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STABILITY AND SCALABILITY OF 2D SWARMING PATTERNS

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Anna L. Lin

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Dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Physics in the Graduate School of Duke University

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ABSTRACT

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Abstract

We study a class of *swarming problems* wherein particles evolve dynamically via pairwise interaction potentials and a velocity selection mechanism. In biology, a swarming system is used to describes the natural aggregation of animals. Via model simulations, we find that such a system undergoes various changes of state as a function of the self-propulsion and interaction potential parameters. While these patterns exhibit different scalability properties, we apply a fundamental theory, called *H*-stability, from statistical mechanics to investigate such a difference. A phase diagram is derived to predict whether the system may scale well or collapse when the number of particles increases. To further analyze the system, we utilize a procedure which, in a definitive way, connects a class of individual-based models to their continuum formulations and determine criteria for the validity of the latter. The results show that H-stability of the interaction potential also plays a fundamental role in determining the validity of the continuum approximation. We then perform a linear stability analysis of the continuum model and compare the results to the simulations of the individual-based one. Using the knowledge gained from natural swarms as building blocks, we design control algorithms for artificial swarms by adapting our swarming model to testbed vehicles. We prove that the vehicles under the adapted algorithms can aggregate at least locally and demonstrate that they retain certain scalability properties from the original model. Testbed experiments show that such algorithms perform well for various tasks with reasonable number of vehicles.

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Contents

A	bstra	let	iv
A	ckno	wledgements	v
Li	st of	Figures	x
1	Intr	oduction	1
2	ΑE	Brief History of Swarming Models	5
	2.1	Population dynamics	5
	2.2	Kinematic models	8
		2.2.1 Aggregation	9
		2.2.2 Biased random walks	12
		2.2.3 The position-jump and the orientation-jump processes \therefore	16
		2.2.4 Advection-diffusion models	20
		2.2.5 Orientation-jump models	26
	2.3	Dynamic models	36
	2.4	Summary	53
3	Ind	ividual-Based Model	55
	3.1	Introduction	55
	3.2	The equations of motion	55
		3.2.1 Numerical method	58
	3.3	Various states of swarming patterns	59
	3.4	The biologically relevant regime	65
	3.5	Quantifying the biologically relevant state transition	69

		3.5.1 Analogy to bulk matter	74
4	H-s	tability	77
	4.1	Introduction	77
	4.2	The basic definition	78
	4.3	The H-stability analysis	79
	4.4	Discussion	83
5	Cor	ntinuum Model	85
	5.1	Introduction	85
	5.2	Derivation of the continuum model	88
		5.2.1 Continuum interaction force	90
		5.2.2 Fluctuation terms	92
		5.2.3 Numerical method	96
	5.3	Comparison to the individual-based model $\ldots \ldots \ldots \ldots \ldots$	96
		5.3.1 Validity of the continuum model	97
	5.4	Linear stability analysis	104
	5.5	Discussion	109
6	ΑE	Brief History of Artificial Swarms	113
	6.1	Walter's turtles	113
	6.2	Recent works	115
7	Cal	tech Multi-Vehicle Wireless Testbed	121
	7.1	Introduction	121
	7.2	Testbed	121
	7.3	Model adaption	122

	7.4	Testbed simulations	124
	7.5	Conclusion	128
8	UC	LA MicroCar Testbed	130
	8.1	Introduction	130
	8.2	Testbed	130
	8.3	Model adaption	131
	8.4	Proof of convergence	135
	8.5	Testbed and computer simulations	138
		8.5.1 Testbed simulations	139
		8.5.2 Computer simulations	139
	8.6	Summary and conclusion	142
9	Sun	nmary and Conclusion	143
B	ibliog	graphy	146
Bi	iogra	phy	155

List of Figures

1.1	The mill formations. Left: A photo of a big eye Trevally school, © 2006 Norbert Wu, <i>http://www.norbertwu.com/</i> . Right: A sketch of the mill formation by ants in Schneirla's book "Army Ants, a Study in Social Organization" [83]. Reprint of Fig. 12.4 on page 283 in T. C. Schneirla, "Army Ants, a Study in Social Organization", Copyright (1971) by W. H. Freeman.	2
2.1	The spread of muskrat (<i>Ondatra zibethica</i> L.) in central Europe [87]. Muskrats were introduced to Europe from North America in 1905. The left figure shows their population distribution during the subsequent years. The right figure shows that the square root of their distribution area increase linear in time, as predicted by Eq. (2.4). Reprint from J. G. Skellam, "Random dispersal in theoretical populations", Biometrika, 1951, Vol. 38, No. 1-2, pp. 196-218, by permission of Oxford University Press.	8
2.2	Examples of Conway's Game of Life [30]. The counter distributions start with the configurations on the left and evolve toward the right until the final steady configurations are reached. These are examples of five initial occupied cells.	10
2.3	Hamilton's selfish-herd model [38, 70]. The line segment AB is the domain of danger of frog c. If by accident, the frog jumps to c', its domain of danger is reduced. Reprint of Fig. 7.16 on page 129 of Ch. 7 "The dynamics of animal grouping" in A. Okubo, (1980) Diffusion and Ecological Problems: Mathematical Models, with kind permission of Springer-Verlag and Business Media	11
2.4	The solutions of the diffusion and the telegraph equations [40]. The initial condition is a delta pulse at the release point. The two figures show snapshots at $t = 5$ (left) and at $t = 7.6$ (right). Reprint of Fig. 1 on page 783 in E. E. Holmes, "Are diffusion models too simple? A comparison with telegraph models of invasion", American Naturalist, Vol. 142, No. 5, pp. 779-795 ©1993 by The University of Chicago.	19

2.5 The effects of an odd kernel (left) and an even one (right) on density movements in Ref. [62]. Reprint of Fig. 1 on page 541 in A. Mogilner and L. Edelstein-Keshet (1999) "A non-local model for a swarm", Journal of Mathematical Biology, Vol. 38, No. 6, pp. 534-570, with kind permission of Springer-Verlag and Business Media.

21

22

24

25

29

- 2.6 A basic simulation for the model of Mogilner and Edelstein-Keshet in Ref. [62]. The horizontal axis represents the one-dimensional spatial coordinate x, while the vertical axis represents density ρ . Reprint of Fig. 4 on page 556 in A. Mogilner and L. Edelstein-Keshet (1999) "A non-local model for a swarm", Journal of Mathematical Biology, Vol. 38, No. 6, pp. 534-570, with kind permission of Springer-Verlag and Business Media.
- 2.7 The incompressible motion of the 2D model in Ref. [94]. The time evolution of the pictures is from top-left to bottom-left, then from top-right to bottom-right. The shaded area represents a constant density evolving on a 2D (x-y) plane. Reprint from C. M. Topaz and A. L. Bertozzi, "Swarming patterns in a two-dimensional kinematic model for biological groups", SIAM Journal on Applied Mathematics, Vol. 65, pp. 152-174, Copyright (2004) by SIAM.
- 2.9 The four different interaction areas defined by Huth and Wissel in Ref. [44]. From inner to outer, the four areas are respectively the repulsion area, the parallel area, the attraction area, and the searching area. Reprinted from Journal of Theoretical Biology, Vol. 156, A. Huth and C. Wissel, "The simulation of the movement of fish schools", pp. 365-385, Copyright (1992) with permission from Elsevier.
 - xi

- 2.10 The functions of behavior patterns adopted by Reuter and Breckling in Ref. [78]. The most preferred swimming distance between neighbors is denoted by *md* in the figure, where the parallel orientation increases to 100% without any attraction or repulsion. The *x*-axis represents the neighbor distance *ma*, and the *y*-axis specifies the percentage of the repulsion, the parallel orientation, and the attraction. Reprinted from Ecological Modelling, Vol. 75-76, H. Reuter and B. Breckling, "Selforganization of fish schools: an object-oriented model", pp. 147-159, Copyright (1994) with permission from Elsevier.
- 2.11 The behavioral zones defined by Couzin et al. in [14]: zor is zone of repulsion, zoo is zone of orientation, and zoa is zone of attraction. The angle α defines a field of perception for an individual, and thus, the region behind an individual, indicated by $(360 \alpha)^{\circ}$, is a blind spot which it cannot see. Reprinted from Journal of Theoretical Biology, Vol. 218, I. D. Couzin, J. Krause, R. James, G. D. Ruxton, and N. R. Franks, "Collective memory and spatial sorting in animal groups", pp. 1-11, Copyright (2002) with permission from Elsevier. 33

31

35

- 2.12 Various swarming states observed in the model of Couzin et al. [14]: (A) Swarm, (B) Torus, (C) Dynamic parallel group, and (D) Highly parallel group. The two figures on the bottom are the phase diagrams of (E) the polarity and (F) the angular momentum. Swarm behavior in (A) corresponds to region a in the phase diagrams; torus behavior in (B) is indicated by region b; regions c and d represent (C) and (D) respectively. Region e has a great chance of resulting in fragmenting groups. Reprinted from Journal of Theoretical Biology, Vol. 218, I. D. Couzin, J. Krause, R. James, G. D. Ruxton, and N. R. Franks, "Collective memory and spatial sorting in animal groups", pp. 1-11, Copyright (2002) with permission from Elsevier.

xii

2.14	Phase diagram of various group patterns of the model in Ref.[81, 91]. The symbols used in the figure is different from those used here: a is α , b is max $ \eta_i $, h is c_0 , and ℓ has the same definition. The figure comes from Ref. [70]. Reprint of Fig. 7.12 on page 126 of Ch. 7 "The dynamics of animal grouping" in A. Okubo, (1980) Diffusion and Ecological Problems: Mathematical Models, with kind permission of Springer-Verlag and Business Media.	39
2.15	The functional form of the forces adopted in the bird flock model of Heppner and Grenander [39]. Left: Velocity regulation. Middle: Homing. Right: Interaction. Reprints of Fig. 1, Fig. 2, and Fig. 3 on page 235 in Ch. 19, F. Heppner and U. Grenander, "A stochas- tic nonlinear model for coordinated bird flocks", The Ubiquity of Chaos, Ed. S. Krasner, ©1990 by American Association for the Advancement of Science.	41
2.16	A series of time frames of the model simulation in Ref. [39]. The encircled cross is the roosting area. The small dots represent indi- vidual birds, on which the line segments indicate the forces acting on the birds. Reprint of Fig. 4 on page 237 in Ch. 19, F. Heppner and U. Grenander, "A stochastic nonlinear model for coordinated bird flocks", The Ubiquity of Chaos, Ed. S. Krasner, ©1990 by American Association for the Advancement of Science	42
2.17	The theoretical curve of the phase transition between polarized and non-polarized groups in Niwa's model [67]. Reprinted from Jour- nal of Theoretical Biology, Vol. 171, HS. Niwa, "Self-organizing dynamic model of fish schooling", pp. 123-136, Copyright (1994) with permission from Elsevier.	43
2.18	Snapshots of the active Brownian particles studied by Erdmann et al. [23]. The counterclockwise mill in the first frame is disrupted in the second frame and then evolves into a clockwise mill in the third frame. Reprinted figure with permission from U. Erdmann, W. Ebeling, and V. S. Anishchenko, Physical Review E, Vol. 65, pp. 061106, 2002. Copyright (2002) by the American Physical Society.	49

2	Snapshots of the transition from a translational state to a rota- tional one in the model of Erdmann et al. [24]. Reprinted figure with permission from U. Erdmann, W. Ebeling, and A. Mikhailov, Physical Review E, Vol. 71, pp. 051904, 2005. Copyright (2005) by the American Physical Society.	50
2	Time frames of a random initial configuration evolving into inter- locking mills in the model of Levine et al. [57]. The line seg- ment at the lower-right corner in Fig. a represents the length of ℓ_a . Reprinted figure with permission from H. Levine, WJ. Rap- pel, and I. Cohen, Physical Review E, Vol. 63, pp. 017101, 2000. Copyright (2000) by the American Physical Society	53
3	Potential characteristic diagram. This diagram specifies several characteristically distinct regimes of the pairwise interaction potential $V(x)$, including the attraction-dominant regime (I, II, III, and IV), the repulsion-dominant regime (V), and the biological relevant regime (VI and VII)	60
3	Biologically relevant swarming patterns: (a) The functional shape of the biologically relevant pairwise interaction potential. (b) The single-mill state. (c) The double-mill state. (d) The coherent flock. (e) The rigid-body rotation	62
3	Attraction-dominant patterns: (a) The functional shapes of the attraction-dominant pairwise interaction potentials of regions I-IV in Fig. 3.1. (b) The clump formation. (c) The ring formation. (d) The clumping-ring formation.	63
3	Ring radius as a function of N from numerical data and from the estimates of Eq. (3.11). The parameters used here are $\alpha = 1$, $\beta = 0.5$, $C_a = \ell_a = 1$, $C_r = 0.6$, and $\ell_r = 0.5$.	65
3	Time variation of the numbers of particles rotating in different directions: The triangles represent the number of CCW particles while the circles are of CW ones. (Top) $\alpha = 1.5$. (Middle) $\alpha = 4.0$. (Lower) $\alpha = 6.0$. The fixed parameters are $\beta = 0.5$, $C_a = 0.5$, $C_r = 1.0$, $\ell_a = 2.0$, $\ell_r = 0.5$, and $N = 500$. All parameters here and throughout the thesis are in arbitrary units	67

3.6	$\alpha_{\rm esc}$ versus the total number of particles in a region VI swarm ($\beta = 0.5, C_a = 0.5, C_r = 1.0, \ell_a = 2.0, \ell_r = 1.5$, dashed line) compared to that of a region VII swarm ($\beta = 0.5, C_a = 0.5, C_r = 1.0, \ell_a = 2.0, \ell_r = 0.5$, dotted line). The solid line is the curve estimated by Eq. (3.12).	68
3.7	Emergence of a rotating single-mill structure from a rigid-body rotation in region VII. The left panel shows the ensemble averaged tangential velocity, $\langle v(r) \rangle_{\text{tang}}$, of particles at a distance r from the center of mass. Each $\langle v(r) \rangle_{\text{tang}}$ figure corresponds to the swarm structure of different values of α on the right panel: from top to bottom are $\alpha = 0.003$, $\alpha = 0.03$, $\alpha = 0.1$, and $\alpha = 0.5$. The other parameters are $\beta = 0.5$, $C_a = 0.5$, $C_r = 1.0$, $\ell_a = 2.0$, $\ell_r = 0.5$, and N = 500	71
3.8	(a) The ordering factor of period 6 versus α and an illustration showing the definition of $\phi_{j,j+1}^{(i)}$. The squares are data points of a region VI swarm while the triangles represent a region VII case. The parameters other than α for both cases are the same as those in Fig. 3.6 with $N = 200$. (b) The distribution of $\phi_{j,j+1}^{(i)}$ for all i and j . (c) Comparison of the ordering factors of different periods Q	73
3.9	The state transition diagram of (a) a region VI swarm and (b) a region VII swarm. The fixed parameters are the same as those in Fig. 3.6 with $N = 200$	75
4.1	H-stability phase diagram	82
5.1	Comparison of the numerical simulations of the individual-based model and the continuum model: The parameters used in both simulations are $C_a = 0.5$, $C_r = 1.0$, $\ell_a = 2.0$, $\ell_r = 0.5$, $\alpha = 1.2$, $\beta = 0.5$, and the total mass $m_{\text{tot}} = 88$. (a) The averaged density profiles along the radial distance from the center of mass. (b) The averaged radial momentum profiles. (c) The averaged tangential momentum profiles. (d) The averaged tangential velocity profiles.	98
5.2	Relative speed fluctuations $\Delta_{\rm K}$ while forming a single mill from random initial conditions. The dashed curve illustrates the nor- malized angular momentum M in Eq. (3.14) while the solid curve represents $\Delta_{\rm K}$. The parameters of this simulation are $\alpha = 1.0$, $\beta = 0.5$, $C_a = 0.5$, $C_r = 1.0$, $\ell_a = 2.0$, $\ell_r = 0.5$, and $N = 500$.	99
	a a a a a a a a a a	50

5.3(a) Flock radius versus number of particles with the total mass fixed at $m_{\text{tot}} = Nm = 500$. The solid circles represent the H-stable flock ($\ell_r = 1.5$), fitted by $R \propto N^{0.41}$, while the solid diamonds represent the catastrophic flock ($\ell_r = 1.3$), fitted by $R \propto N^{0.11}$. The other parameters are $\alpha = 0, \beta = 0.5, C_a = 0.5, C_r = 1.0,$ and $\ell_a = 2.0$. (b) Exponents Z of the power law fitting $R \propto N^Z$ for a range of C and ℓ . The dimensionless parameters C and ℓ are changed by varying C_r and ℓ_r while the other parameters remain the same as above. The gray-scale map indicates the power Z, and darker colors represent higher exponents. The solid curve marks the H-stability boundary with the upper region being H-stable and the lower region being catastrophic. 101 5.4(a) n_s versus ℓ . The upper curve represents the case of s = 10%while the lower curve is for s = 50%. On the upper-right corner is an illustration showing how the significant neighbors are defined. (b) $U_{\rm Eu}$ and $U_{\rm La}$ versus ℓ . Here $\alpha = 0.003$, $\beta = 0.5$, $C_a = 0.5$, $\ell_a = 2.0, C_r = 1.0, \text{ and } N = 500.$ (c) $\triangle U$ and versus $N. \ \ell = 0.65$ for the catastrophic curve while $\ell = 0.75$ for the H-stable curve. The other parameter values are the same as (b). 103The linear stability diagram of the swarming model of Eqs. (3.7)-5.5(3.8).107Left panel: The contours of a density distribution of the continuum 5.6model near the instability onset with (upper) $\alpha = 0$ and (lower) $\alpha = 1$; Right panel: Simulations of the individual-based model using the same parameters and initial conditions as the left figures. The parameter values are $\beta = 0.5$, $C_a = 0.5$, $C_r = 1.0$, $\ell_a = 2.0$, and $\ell_r = 1.35$. 110A picture of Walter's turtle [101]. This is a time-exposure photo-6.1graph in which the turtle starts from the left, avoids the obstacle in the middle, and reaches the target on the right. A lamp is mounted on the turtle, and hence, the photo is able to record its trajectory as well. Reprint from W. G. Walter, "A machine that learns", Scientific American, Vol. 185, No. 2, pp. 61, 1951, photographed by

6.2	Foraging efficiency of Sugawara and Sano's robots [90]. The <i>x</i> -axis is the number of robots, and the <i>y</i> -axis is the inverse of time <i>T</i> by which all pucks are collected. Pucks are distributed homogeneously in Field 1, where robot interaction does not show any significant effect. Field 3 has localized puck distributions, where robots with interaction demonstrate more superior efficiency by increasing the number of robots. Reprinted from Physica D: Nonlinear Phenom- ena, Vol. 100, K. Sugawara and M. Sano, "Cooperative acceleration of task performance: Foraging behavior of interacting multi-robots system", pp. 343-354, Copyright (1997) with permission from El- sevier.	116
6.3	The coordinate system of the steering control law in the paper of Justh and Krishnaprasad [48]. Reprint from E. W. Justh and P. S. Krishnaprasad, "Steering laws and continuum models for planar formation", in Proc. of the 42nd IEEE Conference on Decision and Control, pp. 3609-3614 (© 2003 IEEE)	118
6.4	The vehicular formations in the model of Justh and Krishnaprasad [48]. Left: Rectilinear formation. Right: Circling formation. Reprint from E. W. Justh and P. S. Krishnaprasad, "Steering laws and continuum models for planar formation", in Proc. of the 42nd IEEE Conference on Decision and Control, pp. 3609-3614 (© 2003 IEEE).	119
7.1	The vehicle Kelly at the Caltech MVWT [66]	122
7.2	The coordinate systems of a testbed	124
7.3	Obstacle avoidance using the virtual attractive / repulsive poten- tials [66]. The upper 3D figure illustrates the the shape of the virtual potential designed for this task. The black curve on the potential surface depicts the vehicle trajectory. The lower figure shows the vehicle trajectory with respect to the locations of the obstacles and the target. The blue empty circles illustrate the tra- jectory of Kelly while the red encircled X's represent the stationary obstacles. The initial position of Kelly is denoted by a black star on the left, and the target is the green star to the right. The vehicular parameters are $\alpha = 0.066$ N, and $\beta = 5.05$ kg s ⁻¹ . The obstacle parameters are $C_a = 0$, $C_r = 5$ J, and $\ell_r = 0.15$ m. The target is a pure attractive point with $C_a = 3$ J and $\ell_a = 3.5$ m.	126
	\mathbf{r} = \mathbf	

7.4	(a) - (d): Target searching tasks with Kelly facing toward different initial orientations. (e) - (f): Computer simulations of the target searching tasks in ideal situation and in the realistic situation where an offset exists between the left and right fans [66]. The parameters used in these simulations are $\alpha = 0.6$ N, $\beta = 5.05$ Kg s ⁻¹ , $C_a = 0.06$ Nm, $\ell_a = 4$ m, and $C_r = 0$.	127
7.5	Trajectories of the testbed experiments of target searching and ob- stacle avoidance. Left: One obstacle and one target. Right: Mul- tiple obstacles and one target [66]. The vehicular parameters used in these simulations are $\alpha = 0.6$ N and $\beta = 5.05$ Kg s ⁻¹ , the target parameters are $C_a = 0.06$ Nm, $\ell_a = 4$ m, and $C_r = 0$, and the obstacle parameters are $C_a = 0.5$ Nm, $\ell_a = 0.5$ m, $C_r = 1.0$ Nm, and $\ell_r = 0.3$ m.	128
7.6	Two Kellys performing the greedy search algorithm using virtual potentials [66]. The four squares represent the targets while the triangle and the solid circle represent the vehicles' current locations. The circles surrounding the vehicles represent their communication range, which is 1.5 m in diameter. The vehicular parameters used in these simulations are $\alpha = 2.6$ N and $\beta = 5.05$ Kg s ⁻¹ , the target parameters are $C_a = 1.01$ Nm, $\ell_a = 4$ m, and $C_r = 0$, and the collision avoidance parameters are $C_a = 0$ Nm, $C_r = 2.5$ Nm, and $\ell_r = 1$ m.	129
8.1	UCLA MicroCar Testbed. Left: MicroCars. Middle: Cars on the testbed. Right: Partial screen shot of the processed camera image [42].	131
8.2	Definition of variables for vehicle i in the UCLA MicroCar testbed [11]. The heading is denoted by Θ_i , the angle between its direction of motion and the x axis of the testbed. The interaction force it experiences due to all other vehicles is represented by $\vec{F_i}$. This direction defines an angle γ_i with the heading direction. Vehicle i is at a distance $\vec{r_{i,j}}$ from vehicle j and the angles ϕ_i and ϕ_j here shown are used in the collision avoidance scheme later in Section 8.3. The origin of the reference coordinate system is fixed at the left-lower corner of the testbed. All vehicular angles, γ_i , Θ_i , ϕ_i , are defined in $[\pi, -\pi)$.	133

8.3	Collision avoidance failure [11]. The angles ϕ_i and ϕ_j are too small and vehicles i and j collide even if one of them should pause. An additional algorithm is required to steer the vehicles away from each other and is described in the text. It relies on the angle $\Omega_{i,j}$ here depicted	135
8.4	Multiple interacting vehicles following a leader on the UCLA Mi- croCar testbed [11]. The leader is denoted by a black diamond while the vehicles are represented by a blue triangle, a blue circle and a blue square. In the left figure, two vehicles are following the leader on an ellipse trajectory $(N = 2)$. On the right, the leader is followed by three vehicles $(N = 3)$. The parameters used for the vehicle interaction are $C_a = 10^4$ erg, $C_r = 6 \times 10^3$ erg, $\ell_a = 95.2$ cm, $\ell_r = 5.7$ cm. The virtual leader has $C_a = N \times 10^4$ erg and $C_r = 0$ while the other parameters are the same.	139
8.5	Two observed formations of the computer simulations for Eqs (8.2) - (8.3) [11]	140
8.6	Scaling in the H-stable and catastrophic regimes [11]. The potential parameters are $C = 1.667$, $\ell_a = 95.2 \text{ cm}$. The system is H-stable for $\ell_r > 73.5 \text{ cm}$ (top curve $\ell_r = 76.2 \text{ cm}$), otherwise it is catastrophic (middle curve $\ell_r = 69.02 \text{ cm}$; bottom curve $\ell_r = 35.7 \text{ cm}$). The middle curve is at the threshold. Straight lines are power law fits with powers 0.11, 0.01, and -0.00042 for the top, middle, and bottom sets respectively.	141

Chapter 1

Introduction

The abundant morphological features of animal groups have inspired numerous real-life applications and motivated interdisciplinary investigations. One of the most intriguing phenomena is the complex collective behavior out of rather simple individual responses. The implication in engineering is that, instead of a sophisticated robot, one may design a network of much simpler robots to achieve a complex mission. The idea leads to the recent development of so called "artificial swarms". In biology, certain swarming patterns are strikingly similar across various species and scales. Figure 1.1 shows an example, in which similar "milling patterns" are observed in fish schools (left) and marching ants (right). Such patterns lead scientists to speculate that certain collective behavior in animals may be a physical consequence of their dynamics, instead of a conscious biological decision [76]. Identifying such group-level phenomena as a consequence of individual-level dynamics is important in biology because it helps appropriately interpret collective animal behavior. It is also an interesting problem in physics because a swarming system is an extension of a classical many-particle system, which has been of interest in physics for decades with numerous theories derived. Unlike a classical many-particle system, a swarming system consist of self-propelling particles and thus, is sometimes referred as an "active Brownian system" [21, 23, 24, 25] if noise is also involved. The self-propulsion adds additional parameters to the many-particle system and may completely change the dynamics. Moreover, biological interactions can also be different from the conventional form of physical interactions adopted in classical many-particle systems.



Figure 1.1: The mill formations. Left: A photo of a big eye Trevally school, © 2006 Norbert Wu, *http://www.norbertwu.com/*. Right: A sketch of the mill formation by ants in Schneirla's book "Army Ants, a Study in Social Organization" [83]. Reprint of Fig. 12.4 on page 283 in T. C. Schneirla, "Army Ants, a Study in Social Organization", Copyright (1971) by W. H. Freeman.

The difference may give rise to certain unique patterns that are only observed in systems of organisms but not in classical many-particle systems.

While it is speculated that swarms exhibit different pattern formations based on changes of individuals' responses toward surrounding conditions, a common way to examine such connection is through modeling. Various swarming models have been proposed and applied with a certain extent of success. These models can be divided into two major categories: kinematic models and dynamic ones. In Chapter 2, we explain the difference between these two classes of swarming models and briefly review the origin and the evolution of each class. In Chapter 3, a selfpropelling, interacting, many-particle model is constructed based on individual dynamic movement rules. By simulating this particular individual-based model, we present various swarming patterns, including the frequently observed milling patterns shown in Fig. 1.1. Statistical results are given to illustrate the state transition among these patterns. In Chapter 4, we apply a classical theory in statistical mechanics, called "H-stability", to investigate the scalability of our model. An H-stability phase diagram is derived as a result, and the corresponding H-stability regimes for the patterns in Chapter 3 are discussed.

The first three chapters of this thesis consider discrete models for swarms, based on individual-based formulas. Individual-based models have direct control of each particle's rules of motion and are especially useful for small size aggregates while larger discrete systems are more appropriate for statistical studies. In contrast to individual-based models, there are models that ignore the identity of individuals and instead view the swarm as a density field while formulating the particle movement through density fluxes. Such models are called "continuum models". Because of the greatly reduced degrees of freedom, continuum models present a powerful tool for theoretical analysis, especially for analyzing the stability of various morphologies and emergent patterns. However, most continuum models, especially dynamic ones, are based on heuristic arguments rather than derived from convincing individual rules, and thus, it is difficult to link the analytical results to individual behavior. In Chapter 5, we systematically derive a continuum model from our individual-based model by utilizing a classical procedure in statistical mechanics. The validity of such a derivation is numerically verified and discussed. We also investigate the effect of H-stability after it is revealed that H-stability plays an important role on the validity of the derivation. In the regime where the validity of the continuum model is verified, I present a linear stability analysis for a homogeneous solution of the model and compare the theoretically predicted quantities to simulated results.

In Chapter 6, the history of artificial swarms is briefly reviewed. In Chapter 7, we apply our swarming model to devise a control algorithm for the Multi-Vehicle Wireless Testbed (MVWT) at Cal-Tech. Simple tasks such as target searching and obstacle avoidance are presented. In Chapter 8, our model is modified to adapt various mechanical constraints of the testbed vehicles in the UCLA Mathematics department. Cooperative tasks are experimentally performed with multiple vehicles chasing a virtual leader and interacting with each other. Because of the adaption, H-stability of the modified model is re-evaluated and investigated via computer simulations. We also prove a local convergence theory for the modified system.

This thesis is a collection of collaborative works under the supervision of Prof. Bertozzi. For the individual-based model, I contributed to the studies of selfpropulsion-related state transitions, while the interaction-related state transitions (H-stability) were primarily analyzed by Dr. D'Orsogna with suggestions from Prof. Chayes. To piece together our derivation of the continuum model, Dr. Marthaler made his efforts primarily on the locomotory part of the model, and I worked mostly on the interaction part. Dr. D'Orsogna and I also collaborated in the linear stability analysis of the continuum model. Additionally, I am responsible for all numerical simulations of our models presented in the thesis and further investigate the validity of our continuum derivation. For artificial swarms, I mainly contributed to the model adaption, analysis, and numerical simulations of the control algorithm. The testbed vehicle experiments were conducted by Mr. Nguyen, Mr. Huang, Mr. Hsieh, Mr. Leung, and Mr. Tung.

Chapter 2

A Brief History of Swarming Models

In this chapter, we review the literature leading to current research development of swarming models. The kinematic models are introduced first since they appeared earlier in the literature. The dynamic models appeared later but are directly connected to our model.

