u, v) + B_0^0(u, u, v)$. Using techniques similar to the ones used above, we obtain

$$\left| A_0^0(u, u, w) \right| \leq \frac{1}{2} \left\| u \right\|_{L^2}^2 \left\| w \right\|_{H^1} \left\| \nabla G \right\|_{L^2} ,$$
$$\left| B_0^0(u, u, w) \right| \leq \left\| u \right\|_{L^2}^2 \left\| w \right\|_{H^1} \left\| \nabla G \right\|_{L^2} ,$$

which gives the first estimate of (ii). The second estimate follows from

$$\left| A_0^0(u, v, w) \right| \leq \left\| u \right\|_{L^2} \left\| v \right\|_{H^1} \left\| w \right\|_{L^2} \left\| \nabla G \right\|_{L^2} ,$$
$$\left| B_0^0(u, v, w) \right| \leq C \left\| u \right\|_{L^2} \left\| v \right\|_{H^1} \left\| w \right\|_{L^2} ,$$

which, again, are obtained using techniques similar to the ones used in the proof of (i). □

For $k \geq 1$ we define the space

$$X^k\left(0, T\right) = \left\{ u \in L^\infty \left(0, T; H^k\left(\mathbb{R}^n\right)\right)\ ;\ u_t \in L^\infty \left(0, T; H^{k-1}\left(\mathbb{R}^n\right)\right) \right\} . \tag{2.21}$$

Lemma 2.5 (A Priori Estimate)**.** *Assume G is an acceptable potential. If u belongs to $X^{k+1}(0, T)$, $k \geq 1$, and satisfies (2.5), then*

$$\frac{d}{dt} \left\| u(t) \right\|_{H^k}^2 \leq C \left\| u(t) \right\|_{H^k}^3 \qquad \text{for a.e. } t \in [0, T].$$

The constant C depends only on G, k and n.

Proof. Let α be a multiindex of order less or equal to k. The equality

$$\frac{1}{2} \frac{d}{dt} \left\| D^\alpha u \right\|_{L^2}^2 = (D^\alpha u, D^\alpha u_t) = -T^\alpha(u, u, u),$$

holds in the scalar distribution sense on (0,T) (see for example [16] Lemma 1.2, ch.III, §1). Since $u(t) \in H^{k+1}\left(\mathbb{R}^n\right)$ for a.e. $t \in (0, T)$ we can use statement (i) of Lemma 2.4 to find that $\left| T^\alpha(u, u, u) \right| \leq C \left\| u \right\|_{H^k}^3$. □

Lemma 2.6 (Perturbation)**.** *Assume G is an acceptable potential. If u and v belong to $X^1(0, T)$ and satisfy (2.5), then*

$$\frac{d}{dt} \left\| u - v \right\|_{L^2}^2 \leq C \left(\left\| u \right\|_{H^1} + \left\| v \right\|_{H^1} \right) \left\| u - v \right\|_{L^2}^2 \qquad \text{for a.e. } t \in [0, T].$$

The constant C depends only on G and n.

Proof. Define $w = u - v$. Then w must satisfy the differential equation

$$w_t = -\text{div} \left[w \left(u * \nabla G \right) + v \left(w * \nabla G \right) \right]$$

and therefore, using statements (ii) from Lemma 2.4 we obtain

$$\begin{aligned} (w, w_t) &= -T^0(w, w, u) - T^0(w, v, w) \\ &\leq C \left\| w \right\|_{L^2}^2 \left\| u \right\|_{H^1} + C \left\| w \right\|_{L^2}^2 \left\| v \right\|_{H^1} . \end{aligned}$$

□

Corollary 2.7 (Uniqueness)**.** *Assume f and G satisfy* (2.1)-(2.2)*. Then there is at most one function u satisfying* (2.3)-(2.6)*.*

Proof. Suppose that u_1 and u_2 satisfy (2.3)-(2.6). Then, in particular, u_1 and u_2 belongs to $X^1(0,T)$. Using the previous we obtain

$$\frac{d}{dt}\|u_1-u_2\|_{L^2}^2 \le C\left(\|u_1\|_{L^\infty(0,T;H^1)}+\|u_2\|_{L^\infty(0,T;H^1)}\right)\|u_1-u_2\|_{L^2}^2 ,$$

so the result follows from Gronwall's Lemma. □

2.3 Existence via Successive Approximations

The space $X^k(0,T)$ defined by (2.21) satisfies

$$X^k(0,T)\subset C\left([0,T];H^{k-1}\left(\mathbb{R}^n\right)\right) \tag{2.22}$$

(see for example [10] Lemma 1.2, page 7). Consider the space

$$C_B^k\left(\mathbb{R}^n\right)=\Big\{u\in C^k\left(\mathbb{R}^n\right);\sup_{\substack{|\alpha|\le k\\ x\in\mathbb{R}^n}}|D^\alpha u(x)|<\infty\Big\}.$$

We claim that

$$\begin{aligned}&\text{if}\quad u\in X^k(0,T)\quad\text{and}\quad \nabla G\in L^2\left(\mathbb{R}^n\right),\\ &\text{then}\quad u*\nabla G\in C\left([0,T];C_B^{k-1}(\mathbb{R}^n)\right).\end{aligned} \tag{2.23}$$

This follow from (2.22) together with the fact that,

$$\begin{aligned}&\text{if}\quad \phi_1\in H^{k-1}\left(\mathbb{R}^n\right)\quad\text{and}\quad \phi_2\in L^2\left(\mathbb{R}^n\right),\\ &\text{then}\quad \phi_1*\phi_2\in C_B^{k-1}\left(\mathbb{R}^n\right).\end{aligned}$$

The next lemma gives global existence of solutions of a linear hyperbolic PDE assuming enough smoothness on the coefficients.

Lemma 2.8. *Suppose a_i, b and f are given with*

$$a_i,b\in C\left([0,T];C_B^k\left(\mathbb{R}^n\right)\right),\quad f\in H^k\left(\mathbb{R}^n\right)\quad \textit{for some } k\ge 1.$$

Then there exists a unique $u\in X^k(0,T)$ which satisfies

$$u_t+\sum_{i=1}^n a_i u_{x_i}+bu=0\quad \textit{on } (0,T)\times\mathbb{R}^n,\qquad u(0)=f. \tag{2.24}$$

The proof can be found, for example, in [11] (Theorem 2, Section 12.1). We now use this result to prove global existence for a linearized-localized version of the aggregation equation. The difficulty is that the coefficients are not smooth enough to apply Lemma 2.8 directly.

