Unfolding Square Root Singularities in the 2D Boussinesq Equations

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Abstract

This report provides an unfolding for the Boussineq equations in 2D. By intoduction of a new variable ξ, which is comparable to a square root of the spatial variable, we obtain a nonsingular description for singular solutions of the Boussineq equations. The variable ξ is defined to satisfy $q(x,t,\xi)=0$, so that singularities occur where $q_\xi=0$. This analysis is applicable to motion of singularities in the complex plane for Boussinesq.

1 Boussineq Equations

The Boussinesq equations in stream function - vorticity variables are (for $\mathbf{x}=(\mathbf{r},\mathbf{z})$)

$$\begin{aligned}(\partial_t+\mathbf{u}\cdot\nabla)\rho &= f\\ (\partial_t+\mathbf{u}\cdot\nabla)\zeta &= -\partial_z\rho+g\\ \mathbf{u}=(u,v) &= \nabla^\perp\psi=(-\partial_z\psi,\partial_r\psi)\\ \zeta=\nabla^2\psi &= -\partial_z u+\partial_r v.\end{aligned} \tag{1}$$

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This is similar to axisymmetric flow with swirl in which $\rho = \Omega^2 = (ru_\theta)^2,\ \zeta = -r\omega_\theta$.

Look for a cusp singularity at points $\xi = \xi_R \pm i\xi_I$ defined by

$$q(\xi, \mathbf{x}, \mathbf{t}) = \mathbf{0} \tag{2}$$

with

$$\begin{aligned} q_\xi &= (\xi - (\xi_R + i\xi_I))(\xi - (\xi_R - i\xi_I)) \\ &= (\xi - \xi_R)^2 + \xi_I^2 && (3) \\ q_{\xi\xi} &= 2(\xi - \xi_R) && (4) \\ q_{\xi\xi\xi} &= 2 && (5) \\ q &= \frac{1}{3}(\xi - \xi_R)^3 + \xi_I^2(\xi - \xi_R) + q_0. && (6) \end{aligned}$$

Note that

$$\begin{aligned} q_{\xi\xi}^2 &= 4(q_\xi - \xi_I^2) && (7) \\ q_\xi q_{\xi\xi} &= 2(\xi - \xi_R)^3 + 2\xi_I^2(\xi - \xi_R) \\ &= 6q - 4\xi_I^2(\xi - \xi_R) - 6q_0 \\ &= -2\xi_I^2 q_{\xi\xi} - 6q_0 && (8) \\ q_\xi^2 &= (\xi - \xi_R)^4 + 2(\xi - \xi_R)^2\xi_I^2 + \xi_I^4 \\ &= (\xi - \xi_R)\{3q - (\xi - \xi_R)\xi_I^2 - eq_0\} + \xi_I^4 \\ &= -(\xi - \xi_R)^2\xi_I^2 + \xi_I^4 - 3(\xi - \xi_R)q_0 \\ &= -\xi_I^2 q_\xi + 2\xi_I^4 - \frac{3}{2}q_0 q_{\xi\xi}. && (9) \end{aligned}$$

Also

$$\begin{aligned} \nabla q &= -q_\xi \nabla\xi_R + (\xi - \xi_R)\nabla\xi_I^2 + \nabla q_0 \\ &= -q_\xi \nabla\xi_R + \frac{1}{2}q_{\xi\xi}\nabla\xi_I^2 + \nabla q_0 && (10) \\ \nabla &= -q_\xi^{-1}\nabla q \partial_\xi + \nabla && (11) \\ (\partial_t + u\cdot\nabla) &= -q_\xi^{-1}(q_t + u\cdot\nabla q)\partial_\xi + (\partial_t + u\cdot\nabla). \end{aligned}$$

As a solution ansatz, look for

$$\begin{aligned} u &= u_0 + u_1 q_{\xi\xi} + u_2 q_\xi && (12) \\ \rho &= \rho_0 + \rho_1 q_{\xi\xi} + \rho_2 q_\xi && (13) \\ \zeta &= q_\xi^{-1}(\zeta_1 + \zeta_2 q_{\xi\xi}) + \zeta_0. && (14) \end{aligned}$$

Consider function $F(\xi, \mathbf{x}, \mathbf{t})$ that is a polynomial in ξ. Using the relation $q = 0$, F can be reduced to a quadratic polynomial $\tilde{F}$. Define

$$P_i F = F_i(x,t) \qquad i = 0, 1, 2 \tag{15}$$

in which

$$\tilde{F} = F_0 + q_{\alpha\alpha} F_1 + q_\alpha F_2. \tag{16}$$

2 Incompressiblity

$$\begin{aligned} 0 &= \nabla \cdot u \\ &= -q_\xi^{-1} \nabla q \cdot u_\xi j + \nabla \cdot u \end{aligned} \tag{17}$$

which implies

$$u_\xi \cdot \nabla q = q_\xi \nabla \cdot u. \tag{18}$$

This will be used to cancel singularities in the ζ-equation.