2.1 Population dynamics

The origin of swarming models can be traced back to models of population dynamics in the early 20th century. While physicists had widely adopted mathematical descriptions for physical systems in the late 19th century, biologists rarely formulated mathematical models for biological systems, apparently due to the greater complexity involved [87]. Population dynamics, which studies the long-term expansion and growth of one or more species, was one of the frontiers in ecology that welcomed abstract mathematical models as a tool of research. The famous Lotka-Volterra equations, also known as the predator-prey equations [99], describe the evolution of the population sizes of interactive prey and predator species:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = \alpha_1 N_1 - \lambda_1 N_1 N_2, \qquad (2.1)$$

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = -\alpha_2 N_2 + \lambda_2 N_1 N_2, \qquad (2.2)$$

where N_1 , N_2 respectively represent the total number of species 1 (prey) and 2 (predator); α_1 is the birth rate of the prey while α_2 is the death rate of the predator; λ_1 and λ_2 are the death and the growth rates of each species due to predation.

The fixed points of the equations are $(N_1, N_2) = (0, 0)$ and $(\alpha_2/\lambda_2, \alpha_1/\lambda_1)$, the former a saddle point and the latter a center. Time dependent solutions circulate the center in the phase space, meaning that the numbers of the prey and predator oscillate periodically in time.

Spacial variation was later introduced to population dynamics. One example is the 1937 article "The Wave of Advance of Advantageous Genes" by Fisher [28]. He considered the spread, by generations, of a mutant gene among a group of organisms on a 1D habitat. The 1D habitat is initially occupied only by organisms that contain the original gene. Because of their advantage to survival, organisms with a particular mutant gene may replace those with the original gene in the next generation. Let $\rho(x, t)$ be the probability of finding the mutant gene at position x at time t while $\rho_a(x, t)$ is that of the original gene. The mutant gene has a spatial diffusion coefficient κ , and κ_m is a coefficient of selecting in favor of the mutant gene for the next generation. The spread of the mutant gene can be described by the following reaction-diffusion equation

$$\frac{\partial \rho}{\partial t} = \kappa \frac{\partial^2 \rho}{\partial x^2} + \kappa_m \rho \rho_a, \qquad (2.3)$$

where t is time, and x is the spatial coordinate. While Fisher recognized that the model might be oversimplified, the wave length and the spreading speed of the mutant gene distribution can be analytically related to κ and κ_m .

A similar concept was adopted by Skellam for studying the spread of oaks in Britain during the post-glacial period. Unlike what Fisher did in Ref. [28], which is based on heuristic arguments, Skellam used an individual-based random walk description to model the dispersal / diffusion process in his 1951 paper "Random Dispersal in Theoretical Population" [87]. He concluded that the proportion of the population distributed a distance r away from the origin after t generations of spread can be expressed as

$$\frac{\int_{|\vec{x}|>r} \rho\left(\vec{x}\right) d\vec{x}}{\rho_{\text{tot}}} = \exp\left(-\frac{r^2}{t\kappa^2}\right),\tag{2.4}$$

assuming that the first generation starts at the origin. The population density is denoted by ρ while ρ_{tot} is the total density; κ is the root-mean-square dispersion distance per generation. Note that the functional form of Eq. (2.4) is the ansatz of Eq. (2.3). The spreading speed can be approximated as the speed of the advancing density contour. Skellam used the model to estimate κ for the oaks given their present distribution and examined some hypotheses regarding their population spread. For swarming problems, what is more significant is that Skellam claimed:

The results already deduced can be applied equally well to the disper-

sal of small animals such as earthworms and snails.

Figure 2.1 from Skellam's paper furthermore demonstrates that the spread of muskrat, *Ondatra zibethica* L., in central Europe supports his claim.

In general, population problems involve dispersion and reproduction of the species population when both temporal and spatial dimensions are considered. In contrast, swarming problems investigate individual or collective density movement of self-propelled agents that aggregate through mutual interactions. The time scale is usually much shorter than the life cycle of the swarming organisms. While population dynamics generally explore boundary expansion of an entire population, swarming problems usually study internal structure and pattern formations inside the group. Thus, swarming problems can be regarded as a class of population problems that focus on a shorter time scale and a smaller spatial domain. While Skellam went on to talk about reproduction in Ref. [87], the reproduction is usually negligible on the swarming time scale and thus, neglected.



Figure 2.1: The spread of muskrat (*Ondatra zibethica* L.) in central Europe [87]. Muskrats were introduced to Europe from North America in 1905. The left figure shows their population distribution during the subsequent years. The right figure shows that the square root of their distribution area increase linear in time, as predicted by Eq. (2.4). Reprint from J. G. Skellam, "Random dispersal in theoretical populations", Biometrika, 1951, Vol. 38, No. 1-2, pp. 196-218, by permission of Oxford University Press.

On the other hand, dispersion does play an important role in swarming models, and the emergent patterns of the density distribution are of interest.

2.2 Kinematic models

Kinematic models are named in contrast to dynamic models. Dynamic models are also called the "Newtonian models" because they are constructed based on Newton's laws of motion. In a dynamic model, forces are formulated and serve as the rate of momentum changes; in turn, positions change according to velocities, evaluated from the momenta. On the other hand, kinematic models are based on more abstract rules. In a kinematic model, velocities are directly formulated and serve as the changing rate of positions. Kinematic models consist of up to only the first-order derivatives of positions with respect to time and thus, are also known as first-order models. In contrast, force or acceleration is the second-order derivative of position versus time, and dynamic models are also called secondorder models. Swarming models that directly evolved from population problems usually belong to the class of kinematic models since kinematic descriptions are usually adopted to simulate population dynamics, such as the models of Fisher [28] and of Skellam [87]. As a predecessor to the modern study of swarming problems, the 1951 paper of Skellam hinted that a random dispersal may not be enough for certain animals which tend to move toward more favorable conditions [87]. He suggested that an attraction may be necessary, which later lead to various aggregation mechanism in swarming models. In his 1980 book, Okubo pointed out that the delicate balance between "spreading" and "concentrating" is one of the most important features that separate the animal movements from passive dispersions such as the spreading of plants and non-organisms. [70].

2.2.1 Aggregation

During the early years, possible mechanisms behind animal aggregation were speculated. An unrelated work that serves as an inspiration is Conway's *Game of Life*, studied by Gardner in his 1970 paper "Mathematical Games" [30]. It is a game played on a checkerboard with counters. The counters survive or die according to the number of neighbors surrounding it. They may also give birth to a new counter in an empty cell if there are a certain number of occupied cells surrounding this cell. Figure 2.2 shows some examples of the evolution histories of such a game. These are rather simple rules, but the final steady distribution patterns of the counters are quite fascinating.

Hamilton proposed a graphical model adopting similar rules to those of the Game of Life in his 1971 paper "Geometry for the Selfish Herd" [38]. For the 1D version of his model, an isolated prey (frog) is more likely to encounter and to be



Figure 2.2: Examples of Conway's Game of Life [30]. The counter distributions start with the configurations on the left and evolve toward the right until the final steady configurations are reached. These are examples of five initial occupied cells.



Figure 2.3: Hamilton's selfish-herd model [38, 70]. The line segment AB is the domain of danger of frog c. If by accident, the frog jumps to c', its domain of danger is reduced. Reprint of Fig. 7.16 on page 129 of Ch. 7 "The dynamics of animal grouping" in A. Okubo, (1980) Diffusion and Ecological Problems: Mathematical Models, with kind permission of Springer-Verlag and Business Media.

attacked by a predator (snake). Therefore, he defined a "domain of danger" for each frog as the summation of the half lengths of the distances from its closest neighbors on both sides. The danger of a frog being eaten by the snake is proportional to this domain of danger of the frog. As a result, the frog with a larger domain of danger also has a higher probability to make a random jump, trying to improve its situation. Figure 2.3 illustrates this process. After several iterations, the frogs aggregate as several clusters. The concept for the 2D version of the model is similar, but the domains of danger become polygons rather than line segments. It is known that for example, lions usually attack preys from outside the herd, and the model shows that the prey aggregate in an effort to reduce the domain of danger of each individual.

While it is plausible that minimizing the danger of being attacked can lead to aggregation, others speculate that foraging efficiency also encourages aggregation. Thompson et al. examined such a conjecture in their 1974 paper "Survival Value of Flocking in Birds" [92]. Their full model is composed of several rather complicated logic loops that require a lengthy description to detail. The main idea is that a bird tends to maintain its direction but also has a probability to change direction, depending on environmental factors. Additionally, the birds also have a probability of making social movements, due to interactions, calculated by a "movement response" $\Delta x_{\rm MR}$. The interaction has an attraction and a repulsion: a bird is attracted to the birds which have successfully found food but repelled by the others if their mutual distance becomes less than a threshold. The mathematical formula are composed of pairwise components

$$\Delta x_{\mathrm{MR},i,j} = \begin{cases} 0 & \text{if } r_{i,j} > r_c \text{ and both birds} \\ & \text{searching for prey,} \\ -\frac{(r_{i,j} - r_c)\vec{r}_{i,j}}{r_{i,j}} \mathrm{e}^{-\frac{\left(r_{i,j} - r_c\right)}{\ell}} & \text{otherwise,} \end{cases}$$
(2.5)

and the movement response for bird i in the next iteration is the summation over the other birds

$$\Delta x_{\mathrm{MR},i} = \sum_{j} \Delta x_{\mathrm{MR}}, i, j.$$
(2.6)

Here, $\vec{r}_{i,j}$ is the displacement between bird *i* and bird *j*, and r_c is the threshold distance. The parameter ℓ is determined in their paper by fitting the collected data from earlier experiments. Their results showed that by flocking, birds greatly reduce the risk of not finding any food for a lengthy period of time, which increases the chance of survival.

2.2.2 Biased random walks

The selfish herds of Hamilton and the bird flocks of Thompson et al. are basically extensions of Skellam's random walk model but with inhomogeneous probabilities of individual movements. Such extensions are regarded as a class of "biased random walk models" where the random walks take different probabilities under different conditions. In his 1980 paper "Biased Random Walk Models for Chemotaxis and Related Diffusion Approximations" [2], Alt studied a group of models in which the inhomogeneity is caused by external stimuli. In contrast to Hamilton and Thompson et al., Alt formulate the random walks as the movement of a continuum density field, $\rho(t, \vec{x}, \hat{v})$, which depends not only on time t and position \vec{x} but also on its drift direction \hat{v} . Assuming that the individuals drift at a constant speed v, the random walks are caused by a stochastic probability $f(t, \vec{x}, \hat{v})$ for each individual to change direction to \hat{v} . This is similar to the mechanism used by Thompson et al. for bird flocks. The probability distribution of the drift direction changing from \hat{v} to \hat{u} is denoted by $k(t, \vec{x}, \hat{v}, \hat{u})$. The equation of motion for $\rho(t, \vec{x}, \hat{v})$ is

$$\frac{\partial \rho}{\partial t} + \hat{v} \cdot \vec{\nabla}_{\vec{x}} \left(v\rho \right) = -f\rho + \int_{\mathcal{B}} f\left(t, \vec{x}, \hat{u} \right) \rho\left(t, \vec{x}, \hat{u} \right) k\left(t, \vec{x}, \hat{u}, \hat{v} \right) \mathrm{d}\hat{u}, \tag{2.7}$$

where $\int_{\mathcal{B}} d\hat{u}$ is a surface integral over a unit ball. While Alt concentrated on chemotaxis for the discussion of the turning frequency f and the turn angle distribution k, it is worth noting that the same idea can also be used for interactions, where fk defines the interaction kernel.

A more generalized application of the biased random walks was introduced by Okubo in his 1986 paper "Dynamical Aspects of Animal Grouping: Swarms, Schools, Flocks, and Herds" [71]. Let ρ be a density field that represents the distribution of individuals that are allowed to move to adjacent locations. Okubo's basic 1D equation is of the form

$$\rho(x, t + \delta t) = k_{\rm R} (x - (1 - \Phi) \,\delta x, t) \,\rho(x - \delta x, t) + k_0 (x, t) \,\rho(x, t) \quad (2.8)$$
$$+ k_{\rm L} (x + (1 - \Phi) \,\delta x, t) \,\rho(x + \delta x, t) \,,$$

where x and t are the position coordinate and time with δx and δt denoting infinitesimal increases in x and t. The probability of particles jumping from x to $x + \delta x$ is specified by $k_{\rm R} (x + \Phi \delta x, t)$ while $k_{\rm L} (x - \Phi \delta x, t)$ is that of jumping from x to $x - \delta x$, in which $\Phi \in [0, 1]$. The probability of staying put is denoted by $k_0(x, t)$, and the probabilities satisfy the normalization condition

$$k_{\rm R} \left(x + \Phi \delta x, t \right) + k_{\rm L} \left(x - \Phi \delta x, t \right) + k_0 \left(x, t \right) = 1.$$
(2.9)

Note that the destination of a jump is $x \pm \delta x$, and x is the original location before the jump. Hence, if $\Phi = 1$, $k_{\text{R/L}}$ depend only on the destination, and this case can be interpreted as that the individuals are *attracted* toward the location x (for whatever reasons). Similarly, if $\Phi = 0$, $k_{\text{R/L}}$ depend only on the original location and indicate a *repulsion*. For $0 < \Phi < 1$, it can be viewed as a combination of the attraction and the repulsion. Since $k_{\text{R/L}}$ represent the jump probabilities toward the right and toward the left, the random walk is biased if $k_{\text{R}} \neq k_{\text{L}}$. We may define

$$k(x,t) \equiv k_{\rm R} \left(x + \Phi \delta x, t \right) - k_{\rm L} \left(x - \Phi \delta x, t \right), \qquad (2.10)$$

to indicate such bias. By letting $\delta t, \delta x \to 0$, we may take Eq. (2.9) to its continuum limit and obtain

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x} \left\{ \left[v - (1 - 2\Phi) \frac{\partial \kappa}{\partial x} \right] \rho \right\} + \frac{\partial}{\partial x} \left(\kappa \frac{\partial \rho}{\partial x} \right), \qquad (2.11)$$

where

$$v(x,t) \equiv \lim_{\delta t, \delta x \to 0} k(x,t) \, \delta x / \delta t,$$
 (2.12)

$$\kappa(x,t) \equiv \kappa_0 \left(1 - k_0(x,t)\right), \qquad (2.13)$$

$$\kappa_0 \equiv \lim_{\delta t, \delta x \to 0} \delta x^2 / (2\delta t) \,. \tag{2.14}$$

The diffusion coefficient κ_0 is a constant, and κ is the diffusivity, proportional to the probability of individuals leaving x. The anisotropic jump probabilities result in a non-zero k and hence, a non-zero drift speed v. The advection term in Eq. (2.11) suggests that the spatial inhomogeneity can also cause the individuals to drift; the results show that individuals in a repulsion driven situation (e.g. hazard avoidance) tend to aggregate near the minima of the diffusivity while those in an attraction driven situation (e.g. food searching) tend to aggregate near the diffusivity maxima. A common conjecture is that the diffusion among animals is for the purpose of avoiding overcrowd, which can be classified as a repulsion-induced diffusivity. Therefore, some models choose to construct a diffusivity inversely proportional to the local density field and result in aggregations near where the density is already higher. Note that the mechanism behind such repulsion-induced aggregations is more in line with Hamilton's selfish herds [38] and different from the models of Thompson et al. [92] and of Alt [2], where individuals are actively seeking company. Nonetheless, many kinematic models assume the form of an advection-diffusion equation similar to Eq. (2.11) as a building block [16, 65, 84]. Indeed, an active aggregation due to attraction can be formulated as the biased-induced drift speed v, such as the models discussed in "The Regulation of Inhomogeneous Populations" by Gurney and Nisbet [37] and in "Partial differential equations in ecology: spatial interactions and population dynamics" by Holmes et al. [41]

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left(\frac{c \partial \rho}{\partial x} \rho \right) + \kappa \frac{\partial^2 \rho}{\partial x^2}.$$
(2.15)

Here, κ is homogeneous, and $v \equiv c\partial\rho/\partial x$. If the constant c is chosen greater than zero, the drift is biased toward a higher ρ , and the individuals move upward along the density gradient, resulting in aggregation. On the other hand, the convolution form in Eq. (2.7) provides an alternative idea of handling the active attractions. Although Eq. (2.7) is used for an externally induced drift aggregation in Ref. [2], the convolution can also be applied for internally induced swarming aggregation [62, 94], which is the subject of our research in this thesis.

2.2.3 The position-jump and the orientation-jump processes

Another difference among typical kinematic swarming models is that some models, such as those of Hamilton and of Okubo [38, 71], describe the process as consecutive jumps in positions of individuals, where the length of a jump may vary; others, such as the models of Thompson et al. and of Alt, describe the process as a series of discrete changes in individuals' orientations, where the speed of an individual is usually assumed constant. The former are called *position-jump models* or "kangaroo processes" by Othmer et al. in their 1988 paper "Models of Dispersal in Biological Systems" [74] since the individuals jump like kangaroos. The latter belong to a sub-class of what Othmer et al. call velocity-jump models. They are called *orientation-jump models* here because the most general cases of velocity-jump models in Ref. [74] also include dynamic models, which are discussed in Section 2.3. Position-jump models are the direct descendants of dispersal models in population dynamics while orientation-jump models are variants that lead to dynamic swarming models.

To derive a general form for both position-jump and orientation-jump models, Othmer et al. begin with a waiting time probability function f(t), which specifies the probability of individuals waiting for a period of time t between jumps. The probability of an individual staying put (not making jumps) within t is then

$$\hat{f}(t) = \int_{t}^{\infty} f(s) \, \mathrm{d}s. \tag{2.16}$$

Assuming that the probability of an individual jumping from location \vec{y} to location \vec{x} at time t is $k(\vec{x}, \vec{y}, t)$, the amount of individuals that jump into \vec{x} exactly at

time t can be expressed as

$$\delta\rho(\vec{x},t) = \delta_{t,0}\rho(\vec{x},0) + \int_0^t \int_{\mathbb{R}^d} f(t-s) \, k(\vec{x},\vec{y},t) \, \delta\rho(\vec{y},s) \, \mathrm{d}\vec{y} \, \mathrm{d}s, \qquad (2.17)$$

where $\rho(\vec{x}, 0)$ is the initial density distribution; $\delta_{t,0} = 1$ when t = 0 and = 0 otherwise. Hence, the density distribution at time t is

$$\rho(\vec{x},t) = \int_0^t \hat{f}(t-s_1) \,\delta\rho(\vec{x},s_1) \,\mathrm{d}s \qquad (2.18)$$

$$= \hat{f}(t) \,\rho(\vec{x},0)$$

$$+ \int_0^t \int_0^{s_1} \int_{\mathbb{R}^d} \hat{f}(t-s_1) \,f(s_1-s_2) \,k(\vec{x},\vec{y},s_2) \,\delta\rho(\vec{y},s_2) \,\mathrm{d}\vec{y} \,\mathrm{d}s_2 \,\mathrm{d}s_1.$$

When k is independent of time, Eq. (2.18) can further be reduced to

$$\rho(\vec{x},t) = \hat{f}(t)\,\rho(\vec{x},0) + \int_0^t \int_{\mathbb{R}^d} f(t-s)\,k(\vec{x},\vec{y})\,\rho(\vec{y},s)\,\mathrm{d}\vec{y}\,\mathrm{d}s.$$
(2.19)

The random motion of animals is commonly modeled as a Poisson process, characterized by $f(t) = c \exp(-ct)$, where c is an arbitrary constant. By differentiating Eq. (2.19) with respect to t,

$$\frac{\partial \rho}{\partial t} = -c\rho + c \int_{\mathbb{R}^d} k\left(\vec{x}, \vec{y}\right) \rho\left(\vec{y}\right) \,\mathrm{d}\vec{y} \tag{2.20}$$

for a Poisson process. In one dimension with

$$k(x,y) = \frac{1}{\delta x} k_0 \left(\frac{x-y}{\delta x}, \delta x\right), \qquad (2.21)$$

one obtains an advection-diffusion equation

$$\frac{\partial \rho}{\partial t} = v \frac{\partial \rho}{\partial x} + \kappa \frac{\partial^2 \rho}{\partial x^2} + \mathcal{O}\left(\delta x^3\right), \qquad (2.22)$$
provided that the constants

$$v = \frac{c\delta x^2}{2} \int_{\mathbb{R}} \frac{k_0(r, \delta x)}{\delta x} r \mathrm{d}r, \qquad (2.23)$$

$$\kappa = \frac{c\delta x^2}{2} \int_{\mathbb{R}} k_0(r, \delta x) r^2 \mathrm{d}r, \qquad (2.24)$$

converge in the diffusion limit ($\delta x \to 0, c \to \infty$, and $c\delta x^2 \to a$ finite constant). Note that for non-biased random walk processes, k_0 is symmetric and v vanishes, leaving only the diffusion term in Eq. (2.22). These are consistent with the findings by Okubo in Ref. [71] and further support the use of advection-diffusion equations for position-jump models.

Similar processes can also be applied to orientation-jump models. The equation analogous to Eq. (2.20) for a Poisson process is

$$\frac{\partial \rho}{\partial t} + v\hat{v} \cdot \vec{\nabla}\rho = -c\rho + c \int_{\mathcal{B}^{d-1}} k\left(\hat{v}, \hat{u}\right) \rho\left(\vec{x}, \vec{v}\right) \,\mathrm{d}\hat{u}.$$
(2.25)

Here, k is the probability of an individual changing its orientation from \hat{u} to \hat{v} . Note that Eq. (2.25) becomes the same as Eq. (2.7) derived by Alt in Ref. [2] as c is a constant turning frequency. For a 1D problem,

$$k\left(\hat{v},\hat{u}\right) = \delta_{\hat{u},-\hat{v}} \tag{2.26}$$

because the orientation can only switch to the opposite direction when a jump is made. This reduces Eq. (2.25) to a damped wave equation

$$\frac{\partial^2 \rho}{\partial t^2} + 2c \frac{\partial \rho}{\partial t} = v^2 \frac{\partial^2 \rho}{\partial x^2},\tag{2.27}$$

also known as the *telegraph equation*.

The advection-diffusion models as in Eq. (2.22) are sometimes criticized be-



Figure 2.4: The solutions of the diffusion and the telegraph equations [40]. The initial condition is a delta pulse at the release point. The two figures show snapshots at t = 5 (left) and at t = 7.6 (right). Reprint of Fig. 1 on page 783 in E. E. Holmes, "Are diffusion models too simple? A comparison with telegraph models of invasion", American Naturalist, Vol. 142, No. 5, pp. 779-795 ©1993 by The University of Chicago.

cause they yield solutions with infinite speed of propagation of the support [40, 41, 62]. This is an unphysical situation, and thus, the solutions of the advectiondiffusion equations are often seen as only an approximation to the actual density distribution. On the other hand, the telegraph equation in Eq. (2.27) has a more favorable solution, where the density can only spread with a finite speed. The comparison of the solutions for both models are shown in Fig. 2.4; the solutions of the telegraph equation stay compactly-supported after a finite time while those of the diffusion equation spread indefinitely. However, the derivation of the advection-diffusion equation is more general and can be extended to higher dimensions while the derivation of the telegraph equation really depends on the specific 1D properties [74]. Therefore, the advection-diffusion models are often reasonably assumed for position-jump processes, and the functional forms of the advection and the diffusion terms are conveniently altered on a heuristic ground for various situations or assumptions. The individual jump rules of a general orientation-jump model are usually far more complicated than the 1D case, and its PDE description cannot be obtained as elegantly as in Eq. (2.27). As a result, the advection-diffusion models are usually adopted for analytical studies because of their simplicity. An orientation-jump model often retains its individual-based forms to serve application purposes and may eventually evolve into a dynamic model. Sections 2.2.4 and 2.2.5 discuss more recent developments in each of the categories.

2.2.4 Advection-diffusion models

As previously mentioned, the advection-diffusion models are sometimes criticized for their unconfined solutions that allows individuals to travel at an infinite speed. Attempts were made to provide remedies for this non-physical situation. Holmes compared the solutions of the diffusion and the telegraph equations in her 1993 paper "Are Diffusion Models too Simple? A Comparison with Telegraph Models for Invasion" [40]. It is shown that without population reproduction, both solutions are very close to each other after some transient time and converge toward each other asymptotically (Fig. 2.4). Therefore, it is reasonable to regard the advection-diffusion models as an approximation of the more realistic cases where individuals should only travel at finite speeds.

A further improvement of this situation is provided by Mogilner and Edelstein-Keshet in their 1999 paper "A Non-Local Model for a Swarm" [62]. They begin with the general 1D advection-diffusion form as in Eq. (2.11) with $\Phi = 0.5$ (dispersal due to both attraction and repulsion)

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x} \left(v\rho \right) + \frac{\partial}{\partial x} \left(\kappa \frac{\partial \rho}{\partial x} \right).$$
(2.28)



Figure 2.5: The effects of an odd kernel (left) and an even one (right) on density movements in Ref. [62]. Reprint of Fig. 1 on page 541 in A. Mogilner and L. Edelstein-Keshet (1999) "A non-local model for a swarm", Journal of Mathematical Biology, Vol. 38, No. 6, pp. 534-570, with kind permission of Springer-Verlag and Business Media.

The novel idea is that the drift speed v takes a convolution form

$$v(\rho(x,t)) = k * \rho$$

$$= \int_{y-r}^{y+r} k(x-y) \rho(y,t) dy,$$
(2.29)

introducing non-locality into the dynamics. They found that the even kernels provide a drift motion toward either direction while the odd ones cause the population density to aggregate or to spread, as shown in Fig. 2.5. Knowing the effects of the kernels, a particular kernel is constructed

$$v = k_e \rho + (C_a - C_r \rho) \left(k_o * \rho\right), \qquad (2.30)$$

where k_e , C_a , and C_r are scalar constants and k_o is an odd kernel

$$k_o = \begin{cases} -\frac{x}{2r|x|} & \text{if } |x| \le r \\ 0 & \text{otherwise} \end{cases}$$
(2.31)

Here, r represents an interaction length scale. The drift term $k_e \rho$ is a local function, and the aggregation is modeled by an attraction $C_a k_o * \rho$ and a repulsion $C_r \rho k_o * \rho$, both of which are non-local. The diffusion κ can stabilize the density



Figure 2.6: A basic simulation for the model of Mogilner and Edelstein-Keshet in Ref. [62]. The horizontal axis represents the one-dimensional spatial coordinate x, while the vertical axis represents density ρ . Reprint of Fig. 4 on page 556 in A. Mogilner and L. Edelstein-Keshet (1999) "A non-local model for a swarm", Journal of Mathematical Biology, Vol. 38, No. 6, pp. 534-570, with kind permission of Springer-Verlag and Business Media.

aggregation if it is chosen density-dependent

$$\kappa = \kappa_0 \rho, \tag{2.32}$$

where κ_0 is a positive constant. It is shown that if the diffusion is small enough

$$\kappa_0 < \frac{C_a r}{4} \left(1 - \frac{2k_e}{C_a} \right), \tag{2.33}$$

the compactly-supported solutions are locally stable. A basic simulation of the model is shown in Fig 2.6. A small portion of the density trails behind the main group while the front edge touches down to zero. Although there is no global stability for such "traveling-band" solutions, they last long enough to make their existence meaningful on biologically reasonable time scales.

The studies of the non-local model of Mogilner and Edelstein-Keshet were extended from 1D to 2D by Topaz and Bertozzi in their 2004 paper "Swarming Patterns in a Two-Dimensional Kinematic Model for Biological Groups" [94]. Here, the diffusion κ due to random movement is assumed negligible since local stability exists only for small diffusion. The 2D equations of motion are

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot (\vec{v}\rho), \qquad (2.34)$$

$$\vec{v} = \vec{k} * \rho = \int_{\mathbb{R}^2} \vec{k} \left(|\vec{x} - \vec{y}| \right) \rho \left(\vec{y} \right) \, \mathrm{d}\vec{y}.$$
 (2.35)

Here, the integration kernel \vec{k} is a two-dimensional vector, which can generally be written as

$$\vec{k} \equiv \vec{\nabla}^{\perp} V_N + \vec{\nabla} V_P \tag{2.36}$$

by applying the Hodge decomposition, where $\vec{\nabla}^{\perp} \equiv (-\partial/\partial y, \partial/\partial x)$ in 2D. The first term on the RHS of Eq. (2.36) contributes to density-conserved "incompressible motion" while the second term describes the compressibility of the density distribution, named "potential motion" in the paper. These two component can be investigated independently. For the incompressible motion, the model resembles that of vortex patches in the fluid dynamics. A typical example is shown in Fig. 2.7 by using a Gaussian kernel for V_N . The gray patch is a constant density distribution, which begins to rotate and develops spiral arms. On the other hand, the potential motion is shown in Fig. 2.8 using the same Gaussian kernel but as V_P here. The density gradually concentrates into several groups. The power spectra show that each final distribution has a particular wave number which is related to the interaction length, defined in the kernel V_P . For the case of a general kernel \vec{k} in 2D, the density movement should be a combination of the above two types. While the potential motion depicts the aggregation behavior of animals,



Figure 2.7: The incompressible motion of the 2D model in Ref. [94]. The time evolution of the pictures is from top-left to bottom-left, then from top-right to bottom-right. The shaded area represents a constant density evolving on a 2D (x-y) plane. Reprint from C. M. Topaz and A. L. Bertozzi, "Swarming patterns in a two-dimensional kinematic model for biological groups", SIAM Journal on Applied Mathematics, Vol. 65, pp. 152-174, Copyright (2004) by SIAM.



Figure 2.8: The potential motion of the 2D model in Ref. [94]. In the left panel are the spatial distributions of the density in a $2\pi \times 2\pi$ box. In the right panel are the power spectra of the density distributions to their left. Darker colors mean higher densities. The top figures are the initial distribution, the middle ones are the case of a shorter interaction-length (equaling 0.4), and the bottom ones are of a longer interaction-length (equaling 1.0). Reprint from C. M. Topaz and A. L. Bertozzi, "Swarming patterns in a two-dimensional kinematic model for biological groups", SIAM Journal on Applied Mathematics, Vol. 65, pp. 152-174, Copyright (2004) by SIAM.

the incompressible motion resembles certain circular movements among animals, such as the mill-formation in fish and in ants, discussed later in Chapter 3.

More variations of the advection-diffusion models can be found in Ref. [22, 35, 36, 41, 50, 64, 95]. The advection-diffusion models are usually more favorable for 1D problems because the jump rules are simpler in 1D, and the derivation, as well as the model construction, is rather straightforward. In higher dimensions, it becomes less clear how the functional forms of the advection and the diffusion terms are related to individual rules, and the individual rules are often more so-phisticated. As a result, researchers often model the swarms in higher dimensions as sets of directional rules, which are categorized as orientation-jump models.