Lemma 2.9. *Assume G and f satisfy* (2.1)-(2.2)*. Given $v\in X^k(0,T)$, there exists a unique function $u\in X^k(0,T)$ which satisfies*

$$u_t+div[u\,(v*\nabla G)]=0\quad \textit{on } (0,T)\times\mathbb{R}^n,\qquad u(0)=f. \tag{2.25}$$

Moreover we have the estimate

$$\|u\|_{L^\infty(0,T;H^k)}\le e^{CT\|v\|_{L^\infty(0,T;H^k)}}\|f\|_{H^k}, \tag{2.26}$$

where the constant C depends only on G, k and n.

Proof. Step 1. First assume $v \in X^{k+2}(0,T)$. If we set

$$a_i = v * G_{x_i} \quad \text{and} \quad b = \sum_{i=1}^{n} v_{x_i} * G_{x_i}, \tag{2.27}$$

then (2.25) takes the form (2.24). From (2.23) it follows that

$$a_i \in C\left([0,T], C_B^{k+1}\right) \quad \text{and} \quad b \in C\left([0,T], C_B^{k}\right).$$

Thus Lemma 2.8 shows that there exists $u \in X^k(0,T)$ satisfying (2.25).

Step 2. Assume now that $v \in X^k(0,T)$ and define

$$S_\epsilon v = \rho_\epsilon * v, \qquad S_\epsilon f = \rho_\epsilon * f, \tag{2.28}$$

where ρ_ϵ is a family of smooth mollifiers and the convolution is with respect to the space variable. Obviously we have

$$S_\epsilon v \in X^\infty(0,T) \stackrel{\text{def}}{=} \bigcap_{s \geq 0} X^s(0,T),$$

$$S_\epsilon f \in H^\infty(\mathbb{R}^n) \stackrel{\text{def}}{=} \bigcap_{s \geq 0} H^s(\mathbb{R}^n).$$

Therefore reasoning as in step 1 we find that there exists a function u^ϵ which belongs to $X^\infty(0,T)$ and satisfies

$$u_t^\epsilon + \operatorname{div}\left[u^\epsilon\left(S_\epsilon v * \nabla G\right)\right] = 0 \quad \text{on } (0,T) \times \mathbb{R}^n, \tag{2.29}$$

$$u^\epsilon(0) = S_\epsilon f. \tag{2.30}$$

Our next goal is to prove that there exists a function $u \in X^k(0,T)$ and a subsequence u^{ϵ_j} such that

$$u_t^{\epsilon_j} \stackrel{*}{\rightharpoonup} u_t \quad \text{in } L^\infty\left(0,T;L^2\right) \text{ weak star }, \tag{2.31}$$

$$u^{\epsilon_j} \stackrel{*}{\rightharpoonup} u \quad \text{in } L^\infty\left(0,T;H^1\right) \text{ weak star }, \tag{2.32}$$

$$S_{\epsilon_j} v * \nabla G \to v * \nabla G \quad \text{in } L^\infty\left(0,T;W^{1,\infty}\right) \text{ strongly }, \tag{2.33}$$

$$u^{\epsilon_j}(0) \rightharpoonup u(0) \quad \text{in } L^2 \text{ weakly }. \tag{2.34}$$

This will allow us to pass to the limit in relation (2.29) and (2.30).

An estimate similar to the a priori estimate of Lemma 2.5 together with the fact that $\|S_\epsilon v\|_{H^k} \leq \|v\|_{H^k}$ leads to

$$\frac{d}{dt}\|u^\epsilon(t)\|_{H^k}^2 \leq C\|v(t)\|_{H^k}\|u^\epsilon(t)\|_{H^k}^2.$$

Then we use the Gronwall Lemma and the fact that $\|S_\epsilon f\|_{H^k} \leq \|f\|_{H^k}$ to get

$$\|u^\epsilon\|_{L^\infty(0,T;H^k)} \leq e^{CT\|v\|_{L^\infty(0,T;H^k)}}\|f\|_{H^k}. \tag{2.35}$$

Therefore the family $\{u_\epsilon\}_{\epsilon>0}$ remains in a bounded set of $L^\infty\left(0,T;H^k\right)$. Since $k \geq 1$, this gives (2.32). Letting $\epsilon \to 0$ in (2.35) gives (2.26). From the differential equation (2.29) it follows that

$$\|u_t^\epsilon\|_{H^{k-1}} \leq \|u^\epsilon\|_{H^k}\|S_\epsilon v * \nabla G\|_{W^{k,\infty}} \leq \|u^\epsilon\|_{H^k}\|v\|_{H^k}\|\nabla G\|_{L^2}. \tag{2.36}$$

Considering (2.35) and (2.36) we see that the family $\{u_t^\epsilon\}_{\epsilon>0}$ is in a bounded set of $L^\infty\left(0,T;H^{k-1}\right)$, and, since $k \geq 1$ this implies (2.31). (2.31) and (2.32) implies (2.34). Finally, (2.33) follows from

$$S_\epsilon v \to v \quad \text{in } L^\infty\left(0,T;H^1\right) \text{ strongly}$$

together with the inequality

$$\|(S_\epsilon v - v) * \nabla G\|_{W^{1,\infty}} \leq \|S_\epsilon v - v\|_{H^1}\|\nabla G\|_{L^2}.$$

To prove uniqueness we use similar ideas to the ones used in the proof of Corollary 2.7. ◻

We now prove Theorem 1 using successive approximations.

