Mountain Waves and Gaussian Beams

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SUMMARY

Gaussian beams are approximate solutions to hyperbolic partial differential equations that are concentrated on a curve in space-time. In this paper, we present a method for computing the stationary in time wave field that results from steady air flow over topography as a superposition of Gaussian beams. We derive the system of equations that governs these mountain waves as a linearization of the basic equations of fluid dynamics and show that this system is well-posed. Furthermore, we show that the approximate Gaussian beam stationary solution is close to a true time dependent solution of the linearized system.

KEYWORDS: Orographic Waves Ray Tracing Methods Superpositions of Gaussian Beams

1. Introduction

Mountain waves are stationary atmospheric waves generated by steady air flow over topography. These waves can propagate far in the vertical direction and are in part responsible for transporting momentum to the top of the troposphere. Since these waves exist on a scale that is smaller than the scales resolved by general circulation models, their effect on the large-scale model flow is not directly resolved; it has to be parametrized and incorporated as a correction in simulations of Earth's atmosphere. In this paper, we explore a new method for calculating the stationary wave field generated by steady wind flow over mountains - the method of superposition of Gaussian beams.

Gaussian beams are approximate asymptotically valid solutions to hyperbolic partial differential equations which are concentrated on a single curve through the domain. The existence of such solutions has been known to the pure mathematics community since sometime in the 1960s, and these solutions have been used to obtain results on propagation of singularities in hyperbolic PDEs (Hörmander 1971 and Ralston 1982). We will show that one can use these solutions as building blocks to assemble more general solutions and in particular we will use them to model mountain waves. Gaussian beams are closely related to geometric optics or ray-tracing. Ray-tracing as a method for modeling mountain waves was explored by Broutman *et al.* (2002 and 2004). A common problem with ray-tracing is that the resulting solution does not exist globally. Generally, this break down of the solution occurs when nearby rays cross resulting in a caustic where ray-tracing incorrectly predicts that the amplitude of the solution is infinite. This blowup can be corrected and the solution can be extended past caustics, once they are identified, by Maslov's method. This technique was also explored by Broutman *et al.* (2002); however, as is pointed out by Broutman *et al.* (2004), caustics can occur anywhere along the ray and their correction in numerical ray-tracing is non-trivial. Superpositions of Gaussian beams enjoy an advantage over ray-tracing in that the Gaussian beam solution is valid at caustics and beyond them. Hence, the superposition provides a global solution of the PDE without having to identify and correct for caustics.

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In this paper, we will derive the equations for mountain waves from the basic equations of fluid dynamics and show how one obtains a full description of the atmosphere, including the wind velocity, density and pressure, using a superposition of Gaussian beams. In the appendix we show that the equations that govern mountain waves are well-posed in the sense of Hadamard and we provide a description of the general construction of Gaussian beams. We will also show that the stationary Gaussian beam solution is an approximate solution to the full time dependent formulation of the mountain wave equations.

2. Mountain Waves

(a) *Time Dependent Formulation*

The equations relating the velocity, $\vec{u} = (u, v, w)$, density, ρ, and pressure, P, in three dimensions, $x = (x_1, x_2, x_3)$, are

$$\begin{aligned} \frac{D}{Dt}\vec{u} + \frac{1}{\rho}\nabla_x P + \nabla_x(gx_3) &= 0 \\ \rho_t + \nabla_x \cdot (\rho\vec{u}) &= 0 , \end{aligned}$$

where $\frac{D}{Dt} = \partial_t + \vec{u} \cdot \nabla_x$. These two relations are the mathematical formulation of conservation of momentum and conservation of mass. They give us four equations for the five unknown functions. In order to complete the system, we will use the polytropic gas law $P = A(S)\rho^\gamma$, where γ is a constant, A is some function of the entropy, S. Since $\frac{D}{Dt}S = 0$, we have

$$\frac{D}{Dt}(P\rho^{-\gamma}) = 0 .$$

We will linearize this system of equations about a steady background flow solution which depends only on the height x_3:

$$\vec{u}_0(x_3) = (u_0(x_3), v_0(x_3), 0) , \quad \rho_0(x_3) \quad \text{and} \quad P_0(x_3) .$$

Since this background state needs to satisfy the system of equations, from the third component of the conservation of momentum equation, we have the hydrostatic relationship $\partial_{x_3} P_0 = -g\rho_0$. We treat mountain waves as a perturbation of this state. To carry out the perturbation we linearize the system about the background steady state and obtain the linear system of equations for the perturbation quantities (we will abuse notation here and denote them by $(\vec{u}, \rho, P)$):

$$\begin{aligned} \frac{D}{Dt}\vec{u} + w\frac{\partial\vec{u}_0}{\partial x_3} + \frac{1}{\rho_0}\nabla_x P - \frac{\rho}{\rho_0^2}\nabla_x P_0 &= 0 \\ \frac{D}{Dt}\rho + \rho_0\nabla_x \cdot u + w\frac{\partial\rho_0}{\partial x_3} &= 0 \\ \frac{D}{Dt}P + w\frac{\partial P_0}{\partial x_3} + \gamma P_0\nabla_x \cdot u &= 0 , \end{aligned}$$

where in the last equation we have substituted $\frac{D}{Dt}\rho$ from the previous equation. The topography, $h(x')$, enters the equation as a boundary condition for the vertical velocity, $w(t, x', 0) = \vec{u}_0(0) \cdot \nabla_{x'} h(x')$, where $x' = (x_1, x_2)$ and $\vec{u}_0 = (u_0(x_3), v_0(x_3))$.

In order to apply existence and uniqueness results to this hyperbolic system, it is convenient to write it in symmetric form. After some elementary row operations, this system of equations can be written as

$$\left(A_0\frac{\partial}{\partial t} + B_1\frac{\partial}{\partial x_1} + B_2\frac{\partial}{\partial x_2} + B_3\frac{\partial}{\partial x_3} + C\right)\begin{bmatrix}\vec{u}\\ \rho\\ P\end{bmatrix} \quad = \quad 0\ , \tag{1}$$

where, with $\alpha = \frac{\rho_0^2}{\gamma^2 P_0^2} + \frac{1}{\gamma P_0 \rho_0}$ and $\beta = \frac{\rho_0}{\gamma P_0}$,

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & -\beta\\ 0 & 0 & 0 & -\beta & \alpha \end{pmatrix}, \quad B_1 = \begin{pmatrix} u_0 & 0 & 0 & 0 & \frac{1}{\rho_0}\\ 0 & u_0 & 0 & 0 & 0\\ 0 & 0 & u_0 & 0 & 0\\ 0 & 0 & 0 & u_0 & -\beta u_0\\ \frac{1}{\rho_0} & 0 & 0 & -\beta u_0 & \alpha u_0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} v_0 & 0 & 0 & 0 & 0\\ 0 & v_0 & 0 & 0 & \frac{1}{\rho_0}\\ 0 & 0 & v_0 & 0 & 0\\ 0 & 0 & 0 & v_0 & -\beta v_0\\ 0 & \frac{1}{\rho_0} & 0 & -\beta v_0 & \alpha v_0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{\rho_0}\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{\rho_0} & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & \partial_{x_3} u_0 & 0 & 0\\ 0 & 0 & \partial_{x_3} v_0 & 0 & 0\\ 0 & 0 & 0 & \frac{-\partial_{x_3} P_0}{\rho_0^2} & 0\\ 0 & 0 & \partial_{x_3}\rho_0 - \beta\partial_{x_3} P_0 & 0 & 0\\ 0 & 0 & \alpha\partial_{x_3} P_0 - \beta\partial_{x_3}\rho_0 & 0 & 0 \end{pmatrix}.$$

Note that, since $\alpha > \beta^2$, A_0 is a positive definite matrix and that the eigenvalues of B_3 are $\{0, 0, 0, -1/\rho_0, 1/\rho_0\}$. Thus, Theorem A.4 in section A.(b) guarantees that a solution to this problem exists and is unique, provided that we specify initial data and suitable boundary conditions. An adequate choice for the boundary conditions that satisfies the requirements of Theorem A.4 are

$$M\begin{bmatrix}\vec{u}\\ \rho\\ P\end{bmatrix}_{x_3=0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{bmatrix}u\\ v\\ w\\ \rho\\ P\end{bmatrix}_{x_3=0} = \begin{bmatrix}0\\ 0\\ \vec{u}_0(0)\cdot\nabla_{x'}h(x')\\ 0\\ 0\end{bmatrix}. \tag{2}$$

In other words, we need only specify the boundary values of the vertical wind velocity component.