2.2.5 Orientation-jump models

Orientation-jump models are especially favorable for modeling bird flocks and fish schools, which are known to keep a relatively constant individual speed and to reach intended positions by changing the heading direction. The bird flock model of Thompson et al. [92] in Section 2.2.1 is an earlier example of orientation-jump models. The directional rules of Thompson et al. are too complicated for much theoretical analysis other than behavioral observations. One of the most abstract orientation-jump models is the one proposed by Vicsek et al. in their work "Novel Type of Phase Transition in a System of Self-Driven Particles" in 1995 [98]. Not restricted to any specific species, the model describes a group of general swarming particles that change their orientations to align with each other. The equations of motion are

$$\vec{x}_i \left(t + \Delta t \right) = \vec{x}_i \left(t \right) + \vec{v}_i \left(t \right) \Delta t, \qquad (2.37)$$

$$\vec{v}_i(t) = v \cos \Theta_i(t) \, \hat{x} + v \sin \Theta_i(t) \, \hat{y}, \qquad (2.38)$$

$$\Theta_i \left(t + \Delta t \right) = \langle \Theta \left(t \right) \rangle_{i,\ell} + \eta_{\Theta}, \qquad (2.39)$$

where \vec{x}_i and \vec{v}_i are respectively the position and the velocity of particle *i*; *v* is a constant speed; the orientation is denoted by an angle Θ_i ; $\langle \Theta(t) \rangle_{i,\ell}$ is the ensemble average of the orientations of all particles located within an interaction range ℓ of particle i. A stochastic term η_{Θ} is added to the orientation jump, representing noise in the swarm. Note that for the case of v = 0, the model reduces to an analog of a well-known XY-model [52]. This is an extremely abstract model and thus, particularly suitable for theoretical analysis. Simulations show that for 0.003 < v < 0.3, the results are not affected by the value of v, and thus, Vicsek et al. adopt v = 0.03 for their presentation. The value of η_{Θ} is randomly chosen with a uniform probability distribution from the interval $[-\eta/2, \eta/2]$. The computational domain is an $L \times L$ box with periodic boundary conditions. The number of particles are denoted by N, and thus, $\rho \equiv N/L^2$ is the density. Vicsek et al. found that the aggregation behaviors vary with respect to η and ρ . Examples of their simulations can be found in Fig. 1 of Ref. [98]. It shows that for low density and low noise, the particles aggregate into small groups with locally coherent motion. For high density and high noise, the particles drift randomly, and for high density and low noise, almost all particles move coherently. The degree of coherence can be measure by a quantity P, called the "polarity"

$$P \equiv \frac{1}{Nv} \left| \sum_{i=1}^{N} \vec{v}_i \right|. \tag{2.40}$$

For highly coherent configurations, $P \simeq 1$, and for those randomly distributed, $P \simeq 0$. They discovered that the model exhibits an interesting phase transition

$$P \propto (\eta_c (\rho) - \eta)^{\beta_1}, \qquad (2.41)$$

$$P \propto \left(\rho - \rho_c\left(\eta\right)\right)^{\beta_2}, \qquad (2.42)$$

where η_c and ρ_c are respectively critical noise and critical density, which are evaluated from the simulations. The critical exponents can be fitted to obtain $\beta_1 = 0.45 \pm 0.07$ and $\beta_2 = 0.35 \pm 0.06$.

While the model of Vicsek et al. is not restricted to any specific species, many orientation-jump models are based on real-life examples. In an effort to better describe the movement of organisms, a series of development eventually leads orientation-jump models to dynamic models. The model of Thompson et al. [92] in Section 2.2.1 is an early example that involves sophisticated orientation rules. For the purpose of parametric studies, simplified models are more suitable. Huth and Wissel design an orientation-jump model for fish to test some assumptions in their 1992 paper "The Simulation of the Movement of Fish Schools" [44]. One particular question is whether a fish adjust its orientation according to that of a particular neighbor, or to the averaged direction of multiple neighbors. Let $x_{i,j}$ be the distance between fish i and its neighbor fish j. Following the behavioral patterns suggested by Aoki in his 1982 paper "A Simulation Study on the Schooling Mechanism in Fish" [3], Huth and Wissel categorized the neighbor distances into four different areas: the repulsion area $(x_{i,j} < r_1)$, the parallel area $(r_1 < x_{i,j} < r_2)$, the attraction area $(r_2 < x_{i,j} < r_3)$, and the searching area $(x_{i,j} > r_3)$, as shown in Fig. 2.9. Additionally, there is a blind spot behind the fish, characterized by an angle ω and also categorized as the searching area. When two fish are in the parallel area, they tend to align their orientation. In the repulsion area, fish veer away from each other by each heading toward the perpendicular direction of the other's current orientation. In the attraction area, the fish turn toward each other's current location. In the searching area, a fish just randomly picks an orientation. Let $\delta \Theta_{i,j}$ be a turning angle taken by fish i for the next time step due to the presence of fish j; the orientation rules can be



Figure 2.9: The four different interaction areas defined by Huth and Wissel in Ref. [44]. From inner to outer, the four areas are respectively the repulsion area, the parallel area, the attraction area, and the searching area. Reprinted from Journal of Theoretical Biology, Vol. 156, A. Huth and C. Wissel, "The simulation of the movement of fish schools", pp. 365-385, Copyright (1992) with permission from Elsevier.

formulated as

$$\delta\Theta_{i,j} = \begin{cases} \min\left(\arg\left(\hat{v}_j - \hat{v}_i\right) \pm \frac{\pi}{2}\right) & \text{if } x_{i,j} < r_1 \text{ (attraction area)}, \\ \arg\left(\hat{v}_j - \hat{v}_i\right) & \text{if } r_1 < x_{i,j} < r_2 \text{ (parallel area)}, \\ \arg\left(\frac{\vec{x}_j - \vec{x}_i}{|\vec{x}_j - \vec{x}_i|} - \hat{v}_i\right) & \text{if } r_2 < x_{i,j} < r_3 \text{ (attraction area)}, \\ \text{a random angle} & \text{if } x_{i,j} > r_3 \text{ (searching area)}, \end{cases}$$
(2.43)

where \vec{x}_i and \hat{v}_i are respectively the position and the orientation of fish *i*, and arg (\hat{v}) is the angle of the orientation \hat{v} . For swarms consisting more than two fish, the actual turning angle of fish *i*, $\Delta \Theta_i$, is randomly picked from one of the following two probability distributions

$$f(\Delta\Theta_i) = \sum_{j=1}^{\mu} \frac{6\sqrt{2}}{\pi^2} b_j e^{-\frac{\left(\Delta\Theta_i - \delta\Theta_{i,j}\right)}{2(\pi/12)^2}} \quad (D\text{-model}),$$
(2.44)

$$f(\Delta\Theta_i) = \frac{6\sqrt{2}}{\pi^2} b_j e^{-\frac{\left(\Delta\Theta_i - \langle \delta\Theta_{i,j} \rangle\right)}{2(\pi/12)^2}} \quad (A-\text{model}), \quad (2.45)$$

where $\langle \delta \Theta_{i,j} \rangle = \left(\sum_{j=1}^{\mu} \delta \Theta_{i,j} \right) / \mu$, μ is the number of neighbors, and b_j is a weight factor of fish *j*. *D*-model (Eq. (2.44)) lets a fish randomly decide a neighbor to follow, while in *A*-model (Eq. (2.45)) the turning angle is an ensemble average over all neighbors. After $\Delta \Theta_i$ is determined, the equations of motion are

$$\vec{x}_i \left(t + \Delta t \right) = \vec{x}_i \left(t \right) + \vec{v}_i \left(t + \Delta t \right) \Delta t, \qquad (2.46)$$

$$\vec{v}_i(t) = v(t) \hat{v}_i(t),$$
 (2.47)

$$\hat{v}_i(t + \Delta t) = \begin{pmatrix} \cos \Delta \Theta_i(t) & -\sin \Delta \Theta_i(t) \\ \sin \Delta \Theta_i(t) & \cos \Delta \Theta_i(t) \end{pmatrix} \hat{v}_i(t), \quad (2.48)$$

where $\vec{v_i}$ is the velocity of fish *i*, composed of its magnitude v_i and its orientation $\hat{v_i}$. The magnitude v_i is a random number picked from a Gamma distribution for each time step

$$f_{\Gamma}\left(v_{i}\right) \propto v_{i}^{3} \mathrm{e}^{-3.3v_{i}}.$$
(2.49)



Figure 2.10: The functions of behavior patterns adopted by Reuter and Breckling in Ref. [78]. The most preferred swimming distance between neighbors is denoted by md in the figure, where the parallel orientation increases to 100% without any attraction or repulsion. The x-axis represents the neighbor distance ma, and the y-axis specifies the percentage of the repulsion, the parallel orientation, and the attraction. Reprinted from Ecological Modelling, Vol. 75-76, H. Reuter and B. Breckling, "Selforganization of fish schools: an object-oriented model", pp. 147-159, Copyright (1994) with permission from Elsevier.

By comparing the *D*-model and the *A*-model, Huth and Wissel found that the *A*-model better resembles the group behaviors of real fish schools than the *D*-model and produces more coherent movement among the fish. In their subsequent paper "The Simulation of Fish Schools in Comparison with Experimental Data" in 1994 [45], the internal structures of the fish schools produced by the *A*-model also compare favorably to experimental data. Most swarming models, including ours, calculate neighbor-interactions using ensemble averages, and the results of Huth and Wissel justify this conjecture. A close variant of Huth and Wissel's model is provided by Reuter and Breckling in their 1994 paper "Selforganization of Fish

Schools: an Object-Oriented Model" [78]. In their model, the interaction areas are no longer distinctly separated; instead, the interaction becomes a mixture of various ratios of the repulsion, the parallel orientation, and the attraction, as shown in Fig. 2.10. Comparing to the model of Huth and Wissel, this model is more capable of overcoming randomized initial configurations and environmental disturbance to result in self-organized coherent swarms.

Although kinematic descriptions are reasonably suited for the cases such as fish and birds that feature approximately constant speeds, there are limitations. Most notably, the orientation of the velocity cannot abruptly change by an arbitrarily large angle, as assumed in a kinematic orientation-jump model. A more realistic formulation is imposing a finite turning rate, which is a dynamic feature since it is associated to the changing rate of the velocity. An example of such a hybrid model appears in the 2002 paper "Collective Memory and Spatial Sorting in Animal Groups" by Couzin et al. [14]. The basic idea is similar to that of Huth and Wissel [44]. Several behavioral zones are defined, as shown in Fig. 2.11, including the zone of repulsion (zor), the zone of orientation (zoo), and the zone of attraction (zoa), of which the radii are respectively denoted by r_r , r_o , and r_a . A particular individual tends to move away from the neighbors in zor, to align its orientation with that of the neighbors in zoo, and to move towards the neighbors in zoa. As a result, the "intended" orientation for an individual at the next time step is

$$\hat{v}'_{i} = \frac{\vec{v}'_{i}}{|\vec{v}'_{i}|}, \tag{2.50}$$

$$\vec{v}'_{i} \equiv \sum_{j \in zor} \frac{\vec{x}_{i,j}(t)}{x_{i,j}(t)} + \sum_{j \in zoo} \hat{v}_{i}(t) - \sum_{j \in zoa} \frac{\vec{x}_{i,j}(t)}{x_{i,j}(t)}.$$
(2.51)

An exceptional situation is when there is no neighbor present in any of the be-



Figure 2.11: The behavioral zones defined by Couzin et al. in [14]: *zor* is zone of repulsion, *zoo* is zone of orientation, and *zoa* is zone of attraction. The angle α defines a field of perception for an individual, and thus, the region behind an individual, indicated by $(360 - \alpha)^{\circ}$, is a blind spot which it cannot see. Reprinted from Journal of Theoretical Biology, Vol. 218, I. D. Couzin, J. Krause, R. James, G. D. Ruxton, and N. R. Franks, "Collective memory and spatial sorting in animal groups", pp. 1-11, Copyright (2002) with permission from Elsevier.

havioral zones. In this case, the individual maintains its current orientation, i.e., $\hat{v}'_{i} = \hat{v}_{i}(t)$. Here, $\hat{v}_{i}(t)$ is the orientation of particle *i* at time *t*. The displacement vector from particle i to particle j is denoted by $\vec{x}_{i,j} \equiv \vec{x}_j - \vec{x}_i$, where \vec{x}_i is the position of particle i; $x_{i,j} \equiv |\vec{x}_{i,j}|$ is the scalar distance. Let ω be a finite turning rate and Θ be the angle between the unit vectors $\hat{v}_i(t)$ and \hat{v}'_i . The orientation of particle *i* at the next time step $\hat{v}_i (t + \Delta t) = \hat{v}'_i$ if $\Theta \leq \omega \Delta t$; otherwise, particle *i* turns an angle $\omega \Delta t$ toward the direction of \hat{v}_i . By varying the widths of zoo and zoa, denoted by $\Delta r_o \equiv r_o - r_r$ and $\Delta r_a \equiv r_a - r_o$ respectively, the authors find dramatical transitions among four distinct states, as shown in Fig. 2.12. Such states can be distinguished by two group properties, named the polarity and the angular momentum, which we adopt for our studies and discuss later in Section 3.5. A high polarity indicates a highly aligned arrangement within a group and a high angular momentum points to a highly coherent rotational pattern. The phase diagrams of the polarity and the angular momentum are shown by Fig. 2.12(E) and (F), respectively. The authors call the state in Fig. 2.12 (A) a *swarm*, which has low polarity and low angular momentum, corresponding to region a in the phase diagrams. The pattern in Figure 2.12 (B) is called a *torus*; it has low polarity and high angular momentum, corresponding to region b in the phase diagrams. Figure 2.12 (C) and (D) show respectively a *dynamic parallel group* and a *highly parallel group*; both have much higher polarity and very low angular momentum, corresponding to regions c and d in the phase diagrams. The addition region e in the phase diagram represents a parameter regime where groups tend to fragment. Swarm behaviors are often observed in insects while the torus can be seen in fish schools [70, 75, 76]. While traveling, fish schools also form parallel groups, as do bird flocks. Among the model parameters, the effect of the turning rate ω is also investigated in their paper. It is found that the turning rate affects the internal



Figure 2.12: Various swarming states observed in the model of Couzin et al. [14]: (A) Swarm, (B) Torus, (C) Dynamic parallel group, and (D) Highly parallel group. The two figures on the bottom are the phase diagrams of (E) the polarity and (F) the angular momentum. Swarm behavior in (A) corresponds to region a in the phase diagrams; torus behavior in (B) is indicated by region b; regions c and d represent (C) and (D) respectively. Region e has a great chance of resulting in fragmenting groups. Reprinted from Journal of Theoretical Biology, Vol. 218, I. D. Couzin, J. Krause, R. James, G. D. Ruxton, and N. R. Franks, "Collective memory and spatial sorting in animal groups", pp. 1-11, Copyright (2002) with permission from Elsevier.

structure of a group, where the particles with a higher turning rate tend to be closer to the group center and away from the group front.

The basic assumption of the orientation-jump models is that the particle speed is constant. This is a reasonable approximation for animals in constant movement, such as fish and flying birds, as well as some types of robots in artificial swarms, such as Dubins vehicle [18, 86, 89] (see also Chapter 8). For more general cases, acceleration is involved and the particle speed may not stay homogeneously constant. Fully dynamic descriptions are required to account for such situations. This is especially true for most types of swarming robots which exhibit significant acceleration phases. Dynamic models also offer clearer physical interpretations for the studies of individual and pairwise behaviors, because particle interactions and environment influences are viewed as forces. In the next section, we review the historical development of the dynamic swarming models.

2.3 Dynamic models

As previously mentioned, dynamic descriptions are more favorable for the behavioral studies at the individual-level. Indeed, the dynamic models for swarming problems started with the investigations of the internal structure of swarms, especially how animals manage to keep an optimal distance from one another within a group. The possibility of using Newton's laws to understand these phenomena was first suggested by Breder in his 1951 paper "Studies on the structure of the fish school" [5]. In his 1954 article "Equations descriptive of fish schools and other animal aggregations" [6], he further formulated this idea by proposing an interaction force

$$F(x) = \frac{C_a}{x^m} - \frac{C_r}{x^n},$$
(2.52)

where positive values mean attraction, and negative ones indicate repulsion, although the convention in physics would rather reverse the signs. Eq. 2.52 is discussed and compared with static data analysis.

In two 1973 papers, "A model for group structure and its behavior" by Sakai [81] and "Movement of a group of animals" by Suzuki and Sakai [91], the first truly dynamic model for swarming problems was constructed. The equation of motion is [70]

$$m_{i}\frac{d^{2}\vec{x_{i}}}{dt^{2}} + \beta \frac{d\vec{x_{i}}}{dt} = \alpha \frac{d\vec{x_{i}}/dt}{|d\vec{x_{i}}/dt|} + \frac{1}{N-1} \sum_{j\neq i}^{N} C(x_{i,j}) \frac{\vec{x_{i,j}}}{x_{i,j}} \qquad (2.53)$$
$$+ \frac{1}{\mu} \sum_{x_{i,j} < \ell} c_{0} \left(\frac{d\vec{x_{j}}}{dt} - \frac{d\vec{x_{i}}}{dt} \right) + \eta_{i}(t) ,$$

where m_i and \vec{x}_i are the mass and the position of particle *i*, respectively. Time is denoted by *t*, and β is the friction coefficient. The first term on the RHS of Eq. (2.53) is a forward thrust, or the propulsion, where the parameter α specifies its strength. The second term is a mutual interaction force, where

$$C(x_{i,j}) = \begin{cases} -c_1 x_{i,j} & \text{if } 0 < x_{i,j} < R_0, \\ c_2 & \text{if } R_0 < x_{i,j} < R_1, \\ 0 & \text{if } R_1 < x_{i,j}. \end{cases}$$
(2.54)

The parameters c_1 and c_2 are positive coefficients, and N is the total number of particles in the group. In Eq. (2.54), the mutual interaction force consists of a linear repulsive force at shorter range, a constant attractive force at medium range, and zero force at long range. The third term is an arrayal force that aligns the particle velocities with each other. The arrayal coefficient c_0 is also a positive number, and μ is the number of neighbor particles that qualify the criterion $x_{i,j} < \ell$, in which ℓ is a threshold distance. The fourth term η_i is a stochastic



Figure 2.13: Basic patterns of the model in Ref. [81, 91]. a) Amoebic movement. b) Doughnut pattern. c) Rectilinear movement. The figure comes from Ref. [70]. Reprint of Fig. 7.11 on page 125 of Ch. 7 "The dynamics of animal grouping" in A. Okubo, (1980) Diffusion and Ecological Problems: Mathematical Models, with kind permission of Springer-Verlag and Business Media.

force describing environmental noise. By varying the parameters of the model, the authors observe three major categories of group movement: amoebic movement, doughnut pattern, and rectilinear movement, as shown in Fig. 2.13. Note that the amoebic movement is similar to the swarm behavior in the model of Couzin et al. [14], the doughnut pattern is comparable to their torus behavior, and the rectilinear movement belongs to parallel groups. A phase diagram of pattern transitions regarding some of the model parameters is presented in Fig. 2.14, in which the parameters the authors investigate are the propulsion strength α , the magnitude of noise max $|\eta_i|$, the arrayal coefficient c_0 , and the threshold distance ℓ of the arrayal force. Another earlier dynamic model was proposed by Okubo et al. in their 1977 paper "Studies on the schooling behavior of fish" [72]. They describe fish movement as forced, damped harmonic oscillations, in which the oscillators were coupled under the assumption that a trailing fish would follow



Figure 2.14: Phase diagram of various group patterns of the model in Ref.[81, 91]. The symbols used in the figure is different from those used here: a is α , b is max $|\eta_i|$, h is c_0 , and ℓ has the same definition. The figure comes from Ref. [70]. Reprint of Fig. 7.12 on page 126 of Ch. 7 "The dynamics of animal grouping" in A. Okubo, (1980) Diffusion and Ecological Problems: Mathematical Models, with kind permission of Springer-Verlag and Business Media.

a leading one. This model provided an insight in the spectrum analysis of the trajectory fluctuations due to fish adjusting their equilibrium distances between each other. It is, however, much less relevant to our dynamic model; therefore, we will not discuss it in detail here. These earlier dynamic models are discussed in Okubo's 1980 review book "Diffusion and Ecological Problems: Mathematical Models" [70].

In the next decade, the concept of applying Newton's laws in swarming problems was mostly used for static data analysis, such as individual tendency, spacing between swarmers, and swarmers' relative positions within a swarm [3, 4, 31, 105, 102]. It was not until the 1990s, with the aid of rapid advances in computers, that the group-level dynamics of these models re-emerged as an active area of research. One example is the 1990 article "A stochastic nonlinear model for coordinated bird flocks" by Heppner and Grenander [39]. They model the strongly synchronized movement of certain small bird flocks such as pigeons, starlings, and shorebirds. Since experimental efforts to identify leaders among these flocks have failed, they propose that these phenomena can be a byproduct of some decentralized rules. The equations describing such rules are

$$\frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t} = \vec{v}_i, \tag{2.55}$$

$$\frac{\mathrm{d}\vec{v}_{i}}{\mathrm{d}t} = -\vec{v}_{i} f_{\mathrm{vel}}(v_{i}) - \vec{x}_{i} f_{\mathrm{home}}(x_{i}) + \sum_{j=1}^{N} f_{\mathrm{interact}}(x_{i,j}) \vec{x}_{i,j} + \eta_{i}(t), \quad (2.56)$$

where \vec{x}_i and \vec{v}_i are the position and the velocity of bird *i*, and $\vec{x}_{i,j} \equiv \vec{x}_j - \vec{x}_i$. The functional forms of f_{vel} , f_{home} , and f_{interact} are drawn in Fig. 2.15, from left to right respectively. The first term on the RHS of Eq. (2.56) is a velocity regulation force that expresses the tendency of individual birds to achieve a constant favorite speed. The second term is called a homing force, which describes how the birds



Figure 2.15: The functional form of the forces adopted in the bird flock model of Heppner and Grenander [39]. Left: Velocity regulation. Middle: Homing. Right: Interaction. Reprints of Fig. 1, Fig. 2, and Fig. 3 on page 235 in Ch. 19, F. Heppner and U. Grenander, "A stochastic nonlinear model for coordinated bird flocks", The Ubiquity of Chaos, Ed. S. Krasner, ©1990 by American Association for the Advancement of Science.

are attracted toward their roosting area at the origin. The third term is an interaction force, featuring a short-range repulsion and a long-range attraction. The last term η_i is a stochastic force simulating environmental noise. Computer simulations of the model in Eq. (2.55) - (2.56) exhibit several flock-like behaviors including group hovering and clumping that mimic real birds. One example is shown in Fig. 2.16.

In a 1994 paper, Niwa recognized Breder's and Sakai's works and explicitly laid down a general frame work for dynamic models [67]. He categorized the forces within a swarming system into four components:

- locomotory force: it regulates the velocities of an individual, also know as the self-driving force;
- 2. attraction: coming from other individuals within the swarm, it is usually coupled with a repulsion as well and also known as the interaction force;
- 3. arrayal force: it unifies the velocities among neighbors;
- 4. stochastic force: it is postulated as environmental noise or small errors in the sensory system of individuals.



Figure 2.16: A series of time frames of the model simulation in Ref. [39]. The encircled cross is the roosting area. The small dots represent individual birds, on which the line segments indicate the forces acting on the birds. Reprint of Fig. 4 on page 237 in Ch. 19, F. Heppner and U. Grenander, "A stochastic nonlinear model for coordinated bird flocks", The Ubiquity of Chaos, Ed. S. Krasner, ©1990 by American Association for the Advancement of Science.

Let \vec{x}_i and \vec{v}_i represent the position and velocity of particle *i*; the general equations of motion can be expressed as

$$\frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t} = \vec{v}_i, \tag{2.57}$$

$$\frac{\mathrm{d}\vec{v}_i}{\mathrm{d}t} = \vec{F}_i^{(l)} + \vec{F}_i^{(g)} + \vec{F}_i^{(p)} + \vec{\eta}_i, \qquad (2.58)$$

where $\vec{F}_i^{(l)}$ is the locomotory force, $\vec{F}_i^{(g)}$ is the attraction, $\vec{F}_i^{(p)}$ is the arrayal force, and $\vec{\eta}_i$ is the stochastic force. Note that the locomotory force is called the velocity regulation force in Heppner and Grenander's model; it is also the combination of the forward propulsion and the friction of the Sakai and Suzuki's model. The attraction is often called interaction elsewhere. The specific functionals Niwa



Figure 2.17: The theoretical curve of the phase transition between polarized and non-polarized groups in Niwa's model [67]. Reprinted from Journal of Theoretical Biology, Vol. 171, H.-S. Niwa, "Self-organizing dynamic model of fish schooling", pp. 123-136, Copyright (1994) with permission from Elsevier.

adopted for his model simulations are

$$\vec{F}_i^{(l)} = \alpha \vec{v}_i - \beta v_i^2 \vec{v}_i, \qquad (2.59)$$

$$\vec{F}_{i}^{(g)} = \sum_{j \neq i} \left(C_{a} \frac{\vec{x}_{i,j}}{x_{i,j}} - C_{r} \frac{\vec{x}_{i,j}}{x_{i,j}^{3}} \right), \qquad (2.60)$$

$$\vec{F}_{i}^{(p)} = \frac{c_{0}}{\mu_{i}} \sum_{x_{i,j} < \ell} \left(\vec{v}_{j} - \vec{v}_{i} \right).$$
(2.61)

The parameter α and β are respectively propulsion and friction coefficients while C_a and C_r specify the strengths of attraction and repulsion. The arrayal force only acts on nearby neighbors within a threshold distance ℓ , and the number of such neighbors around particle *i* is denoted by μ_i . The parameter c_0 is the coefficient of the arrayal force. The stochastic force $\vec{\eta}_i$ is randomly picked from a Gaussian distribution satisfying

$$\langle \vec{\eta_i} \left(t \right) \cdot \vec{\eta_i} \left(t' \right) \rangle = \frac{2\epsilon}{N} \delta \left(t - t' \right), \qquad (2.62)$$

where ϵ is a small coefficient, and N is the total number of particles. Niwa

derived a probability function of group velocities and its time evolution from Eqs. (2.57) - (2.62). Similar to the purpose in Ref. [98] of Vicsek et al., he used the probability function to study the phase transition between polarized and nonpolarized groups, as shown in Fig. 2.17. In the figure, the critical parameter is $\sqrt{(1-5\beta\epsilon/\alpha)\alpha/\beta}$, and the normalized speed is $v_{cm}/\sqrt{\alpha/\beta}$, where v_{cm} is the center of mass speed. Note that this normalized speed is a dynamic version of the polarity defined in Eq. (2.40) for the kinematic model of Vicsek et al. Polarized groups result in non-zero normalized speeds, and the critical transition point occurs when the critical parameter equals zero.

Another independent development can also be found in Romey's 1996 paper "Individual differences make a difference in the trajectories of simulated schools of fish" [79]. His model follows the orientation-jump kinematic models in Section 2.2.5. Instead of directly formulating the velocity as in kinematic models, he constructs an expression for the interaction force

$$\vec{F}_{i} = \sum_{j \neq i} \frac{\left(x_{i,j}^{c_{0}} - c_{1}\right) \vec{x}_{i,j}}{\left(x_{i,j}^{c_{2}} + c_{3}\right) x_{i,j}},$$
(2.63)

where $\vec{x}_{i,j} \equiv \vec{x}_j - \vec{x}_i$ is the displacement vector from particle *i* to particle *j*, and \vec{x}_i is the position vector of particle *i*. The parameters c_0 , c_1 , c_2 , and c_3 are four arbitrary positive constants. Since the model is still an orientation-jump model, the equation of motion governs the change in orientation for the next time step

$$\hat{v}_i(t + \Delta t) = \frac{p\hat{v}_i(t) + (1 - p)\vec{F}_i/F_i}{\left|p\hat{v}_i(t) + (1 - p)\vec{F}_i/F_i\right|},$$
(2.64)

where \hat{v}_i denotes the orientation of particle *i*, and $0 \le p \le 1$ is a ratio of how much old momentum is retained. The force \vec{F}_i given in Eq. (2.63) indicates the direction toward which particle i tends to turn. High momentum particles have higher p and make gradual turns while low momentum ones have lower p and turn more sharply. Using the model, he investigated the effects of individual differences on group trajectories and found some predictive trends regarding group structures.

While Romey's model is still not completely dynamic, the model in the 1996 paper "Collective motion in a system of motile elements" by Shimoyama et al. [85] is a further unification of the orientation-jump models and the dynamic models. Their model describes the motion of motile elements that mimic bird flocks. Although the orientation is still the focus in the model, the time variation affects the full velocity, not just the orientation. The equations of motion are

$$m_i \frac{\mathrm{d}\vec{v}_i}{\mathrm{d}t} = \alpha \hat{n}_i - \beta \vec{v}_i + \sum_{j \neq i} c_{i,j} \vec{F}_{i,j} + \vec{G}_i, \qquad (2.65)$$

$$\tau_i \frac{\mathrm{d}\hat{n}_i}{\mathrm{d}t} = \hat{n}_i \times \hat{v}_i \times \hat{n}_i. \tag{2.66}$$

A significant difference of this model is that there are two orientations. One is the direction of the velocity $\hat{v}_i = \vec{v}_i/v_i$, and the other is the heading direction \hat{n}_i . In most of the other models, these are conveniently assumed to be the same. However, if we image a bird gliding through the air, these two directions can be different. As a result, the self-propulsion expressed by the first term on the RHS of Eq. (2.65) is parallel to the heading direction \hat{n}_i , but the friction in the second term is along the velocity direction \hat{v}_i . The third term is the mutual interaction force while the fourth term is the gravitational force. Eq. (2.66) describes that the orientation \hat{n}_i tend to relax toward \hat{v}_i . The parameter m_i is the mass of particle i, and τ_i is the relaxation coefficient associated to its moment. The functional form of the interaction force is chosen as

$$\vec{F}_{i,j} = -C\left[\left(\frac{x_{i,j}}{\ell}\right)^3 - \left(\frac{x_{i,j}}{\ell}\right)^2\right]\left(\frac{\vec{x}_{i,j}}{\ell}\right)e^{(-x_{i,j}/\ell)},\tag{2.67}$$

where $\vec{x}_{i,j}$ is the displacement vector from particle *i* to particle *j*, *C* is the interaction strength parameter, and ℓ is the interaction length. The coefficient $c_{i,j}$ in front of the interaction force enhances the effect of particles in front of particle *i*

$$c_{i,j} = 1 + c_0 \hat{n}_i \cdot \frac{\vec{x}_{i,j}}{x_{i,j}}, \qquad (2.68)$$

where c_0 is a positive constant. In their Fig. 2 of Ref. [85], they observed distinct swarming patterns via model simulations. In addition to the coherent polarized and incoherent non-polarized patterns studied in most previous models, there are coherent rotational patterns that are also observed by Couzin et al. [14] in Section 2.2.5. The authors quantitatively divide these patterns into chaotic (incoherent) and non-chaotic (coherent) classes. By numerical simulations, they find a unique parametric boundary that separates the two classes, which can be found in their Fig. 3 of Ref. [85]. In the figure, the disorder parameter is the standard deviation of particle speeds.