Proof of Theorem 1. Assume G and f satisfy (2.1)-(2.2). Using Lemma 2.9 we construct recursively a sequence $\{u^m\}_{m\geq 0}$ in $X^k(0,T)$ starting with

$$u^0(t) = f \quad \text{for all } t \in [0,T], \tag{2.37}$$

and defining u^m for $m \geq 1$ by

$$u^m_t + \operatorname{div}\left[u^m\left(u^{m-1} * \nabla G\right)\right] = 0 \qquad \text{on } [0,T]\times\mathbb{R}^n, \tag{2.38}$$

$$u^m(0) = f. \tag{2.39}$$

Choose T such that

$$T \leq \frac{\ln(2)}{2C\,\|f\|_{H^k}}, \tag{2.40}$$

then estimate (2.26) implies

$$\|u^m\|_{L^\infty(0,T;H^k)} \leq 2\,\|f\|_{H^k} \quad \text{for all } m\geq 0. \tag{2.41}$$

Thus, for all $m \geq 0$,

$$\begin{aligned}\|u^m * \nabla G\|_{L^\infty(0,T;W^{k,\infty})} &\leq \|u^m\|_{L^\infty(0,T;H^k)}\,\|\nabla G\|_{L^2}\\ &\leq 2\,\|f\|_{H^k}\,\|\nabla G\|_{L^2}\,.\end{aligned}$$

and using the differential equation (2.38),

$$\begin{aligned}\|u^m_t\|_{L^\infty(0,T;H^{k-1})} &\leq \|u^m\|_{L^\infty(0,T;H^k)}\,\|u^m * \nabla G\|_{L^\infty(0,T;W^{k,\infty})}\\ &\leq \left(2\,\|f\|_{H^k}\right)^2\|\nabla G\|_{L^2}\,.\end{aligned}$$

These bounds imply that there exists a function $u \in X^k(0,T)$ and a subsequence u^{m_j} such that

$$u^{m_j}_t \overset{*}{\rightharpoonup} u_t \quad \text{in } L^\infty\left(0,T;L^2\right) \text{ weak star }, \tag{2.42}$$

$$u^{m_j} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\left(0,T;H^1\right) \text{ weak star }, \tag{2.43}$$

$$u^{m_j} * \nabla G \overset{*}{\rightharpoonup} u * \nabla G \quad \text{in } L^\infty\left(0,T;W^{1,\infty}\right) \text{ weak star }, \tag{2.44}$$

$$u^{m_j}(0) \rightharpoonup u(0) \quad \text{in } L^2 \text{ weakly }. \tag{2.45}$$

Our next goal is to prove that

$$u^m \to u \quad \text{in } L^\infty\left(0,T;L^2\right) \text{ strongly }, \tag{2.46}$$

$$u^m * \nabla G \to u * \nabla G \quad \text{in } L^\infty\left(0,T;L^\infty\right) \text{ strongly }. \tag{2.47}$$

This implies that convergence in (2.42)-(2.45) actually holds for the full sequence u^m. Convergence of the full sequence, and not only a subsequence, is required in order to pass to the limit in relation (2.38)-(2.39). The strong convergence in (2.46)-(2.47) also allows us to deal with the nonlinear term. Uniqueness was proven in Corollary (2.7). So the proof of Theorem 1 will be completed once we establish (2.46)-(2.47).

Write $w^m = u^m - u^{m-1}$. Then w^m satisfies

$$\left(w^m\;,\;w^m_t\right) = -T^0\left(w^m, w^m, u^{m-1}\right) - T^0\left(w^m, u^{m-1}, w^{m-1}\right).$$

Using assertion (ii) of Lemma 2.4 we find

$$\frac{d}{dt}\,\|w^m\|^2_{L^2} \leq C\,\|u^m\|_{H^1}\left(\|w^m\|^2_{L^2} + \left\|w^{m-1}\right\|^2_{L^2}\right) \qquad \text{for a.e. } t\in(0,T).$$

From (2.41) it follows that there exists a constant $B > 0$ independent of m such that

$$\frac{d}{dt}\|w^m\|_{L^2}^2 \leq B\left(\|w^m\|_{L^2}^2 + \left\|w^{m-1}\right\|_{L^2}^2\right) \qquad \text{for a.e. } t \in (0,T).$$

Then, using Gronwall's inequality we obtain

$$\|w^m(t)\|_{L^2}^2 \leq Be^{BT}\int_0^t \left\|w^{m-1}(s)\right\|_{L^2}^2 ds \qquad \text{for a.e. } t \in (0,T)$$

and an easy recurrence gives

$$\|w^m(t)\|_{L^2}^2 \leq \frac{\left(Be^{BT}t\right)^m}{m!}\operatorname*{ess\,sup}_{t\in[0,T]}\left\|w^1(t)\right\|_{L^2} \qquad \text{for a.e. } t \in (0,T).$$

So letting $K = Be^{BT}T$ we obtain

$$\|u^m - u^{m-1}\|_{L^\infty(0,T;L^2)} \leq \frac{K^m}{m!}\|w^1\|_{L^\infty(0,T;L^2)}.$$

Since $\sum K^m/m! < \infty$ it follows that u^m is a Cauchy sequence in the space $L^\infty\left(0,T;L^2\right)$ and it must converge to the function u. This proves (2.46). (2.47) follows from (2.46) together with the inequality $\|\phi_1 * \phi_2\|_{W^{1,\infty}} \leq \|\phi_1\|_{H^1}\|\phi_2\|_{L^2}$. □

3 Conservation of Mass and Positivity

The goal of this section is to prove the following theorem.

Theorem 2 (Conservation of Mass and Positivity)**.** *Under the assumption of Theorem 1, and if we assume moreover that*

$$f \in W^{1,1}(\mathbb{R}^n), \tag{3.1}$$

$$f(x) \geq 0 \quad \textit{for a.e. } x \in \mathbb{R}^n, \tag{3.2}$$

then the solution u satisfies

$$u \in L^\infty\left(0,T;W^{1,1}(\mathbb{R}^n)\right), \tag{3.3}$$

$$u(t,x) \geq 0 \textit{ for a.e. } (t,x) \in [0,T]\times\mathbb{R}^n, \tag{3.4}$$

$$\|u(t)\|_{L^1} = \|f\|_{L^1} \quad \textit{for a.e. } t \in [0,T]. \tag{3.5}$$

In Section 3.1, we derive an estimate of the $L^\infty\left(0,T;W^{1,1}\right)$-norm of the smooth solutions of the linearized-localized version of the aggregation equation (see Lemma 3.3). To derive this estimate we use the method of characteristics. It is legitimate to do so since we work with classical smooth solutions. We also prove that smooth solutions don't change sign along their characteristics. In Section 3.2 we note that $L^\infty\left(0,T;W^{1,1}\right)$ is not a dual space and therefore doesn't have a weak star topology. We remedy to this problem by identifying it as a subspace of a dual space. In Section 3.3 we prove that what holds for smooth classical solutions of the linearized-localized problem (i.e.: $L^\infty\left(0,T;W^{1,1}\right)$-bound and conservation of positivity) also holds for weak solution of the nonlinear-nonlocal problem. Conservation of the L^1-norm follows easily.