(b) *Approximate Time-Independent Solutions*

Since we are looking for stationary wave solutions in the far field, it is convenient to rescale the space variables. We let $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) = (x_1/R, x_2/R, x_3/R)$, where R is the large far field parameter. We will use ∇ to denote the gradient with respect to $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, ∇' the gradient with respect to $(\bar{x}_1, \bar{x}_2)$ and $\nabla_{x'}$ the gradient with respect to (x_1, x_2). We now look for approximate solutions of this system that are stationary in time. We assume that the background wind velocity, density and pressure depend only on the rescaled variable $\bar{x}_3$. Let

$$B'_1 = \begin{pmatrix} u_0 & 0 & 0 & 0 & \frac{1}{\rho_0} \\ 0 & u_0 & 0 & 0 & 0 \\ 0 & 0 & u_0 & 0 & 0 \\ 0 & 0 & 0 & u_0 & 0 \\ \frac{1}{\rho_0} & 0 & 0 & 0 & 0 \end{pmatrix} \ , \quad B'_2 = \begin{pmatrix} v_0 & 0 & 0 & 0 & 0 \\ 0 & v_0 & 0 & 0 & \frac{1}{\rho_0} \\ 0 & 0 & v_0 & 0 & 0 \\ 0 & 0 & 0 & v_0 & 0 \\ 0 & \frac{1}{\rho_0} & 0 & 0 & 0 \end{pmatrix} \ , \quad B'_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\rho_0} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\rho_0} & 0 & 0 \end{pmatrix} \ ,$$

$$C'_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{g}{\rho_0} & 0 \\ 0 & 0 & \frac{\partial_{\bar{x}_3}\rho_0}{R} - \frac{\rho_0 \partial_{\bar{x}_3} \log(P_0)}{\gamma R} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \ , \quad C'_2 = \begin{pmatrix} 0 & 0 & \partial_{\bar{x}_3} u_0 & 0 & 0 \\ 0 & 0 & \partial_{\bar{x}_3} v_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial_{\bar{x}_3} \log(P_0)}{\gamma \rho_0} & 0 & 0 \end{pmatrix} \ , \qquad (3)$$

and define

$$L_1 \quad = \quad B'_1 \partial_{\bar{x}_1} + B'_2 \partial_{\bar{x}_2} + B'_3 \partial_{\bar{x}_3} \ .$$

Consider the system

$$\left(\frac{1}{R} L_1 + C'_1 + \frac{1}{R} C'_2\right) \begin{bmatrix} \vec{u}' \\ \rho' \\ P' \end{bmatrix} \quad = \quad 0 \qquad (4)$$

and solutions to it of the form,

$$\begin{bmatrix} \vec{u}' \\ \rho' \\ P' \end{bmatrix} = e^{iR\phi} \left(\begin{bmatrix} \vec{u}_1 \\ \rho_1 \\ P_1 \end{bmatrix} + \frac{1}{R} \begin{bmatrix} \vec{u}_2 \\ \rho_2 \\ P_2 \end{bmatrix} + \dots \right) \ , \qquad (5)$$

called the geometric optics ansatz. Note that solutions of this form will satisfy the linearized time dependent system,

$$\left(A_0 \frac{\partial}{\partial t} + B_1 \frac{\partial}{\partial x_1} + B_2 \frac{\partial}{\partial x_2} + B_3 \frac{\partial}{\partial x_3} + C \right) \begin{bmatrix} \vec{u}' \\ \rho' \\ P' \end{bmatrix} \quad = \quad F' \ ,$$

for some F' which is on the order of $1/P_0 + 1/R^2$. Now, using the results of appendix section A.(b), we can find a unique solution to (1), $(\vec{u}, \rho, P)$, with initial and boundary data given by

$$\begin{bmatrix} u \\ v \\ w \\ \rho \\ P \end{bmatrix}_{t=0} = \begin{bmatrix} u' \\ v' \\ w' + H_e \\ \rho' \\ P' \end{bmatrix}_{t=0} \qquad \text{and} \qquad M \begin{bmatrix} u \\ v \\ w \\ \rho \\ P \end{bmatrix}_{x_3=0} = \begin{bmatrix} 0 \\ 0 \\ \vec{u}_0 \cdot \nabla_{x'} h \\ 0 \\ 0 \end{bmatrix} \ ,$$

where H_e is an extension of $\vec{u}_0 \cdot \nabla_{x'} h - w'|_{x_3=0}$ in the x_3 direction. Taking the difference between this time dependent solution and an asymptotic solution of (4) and using the bounds and notations of appendix section A.(b), we have

$$\left|\!\left|\!\left| \begin{bmatrix} \vec{u} \\ \rho \\ P \end{bmatrix} - \begin{bmatrix} \vec{u}' \\ \rho' \\ P' \end{bmatrix} \right|\!\right|\!\right| \quad \leq \quad C_T \left(|\!|\!| F' |\!|\!| + \left\| w'|_{x_3=0} - \vec{u}_0 \cdot \nabla_{x'} h \right\|_{H^1} \right) \ .$$

Hence, if $(\vec{u}', \rho', P')$ is an approximate solution that approximates the boundary data well, then it will be close to a true solution of (1) in the sense of the inequality above.

We proceed by constructing an approximate solution of (4). Substituting the ansatz (5) into the system of equations (4), we see that to top order

$$(iL_1\phi + C_1') \begin{bmatrix} \vec{u}_1 \\ \rho_1 \\ P_1 \end{bmatrix} = 0 \ ,$$

where $L_1\phi$ denotes the matrix $\phi_{\bar{x}_1}B_1 + \phi_{\bar{x}_2}B_2 + \phi_{\bar{x}_3}B_3$. In order for the above equation to hold, we need to choose ϕ, so that the determinant of the matrix $(iL_1\phi + C_1')$ vanishes and we need to choose $(\vec{u}_1, \rho_1, P_1)$ to belong to the null space of this matrix. Note that this immediately shows us why we need to keep some terms of order $1/R$ in C_1'. If we were to omit them, then the third equation would imply that $\vec{u}_0 \cdot \nabla\phi = 0$. This is a rather strong requirement on the phase function and it will not permit us to approximate the mountain profile well. Thus, we assume that $\vec{u}_0 \cdot \nabla\phi \neq 0$ and compute,

$$\det\left[\begin{pmatrix} i\vec{u}_0\cdot\nabla\phi & 0 & 0 & 0 & i\frac{\phi_{\bar{x}_1}}{\rho_0} \\ 0 & i\vec{u}_0\cdot\nabla\phi & 0 & 0 & i\frac{\phi_{\bar{x}_2}}{\rho_0} \\ 0 & 0 & i\vec{u}_0\cdot\nabla\phi & \frac{g}{\rho_0} & i\frac{\phi_{\bar{x}_3}}{\rho_0} \\ 0 & 0 & \partial_{x_3}\rho_0 + \frac{g\rho_0^2}{\gamma P_0} & i\vec{u}_0\cdot\nabla\phi & 0 \\ i\frac{\phi_{\bar{x}_1}}{\rho_0} & i\frac{\phi_{\bar{x}_2}}{\rho_0} & i\frac{\phi_{\bar{x}_3}}{\rho_0} & 0 & 0 \end{pmatrix}\right]$$
$$= \frac{i}{\rho_0^4}(\vec{u}_0 \cdot \nabla\phi)\left[-g\rho_0\left(\partial_{x_3}\rho_0 + \frac{g\rho_0^2}{\gamma P_0}\right)|\nabla'\phi|^2 - \rho_0^2(\vec{u}_0\cdot\nabla\phi)^2|\nabla\phi|^2\right] \ .$$

Defining the buoyancy frequency $N^2(x_3) = g\partial_{x_3}(\log P_0^{1/\gamma}\rho_0^{-1})$ and using $\partial_{x_3}P_0 = -g\rho_0$, for the determinant to vanish we need

$$\frac{i}{\rho_0^2}(\vec{u}_0\cdot\nabla\phi)\left[N^2|\nabla'\phi|^2 - (\vec{u}_0\cdot\nabla\phi)^2|\nabla\phi|^2\right] = 0 \ .$$

Thus ϕ must satisfy the eikonal type equation,

$$|D\phi||\nabla\phi| - N|\nabla'\phi| = 0 \ , \tag{6}$$

for $D = (u_0\partial_{\bar{x}_1} + v_0\partial_{\bar{x}_2})$. Provided that $\nabla'\phi \neq 0$ and the relation above holds, a simple calculation verifies that the null space of $(iL_1\phi + C_1')$ is 1 dimensional and that it is spanned by

$$\vec{a}_1 = \begin{bmatrix} -\phi_{\bar{x}_3}\phi_{\bar{x}_1} \\ -\phi_{\bar{x}_3}\phi_{\bar{x}_2} \\ |\nabla'\phi|^2 \\ -\frac{i(D\phi)\rho_0|\nabla\phi|^2}{g} \\ (D\phi)\phi_{\bar{x}_3}\rho_0 \end{bmatrix} \ .$$

Since the null space is 1 dimensional, $(u_1, v_1, w_1, \rho_1, P_1) = b_1\vec{a}_1$, for some scalar function $b_1(\bar{x})$.

The next order terms give the equation

$$\left(iL_1\phi + C_1'\right)\begin{bmatrix} \vec{u}_2 \\ \rho_2 \\ P_2 \end{bmatrix} = -\left(L_1 b_1 + b_1 C_2'\right)\vec{a}_1 - b_1(L_1\vec{a}_1) \ . \tag{7}$$

This equation gives us an equation for b_1 in a rather backward fashion. In order for us to be able to solve for $(\vec{u}_2, \rho_2, P_2)$, the rest of the terms must add up to a vector in the range of $(iL_1\phi + C_1')$. One can easily compute that the cokernel of $(iL_1\phi + C_1')$ is spanned by

$$\begin{bmatrix} -\phi_{\bar{x}_3}\phi_{\bar{x}_1} \\ -\phi_{\bar{x}_3}\phi_{\bar{x}_2} \\ |\nabla'\phi|^2 \\ \frac{ig|\nabla'\phi|^2}{(D\phi)\rho_0} \\ (D\phi)\phi_{\bar{x}_3}\rho_0 \end{bmatrix} .$$

Taking the inner product between this vector and (7), we obtain an equation for b_1. Thus, if we solve the resulting equation for b_1 and equation (6) for ϕ, then

$$e^{iR\phi(\bar{x})} b_1(\bar{x}) \vec{a}_1(\bar{x})$$

would give us an approximate solution of (4).

(*c*) *Gaussian Beam Solution*

In the traditional geometric optics approach, one would try to solve equation (6) in all of space for a real valued phase function ϕ. However, in the spirit of Gaussian beams, we will look for a complex valued function ϕ which satisfies this equation to high order on only one curve through space. The imaginary part will be chosen in such a way so that (5) is an approximate solution to (4). For details and a proof that such a construction is possible please see section A.(a).

We proceed by looking at the null bicharacteristics for the eikonal equation (6). Let the bicharacteristics be given by $(\mathcal{X}(s), \xi(s))$ and define

- $\hat{D} = (\xi_1 u_0 + \xi_2 v_0)$,
- $\hat{D}' = (\xi_1 \partial_{\bar{x}_3} u_0 + \xi_2 \partial_{\bar{x}_3} v_0)$,
- $\hat{D}_0 = (u_0(0)\eta_1 + v_0(0)\eta_2)$ and
- $p(\mathcal{X}, \xi) = |\hat{D}||\xi| - N|\xi'|$.