Write out these components as

$$\begin{aligned} u_\xi \cdot \nabla q &= (u_1 q_{\xi\xi\xi} + u_2 q_{\xi\xi}) \cdot (-q_3 \nabla\nabla \xi_R + \frac{1}{2} q_{\xi\xi} \nabla \xi_I^2 + \nabla q_0) \\ &= (2u_1 + q_{\xi\xi} u_2) \cdot \nabla q_0 \\ &+ \frac{1}{2}(2 q_{\xi\xi} u_1 + q_{\xi\xi}^2 u_2) \cdot \nabla \xi_I^2 - q_3 (2u_1 + q_{\xi\xi} u_2) \cdot \nabla \xi_R \\ &= (2u_1 + q_{\xi\xi} u_2) \cdot \nabla q_0 + (q_{\xi\xi} u_1 - 2\xi_I^2 u_2) \cdot \nabla \xi_I^2 \\ &+ q_\xi \{-(2u_1 + q_{\xi\xi} u_2) \cdot \nabla \xi_R + 2u_2 \cdot \nabla \xi_I^2\} \\ &= \{2(u_1 \cdot \nabla q_0 - \xi_I^2 u_2 \cdot \nabla \xi_I^2) + q_{\xi\xi}(u_2 \cdot \nabla q_0 + u_1 \cdot \nabla \xi_I^2)\} \\ &+ q_\xi \{-(2u_1 + q_{\xi\xi} u_2) \cdot \nabla \xi_R + w u_2 \cdot \nabla \xi_I^2\} \\ &= A_1 + q_\xi A_2 \end{aligned} \tag{19}$$

with

$$\begin{aligned} A_a &= A_{10} + q_{\xi\xi} A_{11} \qquad (20) \\ \frac{1}{2} A_{10} &= u_1 \cdot \nabla q_0 - \xi_I^2 u_2 \cdot \nabla \xi_I^2 \\ A_{11} &= u_2 \cdot \nabla q_0 + u_1 \cdot \nabla \xi_I^2 \\ q_x i \nabla \cdot u &= q_\xi (\nabla \cdot u_0 + q_{\xi\xi} \nabla \cdot u_1 + q_\xi \nabla \cdot u_2) \\ &= q_\xi A_3. \qquad (21) \end{aligned}$$

This shows that

$$q_\xi^{-1} A_1 = -A_2 + A_3 \tag{22}$$

which will be useful in the singular terms of the ρ and ζ equations.

We next write the equation in a way that provides equations for u

$$\begin{aligned}
\tilde{A}_2 = q_\xi A_2 &= (-2u_1 \cdot \nabla\xi_R + 2u_2 \cdot \nabla\xi_I^2) q_\xi \\
&+ (-2\xi_I^2 q_{\xi\xi} + 6q_0) u_2 \cdot \nabla\xi_R
\end{aligned} \tag{23}$$

$$\begin{aligned}
\tilde{A}_3 = q_\xi A_3 &= q_\xi \nabla \cdot u_0 + (2\xi_I^2 q_{\xi\xi} + 6q_0) \nabla \cdot u_1 \\
&\qquad + (-\xi_I^2 q_\xi + 2\xi_1^4 - \frac{3}{2} q_0 q_{\xi\xi}) \nabla \cdot u_2 \\
&= q_\xi (\nabla \cdot u_0 - 6q_0 \nabla \cdot u_1 + 2\xi_I^4 \nabla \cdot u_2) \\
&+ q_{\xi\xi}(-2\xi_I^2 - \frac{3}{2} q_0) \nabla \cdot u_2 \\
&+ (-6q_0 \nabla \cdot u_1 + 2\xi_I^4 \nabla \cdot u_2).
\end{aligned} \tag{24}$$