3.1 A Priori Estimate of the $W^{1,1}$-Norm

Define

$$Y^{k,+}(0,T) = \left\{u \in X^k(0,T)\,;\; u \in L^\infty\left(0,T;W^{1,1}(\mathbb{R}^n)\right)\,,\; u(t,x) \geq 0 \text{ for a.e. } (t,x) \in (0,T)\times\mathbb{R}^n\right\}$$

and also

$$Y^{\infty,+}(0,T) = \bigcap_{s\geq 0} Y^{s,+}(0,T).$$

First we derive a careful estimate of the $L^\infty\left(0,T;W^{1,1}\right)$-norm of the smooth solutions of

$$u_t + \sum_{i=1}^{n} a_i u_{x_i} + bu = 0 \quad \text{on } (0,T)\times\mathbb{R}^n, \qquad u(0)=f, \tag{3.6}$$

under the assumptions that

$$a_i, b \in C\left([0,T];C_B^s\right) \qquad \text{for all integers } s, \tag{3.7}$$

$$f \in H^\infty \cap W^{1,1}, \tag{3.8}$$

$$f(x) \geq 0 \quad \text{for all } x\in\mathbb{R}^n. \tag{3.9}$$

Recall that $H^\infty = \cap_{s\geq 0} H^s$. We also introduce the notations

$$|a_x|_\infty = \sup_{\substack{1\leq i,j\leq n\\ (t,x)\in[0,T]\times\mathbb{R}^n}} \left|a_{i,x_j}(t,x)\right|, \qquad |b|_\infty = \sup_{(t,x)\in[0,T]\times\mathbb{R}^n} |b(t,x)|\,.$$

Lemma 3.1. *Assume a_i, b and f satisfy (3.7)-(3.9). Then there exists a unique $u\in Y^{\infty,+}(0,T)$ which satisfies (3.6). Moreover we have the estimate*

$$\|u(t)\|_{W^{1,1}} \leq e^{\left(C|a_x|_\infty + |b|_\infty\right)t}\left(\|f\|_{W^{1,1}} + \int_0^t e^{C|a_x|_\infty s}\,\|u(s)\nabla b(s)\|_{L^1}\,ds\right). \tag{3.10}$$

The constant C depends only on the dimension n.

Proof. Lemma 2.8 already gives the existence of a unique $u\in X^\infty(0,T)$ satisfying (3.6). Assertion (2.22) together with standard Sobolev embedding gives

$$u\in C([0,T];C_B^s\left(\mathbb{R}^n\right)) \qquad \text{for all integers } s. \tag{3.11}$$

Since $C([0,T];C_B^s\left(\mathbb{R}^n\right))$ is an algebra we conclude from (3.6), (3.7) and (3.11) that

$$u_t\in C([0,T];C_B^s\left(\mathbb{R}^n\right)) \qquad \text{for all integers } s. \tag{3.12}$$

Therefore u is a classical solution of (3.6) and we can use the method of characteristics. Let $a=(a_1,\ldots,a_n)$. The ordinary differential equation

$$x'(t) = a(t,x(t)), \qquad x(0)=y \tag{3.13}$$

defining the characteristic has a unique solution $x(t,y)$ which is C^1 on $[0,T]\times\mathbb{R}^n$. If we take (t,y) as new variable instead of (t,x) and set

$$\tilde{u}(t,y) = u(t,x(t,y)), \quad \tilde{b}(t,y) = b(t,x(t,y)),$$

then (3.6) becomes

$$\tilde{u}_t(t,y) + \tilde{b}(t,y)\,\tilde{u}(t,y) = 0, \qquad \tilde{u}(0,y) = f(y). \tag{3.14}$$

This is a linear ordinary differential equation in t. Therefore the function $t\mapsto\tilde{u}(t,y)$ can not change sign. In other words u can not change sign along its characteristics and, since f is nonnegative, then u must also be nonnegative. Using Gronwall's inequality with (3.14) gives

$$|\tilde{u}(t,y)| \leq e^{|b|_\infty t}\,|f(y)|\,,$$

and therefore, integrating over $\mathbb{R}^n$ we get

$$\|\tilde{u}(t)\|_{L^1} \leq e^{|b|_\infty t} \|f\|_{L^1}. \tag{3.15}$$

We then claim that the L^1-norm in the y-variable is equivalent to the L^1-norm in the x-variable:

$$e^{-C|a_x|_\infty t} \|u(t)\|_{L^1} \leq \|\tilde{u}(t)\|_{L^1} \leq e^{C|a_x|_\infty t} \|u(t)\|_{L^1} \tag{3.16}$$

where the constant $C > 0$ depends only on the dimension n. We have already seen that the solution $x(t, y)$ of (3.13) is C^1 on $[0, T] \times \mathbb{R}^n$. Its Jacobian matrix $J(t, y) = \partial x/\partial y(t, y)$ satisfies

$$\frac{d}{dt} J(t, y) = \frac{\partial a}{\partial x} (t, x(t, y)) J(t, y), \qquad J(0, y) = \mathrm{Id}.$$

In particular

$$\det J(t, y) = \exp\left(\int_0^t \mathrm{Tr} \frac{\partial a}{\partial x} (s, x(s, y))\, ds \right).$$

From this formula we see that there exists a constant $C > 0$ depending only on n such that

$$e^{-C|a_x|_\infty t} \leq \det J(t, y) \leq e^{C|a_x|_\infty t} \qquad \text{for all } (t, y) \in [0, T] \times \mathbb{R}^n.$$

This inequality together with the change of variable formula easily leads to (3.16). Combining (3.15) and (3.16) gives

$$\|u(t)\|_{L^1} \leq e^{\left(C|a_x|_\infty + |b|_\infty\right) t} \|f\|_{L^1}. \tag{3.17}$$

We now turn to the estimate of $\|\nabla u\|_{L^1}$. Set $v = \nabla u$ and note that v satisfies the system of differential equation:

$$v_t + \sum_i^n a_i v_{x_i} + (A_x + b)\, v = -u \nabla b, \qquad v(0) = \nabla f,$$

where A_x is the n by n matrix defined by

$$A_x = [\nabla a_1, \ldots, \nabla a_n].$$

Thinking of $-u\nabla b$ as a forcing term and using techniques similar to the one used to derive (3.17) we find that:

$$\|v(t)\|_{L^1} \leq e^{\left(C|a_x|_\infty + |b|_\infty\right) t} \left(\|\nabla f\|_{L^1} + \int_0^t e^{C|a_x|_\infty s} \|u(s) \nabla b(s)\|_{L^1}\ ds \right)$$

where the constant C depends only on the dimension n. This estimate combined with (3.17) gives (3.10). □

Recall that $L^1 \cap L^2$ is a Banach space with norm $\|\phi\|_{L^1 \cap L^2} = \|\phi\|_{L^1} + \|\phi\|_{L^2}$.