The null bicharacteristics satisfy

$$\dot{\mathcal{X}} = \frac{\partial p(\mathcal{X},\xi)}{\partial \xi} = \begin{bmatrix} \frac{\hat{D}\xi_1}{|\xi|} + u_0|\xi| - \frac{\hat{D}\xi_1|\xi|}{|\xi'|^2} \\ \frac{\hat{D}\xi_2}{|\xi|} + v_0|\xi| - \frac{\hat{D}\xi_2|\xi|}{|\xi'|^2} \\ \frac{\hat{D}\xi_3}{|\xi|} \end{bmatrix} \quad \mathcal{X}(0) = y = \begin{bmatrix} y_1 \\ y_2 \\ 0 \end{bmatrix}$$

$$\dot{\xi} = -\frac{\partial p(\mathcal{X},\xi)}{\partial \mathcal{X}} = \begin{bmatrix} 0 \\ 0 \\ -\hat{D}'|\xi| + \frac{\hat{D}|\xi'|\partial_{\bar{x}_3} N}{|\hat{D}|} \end{bmatrix} \quad \xi(0) = \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \frac{|\eta'|\sqrt{N(0)^2 - |\hat{D}_0|^2}}{\hat{D}_0} \end{bmatrix}$$

where $\dot{}$ signifies differentiation with respect to s. The sign of $\xi_3(0)$ is chosen to give upward propagating waves, and the parameter s is chosen so that it increases as $\mathcal{X}_3$ increases. Note that not all choices for η' will lead to a real valued $\xi_3(0)$. Thus, we are implicitly assuming that η' is chosen in such a way so that $N(0) > |\hat{D}_0| > 0$.

The phase function, ϕ, satisfies

$$\dot{\phi} = \hat{D}|\xi| \quad \phi(0) = y_1\eta_1 + y_2\eta_2 .$$

Similarly the Hessian of ϕ, $H = QY^{-1}$, evolves according to

$$\dot{Y} = CQ + BY \quad \dot{Q} = -B^{\mathsf{T}}Q - AY$$
$$Y(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad Q(0) = \begin{pmatrix} i\epsilon & 0 & * \\ 0 & i\epsilon & * \\ * & * & * \end{pmatrix} ,$$

with ϵ a parameter controlling the width of the beam and

$$(A)_{j,k} = \frac{\partial^2 p_m}{\partial x_j \partial x_k} \qquad (B)_{j,k} = \frac{\partial^2 p_m}{\partial \xi_j \partial x_k} \qquad (C)_{j,k} = \frac{\partial^2 p_m}{\partial \xi_j \partial \xi_k} \qquad (H)_{j,k} = \frac{\partial^2 \phi}{\partial x_j \partial x_k} \ .$$

The $*$'s in the initial condition for Q are determined by the requirement that

$$Q(0) = Q(0)^{\mathsf{T}} \quad \text{and} \quad Q(0)\dot{\mathcal{X}}(0) = \dot{\xi} \ .$$

Finally, we have the transport equation

$$\begin{aligned} \dot{b}_1 \;\; = \;\; - & \Bigg[\left(\frac{|\xi|^2 (u_0^2 H_{11} + 2u_0 v_0 H_{12} + v_0^2 H_{22})}{2\hat{D}|\xi|} \right) \\ + & \left(\frac{2u_0(H_{11}\xi_1 + H_{12}\xi_2 + H_{13}\xi_3) + 2v_0(H_{12}\xi_1 + H_{22}\xi_2 + H_{23}\xi_3)}{|\xi|} \right) \\ + & \left(\frac{\hat{D}^2(H_{11}+H_{22}+H_{33}) - N^2(H_{11}+H_{22})}{2\hat{D}|\xi|} \right) \\ + & \left(\frac{\hat{D}\partial_{\bar{x}_3}(\log \rho_0 P_0^{1/\gamma})\xi_3}{2|\xi|} \right) \Bigg] \, b_1 \end{aligned}$$

with some initial data $b_1(0;\eta')$.

We let O be a tubular neighborhood of the x-projection of the bicharacteristic and define the phase and amplitude functions though a Taylor expansion in O. With $\phi = \phi(s(\mathcal{X}))$, $\xi = \xi(s(\mathcal{X}))$, $b_1 = b_1(s(\mathcal{X}))$ and so on, we write the Gaussian beam solution as

$$\psi_y(\bar{x}) b_1 \begin{bmatrix} -\xi_1\xi_3 \\ -\xi_2\xi_3 \\ |\xi'|^2 \\ -\frac{i\hat{D}\rho_0|\xi|^2}{g} \\ \hat{D}\xi_3\rho_0 \end{bmatrix} e^{iR\left(\phi + \xi\cdot(\bar{x}-\mathcal{X}) + \frac{i}{2}(\bar{x}-\mathcal{X})\cdot H(\bar{x}-\mathcal{X})\right)} \ , \tag{8}$$

where ψ_y is a C^∞-function supported in O and equal to 1 in a neighborhood of $\mathcal{X}$. Looking at $\bar{x}_3 = 0$, which corresponds to $s = 0$, we see that the Gaussian beam is given by

$$\psi_y(\bar{x}') b_1(0;\eta') \begin{bmatrix} -\eta_1\eta_3 \\ -\eta_2\eta_3 \\ |\eta'|^2 \\ -\frac{i\hat{D}_0\rho_0|\eta|^2}{g} \\ \hat{D}_0\eta_3\rho_0 \end{bmatrix} e^{iR\left(x'\cdot\eta' + \frac{i\epsilon}{2}|\bar{x}'-y'|^2\right)}$$

at the boundary. This is not of the form of the boundary condition (2); however, as we will see in the next section, this solution can be used as a building block to build solutions that do satisfy the boundary condition.

(*d*) *Boundary Conditions and Superposition of Gaussian Beams*

As described in the previous section, for given (y', η'), provided that $N(0) > |\hat{D}_0| > 0$, we can construct a Gaussian beam of the form (8) with the initial condition for b_1 given by

$$b_1(0;\eta') = \frac{1}{|\eta'|^2} \mathcal{F}\left[\vec{u}_0 \cdot \nabla_{x'} h\right] \ , \tag{9}$$

where $\mathcal{F}$ denotes the Fourier transform from x' to η':

$$\vec{u}_{GB}(\bar{x}; y, \eta) = \psi_y(\bar{x}) b_1(\eta') \begin{bmatrix} -\xi_1\xi_3 \\ -\xi_2\xi_3 \\ |\xi'|^2 \\ -\frac{i\hat{D}\rho_0|\xi|^2}{g} \\ \hat{D}\xi_3\rho_0 \end{bmatrix} e^{iR\left(\phi + \xi\cdot(\bar{x}-\mathcal{X}) + \frac{i}{2}(\bar{x}-\mathcal{X})\cdot H(\bar{x}-\mathcal{X})\right)} \ .$$

We claim that the Gaussian beam superposition,

$$\frac{R\epsilon}{2\pi^2}\int_{\mathbb{R}^2}\int_{N(0)>|\hat{D}_0|>0} u_{GB}(\bar{x};y',\eta')\ d\eta' dy' \tag{10}$$

is a solution to (4). Certainly, differentiation in $\bar{x}$ commutes with these integrals, so this expression is an asymptotic solution to (4). For the boundary condition, we need to only look at vertical velocity since all other quantities are in the null space of the boundary condition matrix M. Now, evaluating the Gaussian beam superposition at $\bar{x}_3=0$ and integrating we have for the vertical velocity

$$\begin{aligned}&\frac{R\epsilon}{2\pi^2}\int_{\mathbb{R}^2}\int_{N(0)>|\hat{D}_0|>0}\psi_y(\bar{x}')|\eta'|^2 b_1(0;\eta')e^{iR\left(\eta'\cdot\bar{x}'+\frac{i\epsilon}{2}|\bar{x}'-y'|^2\right)}d\eta' dy'\\ &=\frac{C}{2\pi}\int_{N(0)>|\hat{D}_0|>0}\mathcal{F}\left[\vec{u}_0\cdot\nabla_{x'}h\right]e^{i\eta'\cdot x'}d\eta'\ ,\end{aligned}$$

where C is a constant which approaches 1 as $R\epsilon\to\infty$ and depends on the support of ψ_y. How well the superposition (10) satisfies the boundary condition (2) depends on how much of the support of the Fourier transform of $\vec{u}_0\cdot\nabla_{x'}h(x')$ is contained inside $N(0)>|\hat{D}_0|$.