Thus far, the development of kinematic orientation models has converged to the evolution of dynamic models. Niwa's Eqs (2.57) - (2.58) are still the general form of dynamic models. However, since Romey found that the effects of arrayal force $\vec{F}^{(p)}$ is minimum, many dynamic models, including ours, have ignored it. Recently, a series of papers by Erdmann et al. has added new insights to swarming problems in the field of physics [21, 23, 24, 25]. They develop the "active Brownian particles" models, which obey a Langevin equation, like normal Brownian particles, but are also self-driven by additional "active forces". Such systems are similar to those of swarming particles. The basic equations of motion are

$$\frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t} = \vec{v}_i, \tag{2.69}$$

$$\frac{\mathrm{d}\vec{v}_i}{\mathrm{d}t} = f^{(l)}(v_i)\vec{v}_i - \vec{\nabla}U(\vec{x}_i) + \eta_i(t). \qquad (2.70)$$

The first term on the RHS of Eq. (2.70) is the self-driving mechanism, i.e., the locomotory force in Niwa's classification, and the most common choice in their various papers is

$$f^{(l)}(v) = \alpha - \beta v^2,$$
 (2.71)

where α and β are positive constants. The second term on the RHS of Eq. (2.70) is the interaction force, written as a potential gradient. The functional form of the potential they usually choose is

$$U(\vec{x}_i) = \sum_{j \neq i} \frac{C}{2} \left(\vec{x}_i - \vec{x}_j \right)^2, \qquad (2.72)$$

which assumes that all particles are attracted toward the center of mass, and the force is proportional to the distance from the center of mass. This is Hooke's Law, and Eq. (2.72) is indeed a spring potential. The third term on the RHS of Eq. (2.70) is the stochastic force, representing noise and satisfying

$$\left\langle \vec{\eta}_{i}\left(t\right)\cdot\vec{\eta}_{i}\left(t'\right)\right\rangle = 2\epsilon\delta\left(t-t'\right),\tag{2.73}$$

where ϵ is a small positive constant. Note that Erdmann et al. also ignore the arrayal force. They focused primarily on noise-induced phenomena, which also has biological significance since most swarming agents reside in noisy environments. One interesting phenomenon observed in their model is that the rotational coherent patterns may randomly switch direction with the presence of noise, as shown

in Fig. 2.18. Noise also induces state transition from a translational state to a rotational one, as shown in Fig. 2.19

All the dynamic models introduced so far are based on individual rules. Like kinematic models in Section 2.2, which feature both individual-based random walk models and continuum advection-diffusion models, continuum dynamic models, albeit rare, also exist. The most noticeable is the contribution from Toner and Tu in their 1995 paper "Long-range order in a two-dimensional XY model: How birds fly together" [93]. By comparing the model of Vicsek et al. [98] to a classical 2D XY spin model [52], they add convection to the classical model and propose the following equations of motion

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\vec{v}\rho) = 0, \qquad (2.74)$$

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = \alpha \vec{v} - \beta v^2 \vec{v} - \vec{\nabla} P \qquad (2.75)$$

$$+\kappa_{0}\vec{\nabla}\left(\vec{\nabla}\cdot\vec{v}\right)+\kappa_{1}\vec{\nabla}^{2}\vec{v}+\kappa_{2}\left(\vec{v}\cdot\vec{\nabla}\right)^{2}\vec{v}+\eta\left(\vec{x},t\right)$$

The density and the velocity fields are denoted by ρ and \vec{v} , respectively. Eq. (2.74) is the continuity equation while Eq. (2.75) is the momentum transport equation. The first term on the RHS of Eq. (2.75) is the self-propulsion, and the second term is the friction or self-deceleration; these two terms combine as the locomotory force classified by Niwa [67]. The third term is a pressure gradient, where the pressure

$$P(\rho) = \sum_{m=1}^{\infty} c_m \left(\rho - \rho_0\right)^m.$$
 (2.76)

It reflects that the density tends to converge to a favorable density ρ_0 , and c_m are expansion coefficient. The parameters κ_0 , κ_1 , and κ_2 in Eq. (2.75) are diffusion



Figure 2.18: Snapshots of the active Brownian particles studied by Erdmann et al. [23]. The counterclockwise mill in the first frame is disrupted in the second frame and then evolves into a clockwise mill in the third frame. Reprinted figure with permission from U. Erdmann, W. Ebeling, and V. S. Anishchenko, Physical Review E, Vol. 65, pp. 061106, 2002. Copyright (2002) by the American Physical Society.



Figure 2.19: Snapshots of the transition from a translational state to a rotational one in the model of Erdmann et al. [24]. Reprinted figure with permission from U. Erdmann, W. Ebeling, and A. Mikhailov, Physical Review E, Vol. 71, pp. 051904, 2005. Copyright (2005) by the American Physical Society.

coefficients, the last term in the equation is a stochastic force, satisfying

$$\left\langle \eta_i\left(\vec{x},t\right)\eta_j\left(\vec{x}',t'\right) = \epsilon \delta_{i,j} \delta^d\left(\vec{x}-\vec{x}'\right)\delta\left(t-t'\right),\tag{2.77}$$

where ϵ is a small positive constant, *i* and *j* represent Cartesian components, *d* is the spatial dimension, and δ is the Dirac delta function. Upon linearizing Eq. (2.74)-(2.75), the authors evaluate the scaling exponents of the model and analytically study the symmetry-breaking states of its solutions. While the model of Toner and Tu is based on heuristic arguments, in this thesis we propose a systematic procedure to derive a dynamic continuum model from an established dynamic individual-based model.

For interests in physics, a more abstract and more generalized swarming model is favorable, especially for analytical studies. The dynamic model proposed by Levine et al. in their 2000 paper "Self-organization in systems of self-propelled particles" is even more simplified than the model of Erdmann et al [57]. It is a deterministic model that ignores the noise term, and the individual-based equations of motion are

$$\frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t} = \vec{v}_i, \qquad (2.78)$$

$$\frac{\mathrm{d}\vec{v}_i}{\mathrm{d}t} = \alpha \hat{n}_i + \beta \vec{v}_i - \vec{\nabla} U\left(\vec{x}_i\right), \qquad (2.79)$$

where \vec{x}_i and \vec{v}_i are respectively the position and the velocity of particle *i*. The first term on the RHS of Eq. (2.79) is the self-propulsion, and the second term is the friction; these two terms combine as the locomotory force in Niwa's classification. The authors propose two self-propulsion mechanisms to construct the unit vector \hat{n}_i . One is a local propulsion for which

$$\hat{n}_i = \hat{v}_i = \frac{\vec{v}_i}{v_i}.$$
(2.80)

The other is a non-local propulsion, where

$$\hat{n}_{i} = \sum_{j \neq i} \frac{\hat{v}_{j} \exp\left(-\left|\vec{x}_{j} - \vec{x}_{i}\right| / \ell_{c}\right)}{\left|\hat{v}_{j} \exp\left(-\left|\vec{x}_{j} - \vec{x}_{i}\right| / \ell_{c}\right)\right|},$$
(2.81)

and ℓ_c is a correlation length. Note that Eq. (2.81) involves velocity averaging and acts also as an arrayal force in Niwa's classification. The third term on the RHS of Eq. (2.79) is an interaction force, expressed by a potential gradient. Instead of the spring potential in Eq. 2.72 adopted by Erdmann et al. and others, Levine et al. choose a more realistic functional form for the interaction. Based on the assumption that the interactions are pairwise, and their strength decreases in distance, the authors adopt a generalized version of Morse potential as their interaction

$$U(\vec{x}_i) = \sum_{j \neq i} V(\vec{x}_j - \vec{x}_i) \equiv \sum_{j \neq i} -C_a e^{-\frac{\left|\vec{x}_i - \vec{x}_j\right|}{\ell_a}} + C_r e^{-\frac{\left|\vec{x}_i - \vec{x}_j\right|}{\ell_r}}.$$
 (2.82)

The characteristic lengths of the attraction and the repulsion are respectively denoted by ℓ_a and ℓ_r while C_a and C_r specify their original strength. Interesting spiraling patterns emerge with the local propulsion in Eq. (2.80), as shown in Fig. 2.20, where some particles rotate clockwise while the others rotate counterclockwise around the same center. By instead implementing the non-local propulsion in Eq. (2.81), i.e., the arrayal force, the rotational directions of such milling patterns quickly unify. The authors continue to construct a continuum version of their individual-based model in Eqs. (2.78) - (2.79)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\vec{v}\rho) = 0, \qquad (2.83)$$

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = \alpha \hat{n} - \beta \vec{v} - \int \rho\left(\vec{y}\right) \vec{\nabla} V\left(\vec{y} - \vec{x}\right) \, \mathrm{d}\vec{y}.$$
 (2.84)



Figure 2.20: Time frames of a random initial configuration evolving into interlocking mills in the model of Levine et al. [57]. The line segment at the lower-right corner in Fig. a represents the length of ℓ_a . Reprinted figure with permission from H. Levine, W.-J. Rappel, and I. Cohen, Physical Review E, Vol. 63, pp. 017101, 2000. Copyright (2000) by the American Physical Society.

With the help of the continuum model, preliminary analytical results are given in the paper and compared to simulations of the individual-based model.

2.4 Summary

In this chapter, we have reviewed the existing literature on models for biological swarming. Such problems emerge as a branch of population dynamics but focus on relatively short time scales and small spatial domains. Earlier swarming models are based on kinematic rules of random-walk particles. A traditional class of such random-walk models involves jumps in particle positions and is called position-jump models. An advection-diffusion model can usually be derived out of a position-jump model as its continuum version, which is suitable for analytical studies since the degrees of freedom in the model are greatly reduced. On the other hand, unphysical phenomena observed in solutions of advection-diffusion
models drives scientists to look for alternative models. Closer observations of individual animals suggest jumps in orientations may better describe the movement inside a typical swarm. Because jumps in orientations are related to changes in velocities, the evolution of orientation-jump models gradually merges with the development of dynamic models. Dynamic swarming models originate from earlier static descriptions of the interactions between swarming individuals by applying Newton's laws. Such models are more favorable when scientists attempt to construct a swarming equation based on the knowledge of individual tendencies. However, unlike the (position-jump) kinematic models, a continuum version of a dynamic (or an orientation-jump) model is not as easy to derive. Some continuum dynamic models are proposed based on heuristic arguments rather than rigorous derivations. Without an understanding of a continuum limit, the dynamic models are mostly understood through statistical studies since the high degrees of freedom obstruct further analytical investigations. In the following chapters, we primarily focus on a dynamic model based on realistic descriptions of individual tendencies. Then we attempt to construct a procedure for deriving a continuum dynamic model from a individual-based dynamic one by using this model as an example.

Also note that the primary subject regarding swarming models has gradually shifted from the formation of polarized groups to the emergence of various swarming patterns or states. Our phenomenological observations in the latter chapters also focus on the pattern formations, and the stability of these patterns are discussed.

Chapter 3

Individual-Based Model

3.1 Introduction

Since there has been some understanding about individual animal tendencies with respect to the presence of other animals, the Newtonian description appears appropriate for formulating such tendencies. A dynamic swarming model is readily obtained based on these individual formula. In this chapter, we construct an individual-based dynamic model according to physical and biological arguments. The equations of motion are put together in Section 3.2. We adopt this model as our research subject and the primary example throughout the discussions in this thesis. In Sections 3.3 and 3.4, we identify several distinct patterns produced by the model over the parametric space. We then quantitatively classify these swarming states and illustrate the transitions between states in Section 3.5.

3.2 The equations of motion

A dynamic model is composed by Newton's second law of motion for each individual

$$\frac{\mathrm{d}\vec{x_i}}{\mathrm{d}t} = \vec{v_i} , \qquad (3.1)$$

$$m_i \frac{\mathrm{d}\vec{v}_i}{\mathrm{d}t} = \vec{F}_i, \qquad (3.2)$$

where m_i , $\vec{x_i}$ and $\vec{v_i}$ denote, respectively, the mass, position, and velocity of particle *i* while $\vec{F_i}$ is the force acting on it. It is suggested by Niwa in Ref. [67],

that \vec{F}_i for a swarming mechanism generally consists of

$$\vec{F}_i \equiv \vec{F}_i^{(l)} + \vec{F}_i^{(g)} + \vec{F}_i^{(p)} + \vec{\eta}_i, \qquad (3.3)$$

Here, $\vec{F}_i^{(l)}$ is a locomotory force that sustains the momentum of a particle; $\vec{F}_i^{(g)}$ is an a group interaction force that describes the interaction among particles; $\vec{F}_i^{(p)}$ is an arrayal force that align the orientation of a particle to that of its neighbors; and $\vec{\eta}_i$ is a stochastic force that imitate environemntal noise. The locomotory force has a forward thrust and a drag that, in combination, regulate the speed of an individual particle. While various formulas can be used for this speed regulation mechanism, the more important factor of swarming problems is that the organisms can reach and sustain a constant speed; the specific details of how they reach such status do not significantly affect the dynamical equilibrium state and are difficult to determine. A widely adopted formula [17, 21, 23, 24, 25, 61, 67, 93] is suggested in Ref. [103, 104, 106] for the studies of fish movement, which can be described by a Rayleigh-type dissipation [77]

$$\vec{F}_i^l \equiv \alpha \vec{v}_i - \beta \left| \vec{v}_i \right|^2 \vec{v}_i. \tag{3.4}$$

Note that the linear thrust $\alpha \vec{v}_i$ and the the cubic drag $\beta |\vec{v}_i|^2 \vec{v}_i$, in combination, give the particles a tendency to approach an equilibrium speed $v_{eq} = \sqrt{\alpha/\beta}$. Eq. (3.4) can generally be used without loss of generality for species other than fish because most swarming phenomena, particularly in steady states, depend only on the equilibrium speed but not the specific functional form of the locomotory force in the model. Indeed, the swarming patterns of our model are also more sensitive to this regulated speed rather than to the specific formula of speed regulation and thus, we adopt this conventional form for our locomotory force as well. The group interaction force $\vec{F}_i^{(g)}$ is usually written as the negative gradient of an interaction potential U_i , where U_i is a summation of pairwise interaction potentials $V_{i,j}$

$$\vec{F}_i^{(g)} \equiv -\vec{\nabla}_{\vec{x}_i} U_i = -\vec{\nabla}_{\vec{x}_i} \sum_{j \neq i} V_{i,j}.$$
(3.5)

One common choice for $V_{i,j}$ is a generalized Morse potential [17, 21, 57, 62],

$$V_{i,j} \equiv V\left(|\vec{x}_i - \vec{x}_j|\right) = -C_a e^{-\frac{\left|\vec{x}_i - \vec{x}_j\right|}{\ell_a}} + C_r e^{-\frac{\left|\vec{x}_i - \vec{x}_j\right|}{\ell_r}}.$$
(3.6)

The pairwise interaction consists of an attraction and a repulsion with C_a , C_r specifying their respective strengths and ℓ_a , ℓ_r their effective interaction length scales. Furthermore, similar behavior is observed with other functional forms of interaction potential characteristically similar to Eq. (3.6). In consistency with the conclusion in Ref. [79], we do not find the arrayal force $\vec{F}_i^{(p)}$ necessary to produce polarized formations as suggested by some earlier models [78, 81, 91], only that the arrayal force shortens the transient period. Therefore, we exclude the arrayal force to simplify our derivation and stability analysis. Another choice to simplify the analysis is to ignore the stochastic force $\vec{\eta}_i$ due to noise and separate deterministic phenomena from stochastic ones. The effects of stochastic forces are studied in many papers [21, 23, 24, 25, 61, 98], which observe that noise affects the swarming patterns only beyond certain thresholds, and thus its consequences are not investigated here. By combining Eqs. (3.1) - (3.6), the swarming model we consider is [10]

$$\frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t} = \vec{v}_i , \qquad (3.7)$$

$$m_{i} \frac{\mathrm{d}\vec{v}_{i}}{\mathrm{d}t} = \alpha \vec{v}_{i} - \beta |\vec{v}_{i}|^{2} \vec{v}_{i} - \vec{\nabla}_{\vec{x}_{i}} \sum_{j \neq i} \left(-C_{a} \mathrm{e}^{-\frac{\left|\vec{x}_{i} - \vec{x}_{j}\right|}{\ell_{a}}} + C_{r} \mathrm{e}^{-\frac{\left|\vec{x}_{i} - \vec{x}_{j}\right|}{\ell_{r}}} \right). \quad (3.8)$$

To understand different regimes of swarming patterns, we non-dimensionalize

the equations of motion by substituting $t' = (m_i/\ell_a^2\beta) t$, $\vec{x}'_i = \vec{x}_i/\ell_a$, and thus, $\vec{v}'_i = (\ell_a\beta/m_i) \vec{v}_i$ into Eqs. (3.7) - (3.8)

$$\frac{\mathrm{d}\vec{x}'_i}{\mathrm{d}t'} = \vec{v}'_i \,, \tag{3.9}$$

$$\frac{\mathrm{d}\vec{v}'_i}{\mathrm{d}t'} = \alpha'\vec{v}'_i - |\vec{v}'_i|^2 \vec{v}'_i - \frac{1}{m_i'} \vec{\nabla}_{\vec{x}'_i} \sum_{j \neq i} \left(-\mathrm{e}^{-\left|\vec{x}'_i - \vec{x}'_j\right|} + C\mathrm{e}^{-\frac{\left|\vec{x}'_i - \vec{x}'_j\right|}{\ell}} \right), \quad (3.10)$$

where $\alpha' = (\alpha \beta \ell_a^2) / m_i^2$, $m_i' = m_i^3 / (\beta^2 C_a \ell_a^2)$, $C = C_r / C_a$, and $\ell = \ell_r / \ell_a$. Hence, the model has 4 parameters, among which m_i' only affects the time scale of the particle interaction. Thus, we keep it fixed and study the effects of the other three parameters C, ℓ , and α' . Note that the dimensional parameter α only appears in the dimensionless parameter α' , which allows us to vary α to change α' without affecting the other three independent parameters, provided that β , ℓ_a , and m_i are fixed during the process. To preserve the original meaning of the model parameters, our results are presented in the dimensional form by using Eqs. (3.7) - (3.8). Also note that the interaction potential only depends on C and ℓ . Therefore, in the ensuing discussion, we design a phase diagram with respect to C and ℓ for the potential and explore the dynamics of of aggregate states as a function of α for the different regimes in the phase diagram.

3.2.1 Numerical method

To integrate Eqs. (3.7) - (3.8) over time, we use the fourth order Runge-Kutta and the four step Adam-Bashforth methods [53]. The time steps are chosen so that the increments in $|\vec{x}|$ are less than $0.1\ell_a$ and the increments in $|\vec{v}|$ are less than $0.1\ell_a\beta/m_i$ for each time step. Unless specified otherwise, we impose free boundary conditions to the model and initiate our simulations with a uniformly random distribution of particle positions and velocities.

3.3 Various states of swarming patterns

In Chapter 4, a statistical property called "H-stability" is investigated and discussed. For our model, H-stability is found associated to the dimensionless parameters C and ℓ , and various aggregation patterns may emerge with respect to different regimes on the H-stability diagram. Before heading into the details of H-stability discussion in the next chapter, let us first explore these distinct patterns with respect to various $C-\ell$ combinations. Figure 3.1 is a phase diagram of C and ℓ , divided into several regimes according to the characteristics of the interaction potential. Region V is classified as the repulsion-dominant regime because the potential has a global minimum at $x \equiv |\vec{x}_i - \vec{x}_j| = \infty$, and interacting particles disperse over the infinite domain. Regions I, II, III, and IV are classified as the *attraction-dominant regime* because their pairwise interaction potentials either have a global minimum at r = 0 that eventually attracts particles together, or have a global maximum at $x = \infty$ that globally drives all particles to aggregate within a finite region. The more interesting regions are VI and VII, categorized as the *biologically relevant regime* [63]. In this regime, the pairwise interactions are characterized by a long-range attraction and a short-range repulsion $(C > 1, \ell < 1)$, meaning that animals tend to seek companies if they are far away from their own kind but would push each other away if they eventually get too close to each other. A general functional shape of the biologically relevant pairwise interaction is shown in Fig. 3.2(a).

In the biologically relevant regime, we find several interesting swarming states of distinct patterns by varying the thrust parameter α while fixing the others. Figure 3.2 (b) and (c) are two typical patterns akin to those observed in various natural swarms: (b) the *single-mill state*, where every particle travels at the same speed v_{eq} around an empty core at the center of the swarm, and (c) the



Figure 3.1: Potential characteristic diagram. This diagram specifies several characteristically distinct regimes of the pairwise interaction potential V(x), including the attraction-dominant regime (I, II, III, and IV), the repulsion-dominant regime (V), and the biological relevant regime (VI and VII).

double-mill state, in which particles travel in both clockwise and counterclockwise directions, also at a uniform speed v_{eq} . In the double-mill state, when viewed as two superimposed mills, the cores of each mill do not exactly coincide but rather fluctuate near each other. Another two states are shown in Fig. 3.2 (d) and (e): (d) the *coherent flock state*, in which all particles travel at a unified velocity while self-organizing into a stable formation, and (e) the *rigid-body rotation state*, which closely resembles the formation of a coherent flock, but instead of traveling at the same velocity, the particles circulate around the swarm center defining a constant angular velocity ω . Unlike the single and double-mill state, where particles swim freely within the swarm, both the coherent flock and the rigid-body rotation states bind particles at fixed relative positions, exhibiting a lattice-type formation. Hence, we also use the term *lattice states* to refer to both the coherent flock and the rigid-body rotation states. Note that the coherent flock is a traveling wave solution of the model, and thus a solution of the following Euler-Lagrange equation

$$\vec{\nabla}U_i = \vec{\nabla}\sum_{j\neq i} V\left(|\vec{x}_i - \vec{x}_j|\right) = 0.$$

It is interesting to note that this equation arises in the context of a gradient flow algorithm for autonomous vehicle control [32, 33, 34, 60]. Thus the flock formations have the shape and structure as equilibria of the gradient flow problem with the same potential.

The coherent flock and the single-mill states are among the most common patterns observed in biological swarms [75, 76, 83]. The double-mill pattern is also occasionally seen; an example is the M. xanthus bacteria at the onset of fruiting body formation [51]. On the other hand, natural occurrences of rigid-body rotation, to the best of our knowledge, have not been reported in the literature. Indeed, the rigid-body rotation, where every particle travels at a constant angular



Figure 3.2: Biologically relevant swarming patterns: (a) The functional shape of the biologically relevant pairwise interaction potential. (b) The single-mill state. (c) The double-mill state. (d) The coherent flock. (e) The rigid-body rotation.



Figure 3.3: Attraction-dominant patterns: (a) The functional shapes of the attraction-dominant pairwise interaction potentials of regions I-IV in Fig. 3.1. (b) The clump formation. (c) The ring formation. (d) The clumping-ring formation.

velocity ω , does not define a rotationally symmetric solution for Eqs. (3.7) - (3.8) and the swarm is observed to drift randomly due to the unbalanced self-driving mechanism. After a transient period, the random drift may eventually break the rotational symmetry and turn the swarm into a coherent flock. Thus, we speculate that this pattern may only be a meta-stable or a transient state. In addition to the above aggregation states, the particles may simply escape from the collective potential field, and no aggregation is observed. We name it the *dispersed state*.

The dispersed state also characterizes the swarming behavior in the repulsiondominant regime $(C \ge 1, \ell \ge 1)$ but for different reasons. It is simply because that the repulsion is dominant, making the particles disperse over the space. On the other hand, the attraction-dominant regime offers other distinct patterns that may have applications in artifical swarms. Figure 3.3 (a) shows the functional shapes of the pairwise interaction potentials in this regime. In region I of Fig. 3.1 $(1 \ge C > \ell)$, the swarm self-organizes into a clump formation, as shown in Fig. 3.3 (b). This phenomena can be explained by the functional shape of its pairwise interaction potential which has a global minimum at a small distance and is mostly attraction dominant. As a result, particles keep a small distance from each other but are still attracted altogether within a finite region, forming clumping groups to circulate around a center as the interaction force provides the necessary centripetal force. In region II of Fig. 3.1 ($1 \ge C = \ell$), the swarm self-organize into a ring formation, as shown in Fig. 3.3 (c). This is because the global minimum is now shifted to x = 0, and thus, the configuration is optimized with particles overlapping each other. By assuming equidistant particle spacing, the ring radius R can be estimated by balancing the centrifugal and the centripetal forces

$$\frac{m\alpha}{2R\beta} = \sum_{n=1}^{N/2} \sin\left(\frac{\pi n}{N}\right) \left[\frac{C_a}{\ell_a} \mathrm{e}^{-2R/\ell_a \sin(\pi n/N)} - \frac{C_r}{\ell_r} \mathrm{e}^{-2R/\ell_r \sin(\pi n/N)}\right].$$
 (3.11)

The estimates match extremely well to those obtained by numerical simulations, as seen in Fig. 3.4. Regions III $(1 \ge \ell > C)$ and IV $(C < 1, \ell > 1)$ of Fig. 3.1 have the same formation, which is the clumping ring formation in Fig. 3.3 (d). In these two regions, the attractive force at x = 0 does not vanish and thus, further attracts particles into some singular points. Unlike the biologically relevant regime, the types of swarm formation is not sensitive to α in the attraction-dominant regime; the formation stays characteristically the same by increasing α until the kinetic energy becomes too large for the interaction potential to sustain the aggregation. Note that the patterns exhibited in the attraction-dominant regime involve extreme collapse, where particles occasionally overlap and run over each



Figure 3.4: Ring radius as a function of N from numerical data and from the estimates of Eq. (3.11). The parameters used here are $\alpha = 1$, $\beta = 0.5$, $C_a = \ell_a = 1$, $C_r = 0.6$, and $\ell_r = 0.5$.

other. In Chapter 4, it becomes clear that the interaction potentials in this regime are not H-stable and unable to prevent indiviuals from collapsing in the large Nlimit. On the other hand, the repulsion-dominant regime is safely H-stable.

3.4 The biologically relevant regime

As stated earlier, we find various swarming patterns by varying α in the biologically relevant regime while the formation types are insensitive to α in the other regimes. Let us return to the biologically-relevant regime to further investigate the α -related transition. The biological regime is divided into two sub-regimes, region VI and region VII, by the H-stability criterion, which is described in Chapter 4. Using numerical simulations, we find that swarms in the H-stable region VI and the non-H-stable region VII undergo a different state transition with respect to α . In both regions VI and VII, the lattice states, as shown in Fig. 3.2 (d) and (e), emerge for low values of α , and thus, of low v_{eq} . In this case, the confining interaction potential is stronger than the kinetic energy of individual particles and tends to bind the particles at specific "crystal" lattice sites. Most initial conditions lead to the coherent flock state while some occasions result in the rigid-body rotation state. In region VI, as α further increases, the particles eventually gain enough kinetic energy to dissolve the aggregation, and the swarm reaches the dispersed state.

On the other hand, the state transition in region VII is not as simple; it is characterized by more behavioral stages. Starting from the lattice states and upon increasing α , the particles gain more kinetic energy through the self-propulsion to reach $v_{\rm eq}$ and are able to break away from the crystal lattice sites. However, unlike the case of region VI, the interaction potential in region VII is still strong enough to aggregate medium-speed particles within a swarm. In this regime, core-free mill states emerge, as shown in Fig. 3.2 (b) and (c). Since all particles travel at a non-zero uniform speed, the centripetal force provided by the collective interaction potential is not strong enough to sustain such particles too close to the rotational center. As a result, the mill core is a particle-free region. At moderate α , a single mill state emerges. At slightly higher α , we observe both single mills and double mills as possible states. In the latter case, the interaction potential gradually loses its effectiveness to unify the clockwise (CW) and counterclockwise (CCW) rotational directions; particles traveling in the opposite direction with respect to the majority tend to not change their direction of motion, and double mills can emerge. The transition from single to double mill is a gradual process. Figure 3.5 shows the number of particles in each rotational direction for various values of α . In the single-mill regime, particles traveling at one direction are quickly assimilated into the other (Fig. 3.5, top). Upon increasing α , the particles no longer settle into a unified rotational direction (Fig. 3.5, middle), and for large



Figure 3.5: Time variation of the numbers of particles rotating in different directions: The triangles represent the number of CCW particles while the circles are of CW ones. (Top) $\alpha = 1.5$. (Middle) $\alpha = 4.0$. (Lower) $\alpha = 6.0$. The fixed parameters are $\beta = 0.5$, $C_a = 0.5$, $C_r = 1.0$, $\ell_a = 2.0$, $\ell_r = 0.5$, and N = 500. All parameters here and throughout the thesis are in arbitrary units.