Lemma 3.2. *Suppose* $1 \leq p \leq 2$. *If* $\phi_1 \in L^2(\mathbb{R}^n)$, $\phi_2 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ *and* $\phi_3 \in L^p(\mathbb{R}^n)$, *then*

$$\|\phi_1 (\phi_2 * \phi_3)\|_{L^1} \leq \|\phi_1\|_{L^2} \|\phi_2\|_{L^1 \cap L^2} \|\phi_3\|_{L^p}.$$

Proof. Using Young's inequality we find that

$$\|\phi_2 * \phi_3\|_{L^p} \leq \|\phi_2\|_{L^1} \|\phi_3\|_{L^p},$$
$$\|\phi_2 * \phi_3\|_{L^r} \leq \|\phi_2\|_{L^2} \|\phi_3\|_{L^p}, \qquad \text{where } \frac{1}{r} = \frac{1}{2} + \frac{1}{p} - 1.$$

The condition $p \in [1, 2]$ implies that $r \in [2, +\infty]$. Therefore $p \leq 2 \leq r$ and we can use interpolation inequalities to obtain

$$\begin{aligned} \|\phi_2 * \phi_3\|_{L^2} &\leq \|\phi_2 * \phi_3\|_{L^p} + \|\phi_2 * \phi_3\|_{L^r} \\ &\leq \left(\|\phi_2\|_{L^1} + \|\phi_2\|_{L^2}\right) \|\phi_3\|_{L^p} \end{aligned}$$

The result follows from the Schwarz inequality. □

Lemma 3.3. *Assume G and f satisfy* (2.1) *and* (3.8)-(3.9). *Given $v \in Y^{\infty,+}(0,T)$, there exists a unique $u \in Y^{\infty,+}(0,T)$ which satisfies*

$$u_t + div[u\,(v * \nabla G)] = 0 \quad \text{on } (0,T)\times\mathbb{R}^n, \qquad u(0) = f. \tag{3.18}$$

Moreover we have the estimate

$$\|u(t)\|_{W^{1,1}} \le e^{CRt}\left(\|f\|_{W^{1,1}} + C\,\|f\|_{L^2}\int_0^t e^{CRs}\,\|\nabla v(s)\|_{L^1\cap L^2}\;ds\right) \tag{3.19}$$

where R stands for $\|v\|_{L^\infty(0,T;H^1)}$ and the constant $C>0$ depends only on G and n.

Proof. Let $a_i = v * G_{x_i}$ and $b = \sum_{i=1}^n v_{x_i} * G_{x_i}$ so that (3.18) takes the form (3.6). We then apply Lemma 3.1 to obtain the existence of a unique $u \in Y^{\infty,+}(0,T)$ which satisfies (3.18). Moreover, u satisfies estimate (3.10). We then claim that

$$\|u(t)\;\nabla b(t)\|_{L^1} \le C\,\|u(t)\|_{L^2}\,\|\nabla v(t)\|_{L^1\cap L^2} \tag{3.20}$$

Assume first that $n \ge 2$. Since G is acceptable we know that $\Delta G \in L^p$ for some $p \in \mathcal{I}_n \subset [1,2]$. We can therefore use Lemma 3.2 to see that

$$\|u\;\nabla b\|_{L^1} = \|u\,(\nabla v * \Delta G)\|_{L^1} \le \|u\|_{L^2}\,\|\nabla v\|_{L^1\cap L^2}\,\|\Delta G\|_{L^p}\,,$$

which gives (3.20) when $n \ge 2$. If $n = 1$, from the definition of an acceptable potential in one dimension we know that $G''(x) = g(x) + c\delta(x)$ where $g \in L^p(\mathbb{R})$ for some $p \in [1,2]$. Proceeding as in the case $n \ge 2$ we get

$$\begin{aligned}\|u\;b'\|_{L^1} &\le \|u\,(v' * g)\|_{L^1} + |c|\,\|uv'\|_{L^1}\\ &\le \|u\|_{L^2}\,\|v'\|_{L^1\cap L^2}\,\|g\|_{L^p} + |c|\,\|u\|_{L^2}\,\|v'\|_{L^2}\end{aligned}$$

which gives (3.20) when $n = 1$.

Therefore estimate (3.10) becomes

$$\|u(t)\|_{W^{1,1}} \le e^{\left(C|a_x|_\infty + |b|_\infty\right)t}\left(\|f\|_{W^{1,1}} + C\int_0^t e^{C|a_x|_\infty s}\,\|u(s)\|_{L^2}\,\|\nabla v(s)\|_{L^1\cap L^2}\;ds\right) \tag{3.21}$$

Note that

$$|a_x|_\infty\,,|b|_\infty \le \|v\|_{L^\infty(0,T;H^1)}\,\|\nabla G\|_{L^2} \le CR \tag{3.22}$$

where R stands for $\|v\|_{L^\infty(0,T;H^1)}$. Using assertion (ii) of Lemma 2.4 and reasoning as in the proof of Lemma 2.5 we find

$$\frac{d}{dt}\,\|u(t)\|_{L^2}^2 \le C\,\|v(t)\|_{H^1}\,\|u(t)\|_{L^2}^2$$

and thus, using Gronwall's Lemma, we obtain

$$\|u(t)\|_{L^2} \le e^{CRt}\,\|f\|_{L^2}\,. \tag{3.23}$$

Combining (3.21) with (3.22) and (3.23) gives (3.19). □

3.2 Compactness and Weak Convergence

$L^\infty\left(0,T;L^1\right)$ is not a dual space and therefore bounded sets of $L^\infty\left(0,T;L^1\right)$ do not posses any obvious compactness properties with respect to some weak star topology. To remedy this problem, we will identify $L^\infty\left(0,T;L^1\right)$ as a subspace of a dual space (see (3.25) below). This procedure is classic: see for example the discussion about the vague topology in [8] or [4] (see [15] for a detailed exposition.)