We make a further remark that will be useful in numerical simulations. If instead of (9), we let the initial condition for b_1 be given by

$$b_1(0;y',\eta')=\frac{\lambda(y')}{|\eta'|^2}\mathcal{F}\left[\vec{u}_0\cdot\nabla_{x'}h\right]\ ,$$

for some smooth compactly supported function λ, then the vertical velocity would be approximately given by

$$\frac{\lambda(\bar{x}')}{2\pi}\int_{N(0)>|\hat{D}_0|>0}\mathcal{F}\left[\vec{u}_0\cdot\nabla_{x'}h\right]e^{i\eta'\cdot x'}d\eta'\ ,$$

as $R\epsilon\to\infty$. To show this, we let

$$Q(\bar{x}')\equiv\frac{1}{2\pi}\int_{N(0)>|\hat{D}_0|>0}\mathcal{F}\left[\vec{u}_0\cdot\nabla_{x'}h\right]e^{iR\eta'\cdot\bar{x}'}d\eta'$$

and compute for a Gaussian beam superposition with the modified b_1 at $\bar{x}_3=0$:

$$\begin{aligned}&\left|\frac{R\epsilon}{2\pi^2}\int_{\mathbb{R}^2}\int_{N(0)>|\hat{D}_0|>0}u_{GB}|_{\bar{x}_3=0}\ d\eta' dy'-\lambda(\bar{x}')Q(\bar{x}')\right|\\ \le\ &\left|\frac{R\epsilon}{\pi}\int_{\mathbb{R}^2}\psi_y(\bar{x}')\lambda(y')e^{-\frac{R\epsilon}{2}|\bar{x}'-y'|^2}Q(\bar{x}')dy'-\lambda(\bar{x}')Q(\bar{x}')\right|\\ \le\ &\left|\frac{R\epsilon}{\pi}\int_{\mathbb{R}^2}\left[\psi_y(\bar{x}')\lambda(y')-\lambda(\bar{x}')\right]e^{-\frac{R\epsilon}{2}|\bar{x}'-y'|^2}Q(\bar{x}')dy'\right|\\ \le\ &\frac{1}{\pi}\left|Q(\bar{x}')\right|\int_{\mathbb{R}^2}\left|\psi_{\bar{x}+z/\sqrt{R\epsilon}}(\bar{x}')\lambda(\bar{x}'+z'/\sqrt{R\epsilon})-\lambda(\bar{x}')\right|e^{-\frac{1}{2}|z'|^2}dz'\ .\end{aligned}$$

Now, expanding $\lambda(\bar{x}'+z'/\sqrt{R\epsilon})$ about $\bar{x}'$ in a Taylor series, we see that the expression that multiplies $|Q(\bar{x}')|$ is bounded from above by

$$\frac{1}{\pi}\int_{\mathbb{R}^2}\left[\left|\left(\psi_{\bar{x}+z'/\sqrt{R\epsilon}}(\bar{x}')-1\right)\lambda(\bar{x}')\right|+\left|\frac{1}{\sqrt{R\epsilon}}z'\cdot\nabla_{\bar{x}'}\lambda(\bar{x}')\right|\right]e^{-\frac{1}{2}|z'|^2}dz'\ ,$$

which goes to 0 as $R\epsilon\to\infty$. A similar argument shows that such a convergence also holds for the L^2 and H^1 norms.

3. Numerical Results

In order to compare the numerical results obtained through the method of superposition of Gaussian beams to the numerical results obtained by other authors (see Shutts 1998 and Broutman *et al.* 2002), we need to make the same assumptions, use the same mountain profile and background flow. The basic approximations that these authors make are the hydrostatic, Boussinesq, and incompressible approximations. They also assume that the buoyancy frequency is constant. In essence this changes the approximate system (4) by deleting some of the entries in the coefficient matrices which are presumably small:

$$B_1'' = \begin{pmatrix} u_0 & 0 & 0 & 0 & \frac{1}{\rho_0} \\ 0 & u_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_0 & 0 \\ \frac{1}{\rho_0} & 0 & 0 & 0 & 0 \end{pmatrix} \;, \quad B_2'' = \begin{pmatrix} v_0 & 0 & 0 & 0 & 0 \\ 0 & v_0 & 0 & 0 & \frac{1}{\rho_0} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_0 & 0 \\ 0 & \frac{1}{\rho_0} & 0 & 0 & 0 \end{pmatrix} \;, \quad B_3'' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\rho_0} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\rho_0} & 0 & 0 \end{pmatrix} \;,$$

$$C_1'' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{g}{\rho_0} & 0 \\ 0 & 0 & \frac{\partial_{\bar{x}_3}\rho_0}{R} - \frac{\rho_0 \partial_{\bar{x}_3} \log(P_0)}{\gamma R} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \;, \quad C_2'' = \begin{pmatrix} 0 & 0 & \partial_{\bar{x}_3} u_0 & 0 & 0 \\ 0 & 0 & \partial_{\bar{x}_3} v_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \;.$$

Note that the results of Theorem A.4 apply to this reduced system as well.

Carrying out the same analysis as we did in section 2.(b), we find that the eikonal equation in this case is

$$|D\phi||\phi_{\bar{x}_3}| - N|\nabla'\phi| = 0 \;,$$

the null space of $(iL_1'\phi + C_1'')$ is spanned by

$$\vec{a}_1 = \begin{bmatrix} -\phi_{\bar{x}_3}\phi_{\bar{x}_1} \\ -\phi_{\bar{x}_3}\phi_{\bar{x}_2} \\ |\nabla'\phi|^2 \\ -\frac{i(D\phi)\rho_0|\phi_{\bar{x}_3}|^2}{g} \\ (D\phi)\phi_{\bar{x}_3}\rho_0 \end{bmatrix} \;,$$

and the transport equation for b_1 is

$$\dot{b}_1 = -\left(\frac{\phi_{\bar{x}_3}^2 D^2\phi + 4\phi_{\bar{x}_3} D\phi D\phi_{\bar{x}_3} + \phi_{\bar{x}_3\bar{x}_3}(D\phi)^2 - N^2 \triangle'\phi}{2D\phi|\phi_{\bar{x}_3}|} \right) b_1 \;.$$

We now look at two cases with different background wind flows. We carry out the calculations for both the reduced system of equations above (examples 1a and 1b) and the system derived in section 2.(b) (examples 2a and 2b). In all cases, the transport equation is initialized with

$$b_1(0; y', \eta') \;=\; \lambda(y')|\eta'|^2 \mathcal{F}\left[\vec{u}_0 \cdot \nabla_{x'} h(x')\right]$$

where the mountain profile, h, is of the form

$$h = \frac{h_0}{\left(1 + \frac{|x'|^2}{a^2}\right)^{3/2}}$$

with $h_0 = 100m$ and $a = 20km$ and the cut-off function $\lambda(y')$ is given by

$$\lambda(y') \;=\; \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{|y'| - 80}{2}\right) \;.$$

This makes the initial condition for b_1 compactly supported in y' and thus the integration in y' much simpler. As described in section 2.(d), how close the Gaussian beam superposition is to a true time dependent solution depends on how well the boundary data is approximated. Since the Fourier transform of this particular choice for h is largely contained inside $N(0) > |\hat{D}_0|$ and the cut-off function λ is such that $\lambda(x')h(x') \approx h(x')$, the Gaussian beam superposition will approximate the boundary data well.

For the full system as derived in section 2.(b) we need the background pressure and density. For the pressure profile we use

$$P_0(x_3) = c_1(c_2 - \delta x_3)^r \ ,$$

with $(c_1, c_2, \delta, r) = (101290, 1.0002, 2.2528 \times 10^{-5}, 5.256)$. This model for the pressure is valid in the troposphere (U.S. Standard Atmosphere, 1976). The density is obtained from the pressure using the hydrostatic relationship $\partial_{x_3} P_0 = -g\rho_0$.

To compare with the results obtained by Shutts (1998), we also compute the vertical displacement quantity, η, defined implicitly through the equation,

$$\vec{u}_0 \cdot \nabla_{x'} \eta \quad = \quad w \ .$$

The computations were carried out using Matlab and more specifically its “ODE23” function was used to solve the system of ODEs.

(a) *Example 1a*

We consider a background wind profile of the form,

$$\vec{u}_0 = U_0[\cos(\pi x_3/H), \sin(\pi x_3/H)] \ ,$$

and use the reduced system of equations to compute the stationary wave field. Figure 1 shows numerical results obtained for a particular case with $U_0 = 10m/s$, $H = 12km$, $N = 0.0113s^{-1}$, $R = 1000$, and $\epsilon = 5 \times 10^{-6}$. Each of the computed quantities were multiplied by $\exp(x_3/2H_\rho)$, $H_\rho = 7.5km$, as in Broutman *et al.* (2002), to correct for the decrease in density with height. The results obtained are almost identical to the results obtained by Broutman *et al.* (2002); see their Figure 3.

(b) *Example 1b*

For this example we also use the reduced system of equations but instead we use a background wind profile given by

$$\vec{u}_0 = [U_0, \Lambda x_3] \ ,$$

and compare with the results obtained by Shutts (1998). The specific values for the wind are $U_0 = 10m/s$ and $\Lambda = 3 \times 10^{-3}s^{-1}$. The buoyancy frequency is $N = 0.01s^{-1}$. The results are shown in Figure 2. Qualitatively, the results are similar to the results shown by Shutts (1998) in his Figure 2 and the Gaussian beam superposition minimum $(-0.0817h_0)$ and maximum $(0.186h_0)$ compare well with those obtained by Shutts, $(-0.085h_0)$ and $(0.213h_0)$.

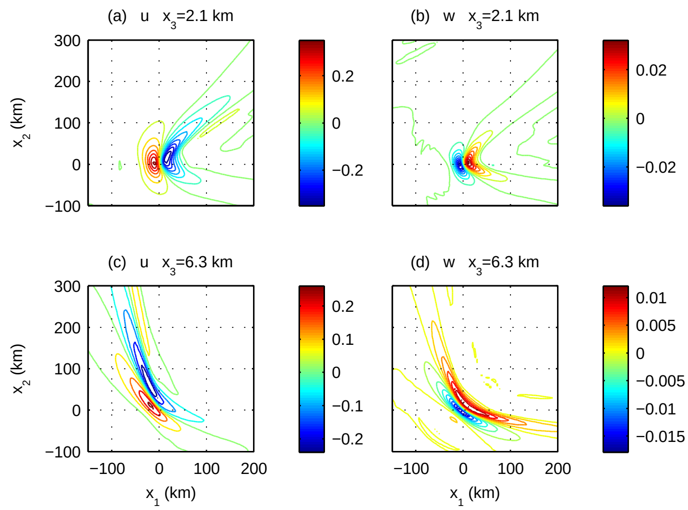


Figure 1. Results obtained using a superposition of Gaussian beams. The contour intervals (CI) and the minimum and maximum values for the plotted quantities are (a) CI=0.05, $(-0.39, 0.36)$, (b) CI=0.004, $(-0.034, 0.034)$, (c) CI=0.05, $(-0.23, 0.22)$, and (d) CI=0.002, $(-0.019, 0.012)$, which are all in good agreement with minimum and maximum values obtained by Broutman *et al.* (2002).