Figure 3.6: $\alpha_{\rm esc}$ versus the total number of particles in a region VI swarm ($\beta = 0.5, C_a = 0.5, C_r = 1.0, \ell_a = 2.0, \ell_r = 1.5$, dashed line) compared to that of a region VII swarm ($\beta = 0.5, C_a = 0.5, C_r = 1.0, \ell_a = 2.0, \ell_r = 0.5$, dotted line). The solid line is the curve estimated by Eq. (3.12).

enough α , approximately the same number of particles travel in each of the CW and CCW directions (Fig. 3.5, bottom). The presence of either a velocity alignment rule or a hard-core repulsive interaction will destroy this double-mill state. The latter case is further discussed in Chapter 4. On the other hand, although the coherent flock state remains a possibility in this regime, the basin of attraction is much smaller than the mill states, and only very polarized initial conditions can lead to a coherent flock formation. As α increases beyond the double-mill regime, particle kinetic energy eventually becomes high enough to break up the swarm. This is the dispersed state, and no aggregation can be found.

Upon fixing the other parameters, the threshold between the aggregation and the dispersed states is described by a critical escape value of α , denoted by α_{esc} . Figure 3.6 shows $\alpha_{\rm esc}$ of swarms in both region VI and VII, where stable rotational formations are initiated and α is increased until the dispersed regime is attained. For the region VI swarm, $\alpha_{\rm esc}$ does not vary significantly with respect to the total particle number of the swarm, denoted by N, due to a related observation that the nearest neighbor distance ($\delta_{\rm NND}$) approach a constant for large N. As a result, the binding potential energy of the interaction force acting over each particle is independent of N. On the other hand, $\alpha_{\rm esc}$ of the region VII swarm varies linearly with respect to N. From our numerical simulations, we observe that the outer and the inner radii of the region VII swarm remain approximately fixed with respect to N while $\alpha \leq \alpha_{\rm esc}$. Based on this observation, we can derive a semi-empirical formula to estimate the value of $\alpha_{\rm esc}$ by assuming that particles are uniformly distributed in a doughnut shape domain. By balancing the centripetal and the interaction forces, we obtain

$$\frac{m\alpha_{\rm esc}}{2\beta} = \frac{N}{2\pi \left(R_{\rm out}^2 - R_{\rm in}^2\right)} \int_{R_{\rm in}}^{R_{\rm out}} V\left(|\vec{r} - R_{\rm out}\hat{x}|\right) \mathrm{d}\vec{r}, \qquad (3.12)$$

where $R_{\rm in}$ and $R_{\rm out}$ denote the inner and the outer radii of the single mill, respectively, and \hat{x} is an arbitrary unit vector. This estimate predicts that $\alpha_{\rm esc}$ should scale linearly with N, which is clearly illustrated in Fig. 3.6, where we use the numerically simulated $R_{\rm out} = 5.2$ and $R_{\rm in} = 1.2$ for a quantitative comparison.

3.5 Quantifying the biologically relevant state transition

In order to quantitatively determine whether the swarm is in a coherent flock state or a single-mill state, Couzin et al. have proposed two measures [14]: the polarity, P, and the (normalized) angular momentum, M, defined as follows

$$P = \left| \frac{\sum_{i=1}^{N} \vec{v}_i}{\sum_{i=1}^{N} |\vec{v}_i|} \right|, \qquad (3.13)$$

$$M = \left| \frac{\sum_{i=1}^{N} \vec{r_i} \times \vec{v_i}}{\sum_{i=1}^{N} |\vec{r_i}| |\vec{v_i}|} \right|,$$
(3.14)

where $\vec{r}_i \equiv \vec{x}_i - \vec{x}_{\rm CM}$, and $\vec{x}_{\rm CM}$ is the position of the center of mass. A perfect coherent flock corresponds to P = 1 and M = 0 while a perfect single-mill pattern corresponds to M = 1 and P = 0. In order to distinguish the double-mill pattern, we propose an additional measure by modifying the normalized angular momentum

$$M_{\rm abs} = \left| \frac{\sum_{i=1}^{N} |\vec{r_i} \times \vec{v_i}|}{\sum_{i=1}^{N} |\vec{r_i}| |\vec{v_i}|} \right|.$$
(3.15)

If a double-mill pattern has perfectly equal numbers of particles going at each direction with the centers of mass of both directions exactly overlap, $M_{\rm abs} = 1$ and M = 0; both M and $M_{\rm abs}$ equal one for a single mill.

Although the presence of the coherent flock that yields $P \simeq 1$ allows us to use P to quantify the transition from lattice to single-mill state, the co-existing rigidbody rotation state, for which $P \simeq 0$, introduces spurious events. Since the rigid body state has a much smaller basin of attraction than the coherent flock, one choice is discarding all rigid-body rotation events and selecting only the coherent flock ones. However, the boundary between a rigid body rotation and a single mill is ambiguous, as shown in Fig. 3.7, where a rigid-body rotation transforms to a single mill by increasing α . Since a constant tangential speed indicates a milling formation, and a constant angular velocity (i.e., a linear tangential speed against r) characterizes a rigid-body rotation, we can see from the figure



Figure 3.7: Emergence of a rotating single-mill structure from a rigid-body rotation in region VII. The left panel shows the ensemble averaged tangential velocity, $\langle v(r) \rangle_{\text{tang}}$, of particles at a distance r from the center of mass. Each $\langle v(r) \rangle_{\text{tang}}$ figure corresponds to the swarm structure of different values of α on the right panel: from top to bottom are $\alpha = 0.003$, $\alpha = 0.03$, $\alpha = 0.1$, and $\alpha = 0.5$. The other parameters are $\beta = 0.5$, $C_a = 0.5$, $C_r = 1.0$, $\ell_a = 2.0$, $\ell_r = 0.5$, and N = 500.

that two states are mixed during the transition: the outer part of the swarm begins to exhibit the milling phenomena while the inner part still remains a rigid body. Indeed, the collective interaction potential is stronger in the inner part of the swarm, and particles need a higher kinetic energy injection from the self-propulsion to escape the binding potential. Since the lattice formation of the rigid-body rotation has an ordered particle distribution, and the milling formation exhibits more disordered distribution, we propose an *ordering factor* of period Qto quantitatively distinguish these two states

$$O_{(Q)} \equiv \frac{1}{N\mu} \left| \sum_{i=1}^{N} \sum_{j}^{\mu} \cos\left(Q \cdot \phi_{j,j+1}^{(i)}\right) \right|, \qquad (3.16)$$

where $\phi_{j,j+1}^{(i)}$ is the angle between $\vec{x}_{i,j}$ and $\vec{x}_{i,j+1}$ with $\vec{x}_{i,j}$ defined as $\vec{x}_j - \vec{x}_i$. The summation index j here represents the j-th nearest neighbor of particle i, and μ denotes the number of neighbors that are taken into consideration for each particle. We also define $\vec{x}_{i,\mu+1} \equiv \vec{x}_{i,1}$ to simplify the formula. If all $\phi_{j,j+1}^{(i)}$ are distributed at $2\pi k/Q$ where k < Q is a positive integer, $O_{(Q)} = 1$, and the particles are distributed on a lattice of period Q. On the other hand, if the distribution is completely random, cancellation occurs in the summation of cosines, and $O_{(Q)} \simeq 0$ for all Q. The number of nearest neighbors of each particle i can be arbitrarily chosen for $\mu \geq 2$. However, note that μ cannot be too large; otherwise, second layer neighbors may be counted, which results in an incorrect Q. For the sake of definiteness, we choose $\mu = 3$. In order to avoid incorrect estimations due to the dispersed state, we also impose that a particle pair must be separated by a distance no larger than $2\ell_a$ for the particles to qualify as neighbors. Figure 3.8 (b) shows the distribution of $\phi_{j,j+1}^{(i)}$ collected for all i and j on a rigid-body formation. Peaks are observed at $k\pi/3$ ($1 \leq k \leq 5$), indicating that the formation is a hexagonal



Figure 3.8: (a) The ordering factor of period 6 versus α and an illustration showing the definition of $\phi_{j,j+1}^{(i)}$. The squares are data points of a region VI swarm while the triangles represent a region VII case. The parameters other than α for both cases are the same as those in Fig. 3.6 with N = 200. (b) The distribution of $\phi_{j,j+1}^{(i)}$ for all *i* and *j*. (c) Comparison of the ordering factors of different periods Q.

lattice. Figure 3.8 (c) shows $O_{(Q)}$ versus Q for the same rigid-body formation. As expected for a hexagonal lattice, the curve peaks at Q = 6. Therefore, $O_{(6)}$ can be used to explore the transition from a hexagonal lattice to a non-lattice mill state.

Using the quantities defined in Eqs. (3.13) - (3.16), different swarming states can be classified. Dramatic changes in P, M, M_{abs} , and $O_{(Q)}$ are observed upon modifying specific parameters in the model and indicate a change in the swarming state. Figure 3.8 (a) shows the transition of a region VII swarm from lattice to single-mill states as $O_{(6)}$ gradually decreases with respect to increasing α . Also shown in the figure are the same quantities for a region VI swarm; note that as α increases, $O_{(6)}$ suddenly drops to zero, corresponding to the sudden dissolution of the hexagonal lattice structure into a dispersed state. The larger value of $O_{(6)}$ in the region VI swarm indicates a more regular hexagonal lattice formation.

For higher values of α , we further consider P, M, and M_{abs} to differentiate the coherent state and the two mill states. Additionally, in order to distinguish the dispersed state from the rest, we calculate the aggregation fraction, f_{agg} , defined as the fraction of the N initial particles that aggregate as a swarm. In Fig. 3.9, we show how region VI and VII swarms are different during the transition between states. Figure 3.9 (a) shows that a region VI swarm is a coherent flock for small α , indicated by $P \simeq 1$. For increasing α , the swarm disperses and $f_{agg} = 0$. Note that P remains close to one when $f_{agg} \neq 0$, indicating that the aggregate goes from the coherent lattice state directly to the dispersed one. Figure 3.9 (b) shows the transition of a region VII swarm, which displays a full four-stage transition: in the small α regime, particles arrange as a coherent lattice with $P \simeq 1$; as α keeps increasing, the single-mill state appears ($P \simeq 0$ and $M \simeq 1$), followed by the double-mill state ($M_{abs} \simeq 1$ and $M \simeq 0$) until the dispersed state ($f_{agg} = 0$) is reached.

3.5.1 Analogy to bulk matter

Drawing an analogy from the state transition of swarming patterns to the phase transition of bulk matter, the lattice states can be regarded as "solid" since interparticle distances are kept constant. The milling state allows particles to "swim" within a finite volume without being bound to a fixed lattice site; thus, it can be regarded as "liquid". Finally, in the dispersed state, particles escape to fill the free space, similar to a "gas". Upon increasing α , a region VII swarm undergoes



Figure 3.9: The state transition diagram of (a) a region VI swarm and (b) a region VII swarm. The fixed parameters are the same as those in Fig. 3.6 with N = 200.

a solid–liquid and liquid–gas transition, which resembles the processes of melting and vaporization. On the other hand, a region VI swarm goes from a solid state directly to a gas one, which is more similar to sublimation.

Consistently with granular media models [7], we may define a "temperature" analog using the variation of the individual particle velocity among the flock: $T_{\rm s} \equiv \langle (\vec{v} - \langle \vec{v} \rangle)^2 \rangle$. Note that $\langle \vec{v} \rangle$ is the velocity of the center of mass, and thus, $T_{\rm s} \simeq 0$ for the coherent flock pattern, while $T_{\rm s} \simeq \alpha/\beta$ for the steady mill states. The swarming patterns change from one state to another by varying $T_{\rm s}$.

Since we have seen the characteristic differences between the region VI and VII swarms, it is natural to ask what contributes to the differences. The functional shapes of their pairwise interaction potential are generally the same as what is shown in Fig. 3.2 (a) for the potentials in the biologically relevant regime. Thus, the differences cannot be easily explained by particle-pair interactions using a qualitative argument as in the other regimes. As stated earlier, the differences are caused by H-stability properties of the interaction potentials in each region. The H-stability analysis of our model is discussed in the next chapter.

Chapter 4

H-stability

4.1 Introduction

Since it was discovered that bulk matter is composed of charged particles, the stability of such particle aggregation was once a center of attention in physics. A conjecture is that the binding potential energy per particle has a lower bound [73]. Otherwise, it can release an unbounded amount of energy by combining bulk matter, for example, pouring several cups of water into one container, and as a consequence, the combined system may collapse in volume. This criterion is later known as *H-stability* because it implies that the Hamiltonian is stable – meaning that the total energy is bounded below proportional to the number of particles [19]. It is also important in thermodynamics where both energy and volume are extensive quantities; without H-stability, equations of state cannot be defined. Indeed, H-stability is a necessary, albeit insufficient, condition for the existence of thermodynamics [27, 59, 80, 82]. Initially taken for granted, H-stability was later proved for hard-core billiard balls by Onsager [73] and for short short-range potentials by Fisher and Ruelle [27]. The proofs of H-stability was then extended and generalized to the more realistic, long-range Coulomb potentials by a series of contributions from Dyson and Lenard [19, 20], Lebowitz and Lieb [54], Lieb and Thirring [59], and Lieb [58]. In Section 3.5.1, we make an analogy between swarms and bulk matter. However, various observed swarming patterns are not as commonly seen in classical charged particle systems and seem to show collapsing characteristics. It is worthwhile examining whether H-stability also holds for the swarming patterns classified in the phase diagram of Fig. 3.1. In this chapter, the basic definitions regarding H-stability are introduced in Section 4.2. H-stability of our Morse-type interaction potential in Eq. (3.6) is analyzed in Section 4.3, followed by a brief discussion in Section 4.4. More discussions are postponed to Chapter 5 where a continuum model is derived, and H-stability shows significant effects when a swarming system approach toward the large-number continuum limit.

4.2 The basic definition

Let us define $V^k(\vec{x}_{i_1}, \vec{x}_{i_2}, ..., \vec{x}_{i_k})$ as a k-body interaction potential among particles $i_1, i_2, ..., i_k$. The total potential U^{tot} of an N-particle system is

$$U^{\text{tot}}\left(\vec{x}_{1},...,\vec{x}_{N}\right) = \sum_{k\geq 2} \sum_{1\leq i_{1}<...< i_{k}\leq N} V^{k}\left(\vec{x}_{i_{1}},\vec{x}_{i_{2}},...,\vec{x}_{i_{k}}\right)$$
(4.1)

The basic criterion for the H-stability is defined as

Definition 4.2.1 H-stability [80].

The interaction potentials V^k 's are H-stable if there exists a constant $B \ge 0$ such that

$$U^{\text{tot}}\left(\vec{x}_{1},...,\vec{x}_{N}\right) \ge -NB \tag{4.2}$$

for all $N \ge 0$ and $\vec{x}_i \in \mathbb{R}^d$.

Definition 4.2.2 Catastrophe.

The interaction potential V^k is called catastrophic if it is not H-stable.

A system composed of catastrophic interaction potentials can lead to non-thermodynamic behavior because the grand partition function

$$\mathcal{Z}_{\Lambda} \equiv 1 + \sum_{N=1}^{\infty} \frac{z^N}{N!} \int_{\Lambda^N} \mathrm{d}\vec{x}_1 ... \mathrm{d}\vec{x}_n \mathrm{e}^{-\beta_{\mathrm{B}}U^{\mathrm{tot}}(\vec{x}_1, ..., \vec{x}_N)}$$
(4.3)

may diverge if U^{tot} is not bounded below. Here $\Lambda \subset \mathbb{R}^d$ and $\beta_{\text{B}} \equiv 1/k_{\text{B}}T$, where k_{B} is the Boltzmann constant and T is the temperature.

4.3 The H-stability analysis

Although the basic definition of the H-stability in Eq. (4.2) is not readily applicable to an arbitrary type of k-body interaction potential, Ref. [80] shows that various equivalent sub-criteria can be deduced for a specific class of pairwise (i.e., 2-body) interaction potentials, which is the isotropic pairwise interaction potential

$$V(\vec{x}_i, \vec{x}_j) = V(|\vec{x}_i - \vec{x}_j|).$$
(4.4)

Recall that our pairwise interaction potential of the Morse type in Eq. (3.6) is indeed an isotropic one. Therefore, we are able to utilize these sub-criteria to determine whether our interaction potential is H-stable or catastrophic.

Proposition 4.3.1 Negative integral.

Assuming that a pairwise interaction potential $V(\vec{x})$ is absolutely integrable, it is catastrophic if

$$\int_{\mathbb{R}^d} V\left(\vec{x}\right) \mathrm{d}\vec{x} < 0. \tag{4.5}$$

As a result, we may integrate Eq. (3.6) to obtain

$$0 > \int_{\mathbb{R}^{d}} V(\vec{x}) d\vec{x} = \int_{\mathbb{R}^{d}} \left(-C_{a} e^{-\frac{|\vec{x}|}{\ell_{a}}} + C_{r} e^{-\frac{|\vec{x}|}{\ell_{r}}} \right) d\vec{x}$$
(4.6)
$$= C_{a} \ell_{a}^{2} \left(\ell^{2} - 1 \right),$$

which leads to our first criterion

Criterion 4.3.2 If

$$C\ell^2 < 1, \tag{4.7}$$

then the Morse potential in Eq. (3.6) is catastrophic.

Proposition 4.3.3 For a pairwise interaction potential satisfying the condition $V(\vec{x}) = V(-\vec{x})$, it is H-stable if and only if

$$\sum_{i=1}^{N} \sum_{j=1}^{N} V\left(\vec{x}_{i} - \vec{x}_{j}\right) \ge 0 \qquad \forall N \ge 0, \, \forall \vec{x}_{1}, \vec{x}_{2}, ..., \vec{x}_{N} \in \mathbb{R}^{d}.$$
(4.8)

In other words, if there exists an N-particle distribution $(\vec{x}_1, \vec{x}_2, ..., \vec{x}_N)$ such that Eq. (4.8) is violated, the pairwise interaction potential is catastrophic. A particular case that we can easily test is when all particles overlap $\vec{x}_1 = \vec{x}_2 = ... = \vec{x}_N$. By substituting Eq. (3.6) into Eq. (4.8) for this particular case,

$$0 \le \sum_{i=1}^{N} \sum_{j=1}^{N} V(0) = \sum_{i=1}^{N} \sum_{j=1}^{N} (-C_a + C_r)$$

$$= N^2 C_a (C-1).$$
(4.9)

If Eq. (4.10) is violated, the Morse type interaction potential is catastrophic. Thus, a second criterion is obtained:

Criterion 4.3.4 If

$$C < 1, \tag{4.10}$$

then the Morse potential in Eq. (3.6) is catastrophic.

Proposition 4.3.5 Positive-type pairwise interaction potential.

Let $\tilde{V}(\vec{q})$ denote the Fourier transform of $V(\vec{x})$. If

$$\tilde{V}(\vec{q}) \ge 0 \qquad \forall \vec{q} \in \mathbb{R}^d,$$
(4.11)

 $V(\vec{x_i} - \vec{x_j})$ belongs to a class of interaction potentials called the positive-type,

which satisfies

$$\sum_{i=1}^{N} \sum_{j=1}^{N} z_i z_j^* V\left(\vec{x}_i - \vec{x}_j\right) \ge 0 \qquad \forall \vec{x}_1, ..., \vec{x}_N \in \mathbb{R}^d \text{ and } \forall z_1, ..., z_N \in \mathbb{C}.$$
(4.12)

A positive-type pairwise interaction potential is always H-stable.

For the Morse-type interaction potential defined in Eq. (3.6), its Fourier transform can be obtained as

$$\tilde{V}(\vec{q}) = 2\pi C_a \left[\frac{C\ell^2}{(1+\ell^2 q'^2)^{3/2}} - \frac{1}{(1+q'^2)^{3/2}} \right],$$
(4.13)

where $q' \equiv q\ell_a = |\vec{q}| \ell_a$. By substituting Eq. (4.13) into Eq. (4.11), we have the following criterion for H-stability.

Criterion 4.3.6 If

$$\left[\frac{C\ell^2}{(1+\ell^2 q'^2)^{3/2}} - \frac{1}{(1+q'^2)^{3/2}}\right] \ge 0 \qquad \forall q' \in \mathbb{R},\tag{4.14}$$

which leads to

$$\ell \ge 1, \quad \ell < C,$$

or $\ell < 1, \quad \ell > \frac{1}{\sqrt{C}},$ (4.15)

then the Morse potential in Eq. (3.6) is H-stable.

Putting together the criteria in Eqs. (4.7), (4.10), and (4.14), the H-stability phase diagram can be obtained in Fig. 4.1. Note that the region of $C < \ell$ and C > 1 is not specified by any of the criteria. Given the fact that the global potential minimum occurs at $x = \infty$ in such a region, dynamical particles tend to diverge asymptotically, which is a H-stable configuration. Thus, we categorize



Figure 4.1: H-stability phase diagram

this region as H-stable as well. For the case of extremely fast dissipative systems, say, a potential gradient flow [32, 33, 34, 60], a great number of initial conditions more likely to form collapsing configurations instead. Later in Chapter 8, we will see such a dissipative system when our model is adapted to be implemented on first-order vehicles. Here, we label this region as H-stable following the classical sense. Nonetheless, this region does not offer interesting patterns to be explored; thus, it is less important whether this region is H-stable or not. Figure 4.1 labels regions V and VI as H-stable while the others are denoted as catastrophic.

4.4 Discussion

Note in Fig. 4.1 that the clump formation in region I, the ring formation in regin II, and the clumping rings in region III and IV are all catastrophic patterns. Furthermore, the H-stability boundary divides the biologically relevant regime into two regions, where region VI is H-stable and region VII is catastrophic. Swarms in the catastrophic region VII also exhibit more abundant pattern transitions that include the lattice, the milling, and the dispersed states, while the H-stable region VI has only the lattice and the dispersed states.

As described in Chapter 3, the presence of a hard-core repulsive interaction will destroy the double-mill state. This is because hard-cores always provide a system with H-stability. Thus, it is clear that for sufficiently many particles, the double mills will ultimately break apart. Notwithstanding, it appears that the double mills are especially sensitive to hard-cores and, even for small cores and moderate N, we have not observed these structures.

While different aggregation morphologies can be studied using the individualbased model, the large number of degrees of freedom involved pose a difficulty for analyzing the dynamics of large N systems. In the following chapter, we develop and investigate a continuum model consistent with the microscopic description of Eqs. (3.7) - (3.8). Additionally, we find that the ghost of H-stability appears yet again in the continuum concept and affects whether the continuum model is valid in a particular regime on the H-stability diagram, which is also discussed in the next chapter.

Chapter 5

Continuum Model

5.1 Introduction

Individual-based models arise naturally in the description of behavioral interactions among organisms. While large simulations of individual-based models are subjects of the statistical analysis, another interesting question is whether these swarming formations are asymptotically stable. However, the large number of degrees of freedom in such many-body systems make it formidable to answer this question analytically. Researchers have been proposing the continuum analogies for certain classes of the individual-based models, which allow them to solve for steady state solutions and to analyze the stability of such solutions [2, 12, 29, 35, 36, 49, 57, 64, 71, 97]. Earlier continuum models are mostly based on a heuristic interpretation of individual-based formula [2, 12, 49, 97]. A more rigorous derivation was presented by Okubo for the simplest case of biased random walks that feature only linear behavioral responses [70], as discussed in Chapter 2. A new approach was proposed by Grünbaum for a more general class of 1D random-walk models that include non-linear behavioral responses in his 1994 paper "Translating stochastic density-dependent individual behavior with sensory constraints to an Eulerian model of animal swarming" [36]. In the paper, the author starts with the following individual-based model

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \beta \left(\frac{\mathrm{d}x}{\mathrm{d}t} - v_0\right) = \eta \left(t\right),\tag{5.1}$$

where m, x, t are mass, position, and time. Eq. (5.1) describes an organism swimming in a flow field of speed v_0 , β is a drag coefficient of the surrounding media, and the stochastic force η represents noise. Assuming that the local velocity is statistically stationary, he then compares the functional form of the velocity autocorrelation function and obtains a continuum PDE via Okubo's results [70]

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2} - \frac{\partial}{\partial x} \left[\rho \left(v_0 + v_a E \left\{ f_d \right\} \right) \right].$$
(5.2)

In Eq. (5.2), the density ρ is a function of x and t. The diffusion coefficient Dand an aggregation speed v_a are evaluated from individual-based arguments, and the drift speed v_0 has been predefined in Eq. (5.1). Note that there is no social interaction in Eq. (5.1). Indeed, the social interaction is considered as a separated kinematic rule, formulated by a "decision function" $f_d(x, t, \rho_1, \rho_2, \rho_3, ..., \rho_{\mu})$, where ρ_i is the density at a predefined position *i*. In 1D, the decision function can be one or minus one, meaning that a particle at position x and time t decides to move toward the right or the left. It depends on the density distribution $\rho_1, \rho_2,$ $..., \rho_{\mu}$. The averaged decision can be evaluated as $E\{f_d\}$, the expected value of f_d , which appears in Eq. (5.2). While non-linear behavioral responses can be represented by non-linear decision functions, the author uses an example to show how to evaluate the expected value for such functions under the assumption that the organisms are Poisson particles, which is generally assumed in large-number swarming problems.

Although Eq. (5.1) takes a dynamic form, the assumption of a statistically stationary local velocity renders the acceleration term negligible. Additionally, the interactions are kinematic. Hence, Grünbaum's derivation is a kinematic approach and essentially an extension of Okubo's work [70]. Other extensions of Okubo's work on kinematic continuum models include two 2005 papers "From individualbased models to partial differential equations. An application to the upstream movement of elvers" by Gómez-Mourelo [35] and "An interacting particle system modelling aggregation behavior: from individuals to populations" by Morale et al. [64]. In both papers, the authors start with a probability density $f(\vec{x}, t)$ of finding particles at position \vec{x} and time t. Each individual particle moves according to a stochastic differential equation

$$d\vec{x}_{i} = \vec{c}_{1}(\vec{x}, t) dt + c_{2}(\vec{x}, t) d\vec{\eta}_{i},$$
(5.3)

where $\vec{\eta}_i$ is a stochastic variable. The two papers choose different behavioral functions \vec{c}_1 and c_2 to describe their own subjects. Nevertheless, by applying Igô's formula, Eq. (5.3) leads to

$$\int_{\mathbb{R}^d} f(\vec{x}, t) \rho(\vec{x}, t) \, \mathrm{d}\vec{x} = \int_{\mathbb{R}^d} f(\vec{x}, 0) \rho(\vec{x}, 0) \, \mathrm{d}\vec{x}$$

$$+ \int_0^t \int_{\mathbb{R}^d} \left(\frac{\partial f}{\partial s} + \vec{c_1} \cdot \vec{\nabla} f + \frac{c_2^2}{2} \vec{\nabla}^2 f \right) \rho \, \mathrm{d}\vec{x} \mathrm{d}s,$$
(5.4)

where $\rho(\vec{x}, t)$ is the particle density distribution. Given specific $\vec{c_1}$ and c_2 , Eq. (5.4) can further be rewritten as a weak form of a partial differential equation of ρ , which is the PDE of the continuum model.

The above examples contain the continuum limit of individual-based kinematic models; our model, however, is a dynamic model. If we want to apply the same procedure to a dynamic model, the velocity or the momentum has to be included as a coordinate on the phase space, and the particle density has to be expressed as $\rho(\vec{x}, \vec{v}, t)$ instead of just $\rho(\vec{x}, t)$. This is a Fokker-Planck approach. The 1999 paper "From individuals to aggregations: the interplay between behavior and physics" by Flierl et al. [29] shows an example of using such an approach to convert an individual-based dynamic model. Since it is easier to understand the spatial pattern of a swarm by expressing the density as $\rho(\vec{x}, t)$, Flierl et al. integrate $\rho(\vec{x}, \vec{v}, t)$ as well as the PDE over \vec{v} . To close out the integration, they have to make some assumptions; most notably, the "preferred velocity", or the equilibrium speed in our terminology, needs to be small with respect to a stochastic noise. The assumption is not applicable to our model since it is a deterministic model; even if we may add a noise term, interesting swarming patterns emerge mostly in a rather high-speed regime, which contradicts the assumption. Flierl et al. focus more on the patchiness phenomena, where the particles concentrate on several small patches, and the assumption is reason for such a problem. For our model and our interest of research, a different approach is needed. In this chapter, we apply a classical method in statistical mechanics that is used to derive fluid and gas-dynamics equations in the 1950 paper "The statistical mechanical theory of transport processes. IV. The equations of hydrodynamics" by Irving and Kirkwood [46]. In Section 5.2, we derive a fluid-like continuum swarming model. In Section 5.3, we compare the results of our continuum model to those of its individual-based counterpart. Additionally, we find that the H-stability properties, discussed in the pervious chapter, affect the validity of our derivation. In Section 5.4, we show a simple linear stability analysis of the continuum model in the regime where our derivation is valid. In Section 5.5, we discuss the choice of using soft-core interaction potentials to simulate swarming patterns. Although there are drawbacks regarding such choice, we propose remedies for this situation.