We say that a continuous function $u:\mathbb{R}^n \to \mathbb{R}$ vanishes at infinity if for every $\epsilon > 0$ the set $\{x : |f(x)| > \epsilon\}$ is compact, and we define

$$C_0(\mathbb{R}^n) = \{u \in C(\mathbb{R}^n) \;\;;\;\; u \text{ vanishes at infinity}\}$$

$C_0(\mathbb{R}^n)$ is a separable Banach space under the sup-norm. We then define

$$E = L^1(0,T;C_0(\mathbb{R}^n))$$

E is also a separable Banach space. For every $u \in L^\infty(0,T;L^1)$ there is a $T_u \in E^*$ defined by

$$\langle T_u, v\rangle_{E^*,E} = \int_0^T \int_{\mathbb{R}^n} u(t,x)v(t,x)\,dx\,dt,$$

and one can check that

$$\|T_u\|_{E^*} = \sup_{\substack{v\in E\\ \|v\|\le 1}} \int_0^T \int_{\mathbb{R}^n} u(t,x)v(t,x)\,dx\,dt = \|u\|_{L^\infty(0,T;L^1)}\,. \tag{3.24}$$

Therefore the mapping $u \mapsto T_u$ is an isometry. In this section we will identify u and T_u and we will write

$$L^\infty(0,T;L^1) \subset E^*. \tag{3.25}$$

Lemma 3.4. *Assume $u^m \in L^\infty(0,T;L^1\cap L^2)$ and $u \in L^\infty(0,T;L^2)$. Assume moreover that*

$$u^m \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;L^2) \text{ weak star} \tag{3.26}$$

$$\|u^m(t)\|_{L^1} \le h(t) \quad \text{for a.e. } t\in[0,T] \text{ and for all } m\ge 0 \tag{3.27}$$

where $h:[0,T]\to(0,+\infty)$ is continuous and increasing. Then u belongs to $L^\infty(0,T;L^1\cap L^2)$ and satisfies

$$\|u(t)\|_{L^1} \le h(t) \quad \text{for a.e. } t\in[0,T]. \tag{3.28}$$

Proof. Fix a $t_0\in[0,T]$ and let $E = L^1(0,t_0;C_0)$. Since h is increasing, then (3.27) implies

$$\|u^m\|_{L^\infty(0,t_0;L^1)} \le h(t_0) \quad \text{for all } m\ge 0.$$

Identifying T_{u^m} and u^m, it follows from (3.24) that

$$\|u^m\|_{E^*} \le h(t_0) \quad \text{for all } m\ge 0.$$

Therefore there exists $\mu\in E^*$ and a subsequence u^{m_j} such that

$$u^{m_j} \overset{*}{\rightharpoonup} \mu \quad \text{in } E^* \text{ weak star} \tag{3.29}$$

$$\|\mu\|_{E^*} \le h(t_0) \tag{3.30}$$

From (3.26) and (3.29) it follows that $\mu = u$, and

$$\|u\|_{L^\infty(0,t_0;L^1)} = \|\mu\|_{E^*} \le h(t_0).$$

(3.28) follows from the fact that t_0 was arbitrary. □

Lemma 3.5. *Assume u and u^m belong to $X^1(0,T)$ and satisfy*

$$u^m \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;H^1) \text{ weak star,} \tag{3.31}$$

$$u^m_t \overset{*}{\rightharpoonup} u_t \quad \text{in } L^\infty(0,T;L^2) \text{ weak star.} \tag{3.32}$$

Suppose further the sequence u^m is nonnegative a.e. on $[0,T]\times\mathbb{R}^n$. Then u is nonnegative a.e. on $[0,T]\times\mathbb{R}^n$.

Proof. (3.31) and (3.32) implies that

$$u^m \text{ is bounded in } L^2(0,T;H^1), \quad u^m_t \text{ is bounded in } L^2(0,T;L^2)$$

This in turn implies that the sequence u^m is bounded in $H^1((0,T)\times\mathbb{R}^n)$. Let Ω be a bounded subset of $\mathbb{R}^n$. By the Rellich-Kondrachov theorem (see [4] or [7]) there exists a subsequence u^{m_j} such that

$$u^{m_j} \to u \quad \text{in } L^2((0,T)\times\Omega) \text{ strongly and almost everywhere.}$$

Since the u^{m_j} are nonnegative a.e. on $(0,T)\times\Omega$, then u is nonnegative a.e. on $(0,T)\times\Omega$. Since Ω was arbitrary, this concludes the proof. □

3.3 Revisiting Section 2.3

In section 2.3 we used two limiting arguments to construct the solution of the aggregation equation. In this section, we show that the $L^\infty\left(0,T;W^{1,1}\right)$-bound and positivity are conserved while passing to these two limits. Conservation of the L^1-norm follows easily. We first revisit Lemma 2.9:

Lemma 3.6. *Assume G and f satisfy* (2.1)-(2.2) *and* (3.1)-(3.2). *Given $v \in Y^{k,+}(0,T)$ there exists a unique $u \in Y^{k,+}(0,T)$ which satisfies*

$$u_t + div\left[u\,(v * \nabla G)\right] = 0 \quad \text{on } (0,T)\times\mathbb{R}^n, \qquad u(0)=f.$$

Moreover we have the estimate

$$\|u(t)\|_{W^{1,1}} \le 2\, e^{CRt}\Big(\|f\|_{W^{1,1}} + C\,\|f\|_{L^2}\int_0^t e^{CRs}\,\|\nabla v(s)\|_{L^1\cap L^2}\;ds\Big) \qquad \text{for a.e. } t\in[0,T] \tag{3.33}$$

where R stands for $\|v\|_{L^\infty(0,T;H^1)}$ and the constant $C>0$ depends only on G and n.