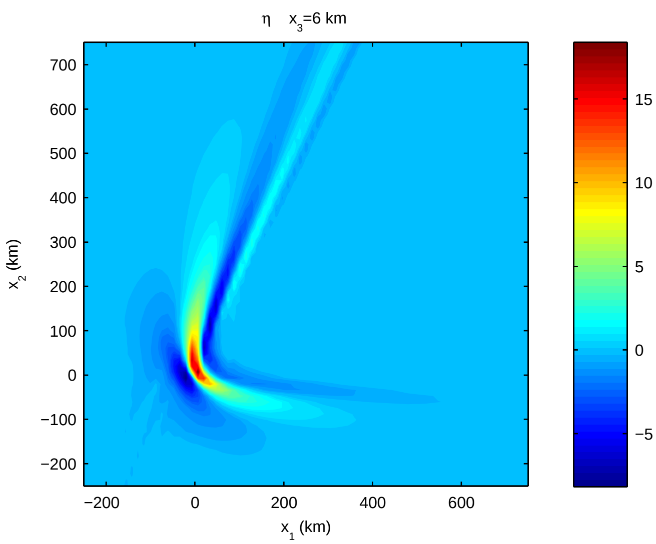


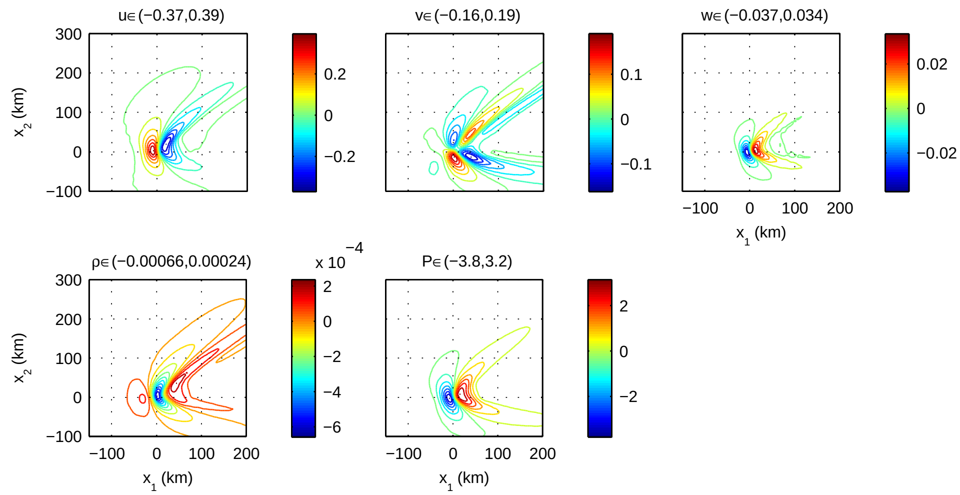
Figure 2. Results obtained using a superposition of Gaussian beams. The minimum and maximum values for η are -0.0817 and 0.186 times the mountain height h_0.

(*c*) *Example 2a*

In this example, we look at the same wind profile as in example 1a, but instead of the reduced equations we use the system derived in section 2.(*b*).

The results obtained are shown in Figure 3. The first column of the figure shows the perturbation quantities (u, v, w, ρ, P) at $x_3 = 2.1km$ and the second

a) Perturbation quantities at $x_3 = 2.1km$



b) Perturbation quantities at $x_3 = 6.3km$

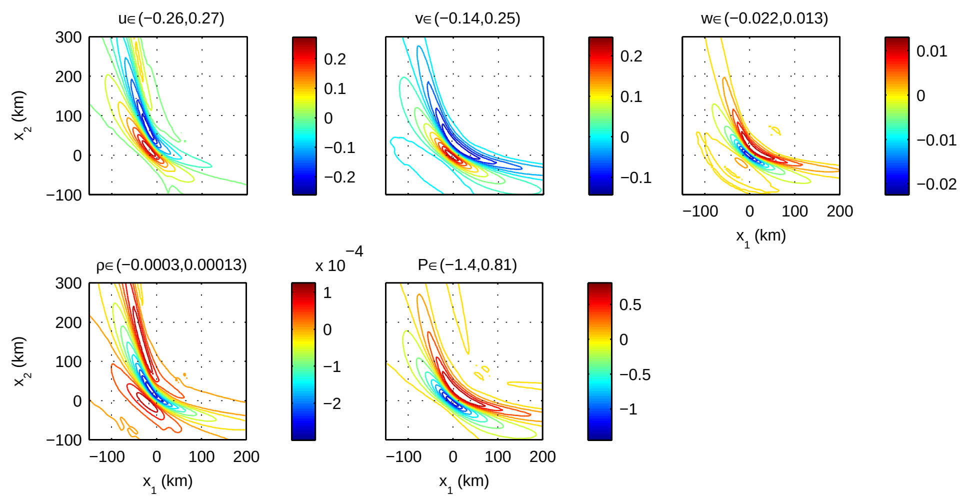


Figure 3. Perturbation quantities computed using a superposition of Gaussian Beams for the system of equations derived in section 2.(b). Each graph has 15 equally spaced contour intervals from the minimum to the maximum values, which are listed at the top of each graph.

column at $x_3 = 6.3km$. Qualitatively the wave field is similar to the field obtained through the reduced system.

(d) *Example 2b*

In this example, we look at the same wind profile as in example 1b, but instead of the reduced equations we use the system derived in section 2.(b). The results are shown in Figure 4. The minimum and maximum values are higher than those computed in example 1b, however, this is expected as in example 1b no corrections were made for the decrease in density with height. If we multiply each of the quantities by the same exponential growth factor as in example 1a, then the two sets of minimums and maximums are in good agreement.

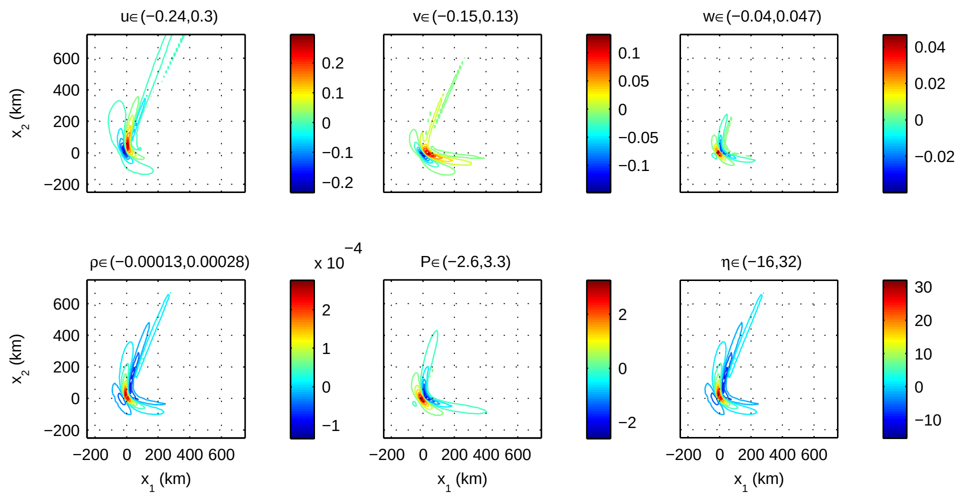


Figure 4. Perturbation quantities computed at $x_3 = 6.0km$ using a superposition of Gaussian beams for the system of equations derived in section 2.(b). Each graph has 15 equally spaced contour intervals from the minimum to the maximum values, which are listed at the top of each graph.

4. Conclusion

We have shown that the method of superposition of Gaussian beams can be used to model the perturbations of the air velocity, density, and pressure that result from steady air flow over topography. The stationary in time solution obtained through this method is globally valid and is close to a true time dependent solution of the linearized system of equations that governs mountains waves. This method has the advantage over previous methods that it provides a global solution without the need to identify and correct for caustics. As mentioned by other authors, this correction is not easily done in numerical simulations. The numerical results obtained by superposition of Gaussian beams are in good agreement with the results obtained in previous studies by other authors.

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A. Appendix

(a) *Construction of Gaussian Beams*

This section is very closely based on section 2.1 in Ralston 1982. For the sake of completeness, however, we review the construction of Gaussian beams here.

Consider the PDE,

$$P(x_0, x', \frac{\partial}{\partial x_0}, \nabla_{x'})u = 0 \qquad \text{for} \qquad x = (x_0, x') \in \mathbb{R} \times \mathbb{R}^n \tag{A.1}$$

where P is a partial differential operator of order m. We would like to construct an asymptotically valid solution, $u(x;k)$, that is concentrated on a single smooth curve, γ. That is to say, $u(x;k)$ is small away from γ and $\|Pu(x;k)\| = O(k^{-M})$, for some appropriate norm and fixed integer M. Of course, this is not possible for all partial differential operators. The restrictions that are necessary for such a solution to exist will become clear in what follows.

The construction of Gaussian beams begins with the geometric optics ansatz,

$$u(x) \quad = \quad e^{ik\phi(x)}\Big[a_0(x) + \frac{a_1(x)}{k} + \ldots + \frac{a_N(x)}{k^N}\Big]. \tag{A.2}$$

The functions $a_0, \ldots, a_N$ and ϕ are all assumed to be smooth, k is the large asymptotic parameter and N is a fixed number. The requirements on the phase function ϕ are slightly different from those of traditional geometric optics. We will require that ϕ is real valued on γ, but away from this curve, ϕ can be complex valued with the restriction that the imaginary part of the Hessian of ϕ is positive definite on planes perpendicular to γ. This will make u look like a Gaussian distribution with variance $1/k$ on planes perpendicular to γ and hence gives the name to these asymptotic solutions.