5.2 Derivation of the continuum model

Here, a continuum model is derived by explicitly calculating the ensemble average of Eqs. (3.7) - (3.8) using a probability distribution function. This classical procedure is described in Ref. [46] where continuum hydrodynamics equations are

derived starting from a microscopic collection of N particles. Let

$$f = f(\vec{x}_1, \vec{x}_2, ..., \vec{x}_N; \vec{p}_1, \vec{p}_2, ..., \vec{p}_N; t)$$
(5.5)

be the probability distribution function on the phase space, defined by position and momentum $(\vec{x}_i, \vec{p}_i), 1 \leq i \leq N$, at time t. The mass density $\rho(\vec{x}, t)$, the ensemble velocity field $\vec{u}(\vec{x}, t)$, and the continuum interaction force $\vec{F}_V(\vec{x}, t)$ can be defined as

$$\rho(\vec{x},t) = m \sum_{i=1}^{N} \langle \delta(\vec{x}_i - \vec{x}); f \rangle, \qquad (5.6)$$

$$\vec{u}(\vec{x},t) = \frac{\vec{p}(\vec{x},t)}{\rho(\vec{x},t)} = \frac{\sum_{i=1}^{N} \langle \vec{p}_i \delta(\vec{x}_i - \vec{x}); f \rangle}{\rho(\vec{x},t)},$$
(5.7)

$$\vec{F}_{V}(\vec{x},t) = \sum_{i=1}^{N} \left\langle -\vec{\nabla}_{\vec{x}_{i}} U(\vec{x}_{i}) \,\delta\left(\vec{x}_{i}-\vec{x}\right);f\right\rangle.$$
(5.8)

We consider the case of identical masses, $m_i \equiv m$. The function $\delta(\vec{x})$ is the Dirac delta function, and $U(\vec{x}_i)$ the collective interaction potential acting on particle *i*. Using the generalized Liouville theorem that incorporates the deformation of phase space due to the non-Hamiltonian nature of the system at hand [96], we obtain the continuum equations of motion

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0, \qquad (5.9)$$

$$\frac{\partial}{\partial t} \left(\rho \vec{u} \right) + \vec{\nabla} \cdot \left(\rho \vec{u} \vec{u} \right) + \vec{\nabla} \cdot \hat{\sigma}_{\mathrm{K}} = \alpha \rho \vec{u} - 2\beta E_{\mathrm{K}} \vec{u} - 2\beta \vec{q}_{\mathrm{K}} + 2\beta \vec{u} \cdot \hat{\sigma}_{\mathrm{K}} + \vec{F}_{V}(5.10)$$

The first is the equation of continuity, and the second is the momentum transport equation. Here, $E_{\rm K} = \rho |\vec{u}|^2 / 2$ is the kinetic energy. The terms $\vec{q}_{\rm K}(\vec{x},t)$ and
$\hat{\sigma}_{\mathrm{K}}\left(\vec{x},t\right)$ are mathematically defined as

$$\vec{q}_{\mathrm{K}}(\vec{x},t) = \sum_{i=1}^{N} \left\langle \frac{m}{2} \left| \frac{\vec{p}_{i}}{m} - \vec{u} \right|^{2} \left(\frac{\vec{p}_{i}}{m} - \vec{u} \right) \delta\left(\vec{x}_{i} - \vec{x} \right); f \right\rangle,$$

$$\hat{\sigma}_{\mathrm{K}}(\vec{x},t) = \sum_{i=1}^{N} m \left\langle \left(\frac{\vec{p}_{i}}{m} - \vec{u} \right) \left(\frac{\vec{p}_{i}}{m} - \vec{u} \right) \delta\left(\vec{x}_{i} - \vec{x} \right); f \right\rangle,$$

and represent the energy flux and the stress tensor due to local fluctuations in particle velocities with respect to $\vec{u}(\vec{x},t)$. The derivation of the term $\vec{\nabla} \cdot \hat{\sigma}_{\rm K}$ can be found in Ref. [46]. Readers can refer to Section 5.2.2 for the derivation of the other terms that are related to $\vec{q}_{\rm K}$ and $\hat{\sigma}_{\rm K}$. By simulating the discrete model, we estimate the magnitude of $\vec{q}_{\rm K}$ and $\hat{\sigma}_{\rm K}$ and find that both fluctuation terms become negligible with respect to the the other terms on the RHS of Eq. (5.10) in the lattice, single-mill, and the dispersed states. Thus, neglecting the fluctuation terms, we obtain

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0, \qquad (5.11)$$

$$\frac{\partial}{\partial t} \left(\rho \vec{u} \right) + \vec{\nabla} \cdot \left(\rho \vec{u} \vec{u} \right) = \alpha \rho \vec{u} - 2\beta E_{\rm K} \vec{u} + \vec{F}_V.$$
(5.12)

5.2.1 Continuum interaction force

In Eq. (5.12), the continuum interaction force can be obtained by substituting the explicit form of the interaction potential Eq. (3.6) into Eq. (5.8)

$$\vec{F}_{V}(\vec{x},t) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left\langle -\vec{\nabla}_{\vec{x}_{i}} V\left(\vec{x}_{i}-\vec{x}_{j}\right) \delta\left(\vec{x}_{i}-\vec{x}\right); f \right\rangle.$$
(5.13)

Using the fact that an arbitrary function $F(\vec{x}_j) \ \forall \vec{x}_j \in \mathbb{R}^d$ can be written as

$$F\left(\vec{x}_{j}\right) = \int_{\mathbb{R}^{d}} F\left(\vec{y}\right) \delta\left(\vec{x}_{j} - \vec{y}\right) \mathrm{d}\vec{y},$$

we can rewrite Eq. (5.13) as

$$\vec{F}_{V}(\vec{x},t) = \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{\mathbb{R}^{d}} d\vec{y} \Big\langle -\vec{\nabla}_{\vec{x}_{i}} V\left(\vec{x}_{i}-\vec{y}\right) \delta\left(\vec{x}_{i}-\vec{x}\right) \delta\left(\vec{x}_{j}-\vec{y}\right); f \Big\rangle$$

$$= \int_{\mathbb{R}^{d}} -\vec{\nabla}_{\vec{x}} V\left(\vec{x}-\vec{y}\right) \sum_{i=1}^{N} \sum_{j=1}^{N} \left\langle \delta\left(\vec{x}_{j}-\vec{y}\right) \delta\left(\vec{x}_{i}-\vec{x}\right); f \right\rangle d\vec{y}$$

$$= \int_{\mathbb{R}^{d}} -\vec{\nabla}_{\vec{x}} V\left(\vec{x}-\vec{y}\right) \rho^{(2)}\left(\vec{x},\vec{y},t\right) d\vec{y}, \qquad (5.14)$$

where the $\rho^{(2)}$ is the pair density

$$\rho^{(2)}(\vec{x}, \vec{y}, t) \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} \langle \delta(\vec{x}_{j} - \vec{y}) \, \delta(\vec{x}_{i} - \vec{x}); f \rangle.$$

Note that we should take the ensemble average on a scale considerably larger than the spacing between particles. If the particles are quite dispersed, the suitable scale may be much larger than the characteristic lengths of the interaction force $(-\vec{\nabla}V \text{ in Eq. (5.14)})$, rendering it localized. In this case, the continuum approach cannot capture the swarming characteristics occurring on the the interaction scale and fails to describe the individual-based model on such a scale. This is what occurs in the H-stable regime, which we further discuss in Section 5.3.

For identical particles, the pair density can be written as

$$\rho^{(2)}(\vec{x}, \vec{y}, t) = \frac{1}{m^2} \rho(\vec{x}, t) \rho(\vec{y}, t) g^{(2)}(\vec{x}, \vec{y}),$$

where the correlation function $g^{(2)}(\vec{x}, \vec{y}) = 1$ when the particles have no intrinsic correlation. Using this assumption,

$$\rho^{(2)}\left(\vec{x}, \vec{y}, t\right) = \frac{1}{m^2} \rho\left(\vec{x}, t\right) \rho\left(\vec{y}, t\right), \qquad (5.15)$$

and

$$\vec{F}_{V}(\vec{x},t) = \int_{\mathbb{R}^{d}} -\vec{\nabla}_{\vec{x}} V(\vec{x}-\vec{y}) \frac{1}{m^{2}} \rho(\vec{x},t) \rho(\vec{y},t) \mathrm{d}\vec{y}.$$
 (5.16)

If we further substitute the interaction potential specified in Eq. (3.6) into the above equation, we get

$$\vec{F}_{V}\left(\vec{x},t\right) = -\rho\left(\vec{x},t\right)\vec{\nabla}\int_{\mathbb{R}^{d}} \left(-\frac{C_{a}}{m^{2}}\mathrm{e}^{-\frac{\left|\vec{x}-\vec{y}\right|}{\ell_{a}}} + \frac{C_{r}}{m^{2}}\mathrm{e}^{-\frac{\left|\vec{x}-\vec{y}\right|}{\ell_{r}}}\right)\rho\left(\vec{y},t\right)\mathrm{d}\vec{y}.$$
 (5.17)

Since we assume that all particles have an identical mass, we may choose m = 1 without loss of generality. In this case, Eq. (5.17) becomes the one proposed in Ref. [57]. Using Eq. (5.11), we may modify Eq. (5.12) and divide by ρ on both sides to obtain a more conventional expression

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0, \qquad (5.18)$$

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = \alpha \vec{u} - \beta \left| \vec{u} \right|^2 \vec{u} - \frac{1}{m^2} \vec{\nabla} \int_{\mathbb{R}^d} V\left(\vec{x} - \vec{y} \right) \rho\left(\vec{y}, t \right) \mathrm{d}\vec{y}.$$
 (5.19)

5.2.2 Fluctuation terms

This section shows the derivation of the fluctuation terms in Eq. (5.10). Since we have argued that these terms are negligible for the phenomena explored in this thesis, the reader can skip this section without loss of essential information for other sections. The derivation here is for future consideration when these terms

are needed and an unambiguous close form can perhaps be defined for other situations.

Following Ref. [46], the momentum transport equation can be obtained by substituting the macroscopic momentum

$$\rho\left(\vec{x},t\right)\vec{u}\left(\vec{x},t\right) = \left\langle \sum_{i=1}^{N} \vec{p}_{i}\delta\left(\vec{x}_{i}-\vec{x}\right);f\right\rangle$$

into the generalized Liouville Equation, valid for non-conserved systems [96],

$$\frac{\partial \left(\rho \vec{u}\right)}{\partial t} = \frac{\partial}{\partial t} \left\langle \sum_{i=1}^{N} \vec{p}_{i} \delta\left(\vec{x}_{i} - \vec{x}\right); f \right\rangle$$
$$= \sum_{k=1}^{N} \left\langle \frac{\vec{p}_{k}}{m} \cdot \vec{\nabla}_{\vec{x}_{k}} \left(\sum_{i=1}^{N} \vec{p}_{i} \delta\left(\vec{x}_{i} - \vec{x}\right) \right) + \dot{\vec{p}}_{k} \cdot \vec{\nabla}_{\vec{p}_{k}} \left(\sum_{i=1}^{N} \vec{p}_{i} \delta\left(\vec{x}_{i} - \vec{x}\right) \right); f \right\rangle.$$

Here f is the probability density function described in Eq. (5.5). Since

$$\begin{aligned} \frac{\vec{p}_k}{m} \cdot \vec{\nabla}_{\vec{x}_k} \left(\sum_{i=1}^N \vec{p}_i \delta\left(\vec{x}_i - \vec{x}\right) \right) &= \frac{\vec{p}_k}{m} \cdot \vec{\nabla}_{\vec{x}_k} \vec{p}_k \delta\left(\vec{x}_k - \vec{x}\right) \\ &= -\vec{\nabla}_{\vec{x}} \cdot \left(\frac{\vec{p}_k \vec{p}_k}{m}\right) \delta\left(\vec{x}_k - \vec{x}\right), \\ \dot{\vec{p}_k} \cdot \vec{\nabla}_{\vec{p}_k} \left(\sum_{i=1}^N \vec{p}_i \delta\left(\vec{x}_i - \vec{x}\right) \right) &= \dot{\vec{p}_k} \delta\left(\vec{x}_k - \vec{x}\right), \end{aligned}$$

the transport equation can further be reduced to

$$\frac{\partial(\rho\vec{u})}{\partial t} = \sum_{k=1}^{N} \left[-\nabla_{\vec{x}} \cdot \left\langle \left(\frac{\vec{p}_k \vec{p}_k}{m}\right) \delta\left(\vec{x}_k - \vec{x}\right); f \right\rangle + \left\langle \dot{\vec{p}}_k \delta\left(\vec{x}_k - \vec{x}\right); f \right\rangle \right] (5.20)$$

The first term on the right hand side can be modified by noting that

$$\sum_{k=1}^{N} m \left\langle \left(\frac{\vec{p}_{k}}{m} - \vec{u}\right) \left(\frac{\vec{p}_{k}}{m} - \vec{u}\right) \delta\left(\vec{x}_{k} - \vec{x}\right); f \right\rangle$$

$$= \sum_{k=1}^{N} \left\langle \left(\frac{\vec{p}_{k}\vec{p}_{k}}{m}\right) \delta\left(\vec{x}_{k} - \vec{x}\right); f\right\rangle - \vec{u} \sum_{k=1}^{N} \left\langle \vec{p}_{k}\delta\left(\vec{x}_{k} - \vec{x}\right); f\right\rangle$$

$$- \sum_{k=1}^{N} \left\langle \vec{p}_{k}\delta\left(\vec{x}_{k} - \vec{x}\right); f\right\rangle \vec{u} + \vec{u}\vec{u} \sum_{k=1}^{N} \left\langle m\delta\left(\vec{x}_{k} - \vec{x}\right); f\right\rangle$$

$$= \sum_{k=1}^{N} \left\langle \left(\frac{\vec{p}_{k}\vec{p}_{k}}{m}\right) \delta\left(\vec{x}_{k} - \vec{x}\right); f\right\rangle - \rho \vec{u}\vec{u},$$

where \vec{u} is the macroscopic velocity defined in Eq. (5.7). Eq. (5.20) then becomes

$$\frac{\partial \left(\rho \vec{u}\right)}{\partial t} + \vec{\nabla}_{\vec{x}} \cdot \left(\rho \vec{u} \vec{u}\right) = -\vec{\nabla}_{\vec{x}} \cdot \hat{\sigma}_{\mathrm{K}}\left(\vec{x}, t\right) + \sum_{k=1}^{N} \left\langle \dot{\vec{p}}_{k} \delta\left(\vec{x}_{k} - \vec{x}\right); f \right\rangle, \quad (5.21)$$

where

$$\hat{\sigma}_{\mathrm{K}} = \sum_{k=1}^{N} m \left\langle \left(\frac{\vec{p}_{k}}{m} - \vec{u} \right) \left(\frac{\vec{p}_{k}}{m} - \vec{u} \right) \delta \left(\vec{x}_{k} - \vec{x} \right); f \right\rangle.$$

We can substitute the explicit form of $\dot{\vec{p}}_k$ from Eqs. (5.7) and (5.8)

$$\dot{\vec{p}}_{k} = \alpha \vec{p}_{k} - \beta \frac{\left|\vec{p}_{k}\right|^{2}}{m^{2}} \vec{p}_{k} - \vec{\nabla} U\left(\vec{x}_{k}\right)$$

into the second term of Eq. (5.21)

$$\begin{split} \sum_{k=1}^{N} \left\langle \dot{\vec{p}}_{k} \delta\left(\vec{x}_{k}-\vec{x}\right); f \right\rangle &= \sum_{k=1}^{N} \left\langle \left(\alpha \vec{p}_{k} - \beta \frac{\left|\vec{p}_{k}\right|^{2}}{m^{2}} \vec{p}_{k} - \vec{\nabla} U\left(\vec{x}_{k}\right) \right) \delta\left(\vec{x}_{k}-\vec{x}\right); f \right\rangle \\ &= \alpha \rho \vec{u} - \sum_{k=1}^{N} \left\langle \left(\beta \frac{\left|\vec{p}_{k}\right|^{2}}{m^{2}} \vec{p}_{k} \right) \right\rangle + \vec{F}_{V}. \end{split}$$

The second term above can be further simplified as

$$\begin{split} \sum_{k=1}^{N} \left\langle \left(\beta \frac{|\vec{p}_{k}|^{2}}{m^{2}} \vec{p}_{k} \right) \delta\left(\vec{x}_{k} - \vec{x}\right); f \right\rangle &= \\ \beta \sum_{k=1}^{N} \left\langle \left(\frac{|\vec{p}_{k}|^{2}}{m^{2}} \vec{p}_{k} \right) \delta\left(\vec{x}_{k} - \vec{x}\right); f \right\rangle - \beta \sum_{k=1}^{N} \left\langle \frac{|\vec{p}_{k}|^{2}}{m} \vec{u} \delta\left(\vec{x}_{k} - \vec{x}\right); f \right\rangle \\ &+ \beta \sum_{k=1}^{N} \left\langle m \left(-2 \frac{\vec{p}_{k}}{m} \cdot \vec{u} + |\vec{u}|^{2} \right) \left(\frac{\vec{p}_{k}}{m} - \vec{u} \right) \delta\left(\vec{x}_{k} - \vec{x}\right); f \right\rangle \\ &+ 2\beta E_{K} \vec{u} - 2\beta \vec{u} \cdot \hat{\sigma}_{K} + \beta \sum_{k=1}^{N} \left\langle m |\vec{u}|^{2} \left(\frac{\vec{p}_{k}}{m} - \vec{u} \right) \delta\left(\vec{x}_{k} - \vec{x}\right); f \right\rangle \\ &= 2\beta \sum_{k=1}^{N} \left\langle \frac{m}{2} \left| \frac{\vec{p}_{k}}{m} - \vec{u} \right|^{2} \left(\frac{\vec{p}_{k}}{m} - \vec{u} \right) \delta\left(\vec{x}_{k} - \vec{x}\right); f \right\rangle + 2\beta E_{K} \vec{u} - 2\beta \vec{u} \cdot \hat{\sigma}_{K} \\ &= 2\beta \vec{q}_{K} + 2\beta E_{K} \vec{u} - 2\beta \vec{u} \cdot \hat{\sigma}_{K}, \end{split}$$

where

$$\vec{q}_{\mathrm{K}} = \sum_{i=1}^{N} \left\langle \frac{m}{2} \left| \frac{\vec{p}_{i}}{m} - \vec{u} \right|^{2} \left(\frac{\vec{p}_{i}}{m} - \vec{u} \right) \delta\left(\vec{x}_{i} - \vec{x} \right); f \right\rangle.$$

As a result, Eq. (5.21) can be written as

$$\frac{\partial}{\partial t} \left(\rho \vec{u} \right) + \vec{\nabla} \cdot \left(\rho \vec{u} \vec{u} \right) + \vec{\nabla} \cdot \hat{\sigma}_{\mathrm{K}} = \alpha \rho \vec{u} - 2\beta E_{\mathrm{K}} \vec{u} - 2\beta \vec{q}_{\mathrm{K}} + 2\beta \vec{u} \cdot \hat{\sigma}_{\mathrm{K}} + \vec{F}_{V},$$

which is the momentum transport equation shown in Eq. (5.10).

5.2.3 Numerical method

For numerical simulations of the continuum model, we use the Lax-Friedrichs method [56] to integrate the partial differential equations (5.11) - (5.12). Unless specified otherwise, the computational domain is a $5\ell_a \times 5\ell_a$ box with boundary conditions of out-going waves, which are equivalent to the free boundary conditions of individual-based simulations. The initial conditions are chosen as a homogeneous density in a $2\ell_a \times 2\ell_a$ box at the center of the computational domain with randomized momentum field. The computational domain is divided into 256×256 grid cells while the time step size is chosen so that the CFL number is 0.98.

5.3 Comparison to the individual-based model

The time-dependent variations of the density $\rho(\vec{x}, t)$ and of the momentum $\vec{p}(\vec{x}, t) \equiv \rho(\vec{x}, t) \vec{u}(\vec{x}, t)$ can be obtained through numerical simulations of Eqs. (5.11) - (5.12). We compare the results of the continuum model to those of the individual-based model of Eqs. (3.7) - (3.8). Figure 5.1 shows the frequently observed single-mill steady state solutions of both models in the catastrophic regime. Both simulations use identical parameter values and the same total mass

$$m_{\rm tot} = \int_{\infty} \rho(\vec{x}) \mathrm{d}\vec{x} = Nm.$$

Consistent with the initial conditions of the continuum model, particles of the individual-based model are initially distributed with random velocities and at random positions in a $2\ell_a \times 2\ell_a$ box. Free boundary conditions are imposed to the individual-based simulation, which allow the particles to move around over

the entire space, The individual-based simulation also adopts an adaptive time step size that keeps the increment in position of each step under $\ell_a/10$ and the increment in velocity under $C_a/5m$. Figure 5.1 (a) illustrates the averaged density $\langle \rho \rangle$ as a function of the radial distance from the center of mass. These two profiles are in good agreement despite the density oscillation shown in the individual-based model, reflecting a multiple-ring ordering of the particle distribution. Figures 5.1 (b) and (c) match the averaged radial and tangential momenta (denoted by $\langle p \rangle_{\rm rad}$ and $\langle p \rangle_{\rm tang}$ respectively) from the simulations of both models. The negligible radial momenta in Fig. 5.1 (b) indicate that there is no net inward or outward mass movement, and thus, the density profile along the radial direction is steady. We can divide the momentum by the density to obtain the velocity field. In Fig. 5.1 (d), we show the averaged tangential velocities, $\langle v \rangle_{\rm tang} \equiv \langle p \rangle_{\rm tang} / \langle \rho \rangle$, from the simulations of both models; it shows that both the individual-based and the continuum swarms are rotating at the same constant speed, which equals to $v_{\rm eq}$.

5.3.1 Validity of the continuum model

The ensemble average implicit in the continuum approach does not allow for double-milling in the continuum limit because the velocity inside a mesh cell is averaged and unified. Local velocity variations, which contribute to $\vec{q}_{\rm K}(\vec{x},t)$ and $\hat{\sigma}_{\rm K}(\vec{x},t)$ in Eq. (5.10), are neglected. We calculate the ratio of the speed variation to the equilibrium speed, $\Delta_{\rm K} \equiv \sqrt{\langle (v - v_{\rm eq})^2 \rangle / v_{\rm eq}^2}$, in order to efficiently estimate the contribution of these local velocity fluctuation terms. Figure 5.2 shows that $\Delta_{\rm K}$ becomes negligible after the swarm has reached the single-mill configuration and $M \simeq 1$. However, during the transient time, $\Delta_{\rm K}$ is significantly larger, which implies that $\vec{q}_{\rm K}(\vec{x},t)$ and $\hat{\sigma}_{\rm K}(\vec{x},t)$ cannot be neglected during this period.



Figure 5.1: Comparison of the numerical simulations of the individual-based model and the continuum model: The parameters used in both simulations are $C_a = 0.5$, $C_r = 1.0$, $\ell_a = 2.0$, $\ell_r = 0.5$, $\alpha = 1.2$, $\beta = 0.5$, and the total mass $m_{\text{tot}} = 88$. (a) The averaged density profiles along the radial distance from the center of mass. (b) The averaged radial momentum profiles. (c) The averaged tangential momentum profiles. (d) The averaged tangential velocity profiles.



Figure 5.2: Relative speed fluctuations $\Delta_{\rm K}$ while forming a single mill from random initial conditions. The dashed curve illustrates the normalized angular momentum M in Eq. (3.14) while the solid curve represents $\Delta_{\rm K}$. The parameters of this simulation are $\alpha = 1.0$, $\beta = 0.5$, $C_a = 0.5$, $C_r = 1.0$, $\ell_a = 2.0$, $\ell_r = 0.5$, and N = 500.

Hence, the continuum model of Eqs. (5.11) - (5.12) can be useful in analyzing the stability of the steady state solution but does not capture the dynamics of the swarm settling into this steady state.

While Figure 5.1 shows good agreement between the steady state solutions of the continuum and the individual-based models in the catastrophic regime, inconsistencies arise as the parameters shift into the H-stable regime. Here, at low particle speeds, the individual-based model results in compactly supported solutions similar to those shown in Fig. 3.2 (d) and (e). Conversely, the corresponding continuum model yields a uniform density distribution spreading over the entire computational domain regardless of domain size. Since a valid continuum model should reflect the large-number limit of the individual-based model, we can investigate how the steady state solutions of the individual-based model evolve by increasing N while keeping all continuum variables and parameters fixed. In particular, we increase N while keeping the macroscopic parameter $m_{\rm tot} = Nm$ fixed. If the individual-based solution has converged to the continuum limit, the solutions should be independent of any microscopic parameter, such as N. In Fig. 5.3 (a), we show the radius R of steady swarms versus N under fixed $m_{\rm tot}$ for an H-stable case and for a catastrophic one. The flock size is indeed independent of N for catastrophic swarms, and the two models yield consistent solutions, as shown in Fig. 5.1. However, in the H-stable regime, the swarm size increases with \sqrt{N} in spite of a fixed $m_{\rm tot}$. This suggests that a compactly supported solution does not exist in the large number limit of an H-stable swarm. Figure 5.3 (b) further illustrates this point by expanding the investigation to a broader parameter space. The H-stability threshold is the solid curve, parting the $C-\ell$ phase space in Fig. 5.3 (b). The flock radius R is approximately independent of N for catastrophic swarms, but when the parameters C and ℓ cross over to the H-stable regime, R scales as N^Z with $Z \simeq 1/2$. As $N \to \infty$, H-stable swarms tend to occupy the entire space.

The cue to the inconsistency between the solutions of the two models in the H-stable regime lies in the derivation of the continuum model. As previously mentioned, the macroscopic variables are obtained as ensemble averages over a large number of microscopic ones. In the catastrophic regime, $\delta_{\text{NND}} \ll \ell_a$, ℓ_r in large N limit, as shown in Fig. 5.3. Hence, as $N \to \infty$, the particle distribution converges to a continuum density on a scale comparable to the interaction length. On the other hand, for an H-stable swarm, δ_{NND} stays non-negligible with respect



Figure 5.3: (a) Flock radius versus number of particles with the total mass fixed at $m_{\text{tot}} = Nm = 500$. The solid circles represent the H-stable flock ($\ell_r = 1.5$), fitted by $R \propto N^{0.41}$, while the solid diamonds represent the catastrophic flock ($\ell_r = 1.3$), fitted by $R \propto N^{0.11}$. The other parameters are $\alpha = 0$, $\beta = 0.5$, $C_a = 0.5$, $C_r = 1.0$, and $\ell_a = 2.0$. (b) Exponents Z of the power law fitting $R \propto N^Z$ for a range of C and ℓ . The dimensionless parameters C and ℓ are changed by varying C_r and ℓ_r while the other parameters remain the same as above. The gray-scale map indicates the power Z, and darker colors represent higher exponents. The solid curve marks the H-stability boundary with the upper region being H-stable and the lower region being catastrophic.

to the characteristic length of the interaction. Hence, Eq. (5.14) does not hold on a scale comparable to the interaction length, and as a result, Eq. (5.16) is not a valid description of the continuum force on such a scale in the H-stable regime. This is also verified in Fig. 5.4. In Fig. 5.4 (a), we define significant neighbors of a particle as neighbors that exhibit a "significant interaction". The quantitative definition is illustrated by the graph on the upper-right corner, in which the pairwise interaction potential V(x) is plotted versus the inter-particle distance x, and the potential well depth is denoted by V_{\min} . We define a distance x_s so that $V(x) > sV_{\min}$ if $x > x_s$, where $0 \le s \le 1$ is a ratio. Then, the number of significant neighbors of each particle is the number of neighbors at a distance x for which $x \leq x_s$. In Fig. 5.4 (a), we count the averaged number of significant neighbors of a particle, denoted by n_s , for s = 0.5 and s = 0.1. In the H-stable regime, n_s is very low and remains steady; it rises rapidly when the parameter crosses over into the catastrophic regime. The results suggest that the H-stable swarms are locally too sparse for Eq. (5.14) (and thus, Eq. (5.16)) to remain valid. Furthermore, we can use ensemble averages to approximate the collective interaction potentials in the two models. If the continuum limit properly describe the individual-based description, these two potential energies should converge as N increases. Let us define U_{Eu} as the continuum ensemble average interaction potential in the Eulerian frame

$$U_{\rm Eu}\left(\vec{x}\right) \equiv \frac{1}{m^2} \int_{\mathbb{R}^d} V\left(\vec{x} - \vec{y}\right) \rho\left(\vec{x}\right) \rho\left(\vec{y}\right) \mathrm{d}\vec{y}.$$

Here $\rho(\vec{x})$ is approximated by the ensemble average of the individual particles during the simulation. We also define U_{La} as the average collective potential calculated in the Lagrangian frame, $U_{\text{La}}(\vec{x}) \equiv \langle U(\vec{x}_i) \rangle_{\vec{x}_i = \vec{x}}$, where $U(\vec{x}_i)$ is defined in Eq. (3.6) of the individual-based model. Since the rigid-body rotation and the



Figure 5.4: (a) n_s versus ℓ . The upper curve represents the case of s = 10% while the lower curve is for s = 50%. On the upper-right corner is an illustration showing how the significant neighbors are defined. (b) $U_{\rm Eu}$ and $U_{\rm La}$ versus ℓ . Here $\alpha = 0.003$, $\beta = 0.5$, $C_a = 0.5$, $\ell_a = 2.0$, $C_r = 1.0$, and N = 500. (c) ΔU and versus N. $\ell = 0.65$ for the catastrophic curve while $\ell = 0.75$ for the H-stable curve. The other parameter values are the same as (b).

single-mill state have rotational symmetry with respect to $\vec{x}_{\rm CM}$, we evaluate $U_{\rm Eu}$ and U_{La} after such states are reached and at position \vec{x} such that $|\vec{x} - \vec{x}_{\text{CM}}| = R/2$, where R is the swarm radius. In Fig. 5.4 (b), U_{Eu} and U_{La} are shown to converge in the catastrophic regime and diverge in the H-stable one. In Fig. 5.4(c), we investigate whether the difference between these two averaged potentials, $\Delta U \equiv$ $U_{\rm Eu} - U_{\rm La}$, vanishes with increasing N. Figure 5.4 (c) shows that ΔU indeed tends to zero for catastrophic swarms by increasing N but remains finite for Hstable ones. For the ensemble average to be valid in the H-stable regime, we may instead choose a scale that is much larger than the characteristic interaction lengths. Under such low resolution, the particle distribution can be seen as a continuous density, and Eq. (5.14) is then valid. This becomes the case of the incompressible fluids in Ref. [46], where the interaction is extremely localized, and hence, the continuum force yields a stress tensor as a function of the local density. However, the swarming patterns which we am interested in emerge on a much smaller scale. In contrast, when the particles are in the dispersed state, they are far away from each other; thus, particle-particle interaction is very weak and dominated by velocity fluctuations. The continuum force then yields a scalar pressure, which gives the gas dynamics equations [46].