Proof. The function u can be approximated by the family u^ϵ defined by (2.29)-(2.30). Using Lemma 3.3 we see that u^ϵ belongs to $Y^{\infty,+}(0,T)$ and satisfies

$$\|u^\epsilon(t)\|_{W^{1,1}} \le e^{CRt}\Big(\|f\|_{W^{1,1}} + C\,\|f\|_{L^2}\int_0^t e^{CRs}\,\|\nabla v(s)\|_{L^1\cap L^2}\;ds\Big) \qquad \text{for a.e. } t\in[0,T] \tag{3.34}$$

where R stands for $\|v\|_{L^\infty(0,T;H^1)}$. Here we have used the facts that $\|S_\epsilon\phi\|_{H^k} \le \|\phi\|_{H^k}$ and $\|S_\epsilon\phi\|_{W^{1,1}} \le \|\phi\|_{W^{1,1}}$. Since the right hand side of (3.34) is a continuous and increasing function of t, we can use Lemma 3.4 together with (2.32) to see that u satisfies (3.33). Of course this implies that u belongs to $L^\infty\left(0,T;W^{1,1}\right)$. Since the family u^ϵ is nonnegative, (2.31), (2.32) and Lemma 3.5 implies that u is nonnegative. □

Proof of Theorem 2. Assume G and f satisfy (2.1)-(2.2) and (3.1)-(3.2). Consider the sequence $\{u^m\}_{m\ge0}$ defined by (2.37)-(2.39). From Lemma 3.6 this sequence belongs to $Y^{k,+}(0,T)$. Choose T satisfying (2.40). Then, as we have already seen,

$$\|u^m\|_{L^\infty(0,T;H^k)} \le 2\,\|f\|_{H^k} \quad \text{for all } m\ge0. \tag{3.35}$$

From this bound and (3.33), it follows that there exists two positive constants A and B independent of m such that

$$\left\|u^{m+1}(t)\right\|_{W^{1,1}} \le A + B\int_0^t \|u^m(s)\|_{W^{1,1}}\;ds \quad \text{for a.e. } t\in[0,T] \qquad \text{and for all } m\ge0$$

An easy recurrence then gives

$$\|u^m(t)\|_{W^{1,1}} \le Ae^{Bt} \quad \text{for a.e. } t\in[0,T] \text{ and for all } m\ge0.$$

Assertion (3.3) then follows from (2.43) and Lemma 3.4. Assertion (3.4) follows from (2.42), (2.43) and Lemma 3.5.

We now turn to the proof of (3.5). Since $u\in L^\infty\left(0,T;H^1\cap W^{1,1}\right)$ and $\nabla G\in L^2$ we obtain

$$\|u\,(u*\nabla G)\|_{W^{1,1}} \le \|u\|_{W^{1,1}}\,\|u*\nabla G\|_{W^{1,\infty}} \le \|u\|_{W^{1,1}}\,\|u\|_{H^1}\,\|\nabla G\|_{L^2}\,.$$

Thus

$$\operatorname{div}\left[u\,(u*\nabla G)\right] \in L^\infty\left(0,T;L^1\left(\mathbb{R}^n\right)\right) \tag{3.36}$$

Using (2.6) we see that u_t also belongs to $L^\infty\left(0,T;L^1\left(\mathbb{R}^n\right)\right)$. It is therefore legitimate to integrate (2.6) over $\mathbb{R}^n$ and to use the divergence theorem. This leads to $\int_{\mathbb{R}^n} u_t(t,x)\,dx = 0$ for a.e. $t\in[0,T]$. This in turn leads to

$$\frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)\;dx = 0 \tag{3.37}$$

in the scalar distribution sense on $[0,T]$ (see [16] Lemma 1.1, ch.III,§1). Then (3.5) follows from the fact that u is nonnegative. □

4 Global Existence for Smoother Potential

The goal of this section is to prove the following theorem:

Theorem 3 (Global Existence)**.** *Assume G and f satisfy:*

$$G \text{ is acceptable and belongs to } W^{k+2,\infty}\left(\mathbb{R}^n\right), \tag{4.1}$$

$$f \in H^k\left(\mathbb{R}^n\right) \cap W^{1,1}\left(\mathbb{R}^n\right), \tag{4.2}$$

$$f(x) \geq 0 \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{4.3}$$

where $k \geq 1$. Then, for all $T > 0$, there exists a unique function u satisfying

$$u \in L^\infty\left(0, T; H^k\left(\mathbb{R}^n\right)\right), \quad u_t \in L^\infty\left(0, T; H^{k-1}\left(\mathbb{R}^n\right)\right), \tag{4.4}$$

$$u_t + div[u\,(u * \nabla G)] = 0 \quad \text{in } (0,T) \times \mathbb{R}^n, \tag{4.5}$$

$$u(0) = f. \tag{4.6}$$

Moreover,

$$u \in L^\infty\left(0, T; W^{1,1}\left(\mathbb{R}^n\right)\right), \tag{4.7}$$

$$u(t,x) \geq 0 \text{ for a.e. } (t,x) \in [0,T] \times \mathbb{R}^n, \tag{4.8}$$

$$\|u(t)\|_{L^1} = \|f\|_{L^1} \quad \text{for a.e. } t \in [0,T]. \tag{4.9}$$

Note that that the potentials

$$G_3(x) = e^{-|x|^2} \qquad \text{and} \qquad G_4(x) = \left(1 + |x|^2\right)^{-\alpha/2}, \quad \alpha > n/2 - 1.$$

satisfy (4.1) for every integer k. Therefore for such potentials we have global existence.

Lemma 4.1. *Suppose $\phi_1 \in L^2\left(\mathbb{R}^n\right)$, $\phi_2 \in L^2\left(\mathbb{R}^n\right)$, $\phi_3 \in L^1\left(\mathbb{R}^n\right)$ and $\phi_4 \in L^\infty\left(\mathbb{R}^n\right)$. Then*

$$\left|\int_{\mathbb{R}^n} \phi_1\ \phi_2\ \left(\phi_3 * \phi_4\right)\ dx\right| \leq \|\phi_1\|_{L^2} \|\phi_2\|_{L^2} \|\phi_3\|_{L^1} \|\phi_4\|_{L^\infty}.$$

Proof. Use the inequality $\|\phi_3 * \phi_4\|_{L^\infty} \leq \|\phi_3\|_{L^1} \|\phi_4\|_{L^\infty}$ together with Schwarz inequality. □

Next we derive a new estimate for the trilinear form T^α defined by (2.9):

Lemma 4.2. *Assume α is a multiindex of order $k \geq 0$ and $G \in W^{k+2,\infty}\left(\mathbb{R}^n\right)$. Then*

$$|T^\alpha(u,u,w)| \leq C\,\|u\|_{H^k}^2 \|w\|_{L^1} \quad \forall u \in H^{k+1}\left(\mathbb{R}^n\right) \text{ and } \forall w \in L^1\left(\mathbb{R}^n\right). \tag{4.10}$$

The constant $C > 0$ depends only on α, G and n.