The first step is to substitute the ansatz into the PDE:

$$Pu \quad = \quad k^m p_m(x, \nabla\phi) a_0(x) e^{ik\phi(x)} + O(k^{m-1}) \ .$$

Here, $p_m(x,\xi)$ is the principal symbol of P and is assumed to be real valued. Recall that this allows us to write P as $p_m(x, i\nabla)$ plus derivatives of order lower than m. We must choose ϕ in such a way so that the contribution of the first term is order less than k^m. The idea of the construction is to choose the Taylor series of ϕ on γ in such a way that $p_m(x,\nabla\phi)$ vanishes to high order on γ. In combination with the decay from the imaginary part of ϕ, we will make the contribution of first term small. To accomplish this, we must have p_m and a number of its derivatives vanishing on γ. We require that

$$p_m = 0, \quad \tfrac{\partial p_m}{\partial x_i} = 0, \quad \tfrac{\partial^2 p_m}{\partial x_i \partial x_j} = 0, \quad \ldots$$

on the curve γ. We let $x(s)$ parametrize this curve and define $\xi(s) = \nabla\phi(x(s))$. Using the summation convention and expanding the first two equations,

$$p_m \quad = \quad 0, \tag{A.3}$$

$$\frac{\partial p_m}{\partial x_i} + \frac{\partial p_m}{\partial \xi_l}\frac{\partial^2 \phi}{\partial x_i \partial x_l} \quad = \quad 0, \tag{A.4}$$

we can make the following conclusion about the curve $x(s)$:

Theorem A.1: *The construction of an asymptotic solution of the form* (A.2) *with the restrictions*

(i) $x(s)$ *is a smooth curve,*
(ii) ϕ *is real on* $x(s)$*,*
(iii) $\mathrm{Im}\left[\frac{\partial^2\phi}{\partial x_i \partial x_j}\right]$ *is positive definite on* $\dot{x}^{\perp}(s)$*,*

is only possible if (up to a rescaling of s)

a) $\dot{x}(s) = \frac{\partial p_m}{\partial \xi}(x(s),\xi(s)) \qquad \dot{\xi}(s) = -\frac{\partial p_m}{\partial x}(x(s),\xi(s)),$

b) $p_m(x(0), \xi(0)) = 0$,
c) $\frac{\partial p_m}{\partial \xi} \neq 0$.

The curve $(x(s), \xi(s))$ in phase space is referred to as a null bicharacteristic of the operator P.

Proof Examining equation (A.4) and noting that p_m is real, we have

$$\frac{\partial p_m}{\partial \xi_l} \mathrm{Im}\left[\frac{\partial^2 \phi}{\partial x_i \partial x_l}\right] = 0.$$

Thus $\frac{\partial p_m}{\partial \xi}$ is in the null space of $\mathrm{Im}\left[\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right]$ and hence $\frac{\partial p_m}{\partial \xi}$ cannot belong to $\dot{x}^{\perp}$. Therefore, for some constant α we must have

$$\dot{x}(s) = \alpha \frac{\partial p_m}{\partial \xi}.$$

Substituting for $\dot{x}(s)$ in (A.4) and using the compatibility condition $\dot{\xi}_i(s) = \frac{\partial^2 \phi(x(s))}{\partial x_i \partial x_j} \dot{x}_j(s)$ we get,

$$\dot{\xi} = -\alpha \frac{\partial p_m}{\partial x}.$$

Finally, since $p_m = 0$ and $\dot{p}_m = 0$, the curve $(x(s), \xi(s))$ is a null bicharacteristic of the equation. Since we require $x(s)$ to be a smooth path, $\frac{\partial p_m}{\partial \xi} \neq 0$. ∎

Next we look at the second order derivative of p_m. We have to satisfy $\frac{\partial^2 p_m}{\partial x_i \partial x_j} = 0$ on $x(s)$:

$$\frac{\partial^2 p_m}{\partial x_i \partial x_j} + \frac{\partial^2 p_m}{\partial x_i \partial \xi_k}\frac{\partial^2 \phi}{\partial x_k \partial x_j} + \frac{\partial^2 p_m}{\partial x_j \partial \xi_k}\frac{\partial^2 \phi}{\partial x_i \partial x_k} + \\ + \frac{\partial^2 p_m}{\partial \xi_k \partial \xi_l}\frac{\partial^2 \phi}{\partial x_i \partial x_k}\frac{\partial^2 \phi}{\partial x_j \partial x_l} + \frac{\partial p_m}{\partial \xi_k}\frac{\partial^3 \phi}{\partial x_i \partial x_k \partial x_j} = 0.$$

Recognizing that $\frac{\partial p_m}{\partial \xi_k}\frac{\partial^3 \phi}{\partial x_i \partial x_k \partial x_j} = \frac{d}{ds}\frac{\partial^2 \phi}{\partial x_i \partial x_j}$ and defining

$$(A)_{i,j} = \frac{\partial^2 p_m}{\partial x_i \partial x_j} \qquad (B)_{i,j} = \frac{\partial^2 p_m}{\partial \xi_i \partial x_j} \qquad (C)_{i,j} = \frac{\partial^2 p_m}{\partial \xi_i \partial \xi_j} \qquad (H)_{i,j} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$$

the last equation becomes

$$A + B^{\mathsf{T}} H + HB + HCH + \dot{H} = 0. \tag{A.5}$$

This equation is known as a matrix Ricatti equation and, since it is non-linear, the solution might not exist for all s. In our case, however, the choice of initial data will guarantee that a solution exists for all s.

In order to satisfy the requirements that we want to impose on ϕ, the Hessian of ϕ must satisfy

i) $H(s) = H^{\mathsf{T}}(s)$,
ii) $H(s)\dot{x}(s) = \dot{\xi}(s)$,

iii) $\mathrm{Im}[H(s)]$ must be positive definite on $\dot{x}^{\perp}$.

The initial condition $H(0)$ must also satisfy these conditions and we choose it as follows. Without loss of generality, assume that $\dot{x}_0(0) = 1$. Choose $\mathrm{Im}[H(0)]_{i,j}$ for $1 \le i, j \le n$ to be a symmetric positive definite $n \times n$ matrix and choose the rest of $H(0)$ so that (i) and (ii) hold. This procedure determines all of $H(0)$ and since $\dot{x}_0(0) = 1$, all three conditions will be satisfied for $s = 0$. We obtain a global solution of (A.5) in the following way:

Theorem A.2: *Let $N(s)$ and $Y(s)$ be the (global) solutions of*

$$\dot{Y}(s) = CN + BY \quad \dot{N}(s) = -B^{\mathsf{T}}N - AY \tag{A.6}$$

with the initial conditions (I is the identity matrix) :

$$Y(0) = I \quad N(0) = H(0). \tag{A.7}$$

We obtain the following results:

a) $(\dot{x}(s), \dot{\xi}(s)) = (Y(s)\dot{x}(0), N(s)\dot{x}(0))$,
b) *$Y(s)$ is invertible for all s,*
c) *$H(s) = N(s)Y^{-1}(s)$ satisfies* (A.5) *and the conditions* i), ii) *and* iii) *above.*

Proof *a)* Differentiating $\dot{x}$ and $\dot{\xi}$ with respect to s we see that they satisfy (A.6). Noting that $Y(s)\dot{x}(0)$ and $N(s)\dot{x}(0)$ also satisfy (A.6) and that $\dot{x}(0) = Y(0)\dot{x}(0)$ and $\dot{\xi}(0) = N(0)\dot{x}(0)$, we have that $\dot{x}(s) = Y(s)\dot{x}(0)$ and $\dot{\xi}(s) = N(s)\dot{x}(0)$ for all s, by uniqueness for ODEs.

b) Let $\psi_1(s) = (y^1(s), \eta^1(s))$ and $\psi_2(s) = (y^2(s), \eta^2(s))$ be two vector valued solutions of (A.6). Define:

$$\sigma(\psi_1, \psi_2) = y^2 \cdot \eta^1 - y^1 \cdot \eta^2$$

By differentiating this expression in s and using the fact that $A = A^{\mathsf{T}}$ and $C = C^{\mathsf{T}}$, we see that σ is constant for solutions of (A.6).

Suppose that $Y(s)$ fails to be invertible for some $s = s_0$. Then there exists a non-zero $c \in \mathbb{C}^{n+1}$, such that $Y(s_0)c = 0$. Certainly, $\psi(s) = (Y(s)c, N(s)c)$ and $\overline{\psi(s)}$ are two vector valued solutions to (A.6) and so we have

$$\begin{aligned} 0 &= \overline{Y(s_0)c} \cdot N(s_0)c - Y(s_0)c \cdot \overline{N(s_0)c} = \sigma(\psi(s_0), \overline{\psi(s_0)}) \\ &= \sigma(\psi(0), \overline{\psi(0)}) = \overline{c} \cdot H(0)c - c \cdot \overline{H(0)c} \\ &= 2i\overline{c} \cdot \mathrm{Im}[H(0)]c \end{aligned}$$

Since $\mathrm{Im}[H(0)]$ is positive definite on $\dot{x}(0)^{\perp}$, $c = \alpha\dot{x}(0)$. By part (a), $\alpha\dot{x}(s_0) = \alpha Y(s_0)\dot{x}(0) = Y(s_0)c = 0$ and since $\dot{x}(s)$ is non vanishing, $\alpha = 0$. This is a contradiction, since c was assumed to be non-zero. Thus, $Y(s)$ is invertible for all s.

c) By substituting $N(s)Y^{-1}(s)$ into (A.5) instead of $H(s)$, one directly verifies that it satisfies the equation.

c.i) Since H^{T} also satisfies (A.5) and $H(0) = H^{\mathsf{T}}(0)$, it is clear that $H(s) = H^{\mathsf{T}}(s)$.

c.ii) From parts *a)* and *b)* we deduce that $\dot{x}(0) = Y^{-1}(s)\dot{x}(s)$ and so $\dot{\xi}(s) = N(s)Y^{-1}(s)\dot{x}(s) = H(s)\dot{x}(s)$.

c.iii) With the definitions in part *b*), for an arbitrary $c \in \mathbb{C}^{n+1}$ we have that

$$\begin{aligned} 2i\overline{c}\cdot \mathrm{Im}[H(0)]c &= \sigma(\psi(0),\overline{\psi(0)}) = \sigma(\psi(s),\overline{\psi(s)}) \\ &= \overline{Y(s)c}\cdot N(s)c - Y(s)c\cdot \overline{N(s)c} \\ &= \overline{Y(s)c}\cdot H(s)Y(s)c - Y(s)c\cdot \overline{H(s)Y(s)c} \\ &= 2i\overline{Y(s)c}\cdot \mathrm{Im}[H(s)]Y(s)c. \end{aligned}$$

Since $Y(s)c \in \dot{x}^{\perp}(s)$ implies $c \in \dot{x}^{\perp}(0)$, we have that $\mathrm{Im}[H(s)]$ is positive definite on $\dot{x}^{\perp}(s)$.