5.4 Linear stability analysis

That the solutions of the continuum model of Eqs. (5.11) and (5.12) relax toward a uniform density distribution in the H-stable regime can also be shown by the linear stability analysis of its homogeneous solution. Let us first consider a more general case for a 2D self-driving continuum model with a non-local interaction

$$\frac{\partial \rho}{\partial t} - \vec{\nabla} \cdot (\rho \vec{u}) = 0; \qquad (5.22)$$

$$\frac{\partial}{\partial t} (\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u} \vec{u}) = f^{(l)} (|\vec{u}|) \rho \vec{u} - \rho \vec{\nabla} \int_{\mathbb{R}^d} V (|\vec{x} - \vec{y}|) \rho (\vec{y}) \, \mathrm{d}\vec{y},$$

where $f^{(l)}(|\vec{u}|)$ is a scalar function specifying the self-driving mechanism, and the non-local interaction is expressed by the convolution term. For our model,

$$f^{(l)}\left(\left|\vec{u}\right|\right) = \alpha - \beta \left|\vec{u}\right|^{2}.$$

The possible homogeneous steady state solutions can be written as $\rho(\vec{x},t) = \rho_0$ and $\vec{u}(\vec{x},t) = v_0 \hat{v}$, where \hat{v} is a unit vector and v_0 can be 0 or any of the roots of $f^{(l)}(v_0) = 0$. For our Rayleigh-type dissipation, $v_0 = \sqrt{\alpha/\beta}$ or 0. We perturb the steady state solution using $\rho(\vec{x},t) = \rho_0 + \delta\rho \exp(\sigma t + i\vec{q}\cdot\vec{x})$ and $\vec{u}(\vec{x},t) =$ $v_0 \hat{v} + (\delta u \hat{u} + \delta v \hat{v}) \exp(\sigma t + i\vec{q}\cdot\vec{x})$, where $\delta\rho$, δu , $\delta v \ll 1$ are small amplitudes. The unit vector \hat{u} points to the direction perpendicular to \hat{v} on the 2D space. The wave vector is denoted by \vec{q} while $\sigma = \sigma(\vec{q})$ represents its growth rate. By substituting this ansatz into Eq. (5.22), the dispersion relation is

$$\sigma'\begin{pmatrix}\delta\rho\\\delta u\\\delta v\end{pmatrix} = \begin{pmatrix}0 & -\mathrm{i}\rho_0 q\sin\theta & -\mathrm{i}\rho_0 q\cos\theta\\-\mathrm{i}q\tilde{V}\sin\theta & f^{(l)}(v_0) & 0\\-\mathrm{i}q\tilde{V}\cos\theta & 0 & f^{(l)}(v_0) + v_0 f^{(l)'}(v_0)\end{pmatrix}\begin{pmatrix}\delta\rho\\\delta u\\\delta v\end{pmatrix}, (5.23)$$

where $\sigma' \equiv \sigma + iv_0 \hat{v} \cdot \vec{q}$, and $\tilde{V} = \tilde{V}(\vec{q})$ is the Fourier transform of the pairwise interaction potential $V(\vec{x})$. The angle between the wave vector \vec{q} and the unit vector \hat{u} is denoted by θ .

For the case of $v_0 = 0$, the solution is isotropic, and we can arbitrarily choose the unit vector \hat{v} . If the wave vector \vec{q} is parallel to the arbitrarily chosen \hat{v} , Eq. (5.23) reduces to

$$\sigma \begin{pmatrix} \delta \rho \\ \delta u \\ \delta v \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i\rho_0 q \\ 0 & f^{(l)}(0) & 0 \\ iq\tilde{V}(\vec{q}) & 0 & f^{(l)}(0) \end{pmatrix} \begin{pmatrix} \delta \rho \\ \delta u \\ \delta v \end{pmatrix},$$

and
$$\sigma = f^{(l)}(0)$$
 or $\left(f^{(l)}(0) \pm \sqrt{f^{(l)}(0)^2 - 4\rho_0 q^2 \tilde{V}(\vec{q})}\right)/2$. If $f^{(l)}(0) > 0$, the

homogeneous solution is always unstable. If $f^{(l)}(0) < 0$, the homogeneous solution is stable only when $\rho_0 q^2 \tilde{V}(\vec{q}) > 0$. Since ρ_0 and q^2 are both non-negative, the criterion can be reduced to

$$V(\vec{q}) > 0.$$
 (5.24)

For our Rayleigh-type dissipation, $f^{(l)}(0) = \alpha$. Since α is positive, the uniform density solution with zero speed is an unstable steady state solution.

For the case of $v_0 \neq 0$ satisfying $f^{(l)}(v_0) = 0$, Eq. (5.23) becomes

$$\sigma'\begin{pmatrix}\delta\rho\\\delta u\\\delta v\end{pmatrix} = \begin{pmatrix}0 & -\mathrm{i}\rho_0 q\sin\theta & -\mathrm{i}\rho_0 q\cos\theta\\-\mathrm{i}q\tilde{V}(\vec{q})\sin\theta & 0 & 0\\-\mathrm{i}q\tilde{V}(\vec{q})\cos\theta & 0 & v_0 f^{(l)'}(v_0)\end{pmatrix}\begin{pmatrix}\delta\rho\\\delta u\\\delta v\end{pmatrix}.$$

We thus obtain the growth rate by solving the following eigenvalue equation

$$\sigma^{\prime 3} - v_0 f^{\prime}(v_0) \, \sigma^{\prime 2} + \sigma^{\prime} \rho_0 q^2 \tilde{V}(\vec{q}) - v_0 f^{(l)\prime}(v_0) \, \rho_0 q^2 \tilde{V}(\vec{q}) \sin^2 \theta = 0. \tag{5.25}$$

Let us consider the two cases of the wave vectors parallel and perpendicular to the \hat{v} -direction. For the parallel case, i.e., $\theta = 0$,

$$\sigma' = 0 \text{ or } \sigma' = \frac{1}{2} \left[v_0 f^{(l)'}(v_0) \pm \sqrt{\left(v_0 f^{(l)'}(v_0) \right)^2 - \rho_0 q^2 \tilde{V}(\vec{q})} \right].$$

On the other hand, in the perpendicular case $(\theta = \pi/2)$, $\sigma' = v_0 f^{(l)'}(v_0)$ or $\sigma' = \pm \sqrt{-\rho_0 q^2 \tilde{V}(\vec{q})}$. If $f^{(l)'}(v_0) > 0$, the homogeneous solutions are always



Figure 5.5: The linear stability diagram of the swarming model of Eqs. (3.7) - (3.8).

unstable. For our Rayleigh-type dissipation, $f^{(l)'}(v_0) = -2\beta v_0 < 0$; hence, the homogeneous solution is stable only when $\rho_0 q^2 \tilde{V}(\vec{q}) > 0$, which is the same as the criterion in Eq. (5.24). Further analysis shows that for a general angle θ , Eq. (5.25) can be rewritten as

$$\left(\sigma' - v_0 f^{(l)'}(v_0)\right) \left(\sigma'^2 + \rho_0 q^2 \tilde{V}(\vec{q})\right) + \Gamma \cos^2 \theta = 0,$$

where $\Gamma \equiv v_0 f^{(l)'}(v_0) \rho_0 q^2 \tilde{V}(\vec{q})$. In our model, $\Gamma > 0$ whenever the homogeneous solution is unstable. Thus, an inspection of the above equation shows that its largest root, i.e., the fastest growth rate, is at $\theta = \pi/2$. As a result, perturbations on the direction perpendicular to the swarm velocity are the fastest growing mode, and their rate is $\sqrt{-\rho_0 q^2 \tilde{V}(q)}$ for a given q. Substituting Eq. (3.6) into Eq. (5.24), the linear stability criterion for our swarming model can be explicitly obtained as

$$\tilde{V}(q) \equiv 2\pi \left[-\frac{1}{\left(1+q'^2\right)^{3/2}} + \frac{C\ell^2}{\left(1+\ell^2 q'^2\right)^{3/2}} \right] > 0,$$
(5.26)

where $q' \equiv q\ell_a$. Since the above criterion has to hold for all $q' \in \mathbb{R}$, stability is attained at

$$\begin{array}{ll} C\ell^2 > 1 & \mbox{if } \ell < 1, \\ C > \ell & \mbox{if } \ell \geq 1, \end{array}$$

The linear stability diagram is shown in Fig. 5.5. Note the close connection between the different regimes shown here and in the H-stability diagram of Fig. 4.1. When the homogeneous solution is linearly stable, the interaction potential is also H-stable. This is because the condition of Eq. (5.24) is also sufficient, but not necessary, for H-stability [80]. Further study on the dispersion relation in Eq. (5.25)reveals that σ' increases as $q^{2}\tilde{V}(\vec{q})$ decreases, and the maximum of σ' occurs when the minimum of $q^2 \tilde{V}(\vec{q})$ is reached. As a result, we amable to evaluate the wavelength of the fastest growth mode and categorize the long-wave and the short-wave instability regions in the parameter space. Furthermore, we compare the fastest growth wavelength to the pattern of the fully nonlinear continuum model near the onset of the instability, shown in the left panel of Fig. 5.6. The simulations are initiated with a homogeneous density distribution and computed on a periodic domain of a 206.8×206.8 box. The wavenumber of the fastest growth mode is calculated as the minimum of Eq. (5.26). For the parameters chosen in Fig. 5.6, $|\vec{q}| = 0.121$, which corresponds to a wavelength $\lambda = 51.87$. This value matches the density aggregation patterns quite well. In the upper figure, $\alpha = 0$; the steady state density has zero velocity, and the x-y directions are isotropic.

In the lower figure, $\alpha \neq 0$, and the velocity field of the swarm is initiated as $\vec{u} (t = 0) = \sqrt{\alpha/\beta}\hat{y}$. The direction of the stripes indicates that the fastest growth mode is indeed perpendicular to the initial velocity, which is also consistent with the theoretical prediction. The results can also be compared to the simulation of the individual-based model by using the same parameter values and equivalent initial and boundary conditions. Since V(r) decays rapidly in r, $U(\vec{x}_i)$ can be well approximated by including only the adjacent eight boxes surrounding the computational domain. The steady particle distributions of the individual-based simulations are shown on the right panel of Fig. 5.6. The theoretically predicted wavelength agrees with the patterns seen in the simulations of the continuum and the individual-based models.

5.5 Discussion

Soft-core interactions are widely adopted in the swarming literature [14, 17, 21, 23, 24, 25, 57, 61, 63, 79, 93]. Our investigations, with the Morse potential of Eq. (3.6), reveal that the commonly observed core-free mill patterns only exist in the catastrophic regime and not in the H-stable one. In this latter case, particles arrange in rigid-body-like structures, similar to the solid configurations in molecular dynamics simulations using a Lennard-Jones potential. The morphology richness associated with soft-core catastrophic potentials has led to their popularity in the literature of swarming patterns. One drawback to soft-core interactions for swarming models is that particles could occupy the same space, which is an unphysical situation. However, one need not abandon such models completely; it can be modified locally and still retain large scale features of the swarm. One modification is to consider that animals usually flock on a reduced dimension and thus, can use the extra dimension to avoid actually occupying the



Figure 5.6: Left panel: The contours of a density distribution of the continuum model near the instability onset with (upper) $\alpha = 0$ and (lower) $\alpha = 1$; Right panel: Simulations of the individual-based model using the same parameters and initial conditions as the left figures. The parameter values are $\beta = 0.5$, $C_a = 0.5$, $C_r = 1.0$, $\ell_a = 2.0$, and $\ell_r = 1.35$.

same space. For example, ants can crawl over each other; therefore, they can use z-direction to "pass through" each other when viewed as a swarm on the x-yplane. Another way of modification is using separation of scales. We may actually add an additional hard-core repulsion solely to prevent overlapping. In other words, there is a soft-core potential that defines an equilibrium distance between particles and gives rise to the swarming patterns, and there is also a hard-core potential on a much smaller scale that specifies a forbidden distance and prevent particles from penetrating each other. We find that the presence of the hard-core potential affects the swarming pattern only when N is large enough, and hence the equilibrium distance between nearest neighbors, determined by the collective soft-core interaction, collapses to the vicinity of the hard-core forbidden distance. Otherwise, flocks exhibit the same soft-core steady state patterns for small to moderate N, except for the double-mill state, which is apparently very sensitive to hard-cores and, in our simulations, are absent altogether. As N increases, the equilibrium δ_{NND} at first decreases; the flock size increases with N only when the equilibrium δ_{NND} becomes close to the forbidden hard-core zone and cannot decrease further. Thus, a swarming flock at moderate N can have soft-core patterns in spite of the existence of a local hard-core repulsion.

Despite the natural tendency to keep a reasonable distance between each other, animals may still come close and occasionally touch each other while moving in a biological swarm. Thus, the natural repulsive tendency can be realized as a soft-core repulsion while the body length of the swarming animals can be viewed as a hard-core forbidden zone. For biological swarms, the equilibrium δ_{NND} is visibly larger than the hard-core forbidden zone, which supports the description given in the previous paragraph. In contrast, the Lennard-Jones potential, used for physical systems of molecules, defines an equilibrium distance very close to where the potential rapidly rises toward infinity. In other words, the equilibrium distance is nearly the same as the hard-core forbidden zone. Compressibility is perhaps the reason why various catastrophic patterns, which are not observed in the condensed phases of classical matter, can exist in the aggregation states of natural swarms. In artificial swarms, such as the applications in Chapter 8, the hard-core repulsion can be understood as a collision avoidance strategy. If the distance to invoke the collision avoidance is much shorter than the equilibrium spacing between agents, various collapsing patterns shown in Ref. [17] become possible and might even be engineered for artificial swarming of vehicles.

Chapter 6

A Brief History of Artificial Swarms

While various patterns in biological swarms are observed, the foraging efficiency and defensive advantage are especially of interest to engineers. Inspired by such natural phenomena, engineers have built systems of multiple autonomous robots that performs cooperative strategies, in some cases attempting to imitate natural swarms. Such studies serve two purposes. On the one hand, they provides reallife simulations for testing conjectures related to animal group behaviors. On the other hand, the advantage shown in cooperative animal groups can be used as as design paradigm for complicated tasks with multiple independent robots. In this chapter, we first review some pioneering work of cooperative autonomous robots, followed by some recent results that are more relevant to our model in previous chapters and its applications in the next two chapters.

6.1 Walter's turtles

The famous neurophysiologist and robotician W. Grey Walter is believed to be the first to bring this idea to reality. In late 1940s, he started building several autonomous robots, nicknamed "turtles" because of their shapes [100]. A turtle is a self-propelled three-wheel vehicle with a steering motor to change its direction. A light sensor is installed on the vehicle to detect light sources; touch sensors are also installed in order to detect and avoid obstacles, walls, or another vehicle. On-board vacuum tube computers allow the turtles to make simple logical computations. The picture in Fig. 6.1 shows one of the turtles heading for a target point while avoiding an obstacle on its way. Using light sources as the target points,



Figure 6.1: A picture of Walter's turtle [101]. This is a time-exposure photograph in which the turtle starts from the left, avoids the obstacle in the middle, and reaches the target on the right. A lamp is mounted on the turtle, and hence, the photo is able to record its trajectory as well. Reprint from W. G. Walter, "A machine that learns", Scientific American, Vol. 185, No. 2, pp. 61, 1951, photogrphed by Desmond Tripp.

a series of experiments were conducted on these machines, which include singlevehicle tasks and multi-vehicle behaviors. The observations were presented in two of his papers "An imitation of life" and "A machine that learns" [100, 101]. Based on his research of brain activities, his primary purpose is to show that such a robotic system is capable of giving rise to complex behaviors, much like what brain cells are believed to do. Nevertheless, his contribution opened up a promising field that has generated great interests at present.

6.2 Recent works

Sugawara and Sano, two authors of the theoretical swarming model in Ref. [85] which we have reviewed in Section 2.3, published the paper "Cooperative acceleration of task performance: Foraging behavior of interacting multi-robots system" in 1997 [90]. Their robots are similar to Walter's turtles; they are self-propelled vehicles equipped with a light sensor and two touch sensors. They are able to steer left or right depending on which side touches an obstacle. The authors investigated the efficiency of collecting pucks on a testbed where multiple robots are deployed. They test a simple interaction rule similar to that of the foraging bird flocks in Ref. [92]. According to the rule, every robot that just picks up a puck will turn on a lamp, mounted on the robot, for a period of time. Other robots that have not found a puck will respond to this light and head toward this location. For a homogeneous distribution of the pucks, the robots perform similarly with or without interaction. However, when the pucks distribute locally, a more realistic situation for animals, the foraging efficiency is greatly enhanced just by such a simple interacting rule. The comparison is shown in Fig. 6.2, where the collecting time become much shorter for interacting robots when the number of robots increases.



Figure 6.2: Foraging efficiency of Sugawara and Sano's robots [90]. The x-axis is the number of robots, and the y-axis is the inverse of time T by which all pucks are collected. Pucks are distributed homogeneously in Field 1, where robot interaction does not show any significant effect. Field 3 has localized puck distributions, where robots with interaction demonstrate more superior efficiency by increasing the number of robots. Reprinted from Physica D: Nonlinear Phenomena, Vol. 100, K. Sugawara and M. Sano, "Cooperative acceleration of task performance: Foraging behavior of interacting multi-robots system", pp. 343-354, Copyright (1997) with permission from Elsevier.

As the capability of on-board computing for robots has continuously progressed, more sophisticated interaction rules are implemented for later development of artificial swarms. Leonard et al. applied the idea of interaction forces by constructing virtual potentials to guide their underwater vehicles [26, 55, 68, 69, 88]. One example is in the 2001 paper "Virtual leaders, artificial potentials and coordinated control of groups" by Leonard and Fiorelli [55]

$$V_{i,j} = \begin{cases} -C\left(\ln\left(x_{i,j}\right) + \frac{\ell}{x_{i,j}}\right) & \text{if } 0 < x_{i,j} < \ell_c, \\ -C\left(\ln\left(\ell_c\right) + \frac{\ell}{\ell_c}\right) & \text{if } x_{i,j} \ge \ell_c, \end{cases}$$
(6.1)

where C, ℓ , and ℓ_c are positive constants. The force is then given by $\vec{F_i} = -\vec{\nabla} \sum_{j \neq i} V_{i,j}$. They showed that underwater vehicles can self-organize into a rigid-body formation, through interaction forces defined by virtual potentials, and proved that the formation is stable [55, 68, 88]. While the ability of obstacle avoidance was theoretically discussed in literature [8, 69], Leonard et al. also

conducted field tests with their autonomous underwater vehicles for problems involving ocean sampling [26]. We adopt the idea of virtual potential and formulate our control algorithm in Chapters 7 and 8 by converting our swarming model in Eqs (3.7) - (3.8).

Some vehicles have a fixed speed, and thus, steering control is another common scheme for artificial swarms. An example of such a vehicle is discussed in Chapter 8. In their 2003 conference paper "Steering laws and continuum models for planar formations" [48], Justh and Krishnaprasad propose a steering control law for such vehicles:

$$\frac{\mathrm{d}\vec{r_i}}{\mathrm{d}t} = \hat{x}_i, \tag{6.2}$$

$$\frac{\mathrm{d}\hat{x}_i}{\mathrm{d}t} = \hat{y}_i \omega_i, \tag{6.3}$$

$$\frac{\mathrm{d}\hat{y}_i}{\mathrm{d}t} = -\hat{x}_i\omega_i, \tag{6.4}$$

where \hat{x}_i and \hat{y}_i are respectively a tangential and a normal unit vector with respect to the trajectory of vehicle *i*, while \vec{r}_i is the displacement vector from the vehicle to a reference point, as shown in Fig. 6.3. Note that the speed does not appear in Eqs. (6.2) - (6.4) because they assume the unit speed in their model. In polar coordinates, $d\vec{r}_i/dt = (\cos \Theta_i, \sin \Theta_i)$ and $d\Theta_i/dt = \omega_i$. The steering control parameter is the angular velocity ω_i . In the paper, Justh and Krishnaprasad design two control laws that produce distinctly different swarming formations. One of them is called the "rectilinear control law"

$$\omega_{i} = \frac{1}{N} \sum_{j \neq i} \left[c_{0} \left(\frac{\vec{r}_{i,j}}{r_{i,j}} \cdot \hat{x}_{i} \right) \left(\frac{\vec{r}_{i,j}}{r_{i,j}} \cdot \hat{y}_{i} \right) - V\left(r_{i,j}\right) \left(\frac{\vec{r}_{i,j}}{r_{i,j}} \cdot \hat{y}_{i} \right) + c_{1} \hat{x}_{j} \cdot \hat{y}_{i} \right], \quad (6.5)$$

where $\vec{r}_{i,j} \equiv \vec{r}_j - \vec{r}_i$, N is the number of vehicles, and c_0 , c_1 are positive constants.



Figure 6.3: The coordinate system of the steering control law in the paper of Justh and Krishnaprasad [48]. Reprint from E. W. Justh and P. S. Krishnaprasad, "Steering laws and continuum models for planar formation", in Proc. of the 42nd IEEE Conference on Decision and Control, pp. 3609-3614 (© 2003 IEEE).

The first term in the summation gives the vehicles a tendency to form a rectangular formation, and the third term unifies the orientation of the vehicles. The function f in the second term is chosen to define an equilibrium distance between the vehicles. The rectilinear formations are shown in the left figure of Fig. 6.4. The second control law is called the "circling control law"

$$\omega_{i} = \frac{1}{N} \sum_{j \neq i} \left[c_{2} \left(\frac{\vec{r}_{i,j}}{r_{i,j}} \cdot \hat{x}_{i} \right) - V \left(r_{i,j} \right) \left(\frac{\vec{r}_{i,j}}{r_{i,j}} \cdot \hat{y}_{i} \right) \right], \tag{6.6}$$

where the parameter c_2 is also a positive constant. The circling patterns are shown in the right figure of Fig. 6.4. The proofs of convergence of such a multi-vehicle system are also given in the paper. In Chapter 8, we show applications of these steering control laws on our testbed vehicles.

Convergence is an important property for multiple autonomous vehicles to be



Figure 6.4: The vehicular formations in the model of Justh and Krishnaprasad [48]. Left: Rectilinear formation. Right: Circling formation. Reprint from E. W. Justh and P. S. Krishnaprasad, "Steering laws and continuum models for planar formation", in Proc. of the 42nd IEEE Conference on Decision and Control, pp. 3609-3614 (© 2003 IEEE).

viewed as a swarm. An application of virtual potentials that is especially useful in proving convergence is the "potential flow" model, expressed by the following kinematic description

$$\frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t} = -\vec{\nabla} \sum_{j \neq i} V\left(x_{i,j}\right),\tag{6.7}$$

where V is a pairwise interaction potential depending on the distance between particles i and j, denoted as $x_{i,j}$. In a series of papers [32, 33, 34, 60], Gazi and Passino proves global convergence for a particular potential flow model, in which

$$V(x) = c_0 x^2 + c_1 \exp\left(-\frac{x^2}{\ell^2}\right).$$
 (6.8)

Here, c_0 , c_1 , and ℓ are positive constants. The first term on the RHS of Eq. (6.8) is an attraction while the second term is a repulsion. Note that the attraction is a spring potential, which is unphysical for general vehicular communications because radio signals usually decay in distance. The proof of global convergence takes advantage of this strong interaction at a distance. Although we are not able to prove the same global convergence for our virtual potential, the works of Gazi

and Passino inspire us to instead prove local convergence, which is discussed in Chapter 8.

Chapter 7

Caltech Multi-Vehicle Wireless Testbed

7.1 Introduction

The Multi-Vehicle Wireless Testbed built by Murray et al. provided numerous interesting applications of artificial swarms [15, 47]. The testbed consists of secondorder vehicles, nicknamed "Kelly", with ducted fans and is capable of performing cooperative control algorithms. It was originally designed for RoboFlag project [9] but has also been used to demonstrate many other tasks [1, 43, 66]. In this chapter, we show applications of using the idea of virtual interaction potentials to guide the vehicles. Sections 7.2 - 7.4 review our collaborative work in the paper "Virtual attractive-repulsive potentials for cooperative control of second order dynamic vehicles on the Caltech MVWT" [66]. In Section 7.2, we describe the composition of the testbed. In Section 7.3, we present a model adaption that converts Eqs (3.7) - (3.8) into a virtual-potential control algorithm. After the control algorithm is obtained, in Section 7.4 we show the experimental results of our testbed applications.

7.2 Testbed

Figure 7.1 shows the Caltech MVWT vehicles, Kelly. They sit on casters, and each consists of an onboard computer, a pair of fans, and wireless network connection. The positions of the vehicles are reported by overhead cameras to an offboard computer and then passed along to each vehicle's onboard computer via the wireless network. Readers can refer to Ref. [15, 43, 66] for more detailed hardware setup of the testbed. Thanks to the wireless network connections, each Kelly



Figure 7.1: The vehicle Kelly at the Caltech MVWT [66].

is able to obtain real-time tracking information of its own position and that of the others. Each Kelly is driven by a pair of fans mounted on each side of the caster. Their relative power outputs determine the magnitude and the direction of the vehicle's acceleration. The primary challenge of applying our model is to translate the equations of motion to the output control of the fans, which is detailed in Section 7.3.

7.3 Model adaption

The motion of vehicle *i* in the Caltech MVWT system is described by its position $\vec{x}_i = x_i \hat{x} + y_i \hat{y}$ and an orientation angle Θ_i , as shown in Fig. 7.2. In the figure, the *x-y* coordinate system is fixed on the testbed while Θ_i is measured with respect to the *x*-axis. The basic equations of motion of Kelly is

$$m_i \frac{\mathrm{d}^2 \vec{x}_i}{\mathrm{d}t^2} = -\beta \vec{v}_i + (F_{\mathrm{R},i} + F_{\mathrm{L},i}) \left(\cos \Theta_i \hat{x} + \sin \Theta_i \hat{y}\right)$$
(7.1)

$$J_{i} \frac{\mathrm{d}^{2} \Theta_{i}}{\mathrm{d}t^{2}} = -\psi \omega_{i} + (F_{\mathrm{R},i} - F_{\mathrm{L},i}) r_{1/2}, \qquad (7.2)$$

where ω_i is the angular speed; a positive ω represents a counter-clockwise spin while a negative ω represents a clockwise one. The linear and the rotational friction coefficients are respectively denoted by β and ψ while J_i is the moment and $r_{1/2}$ is the half length of the vehicle. The most important quantities in the equations are $F_{\mathrm{R},i}$ and $F_{\mathrm{L},i}$, the output forces by the right and the left fans, because these are the only thing that we can directly control. By writing down Eqs. (3.7) -(3.8) and (7.1) - (7.2) in the same coordinate system, we obtain formulas for $F_{\mathrm{R},i}$ and $F_{\mathrm{L},i}$

$$F_{\mathrm{R},i} = \frac{\alpha}{2} - \sum_{j \neq i} \left[\left(-C_a \mathrm{e}^{-\frac{\left|\vec{x}_i - \vec{x}_j\right|}{\ell_a}} + C_r \mathrm{e}^{-\frac{\left|\vec{x}_i - \vec{x}_j\right|}{\ell_r}} \right) \right] \\ \cdot \frac{(x_i - x_j) \left(\cos \Theta_i - \sin \Theta_i\right) + (x_i - x_j) \left(\sin \Theta_i + \cos \Theta_i\right)}{\left|\vec{x}_i - \vec{x}_j\right|} \right], \quad (7.3)$$

$$F_{\mathrm{L},i} = \frac{\alpha}{2} - \sum_{j \neq i} \left[\left(-C_a \mathrm{e}^{-\frac{\left|\vec{x}_i - \vec{x}_j\right|}{\ell_a}} + C_r \mathrm{e}^{-\frac{\left|\vec{x}_i - \vec{x}_j\right|}{\ell_r}} \right) \right] \\ \cdot \frac{(x_i - x_j) \left(\cos \Theta_i + \sin \Theta_i\right) + (x_i - x_j) \left(\sin \Theta_i - \cos \Theta_i\right)}{\left|\vec{x}_i - \vec{x}_j\right|} \right]. \quad (7.4)$$

Here, we adopt the self-driving model $\alpha \vec{v_i} / |\vec{v_i}| - \beta \vec{v_i}$ of Ref. [57] because the friction is linear to the velocity. Also note that the rotational friction term $-\psi\omega$ cannot be obtain from a point-mass swarming model such as Eqs. (3.7) - (3.8). Nevertheless, the presence of the dissipative friction in the testbed environment actually helps to stabilize the angular speed and does not affect the behavior substantially. Our interaction in Eq. (3.6) decays in distance, implying that the vehicles have limited sensor range and do not necessarily know the entire configuration.



Figure 7.2: The coordinate systems of a testbed.

7.4 Testbed simulations

The Caltech MVWT consists of second-order vehicles that can be controlled through fan power outputs using Eqs. (7.3) - (7.4). However, there is a central structural pole on the testbed floor which results in a real obstacle to avoid while performing flocks. Therefore, we initially considered single-vehicle tasks and applied our models to some fundamental problems. Two such problems are *target* searching and obstacle avoidance, which combine as a path planning problem. We used the Morse potential in Eq. (3.6) to create a virtual field that leads the vehicle to the target and keeps it away from obstacles. One feature of this kind of control algorithm is that it only depends on real-time displacements of the vehicle to the targets and the obstacles; the vehicle does not need to know global coordinate and can rely only on sensors for these tasks. However, in our MVWT testbed simulations, we did not use onboard sensors but used the offboard computer to simulate such processes for convenience because the existing control subroutines for RoboFlag were in the offborad computer and were readily modified for our model. This introduces an additional delay due to wireless communication between the offboard computer and the onboard computer, but the effect is neglible since our testbed experiments are at very low vehicular speed. The computation simulation in Fig. 7.3 illustrates the idea of using virtual potentials for path planning to search for a target while avoiding obstacles. The upper figure shows that the vehicle circumvents the potential "hills" created by the obstacles and heads toward the potential "valley" where the target is located. The lower figure is the same task drawn on a 2D plane, where the starting point is denoted by a black star; the vehicle (blue circles) navigates through the obstacles (encircled red X's), and reached the target (green star) on the bottom. Note that we use different potential parameters for the obstacles and the targets.