Proof. Let A_γ^α and B_γ^α be the trilinear forms defined by (2.10) and (2.11). They can be rewritten as

$$A_\gamma^\alpha(u,v,w) = \int_{\mathbb{R}^n} \left(D^\alpha u\right)\ \left(D^\gamma \nabla v\right) \cdot \left(w * D^{\alpha-\gamma} \nabla G\right)\ dx,$$

$$B_\gamma^\alpha(u,v,w) = \int_{\mathbb{R}^n} \left(D^\alpha u\right)\ \left(D^{\alpha-\gamma} v\right)\ \left(w * D^\gamma \Delta G\right)\ dx.$$

From Lemma 4.1 one finds

$$\left|A_\gamma^\alpha(u,u,w)\right| \leq \|u\|_{H^k}^2 \|w\|_{L^1} \|\nabla G\|_{W^{k,\infty}} \qquad \text{if } |\gamma| \leq k-1, \tag{4.11}$$

$$\left|B_\gamma^\alpha(u,u,w)\right| \leq \|u\|_{H^k}^2 \|w\|_{L^1} \|\Delta G\|_{W^{k,\infty}} \qquad \text{for all } \gamma\ . \tag{4.12}$$

(2.15) can be rewritten as

$$A_\alpha^\alpha(u,u,w) = -\frac{1}{2}\int_{\mathbb{R}^n} (D^\alpha u)^2 \; (w * \Delta G)\; dx.$$

which leads to

$$|A_\alpha^\alpha(u,u,w)| \le (1/2)\,\|u\|_{H^k}^2 \,\|w\|_{L^1}\,\|\Delta G\|_{L^\infty}\,. \tag{4.13}$$

Combining (4.11), (4.12) and (4.13) gives (4.10). □

Lemma 4.3 (A Priori Estimate)**.** *Assume G and f satisfy* (4.1)-(4.3)*. If u belongs to $Y^{k+1,+}(0,T)$, $k \ge 0$, and satisfies (4.5)-(4.6), then*

$$\frac{d}{dt}\,\|u(t)\|_{H^k}^2 \le C\,\|f\|_{L^1}\,\|u(t)\|_{H^k}^2 \quad \text{for a.e. } t \in (0,T). \tag{4.14}$$

The constant C depends only on G, k and n.

Proof. Using the previous lemma, one proves that

$$\frac{d}{dt}\,\|u(t)\|_{H^k}^2 \le C\,\|u(t)\|_{L^1}\,\|u(t)\|_{H^k}^2$$

in exactly the same way as in Lemma 2.5. But since u belongs $Y^{k+1,+}(0,T)$ and satisfies the initial value problem (4.5)-(4.6) we know that $\|u(t)\|_{L^1} = \|f\|_{L^1}$. □

The a priori estimate (4.14) is linear, and therefore we can use it to prove global existence.

Proof of Theorem 3. The proof follows the same steps as in Theorem 2. One just replaces (3.35) with

$$\|u^m\|_{L^\infty(0,T;H^k)} \le e^{C\|f\|_{L^1}T}\,\|f\|_{H^k} \quad \text{for all } m \ge 0. \tag{4.15}$$

This estimate is valid for all $T > 0$. One obtains (4.15) as follows. Recall that the sequence u^m was defined by (2.37)-(2.39). Using the same techniques as in Lemma 4.3 we find

$$\frac{d}{dt}\,\|u^m(t)\|_{H^k}^2 \le C\,\|f\|_{L^1}\,\|u^m(t)\|_{H^k}^2\,.$$

Then (4.15) follows from Gronwall's Lemma. □

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References

[1] Luis Alvarez and Luis Mazorra, *Signal and image restoration using shock filters and anisotropic diffusion*, SIAM J. Numer. Anal. **31** (1994), no. 2, 590–605.

[2] M. Bodnar and J. J. L. Velazquez, *Derivation of macroscopic equations for individual cell-based models: a formal approach*, Math. Methods Appl. Sci. **28** (2005), no. 15, 1757–1779.

[3] ______, *An integro-differential equation arising as a limit of individual cell-based models*, J. Differential Equations **222** (2006), no. 2, 341–380.

[4] Haïm Brezis, *Analyse fonctionnelle*, Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree], Dunod, Paris, 1999, Théorie et applications. [Theory and applications].

[5] Martin Burger and Marco Di Francesco, *Large time behavior of nonlocal aggregation models with nonlinear diffusion*, RICAM-Report, Johann Radon Institute for Computational an Applied Mathematics, Austrian Academy of Sciences (2006), no. 15.

[6] Francine Catté, Pierre-Louis Lions, Jean-Michel Morel, and Tomeu Coll, *Image selective smoothing and edge detection by nonlinear diffusion*, SIAM J. Numer. Anal. **29** (1992), no. 1, 182–193.

[7] Lawrence C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[8] Gerald B. Folland, *Real analysis*, second ed., Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1999, Modern techniques and their applications, A Wiley-Interscience Publication.

[9] Darryl D. Holm and Vakhtang Putkaradze, *Aggregation of finite size particles with variable mobility*, Preprint (2005).

[10] Jacques-Louis Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.

[11] Robert C. McOwen, *Partial differential equations, methods and applications*, second ed., Prentice Hall, 2003.

[12] Alexander Mogilner and Leah Edelstein-Keshet, *A non-local model for a swarm*, J. Math. Biol. **38** (1999), no. 6, 534–570.

[13] Daniela Morale, Vincenzo Capasso, and Karl Oelschläger, *An interacting particle system modelling aggregation behavior: from individuals to populations*, J. Math. Biol. **50** (2005), no. 1, 49–66.

[14] Stanley Osher and Leonid Rudin, *Feature-oriented image enhancement using shock filters*, SIAM J. Numer. Anal. **27** (1990), no. 4, 919–940.

[15] Laurent Schwartz, *Analyse. III*, Collection Enseignement des Sciences [Collection: The Teaching of Science], vol. 44, Hermann, Paris, 1993, Calcul Intégral. [Integral Calculus], With the collaboration of K. Zizi.

[16] Roger Temam, *Navier-Stokes equations*, third ed., Studies in Mathematics and its Applications, vol. 2, North-Holland Publishing Co., Amsterdam, 1984, Theory and numerical analysis, With an appendix by F. Thomasset.

[17] Chad M. Topaz and Andrea L. Bertozzi, *Swarming patterns in a two-dimensional kinematic model for biological groups*, SIAM J. Appl. Math. **65** (2004), no. 1, 152–174 (electronic).

[18] Chad M. Topaz, Andrea L. Bertozzi, and Mark A. Lewis, *A nonlocal continuum model for biological aggregation*, Bulletin of Mathematical Biology (2006).