∎

This concludes the difficult part of the construction. For higher order derivatives of p_m on γ, we note that the equations that must be solved are linear first order inhomogeneous ordinary differential equations for the 3^{rd} and higher order derivatives of ϕ. The solutions to these equations will exist for all s and thus we can make p_m vanish to any prescribed order on γ.

We now look to Pu to determine the amplitude functions $a_0, \ldots, a_N$. Once again substituting the ansatz into the equation and collecting powers of k, we have:

$$Pu = \sum_{j=-m}^{N-m} k^{-j} c_j(x) e^{ik\phi(x)}, \tag{A.8}$$

where the c_j's are defined as

$$\begin{aligned} c_{-m} &= p_m(x,\nabla\phi)a_0 \\ c_{-m+1} &= \frac{1}{i}\frac{\partial p_m}{\partial \xi_l}\frac{\partial a_0}{\partial x_l} + \frac{1}{2i}\frac{\partial^2 p_m}{\partial \xi_l \partial \xi_j}\frac{\partial^2 \phi}{\partial x_l \partial x_j}a_0 + \\ &\quad + p_{m-1}(x,\nabla\phi)a_0 + p_m(x,\nabla\phi)a_1 \\ &\equiv La_0 + p_m(x,\nabla\phi)a_1 \\ c_{-m-r+1} &= La_r + p_m(x,\nabla\phi)a_{r+1} + g_r \end{aligned}$$

and g_r is a known function of $\phi, a_0, \ldots, a_{r-1}$. Just as in the case for ϕ, we will solve these equations for the amplitude functions so that the c_j's vanish on γ.

If $p_m(x,\nabla\phi)$ vanishes to order R on γ and we recursively solve

$$La_0 = 0, \qquad \frac{\partial}{\partial x_i}La_0 = 0, \qquad \ldots$$

up to derivatives of order $R-2$, c_{-m+1} will vanish to order $R-2$. Note that on γ,

$$La_r = \frac{1}{i}\frac{da_r}{ds} + \left[\frac{1}{2i}\frac{\partial^2 p_m}{\partial \xi_l \partial \xi_j}\frac{\partial^2 \phi}{\partial x_l \partial x_j} + p_{m-1}\right]a_r,$$

so the equations that we are trying to solve are linear ODEs and they will have solutions for all s. Also, note that we cannot solve for derivatives of a_0 of order higher than $R-2$ because those equations require derivatives of ϕ of order higher than R.

The rest of the amplitude functions are constructed in a similar way. The equations that they satisfy are linear inhomogeneous ODEs and so their solutions will exist for all s as well. Through this process, we get a_r and its derivatives of order up to $R-2(r+1)$ on γ, making c_j vanish to order $R-2(m+j)$ on γ. We remark here that number, N, in the Ansatz (A.2) is closely related to R. In fact, $2 \le R-2N \le 3$. This expression comes from a balance between having enough information to determine a_N on γ and recognizing that it is useless to have c_{-m+N} vanish to anything beyond order 1 (otherwise the remainder terms would contribute more than it to the whole sum).

To extend the phase and amplitude functions beyond γ, we use a Taylor series. Using $x_0(s)=s$, we define these functions by

$$\begin{aligned} \phi(x) &= \sum_{|\alpha|=0}^{R} \frac{1}{\alpha!}\frac{\partial^\alpha \phi(x(x_0))}{\partial x^\alpha}(x-x(x_0))^\alpha \\ a_r(x) &= \sum_{|\alpha|=0}^{R-2r-2} \frac{1}{\alpha!}\frac{\partial^\alpha a_r(x(x_0))}{\partial x^\alpha}(x-x(x_0))^\alpha, \end{aligned}$$

where α is a multi-index. Clearly these functions are smooth. Note that since the Taylor series of the phase and amplitude functions depend continuously on the initial data, the asymptotic solution given by equation (A.2) will depend continuously on the initial data.

The final step is to multiply each of amplitudes by a smooth function supported on U and equal to 1 near γ. The set U is chosen so that $U \cap \{|x_0|<T\}$ is compact and that $\mathrm{Im}[\phi(x)] > ad^2(x,\gamma)$ for $x \in \{|x_0|<T\} \cap U$. Here $d(x,\gamma)$ the distance function from x to γ and a is some positive fixed constant.

It remains to show that this construction gives us an asymptotic solution. The key result is contained within the following lemma:

Lemma A.3: *Assume that $c(x)$ vanishes to order $S-1$ on γ, supp$(c) \cap \{|x_0|<T\}$ is compact and that $\mathrm{Im}[\phi(x)] > ad^2(x,\gamma)$ on $\mathrm{supp}(c)\cap\{|x_0|<T\}$. Then*

$$\int_{|x_0|<T} |c(x)e^{ik\phi(x)}|^2 dx \le Ck^{-S-n/2}.$$

Proof Let z be k-independent local coordinates for the curve γ such that the curve is traced out by $z_0=s,\ z_i=0$ and such that $d^2(x(z),\gamma) \ge z_1^2+..+z_n^2$. Rescalling these coordinates so that $y_0=z_0$ and $y_i=k^{1/2}z_i$ we obtain that $kd^2(x(y),\gamma) \ge y_1^2+\ldots+y_n^2$. By making this change of variables in the integral, we have that the new integrand is bounded by a constant times

$$k^{-S-n/2}|k^{S/2}c(x(y_0,k^{-1/2}y'))|^2 e^{-2a|y'|^2}.$$

Since c vanishes to order $S-1$, $|k^{S/2}c(x(y_0,k^{-1/2}y'))|^2$ is bounded on $\{|y_0|<T\}$ as $k\to\infty$. Hence,

$$\int_{|x_0|<T} |c(x)e^{ik\phi(x)}|^2 dx \le Ck^{-S-n/2}.$$

∎

Finally, we estimate the Sobolev s-norm of Pu on $|x_0|<T$ with the help of the lemma. Since differentiation of (A.8) will either lower the order to which c_j

vanishes on γ or multiply c_j by k, repeated application of the lemma shows that

$$\|Pu\|_s \leq Ck^{m+s-(R+1)/2-n/4}.$$

Thus $\|Pu\|_s = O(k^{-M})$ for an appropriate choice of R.

(b) Well Posedness of The Symmetric Hyperbolic System

In this section, we prove that the time dependent problem is well-posed. Before we begin, we introduce the following notations which will be used in this section

- $I = [0, T]$, $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+$
- $|\!|\!|\cdot|\!|\!|$ denotes the L^2 norm in $(t, x) \in I \times \mathbb{R}^n_+$,
- $\|\cdot\|$ denotes the L^2 norm in $x \in \mathbb{R}^n_+$,
- $\langle\cdot,\cdot\rangle$ denotes the vector dot product in $\mathbb{R}^k$,
- "boundary conditions" refers to both the $x_n = 0$ boundary and the $t = 0$ boundary,
- the Einstein convention of summing over repeated indexes, i.e. $B_j \partial_{x_j} = \sum_{j=1}^n B_j \partial_{x_j}$.

Theorem A.4: *The symmetric hyperbolic system,*

$$\begin{aligned} A_0\partial_t u + B_j \partial_{x_j} u + Cu \equiv Lu = F(t, x) &\qquad \text{in } I \times \mathbb{R}^n_+ \\ M(u - h) = 0 &\qquad \text{for } x_n = 0 \\ u = f &\qquad \text{for } t = 0 \ , \end{aligned} \tag{A.9}$$

where

- *$A_0(x)$, $B_j(x)$ are $k \times k$ symmetric and $\langle A_0(x)u, u\rangle \geq d|u|^2$, $d > 0$,*
- *$A_0(x)$, $B_j(x)$ and $C(x)$ are smooth and bounded along with their derivatives,*
- *F, h and f are C^∞-functions in L^2, H^1 and H^1 respectively,*
- *B_n is of constant rank near $x_n = 0$,*
- *$M(x')$ is a projection and $M(x')\mathbb{R}^k$ is a non-negative subspace for $B_n(x')$, that is $\langle B_n(x')M(x')w, M(x')w\rangle \geq 0$ for all w and x' ,*
- *$M(x')\mathbb{R}^k$ is maximal in the sense that it is not properly contained in any other non-negative subspace for $B_n(x')$, and*
- *$h(0, x') = Mf(x', 0)$,*

is well-posed in the sense that a unique solution exists and its norm is controlled by the norms of the initial data:

$$|\!|\!|u|\!|\!| \quad \leq \quad C_T \left(|\!|\!|F|\!|\!| + \|f\|_{H^1} + \|h\|_{H^1}\right) \ .$$

Before we prove this theorem, we need a result which is proved by Rauch (1985).