To implement virtual potentials on the actual testbed, we start with the simplest case where only an attractive point (target) exists without any obstacles. The vehicle is placed approximately four meters away from the target, and different initial orientations of the Kelly vehicle are tested. The Kelly trajectories are shown in the left panel of Fig. 7.4. Immediately, a deviation from ideal is noticed: the left and right fans do not generate the same propulsion given the same power input. It is especially evident by Fig. 7.4 (a) where the vehicle starts by facing directly toward the target but still drifts toward the right before making left turns to hit the target. This case is verified by the computer simulations in the right panel of Fig. 7.4. Figure 7.4 (e) is the ideal path that directly leads to the target; Figure 7.4 (f) is is the path for a vehicle with a fan-output offset, which deviates from its supposed trajectory, similar to the one on the testbed shown in Fig. 7.4 (a) of the left panel. The phenomena, however, demonstrate an advantage of using virtual potentials for path planning. The vehicles have the capability of overcoming a systematic error and making corrections to find their way back to a desired target point, provided that the target is not out of the interaction range of the vehicles. Figure 7.5 show the actual testbed experiments of a vehicle searching


Figure 7.3: Obstacle avoidance using the virtual attractive / repulsive potentials [66]. The upper 3D figure illustrates the the shape of the virtual potential designed for this task. The black curve on the potential surface depicts the vehicle trajectory. The lower figure shows the vehicle trajectory with respect to the locations of the obstacles and the target. The blue empty circles illustrate the trajectory of Kelly while the red encircled X's represent the stationary obstacles. The initial position of Kelly is denoted by a black star on the left, and the target is the green star to the right. The vehicular parameters are $\alpha = 0.066$ N, and $\beta = 5.05$ kg s⁻¹. The obstacle parameters are $C_a = 0$, $C_r = 5$ J, and $\ell_r = 0.15$ m. The target is a pure attractive point with $C_a = 3$ J and $\ell_a = 3.5$ m.



Figure 7.4: (a)-(d): Target searching tasks with Kelly facing toward different initial orientations. (e)-(f): Computer simulations of the target searching tasks in ideal situation and in the realistic situation where an offset exists between the left and right fans [66]. The parameters used in these simulations are $\alpha = 0.6$ N, $\beta = 5.05$ Kg s⁻¹, $C_a = 0.06$ Nm, $\ell_a = 4$ m, and $C_r = 0$.

for a target while avoiding obstacles on its way. Despite the fan-output offset, the vehicle performs well on these tasks. Figure 7.6 shows a simple experiment, in which two vehicles search for multiple targets using a greedy search algorithm [13] while avoiding colliding with each other. In this case, a vehicle sees the other one as a moving obstacle. In the algorithm, the vehicles search a list of targets of known locations. Upon reaching the current target location, the vehicle automatically select the closest unfinished target from the list as its next target. Additionally, the vehicles cooperate with each other through communication. If two vehicles have selected the same target, the one farther away from the target will select another target. The vehicles also exchange the current and finished target list when they are in communication range, which is 1.5 m in this experiment. In Fig. 7.6, the vehicles start at (a) in which they are within each other's communication range. As they find out that both have selected the upper-left target as their current target, one of the vehicles switches to the lower-left target.



Figure 7.5: Trajectories of the testbed experiments of target searching and obstacle avoidance. Left: One obstacle and one target. Right: Multiple obstacles and one target [66]. The vehicular parameters used in these simulations are $\alpha = 0.6$ N and $\beta = 5.05$ Kg s⁻¹, the target parameters are $C_a = 0.06$ Nm, $\ell_a = 4$ m, and $C_r = 0$, and the obstacle parameters are $C_a = 0.5$ Nm, $\ell_a = 0.5$ m, $C_r = 1.0$ Nm, and $\ell_r = 0.3$ m.

Knowing that the lower-left target has been selected by the circle vehicle, the triangle vehicle goes to the upper-right target after finishing the upper-left one, as shown in Fig. (c). Finally, in Fig. (d), the lower-right target has been finished by the circle vehicle; both vehicles stop when they are in communication again and realize that all targets have been searched.

7.5 Conclusion

In this chapter, we present a decentralized control method for nonlinear secondorder dynamic vehicles. We show that virtual potentials provide a useful mechanism for the vehicles to guide themselves toward target locations or to avoid obstacles. Due to time and space constraint, we were unable to test our algorithm extensively on the MVWT testbed. Subsequently, a multi-vehicle testbed was built in the Applied Mathematics Laboratory at UCLA. That testbed is introduced in the next chapter, and more complex theoretical and experimental results are also discussed.



Figure 7.6: Two Kellys performing the greedy search algorithm using virtual potentials [66]. The four squares represent the targets while the triangle and the solid circle represent the vehicles' current locations. The circles surrounding the vehicles represent their communication range, which is 1.5 m in diameter. The vehicular parameters used in these simulations are $\alpha = 2.6$ N and $\beta = 5.05$ Kg s⁻¹, the target parameters are $C_a = 1.01$ Nm, $\ell_a = 4$ m, and $C_r = 0$, and the collision avoidance parameters are $C_a = 0$ Nm, $C_r = 2.5$ Nm, and $\ell_r = 1$ m.

Chapter 8

UCLA MicroCar Testbed

8.1 Introduction

The MicroCar Testbed at UCLA was built in 2005, motivated by the collaboration with Caltech on the MVWT. While the Caltech MVWT system is capable of performing a wide range of tasks, the MicroCar Testbed at UCLA is a costdown design, which also consumes a much smaller space and thus, can potentially accommodate more vehicles. However, the vehicles present more mechanical constraints, thus, requiring modification of the model. In this chapter, we review our collaborative works in the papers "An economical Micro-Car Testbed for validation of cooperative control strategies" [42] and "Multi-vehicle flocking: scalability of cooperative control algorithms using pairwise potentials" [11]. In Section 8.2, we describes the composition of the testbed. The model adaption is discussed in Section 8.3, where an analogous potential-flow algorithm is also introduced. In section 8.4, we prove the aggregation properties of a general potential-flow model. In section 8.5, we present our results in both testbed experiments and computer simulations.

8.2 Testbed

The UCLA MicroCar Testbed, as designed in 2005, features multiple converted 1/64th scale radio-controlled cars, a position-tracking system using overhead cameras, and offboard computers that receive position information and send radio signals. The left picture in Fig. 8.1 shows a side view of two UCLA MicroCars, and the middle picture is an overhead view of three vehicles on the testbed. The



Figure 8.1: UCLA MicroCar Testbed. Left: MicroCars. Middle: Cars on the testbed. Right: Partial screen shot of the processed camera image [42].

right picture shows an image taken by one of the overhead CDC cameras and processed by the OpenCV¹ computer software, which can detect the shapes of the objects in an image by using a contour searching function. The position and the identity of each vehicle on the testbed are obtained by detecting a black and white two dimensional bar code on top of each vehicle, similar to that used on the CalTech MVWT [15, 43, 66]. More detailed description of the hardware can be found in Ref. [42]. Because the cars do not have onboard computations, the individual decision-making processes are imitated by offboard computations. Even with a chip installed, the cars should still only be capable of doing rather simpler calculations and rely on communication and interaction to produce complex behavior, much in line with natural swarming systems.

8.3 Model adaption

The models in Eqs. (3.7) - (3.8) need to be modified due to mechanical constraints of the UCLA MicroCar testbed vehicles. The testbed consists of Dubins microcars, which are better described by a first-order kinematic model [18, 86, 89]. Virtual potentials can be implemented as a controller for a first-order vehicle

 $^{^{1}}http://www.intel.com/technology/computing/opencv/index.htm$

through the motion rule

$$\frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t} = -c\vec{\nabla}_i \sum_{j\neq i} V(x_{i,j}).$$
(8.1)

Here, $\vec{x}_i = (x_i, y_i)$ is the position of vehicle *i*, and *t* is time. The distance between vehicles *i* and *j* are denoted as $x_{i,j}$; *V* is a pairwise interaction potential, and *c* is a constant. Without loss of generality, let us set c = 1. The Morse-type potential as in Eq. 3.6 can be adopted here for the interaction potential *V*. Note that the steady state solution of Eq. 8.1 is the same as the coherent-flock solution of the dynamic model Eqs. (3.7) - (3.8), as pointed out in Section 3.3. However, Dubins vehicles have more constraints than general first-order vehicles. Dubins vehicles have bounds on their turning radii and can only move forward. Additionally, the UCLA MicroCars can only turn at fixed turning radii and move with a fixed speed, which is even more strict than general Dubins vehicles. Due to such constraints, Eq. 8.1 is not realistic enough for the UCLA MicroCars. Further adaption results in the following control algorithm

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \alpha \cos \Theta_i, \quad \frac{\mathrm{d}y_i}{\mathrm{d}t} = \alpha \sin \Theta_i \tag{8.2}$$

$$\frac{\mathrm{d}\Theta_i}{\mathrm{d}t} = \begin{cases} \frac{\alpha}{R_{\mathrm{L}}} & \text{if } \gamma_i > \Gamma_{\mathrm{t}} & (\text{left turn}), \\ -\frac{\alpha}{R_{\mathrm{R}}} & \text{if } \gamma_i < -\Gamma_{\mathrm{t}} & (\text{right turn}), \\ \frac{\alpha}{R_{\mathrm{S}}} & \text{otherwise.} \end{cases}$$
(8.3)

Here, α represents the fixed vehicle speed, and $R_{\rm L}$, $R_{\rm R}$ are the left and right turning radii, respectively. The only independent variable in Eqs. (8.2) - (8.3) is the orientation Θ_i , defined similarly as in the Caltech MVWT controller and illustrated in Fig. 7.2. Due to the fixed turning radii and speed, the potential gradient in Eq. (8.1) can dictate neither the magnitude nor the orientation of the



Figure 8.2: Definition of variables for vehicle *i* in the UCLA MicroCar testbed [11]. The heading is denoted by Θ_i , the angle between its direction of motion and the *x* axis of the testbed. The interaction force it experiences due to all other vehicles is represented by $\vec{F_i}$. This direction defines an angle γ_i with the heading direction. Vehicle *i* is at a distance $\vec{r_{i,j}}$ from vehicle *j* and the angles ϕ_i and ϕ_j here shown are used in the collision avoidance scheme later in Section 8.3. The origin of the reference coordinate system is fixed at the left-lower corner of the testbed. All vehicular angles, γ_i , Θ_i , ϕ_i , are defined in $[\pi, -\pi)$.

vehicle velocity. Instead, in Eq. (8.3), the potential gradient vector provides an direction, indicating how the vehicle should turn. For each vehicle, we measure the angle γ_i between vehicle heading and the potential gradient $\vec{F_i}$ associated to vehicle *i*, as shown in Fig. 8.2. Vehicle *i* then changes direction only if $|\gamma_i| > \Gamma_t$, where Γ_t is an angular threshold $0 \leq \Gamma_t \leq \pi$. If the turning commands are not given, the vehicle proceeds along the direction specified by the heading parameter Θ_i .

An additional parameter $R_{\rm S}$ is also introduced in Eq. (8.3) to take a mechanical defect into account. Due to problems in steering alignment, the vehicle may not run on a straight path when it is ordered to; the actual path can be described as a portion of the perimeter of a circle with radius $R_{\rm S}$. In the ideal case, $R_{\rm L} = R_{\rm R}$, and $R_S = \infty$. In general, the alignment asymmetries cause $R_L \neq R_R$, and $|R_S| < \infty$.

Collision avoidance

A crucial point for practical applications is that the interaction potential in Eq. (3.6) is soft-core and does not prevent vehicles from colliding. In fact, even hard-core potentials cannot avoid collisions due to various practical issues, such as communication delays, errors in position information, and the finite turning radius of the vehicles. Because of the low vehicle speed in our Caltech MVWT experiments, the repulsive potential was enough to avoid collision. The same thing can not be said for the UCLA MicroCar testbed vehicles since they have fixed speeds, which are rather fast. The repulsive range may be increased to initiate turning at larger inter-vehicle distances. However, this would significantly affect pattern formation, and the emergence of cooperative aggregates would be unlikely. As a result, we instead add an additional collision avoidance algorithm to address short range interactions for the UCLA MicroCars. Note that we discuss the implication of adding such a collision avoidance strategy in Section 5.5. Here, we use a 'wait and go' scheme for vehicles closer than a cutoff distance $r_{\rm c}$. For vehicles i, j at distance $\vec{r}_{i,j}$ such that $r_{i,j} < r_c$, we define the angles ϕ_i, ϕ_j between their main axis and $\vec{r}_{i,j}$, as shown in Fig. 8.2. If $\phi_i < \phi_j$ vehicle *i* will pause while vehicle j veers away, until $r_{i,j} > r_c$. The cutoff distance r_c in the control algorithm acts as an effective hard-core potential, as discussed in Section 5.5. If $\phi_i = \phi_j$, one of the vehicles (in our simulations the one with a higher labeling index) will pause and let the other proceed. When $\phi_i, \phi_j \simeq 0$ the 'wait and go' scheme cannot avoid collision as shown in Fig. 8.3, and an alternate algorithm is invoked. For vehicles i and j we define the angle $\Omega_{i,j}$ between $\vec{r}_{i,j}$ and the segment joining their opposite front edges measured from $\max\{\phi_i, \phi_j\}$ as shown in Fig. 8.3. If $\max\{\phi_i, \phi_j\}$ $\phi_j \} < \Omega$, where Ω is an angular threshold $0 \le \Omega \le \pi/2$, then the vehicle closer to the center of the testbed is veered towards the center and the other in the



Figure 8.3: Collision avoidance failure [11]. The angles ϕ_i and ϕ_j are too small and vehicles *i* and *j* collide even if one of them should pause. An additional algorithm is required to steer the vehicles away from each other and is described in the text. It relies on the angle $\Omega_{i,j}$ here depicted.

opposite direction.

8.4 **Proof of convergence**

In Ref. [32, 33, 34, 60], Gazi and Passino prove a global convergence for particles to aggregate under the potential-flow algorithm in Eq. (8.1) using a particular functional form of the potential V. The particular functional form they adopt is asymptotically quadratic, i.e., a spring potential at long range, and their proof of global convergence heavily relies on the fact that the potential diverges at a distance. Therefore, the global convergence is restricted to only a class of interaction potentials which is rather artificial than natural or physical because most communication signals generally dissipate in distance, including our Morse potential in Eq. (3.6). However, we can still prove a local convergence for such physical interaction potentials. First let us mathematically define

Definition 8.4.1 Diffused state. A flock or is in a diffused state if $x_{i,j} > \delta \forall i \neq j$, where δ is the repulsive range such that V'(x) > 0 for all $x > \delta$.

Note that in order to be in a diffused state, the potential must yield only attraction outside of a certain radius. The following Lemma shows that, regardless of the specific form of the potential, a diffused state always shrinks. **Lemma 8.4.2 Weak maximum principle.** Define the flock radius as $R \equiv \sup_i x_i$, where $x_i = |\vec{x}_i|$. For a flock in the diffused state, $\dot{R} \leq 0$.

Proof : Let $R = x_i$ and define $\hat{x}_{i,j} \equiv \vec{x}_{i,j}/x_{i,j}$. Then

$$\frac{\dot{x_i}^2}{2} = \vec{x_i} \cdot \dot{\vec{x_i}} = -\vec{x_i} \cdot \sum_{j \neq i}^N \hat{x_{i,j}} V'(x_{i,j})$$
(8.4)

$$= \sum_{j \neq i}^{N} \frac{(\vec{x}_i \cdot \vec{x}_j - x_i^2)}{x_{i,j}} V'(x_{i,j}) \le 0$$
(8.5)

since $x_i^2 \ge \vec{x}_i \cdot \vec{x}_j$ and V' > 0 in the diffused state. Thus x_i^2 , and x_i , are decreasing functions and $\dot{R} \le 0$. \Box

Note that the proof of Lemma 8.4.2 implies that the swarm size decreases even if only the outermost agents are in the diffused state. This is due to the fact that the proof only uses an estimate for the farthest agents of the swarm. Using this lemma, we can prove a local stability limit for a general interaction V. We can also find conditions for particles initially constrained to a local region of radius R, to evolve into a more compact ball of radius $R^* < R$. The proof uses a Lyapunov function discussed in [32, 34].

Theorem 8.4.3 Consider N particles located at \vec{x}_i with $x_i \leq R \ \forall i, 1 \leq i \leq N$. If a finite constant value c > 0 exists such that $\max_{\{0 \leq x \leq 2R\}} |cx - V'(x)| < cR$, then asymptotically $x_i \leq R^*$, with $R^* < R$. **Proof**: We choose the Lyapunov function $\mathcal{L}_i \equiv x_i^2/2$. Its time derivative obeys

$$\dot{\mathcal{L}}_i = -\vec{x}_i \cdot \vec{\nabla}_i \sum_{j \neq i} V(x_{i,j})$$
(8.6)

$$= -\vec{x}_i \cdot \sum_{j \neq i} \hat{x}_{i,j} V'(x_{i,j})$$

$$(8.7)$$

$$\leq -cNx_i^2 + x_i(N-1)\eta, \qquad (8.8)$$

where $\eta \equiv \max_{\{0 \le x \le 2R\}} |cx - V'(x)|$. In going from Eq. (8.7) to Eq. (8.8), we have added and subtracted $cx_{i,j}$ where c > 0 is an arbitrary constant. We also used the fact that $\vec{x}_i \cdot \sum_{j \ne i}^N \vec{x}_{i,j} = Nx_i^2$. Also note that $x_{i,j} \le 2R$ since by assumption $x_i \le R$. Asymptotically then

$$x_i \le \frac{N-1}{N} \frac{\eta}{c} \le \frac{\eta}{c} \equiv R^*, \tag{8.9}$$

and we require $\eta < cR$ for this bound to be more stringent than the initial radius R. \Box

Our control algorithm adopts the generalized Morse potential in Eq. (3.6), for which a sufficient condition for Theorem 8.4.3 is

$$\left(\frac{C_a}{\ell_a}e^{-\frac{2R}{\ell_a}} - \frac{C_r}{\ell_r}\right) < 2cR < 2\left(\frac{C_a}{\ell_a}e^{-\frac{2R}{\ell_a}} - \frac{C_r}{\ell_r}\right).$$
(8.10)

Recalling that $\ell \equiv \ell_r/\ell_a$ and $C \equiv C_r/C_a$, Eq. (8.10) is satisfied only if $\ell > C$ so that R can be chosen as $2R < \ell_a \ln (\ell/C)$. It is interesting to compare this sufficient (but not necessary) condition with the H-stability phase diagram in Fig. 4.1 for the same potential in the second order model of Eq. (3.7) - (3.8). In the phase diagram, the region $\ell > C$ with C < 1 is classified as catastrophic, where particles converge and becoming denser as $N \to \infty$. This is consistent with the results proven here that N particles initially in a ball of radius R get "squeezed" into a tighter one. However, the region $\ell > C$, with C > 1 is classified as stable in Fig. 4.1, with no possible squeezing effects in the long time limit, which is not consistent with the results proven here. In this region, the pairwise potential has a positive, local minimum for $r_{i,j} = 0$ and a barrier at $r_{i,j} = r_{max} > \ell_a \ln(\ell/C)$, before decaying to zero as $r_{i,j} \to \infty$. The discrepancy is due to, as mentioned in Ch. 4, that the potential-flow equation in Eq. (8.1) is purely dissipative. Once the particles start within the threshold radius $\ell_a \ln(\ell/C)$, their energy can only dissipate under the potential flow, and there is no chance such particles can gain energy to eventually overcome the barrier and escape the attraction.

For other functional form of potentials, the specific criteria of convergence can similarly be derived either analytically or numerically by applying Theorem 8.4.3. The theorem can be regarded as a local version of Gazi and Passino's results in Ref. [32, 34]; it can only prove local convergence but is applicable to a broader class of potentials.

8.5 Testbed and computer simulations

The potential flow equation in Eq. (8.1) for general first-order vehicles cannot be directly applied to the UCLA MicroCar Testbed, and thus, an adapted version is devised as in Eqs. (8.2)-(8.3). After the model is adapted to hardware constraints and formulated into control algorithms, we are able to explore the corresponding behaviors of the vehicles on the testbeds. Since our second-order model is greatly modified to a first-order model to adapt various hardware constraints of the UCLA MicroCars, we shall also investigate the model behaviors through computer simulations. They are presented in Sections 8.5.1 and 8.5.2, respectively.



Figure 8.4: Multiple interacting vehicles following a leader on the UCLA MicroCar testbed [11]. The leader is denoted by a black diamond while the vehicles are represented by a blue triangle, a blue circle and a blue square. In the left figure, two vehicles are following the leader on an ellipse trajectory (N = 2). On the right, the leader is followed by three vehicles (N = 3). The parameters used for the vehicle interaction are $C_a = 10^4 \text{ erg}$, $C_r = 6 \times 10^3 \text{ erg}$, $\ell_a = 95.2 \text{ cm}$, $\ell_r = 5.7 \text{ cm}$. The virtual leader has $C_a = N \times 10^4 \text{ erg}$ and $C_r = 0$ while the other parameters are the same.

8.5.1 Testbed simulations

The UCLA MicroCar testbed consists of Dubins vehicles that can be controlled through Eqs. (8.2) - (8.3). The size of these vehicles are much smaller than the Kellys at Caltech, allowing us to observe dynamic formations of multiple vehicles on the testbed. Figure 8.4 shows a group of vehicles following a virtual leader while interacting with each other. In the left panel, two vehicles are following the leader while the interaction keeps an equilibrium distance between them; in the right panel, three vehicles adjust to triangle formations while following the leader. These experiments show that the testbed vehicles can attain cohesive movements via the adapted control algorithm of Eqs. (8.2) - (8.3). For this model, we may investigate behavior of larger groups via computer simulations.

8.5.2 Computer simulations

Unlike our applications to the Caltech MVWT testbed, in which the equations of motion are only adapted but unchanged, the control algorithms of the UCLA MicroCar testbed is a modified version of the original swarming model. In par-



Figure 8.5: Two observed formations of the computer simulations for Eqs (8.2) - (8.3) [11].

ticular, the interaction "force" does not provide acceleration for the vehicles but instead gives directional orders. As a result, the large-group swarming behavior may not be the same as those described in Chapter 3 and 4. We first calibrate the computer model using the experimental data obtained from the openloop tests of the UCLA MicroCar testbed. In line with the experiments in Section 8.5.1, we simulate the behaviors of a large group of vehicles (N = 100) with the presence of a leader. Figure 8.5 shows two observed formations in the computer simulations. In the left panel, the vehicles fall out of leader's path and self-organize into circulating mills, similar to the formations in Fig. 3.2 (b) and (c). In the right panel, the vehicles successfully follow the leader and form a polarized group much similar to the coherent pattern in Fig. 3.2 (d). Note that the symmetry is broken due to the modified model. The vehicles are less capable of adjusting relative positions due to fixed turning radii, and it is also less likely for a vehicle to catch up others in front because of the constant vehicular speed. As a result, the coherent formation exhibits larger gaps along the traveling path and denser packs along the lateral direction.

The next thing we explore is whether H-stability also has effects on this mod-



Figure 8.6: Scaling in the H-stable and catastrophic regimes [11]. The potential parameters are C = 1.667, $\ell_a = 95.2$ cm. The system is H-stable for $\ell_r > 73.5$ cm (top curve $\ell_r = 76.2$ cm), otherwise it is catastrophic (middle curve $\ell_r = 69.02$ cm; bottom curve $\ell_r = 35.7$ cm). The middle curve is at the threshold. Straight lines are power law fits with powers 0.11, 0.01, and -0.00042 for the top, middle, and bottom sets respectively.

ified first-order model. Similarly, we investigate the scaling properties with respect to H-stable and catastrophic parameter values, as shown in Fig. 8.6. In the figure, we adopt the same parameters as the experiments in Section 8.5.1: $C_a = 10^4 \text{ erg}$, $C_r = 6 \times 10^3 \text{ erg}$, and $\ell_a = 95.2 \text{ cm}$. Using Eqs. (4.7) and (4.15), we obtain that the Morse potential is H-stable if $\ell_r > 73.5 \text{ cm}$. The top curve in the figure is for $\ell_r = 76.2 \text{ cm}$, which is in the H-stable regime; the middle curve is for $\ell_r = 69.0 \text{ cm}$, just below the H-stability threshold; the bottom curve is for $\ell_r = 35.7 \text{ cm}$, a catastrophic case. Even though the modified model is less capable of adjusting inter-distances between the vehicles, we still see that the flock radius of H-stable swarms scales up against increasing N while the catastrophic swarms have a fixed swarm size.

8.6 Summary and conclusion

We consider a well-known first-order potential gradient flow model analogous to the second-order dynamic model we discussed in previous chapters. We prove a criterion of local convergence for a general class of potentials. The criterion estimates the radius of a ball inside which the particles asymptotically collapse into a even smaller ball. Such radius is found independent of the number of particles, implying that the swarm density can go to infinity upon increasing number of particles. This scaling property is very important in designing large swarming algorithm.

We then adapt the model to a system of Dubins vehicles and program the vehicles to follow a virtual leader in swarming formations. For small numbers of vehicles, the testbed verifies some facts about the stability with respect to the algorithm parameters. For larger groups, we use computer simulations to investigate their scaling properties. The results show that the swarm in our model can maintain its cohesion while the system size scales up.

In Chapters 7 and 8, we adapt and apply our model respectively to secondorder and to first-order vehicles using the control algorithm of virtual potentials. On the testbeds, small numbers of vehicles perform cooperative tasks and achieve cohesive formations. Larger systems are studied via computer simulations and show similar scaling stability to that discussed in Chapter 4 for the original theoretical model. While our testbed continues to improve and upgrade, these results paint a promising future for further applications of our swarming model.

Chapter 9

Summary and Conclusion

In Chapter 2, we briefly review the historical development of swarming models and distinguish two major classes of the models: kinematic models and dynamic ones. Kinematic models evolved from population dynamics while dynamic models originated from individual descriptions of organism interactions. Because of the population dynamics origin, kinematic models have better developed continuum descriptions, and are generally used to explore polarization, segregation, and coarsening of swarming groups. In contrast, dynamic models are based on realistic individual descriptions and are more suitable to study small-scale phenomena, such as internal structure and pattern formation of a swarming group. In this thesis, we focus on aggregation patterns of swarms and thus, adopt a dynamic description for modeling. In Chapter 3, the individual-based dynamic model is constructed. Through computer simulations, we observe various distinct patterns including coherent flocks, rigid-body rotations, single mills, double mills, rings, clumps, and dispersed configurations. Statistical properties are numerically calculated to investigate the parametric boundaries between the patterns. Many of these patterns do not appear to scale well with increasing number of swarmers. To better understand the scalability of the swarming patterns, in Chapter 4 we apply H-stability analysis from classical statistical mechanics to the interaction of our swarming model. A phase diagram given by the analysis shows that most swarming patterns are in non-H-stable (catastrophic) regimes and do not scale with respect to number of swarmers, while H-stable regimes feature only lattice-type patterns (coherent flocks and rigid-body rotations) in addition to the dispersed state. Most notably, the single-mill pattern, frequently observed in natural swarms, is a catastrophic state. It helps to explain why models used to study milling formations always adopt non-H-stable interactions, while studies focusing on group coherence and segregation usually do not experience any difference with respect to H-stability. To further investigate analytical properties of the model, a continuum description of the model is necessary. Unlike kinematic models, the connection between the individual-based and the continuum descriptions of dynamic models has not been rigorously established. In Chapter 5, we apply a classical method in statistical mechanics and propose a systematic derivation for the continuum description of an individual-based dynamic model. The validity of such derivation is verified by comparing the solutions of the individual-based and the continuum models. We find that H-stability of the interaction potential also plays an important role here. Our examinations show that on the scale of a typical swarming pattern, our proposed continuum limit is only appropriate for catastrophic potentials. In the H-stable regime, the continuum limit exists only on a very large scale where swarming patterns are indistinguishable, and this case converges to the classical fluid or gas dynamics. Despite such a limitation, we may analyze the continuum swarming model in the catastrophic regime. We present a linear stability analysis for the model and compare the predicted characteristic wave length to both the simulated patterns of the continuum and the individual-based models. The implication of H-stability and the choices for interaction potentials are discussed first briefly in the end of Chapter 4 and then more thoroughly in the end of Chapter 5.

The defensive advantage and foraging efficiency of swarming organisms inspires engineers to apply such models to autonomous robotic systems. In Chapter 6, we briefly review some of the earlier literature on such artificial swarms. In Chapter 7, we apply our model to a second-order multi-vehicle testbed housed at Cal-Tech. During the summer of 2004, we utilized our swarming model and derived a virtualpotential control algorithm for the testbed vehicles. Using this control algorithm, the vehicles successfully performed target searching and obstacle avoidance tasks. In Chapter 8, a first-order multi-vehicle testbed is built at UCLA for further exploration of artificial swarms. In that chapter, we derive a first-order control algorithm from our second-order model and prove local convergence for such an algorithm. Cooperative tasks of multiple vehicles are tested with three vehicles chasing a virtual leader while interacting with each other. The interaction keeps the vehicles on a formation and prevents them from collisions. The scalability of this first-order algorithm is also investigated via computer simulations for a large number of vehicles. The results show that the scaling exponents are not the same as the full second-order model; however, there still exist distinct differences between the H-stable and the catastrophic regimes.

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Biography

My name is Yao-li Chuang. I was born on Feb. 28, 1974 in Taipei, Taiwan. I went to National Taiwan University for my undergraduate study, in the Physics major. I obtained my B.S. degree in Jun. 1996 and went on to obtain a master degree in physics from National Taiwan University in Jun. 1998. While pursuing the Ph. D. degree at Physics Department, Duke University, I have published the following articles:

- Y.-L. Chuang, R. Oren, A. L. Bertozzi, N. Phillips, and G. G. Katul, "The porous media model for the hydraulic system of a conifer tree: from sap flux data to transpiration rate", Ecological Modelling, Vol. 191, pp. 447-468 (2006).
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- B. Q. Nguyen, Y.-L. Chuang, D. Tung, C. Hsieh, Z. Jin, L. Shi, D. Marthaler, A. L. Bertozzi, and R. M. Murray "Virtual Attractive-Repulsive Potentials for Cooperative Control of Second Order Dynamic Vehicles on the Caltech MVWT", Proceedings of the 2005 American Control Conference, pp. 1084-1089 (2005).
- M. R. D'Orsogna, Y.-L. Chuang, A. L. Bertozzi, and L. S. Chayes, "Self-Propelled Particles with Soft-Core Interactions: Patterns, Stability, and Collapse", Physical Review Letters, Vol. 96, pp. 104302 (2006).

Articles submitted include:

- Y.-L. Chuang, Y. R. Huang, M. R. D'Orsogna, and A. L. Bertozzi, "Multivehicle flocking: scalability of cooperative control algorithms using pairwise potentials", 2007 IEEE International Conference on Robotics and Automation (2007).
- Y.-L. Chuang, M. R. D'Orsogna, D. Marthaler, A. L. Bertozzi, and L. S. Chayes, "State Transitions and the Continuum Limit for a 2D Interacting, Self-Propelled Particle System" Physica D (2006).