Equation 1, $q_{\xi\xi}$ and q_ξ terms in $A_1 + \tilde{A}_2 = \tilde{A}_3$ leads to

$$2(u_1 \cdot \nabla q_0 - \xi_I^2 u_2 \cdot \nabla\xi_I^4) + 6q_0 u_2 \cdot \nabla\xi_R = (-6q_0 \nabla \cdot u_1 + 2\xi_I^4 \nabla \cdot u_2) \tag{25}$$

$$(u_2 \cdot \nabla q_0 + u_1 \cdot \nabla\xi_I^2) + 2\xi_I^2 u_2 \cdot \nabla\xi_R = (-2\xi_I^2 - \tfrac{3}{2} q_0) \nabla \cdot u_2 \tag{26}$$

$$(-2u_1 \cdot \nabla\xi_R + 2u_2 \cdot \nabla\xi_I^2) = \nabla \cdot u_0 - 6q_0 \nabla \cdot u_1 + 2\xi_I^4 \nabla \cdot u_2. \tag{27}$$

These are 3 of the final equations.

3 Definition of ζ

$$\begin{aligned}
\zeta &= \nabla^\perp \cdot u \\
&= -q_\xi^{-1} \nabla^\perp q \cdot u_\xi + \nabla^\perp \cdot u \\
&= -q_\xi^{-1}(-q_\xi \nabla^\perp \xi_R + \frac{1}{2} q_{\xi\xi} \nabla^\perp \xi_I^2 + \nabla^\perp q_0) \cdot (2u_1 + q_{\xi\xi} u_2) \\
&\qquad + (\nabla^\perp \cdot u_0 + q_{\xi\xi} \nabla^\perp \cdot u_1 + q_3 \nabla^\perp \cdot u_2) \\
&= -q_\xi^{-1}\{(-q_\xi \nabla^\perp \xi_R + \frac{1}{2} q_{\xi\xi} \nabla^\perp \xi_I^2 + \nabla^\perp q_0) \cdot 2u_1 \\
&\qquad + (-q_\xi q_{\xi\xi} \nabla^\perp \xi_R + \frac{1}{2} q_{\xi\xi}^2 \nabla^\perp \xi_I^2 + q_{\xi\xi} \nabla^\perp q_0) \cdot u_2
\end{aligned}$$

$$
\begin{aligned}
& \qquad -(q_3\nabla^{\perp}\cdot u_0 + q_\xi q_{\xi\xi}\nabla^{\perp}\cdot u_1 + q_\xi^2\nabla^{\perp}\cdot u_2)\} \\
= \;& -q_\xi^{-1}\{(-q_\xi\nabla^{\perp}\xi_R + \frac{1}{2}q_{\xi\xi}\nabla^{\perp}\xi_I^2 + \nabla^{\perp}q_0)\cdot 2u_1 \\
& \qquad +(2\xi_I^2 q_{\xi\xi} + 6q_0)\nabla^{\perp}\xi_R + 2(q_\xi - \xi_I^2)\nabla^{\perp}\xi_I^2 + q_{\xi\xi}\nabla^{\perp}q_0)\cdot u_2 \\
& \qquad -(q_\xi\nabla^{\perp}\cdot u_0 - (2\xi_I^2 q_{\xi\xi} + 6q_0)\nabla^{\perp}\cdot u_1 + 4(q_3 - \xi_I^2)\nabla^{\perp}\cdot u_2)\}
\end{aligned} \tag{28}
$$

$$
\zeta_0 = 2\nabla^{\perp}\xi_R\cdot u_1 - 2\nabla^{\perp}\xi_I^2\cdot u_2 + \nabla^{\perp}\cdot u_0 + 4\nabla^{\perp}\cdot u_2 \tag{29}
$$

$$
\zeta_1 = -2\nabla^{\perp}q_0\cdot u_1 - 6q_0\nabla^{\perp}\xi_R\cdot u_2 + 2\xi_I^2\nabla^{\perp}\xi_I^2\cdot u_2 - 6q_0\nabla^{\perp}\cdot u_1 - 4\xi_I^2\nabla^{\perp}\cdot u_2 \tag{30}
$$

$$
\zeta_2 = -\nabla^{\perp}\xi_I^2\cdot u_1 - 2\xi_I^2\nabla^{\perp}\xi_R\cdot u_2 - \nabla^{\perp}q_0\cdot u_2 - 2\xi_I^2\nabla^{\perp}\cdot u_1. \tag{31}
$$