Lemma A.5: *Suppose that*

- *$A_0(x)$ is symmetric positive definite,*
- *$B_j(x)$ are symmetric and B_n is of constant rank near $x_n = 0$,*
- *$N(x')$ is a maximal non-positive subspace for $B_n(x')$,*
- *A_0, B_j, and C are assumed to be smooth and bounded along with their derivatives and G is in L^2.*

Then there exists a weak solution to

$$\begin{aligned} A_0\partial_t u + B_j\partial_{x_j} u + Cu \equiv Lu = G(t,x) \qquad & \textit{in } I\times\mathbb{R}^n_+ \\ u \in N(t,x') \qquad & \textit{for } x_n = 0 \\ u = 0 \qquad & \textit{for } t = 0\ , \end{aligned} \tag{A.10}$$

i.e., there exists a $u\in L^2$*, such that for all piecewise continuously differentiable* $v\in C([0,T],C_0(\mathbb{R}^n_+))$*,* $v\in N^*(x') = [B_n(x')N(x')]^\perp$*, and* $v(T)=0$*,*

$$(L^*v,u) \quad = \quad (v,G)\ .$$

Proof The adjoint operator L^* is given by,

$$L^* \quad \equiv \quad -A_0\partial_t - (\partial_{x_j}B_j) - B_j\partial_{x_j} + C^*\ .$$

Let

$$Z^* \equiv -\frac{L+L^*}{2} = -\frac{C+C^*}{2} + B_j\partial_{x_j}\ .$$

Since A_0 is a symmetric positive definite matrix, $\langle A_0 u,u\rangle \ge d|u|^2$ with $d>0$ and $A_0=S^2$ for some S. Now, for piecewise continuously differentiable $v\in C([0,T],C_0(\mathbb{R}^n_+))$, $v\in N^*$, and $v(T)=0$, we compute

$$\begin{aligned} & \partial_t \left\|Sv(T-t)\right\|^2_{L^2(\mathbb{R}^n_+)} + 2(v(T-t),Z^*v(T-t))_{L^2(\mathbb{R}^n_+)} \\ = \quad & 2\mathrm{Re}(-A_0v_t,v) + 2\mathrm{Re}(-\partial_{x_j}(B_jv) + C^*v,v) - 2\int_{x_n=0}\langle B_nv,v\rangle dx' \\ \le \quad & 2\mathrm{Re}(L^*v,v))\ , \end{aligned}$$

as $\langle B_nv,v\rangle \ge 0$ for $v\in N^*$. Furthermore, as Z^* is a bounded operator,

$$\begin{aligned} \partial_t \left\|Sv(T-t)\right\|^2 \quad & \le \quad \frac{2}{d^{1/2}}\left\|L^*v(T-t)\right\| \ \left\|Sv(T-t)\right\| + 2\frac{C_Z}{d}\left\|Sv(T-t)\right\|^2 \\ \partial_t \left\|Sv(T-t)\right\| \quad & \le \quad C\left[\left\|L^*v(T-t)\right\| + \left\|Sv(T-t)\right\|\right]\ . \end{aligned}$$

By Gronwall's inequality, we obtain the estimate

$$\sup_{t\in I}\left\|Sv(t)\right\| \le C_T\left[\left\|Sv(T)\right\| + \int_0^T \left\|L^*v(T-s)\right\|ds\right]\ . \tag{A.11}$$

Let

$$\mathcal{B} = \{v \mid v\in C(I;C_0(\mathbb{R}^n_+)),\ v \text{ piecewise } C^1,\ v\in N^*,\ v(T)=0\}$$

and let $\mathcal{R} = L^*\mathcal{B}$. The estimate (A.11) shows that L^* maps $\mathcal{B}\to\mathcal{R}$ one-to-one, since for $w_1,w_2\in\mathcal{B}$ and $L^*w_1 = L^*w_2$, we have

$$\sup_{t\in I}\left\|S(w_1-w_2)\right\| \le C\int_0^T \left\|L^*w_1(s) - L^*w_2(s)\right\|ds = 0$$

Consequently, for a fixed $G\in L^2(I\times\mathbb{R}^n_+)$ we can define a linear functional on $\mathcal{R}$ by

$$l(r) = l(L^*w) = \int_I\int_{\mathbb{R}^n_+}\langle w,G\rangle \equiv (w,G)\ .$$

Now

$$|l(r)| \leq \|w\| \, \|G\| \leq C \, \|Sw\| \, \|G\| \leq C \, \|L^*w\| \, \|G\| = C \, \|r\| \, \|G\| \ ,$$

so l is a bounded linear functional on $\mathcal{R}$. As $\mathcal{B}$ is a subspace of L^2, the Hahn-Banach Theorem allows us to extend l to all of L^2 and the Riesz Representation Theorem guarantees the existence of a function $u \in L^2$ such that

$$l(w) = (w, u) \ .$$

For $v \in \mathcal{B}$

$$(L^*v, u) = l(L^*v) = (v, G) \ .$$

Hence, u is a weak solution of (A.10) . ∎

The classical results of P. Lax and R. Phillips (1960), section 4, can be almost directly applied to obtain that u is a semi-strong solution of (A.10). We restate the relevant theorem (Theorem 2.1) from their paper and a brief outline of the proof.

Lemma A.6: *If u satisfies the equation $Lu = G$ and the boundary conditions in the weak sense then u also satisfies the equation and the boundary conditions in the semi-strong sense. That is, there exist u_ϵ such that*

- *u_ϵ is continuously differentiable away from the boundaries*
- *$B_n u_\epsilon$ is continuously differentiable up to $x_n = 0$ and orthogonal to $N^*(x')$ at $x_n = 0$,*
- *$A_0 u_\epsilon$ is continuously differentiable up to $t = 0$ and $u_\epsilon = 0$ at $t = 0$,*
- *$u_\epsilon \to u$ in L^2,*
- *there exist functions $g_\epsilon \in L^2$ such that $Lu_\epsilon = g_\epsilon$ weakly,*
- *$g_\epsilon \to G$ in L^2,*
- *first order derivatives in (t, x') variables of u_ϵ are in L^2, and*
- *$\partial_{x_n} B_n u_\epsilon \in L^2$.*

Proof Outline.
The proof begins by partitioning the domain into a region close to the boundary (both the $t = 0$ and $x_n = 0$ boundaries) and a region away from the boundary. Then, u is broken up by a partition of unity into two pieces. Away from the boundary we obtain the result by mollifying u in all variables.

Near the boundary, we extend the weak solution for $t \in [-T, T]$ by defining it to be equal to 0 for $t < 0$. Also, we extend G in the same way and consider L for all $t \in [-T, T]$ since it is independent of t. Certainly, the extended u is a weak solution of $Lu = G$ in $[-T, T] \times \mathbb{R}^n_+$. We then obtain a regularity result in the x_n direction for the components of u which are not annihilated by B_n (see Lemma 2.1 in Lax and Phillips 1960), which states that $B_n u$ is a continuous function of x_n taking values in H^{-1}. Also, we obtain that $B_n u = 0$ at $x_n = 0$ in the distribution sense.

Next, we look at the (t, x') variables. Consider the shifted mollifier $\tilde{J}_\epsilon$ in the (t, x') coordinates given by the convolution kernel

$$\tilde{j}_\epsilon(t, x') = \frac{1}{\epsilon^n} j \left(\frac{t}{\epsilon} - 2, \frac{x'}{\epsilon} \right) \ ,$$

where $j \in C_0^\infty(\mathbb{R}^n)$, $j \geq 0$, $\text{supp}(j) \subset B_1(0)$, and $\|j\|_{L^1} = 1$. Note that this particular choice of mollifier ensures that $\tilde{J}_\epsilon^* v \in N^*$ and $\tilde{J}_\epsilon^* v = 0$ for $t = T$ whenever v

enjoys these properties. Using this property, the definition of the shifted mollifier, and the regularity of u in the x_n direction, one can prove that $u_\epsilon = \tilde{J}_\epsilon u$ satisfies all of the conditions required to make u a semi-strong solution.

∎

We now have all of the tools to prove that the solution to (A.10) is unique. Suppose that there are two solutions. Denote their difference by w. This difference satisfies (A.10) with $G \equiv 0$. We can use the above to find a sequence of functions w_ϵ converging to w and Lw_ϵ converging to $G \equiv 0$. Re-deriving the estimate (A.11) for L, we find that

$$\sup_{t\in I} \|Sw_\epsilon\| \leq C_T \int_0^T \|Lw_\epsilon(s)\| \, ds \ .$$

In the limit, this shows that $w \equiv 0$. Hence the solution is unique.

Proof [Proof of Theorem A.4] We reduce to the $f = 0$ and $h = 0$ case by subtracting the following H^1 function u_0 from u:

$$u_0(x,t) \quad = \quad \frac{x_n \chi(t)}{t + x_n} M f(x) + \frac{t\chi(x_n)}{t + x_n} h(x', t) + \chi(t) M^\perp f(x) \ ,$$

where χ is a non-negative $C_0^\infty(\mathbb{R})$-function such that $\chi(s) = 1$ for $|s| < 1$ and $\chi(s) = 0$ for $|s| > 2$. The function u_0 is such that $u_0|_{t=0} = f(x)$ and $Mu_0|_{x_n=0} = h(t, x')$. We now apply the above existence arguments to obtain a solution w for $G = F - Lu_0$ and $N(x') = \left[M(x')\mathbb{R}^k\right]^\perp$. Hence $u = w + u_0$ is solution to (A.9). Uniqueness follows since the difference of two solutions solves (A.10) with $f = h = F = 0$. Using the approximations w_ϵ for w, we have

$$\begin{aligned}
\|u\| &\leq \|w\| + \|u_0\| \leq \lim_{\epsilon\to 0} \|w_\epsilon\| + C\left(\|f\| + \|h\|\right) \\
&\leq T \lim_{\epsilon\to 0} \sup_{t\in I} \|w_\epsilon(t)\| + C\left(\|f\| + \|h\|\right) \\
&\leq C_T \lim_{\epsilon\to 0} \int_0^T \|g_\epsilon(s)\| \, ds + C\left(\|f\| + \|h\|\right) \\
&\leq C_T \lim_{\epsilon\to 0} \|g_\epsilon\| + C\left(\|f\| + \|h\|\right) \\
&\leq C_T\left(\|F - Lu_0\| + \|f\| + \|h\|\right) \\
&\leq C_T\left(\|F\| + \|f\|_{H^1} + \|h\|_{H^1}\right)
\end{aligned}$$

Therefore, the hyperbolic system (A.9) is well-posed.

∎

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