

Local Scales and Multiscale Image Decompositions

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Abstract

This paper is devoted to the study of local scales or oscillations in images and use the knowledge of local scales for image decompositions. Denote by

$$K_t(x) = \left(e^{-2\pi t |\xi|^2} \right)^\vee (x) = (2t)^{-n/2} e^{-2\pi \frac{|x|^2}{4t}}, \quad t > 0,$$

the Gaussian kernel. Motivated from the Triebel-Lizorkin function space $\dot{F}_{p,\infty}^\alpha$, we define a local scale of f at x to be $t(x) \geq 0$ such that

$$|Sf(x, t)| = \left| t^{1-\alpha/2} \frac{\partial K_t}{\partial t} * f(x) \right|$$

is a local maximum with respect to t for some $\alpha < 2$. The choice of α will be discussed. We also define a nontangential local scale (a smooth version of the previous local scale) of f at x to be $t^*(x)$ such that $S^*f(x, t^*)$ is a local maximum in t , where

$$S^*f(x, t) = \sup_{|x-y|<t} \left| Sf(y, t) e^{-\frac{|x-y|^2}{2t}} \right|.$$

We then extend the work in [14] to decompose f into $u + v$, with u being piecewise-smooth and v being texture, via the minimization problem,

$$\inf_{u \in BV} \left\{ \mathcal{K}(u) = |u|_{BV} + \lambda \|K_{\bar{t}(\cdot)} * (f - u)(\cdot)\|_{L^1} \right\},$$

where $\bar{t}(x)$ is some appropriate choice of a (nontangential) local scale to be captured in the oscillatory part v at x .

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1 Introduction and Motivation

Given an image f defined on \mathbb{R}^n or $\Omega = [0, 1]^n \subset \mathbb{R}^n$. When f is defined on Ω , we assume that f is periodic and Ω is the fundamental domain. Denote the function space defined on \mathbb{R}^n by $X(\mathbb{R}^n)$ and the function space defined on Ω by $X(\Omega)$. When we make no distinction between $X(\mathbb{R}^n)$ or $X(\Omega)$, we write X to mean either $X(\mathbb{R}^n)$ or $X(\Omega)$. An important problem in image analysis is the decomposition of f into $u + v$, where u is piecewise-smooth containing the geometric components of f and v is oscillatory, typically texture or noise. A variational approach to this image decomposition problem is by solving the following variational problem,

$$\inf_{(u,v) \in (X_1, X_2)} \{ \mathcal{K}(u) = F_1(u) + \lambda F_2(v), f = u + v \}, \quad (1)$$

where F_i 's are functionals on the function spaces X_i 's, and $\lambda > 0$ is a fixed tuning parameter. Given the desired properties of u and v , a good model for (1) should have $F_1(u)$ and $F_2(v)$ small in X_1 and X_2 respectively. Note that if F_i is a norm on X_i , then $\mathcal{K}(u)$ is the usual Peetre K -functional. We recall a few models of this type.

The weak formulation of the Mumford-Shah model [30] introduced by E. De Giorgi, M. Carriero and A. Leaci [11] has $X_1 = SBV$ (the space of Special functions of Bounded Variation [10]) and $X_2 = L^2$ with

$$F_1(u) = \int_{\Omega \setminus S_u} |\nabla u|^2 dx + \beta \mathcal{H}^{n-1}(S_u), \text{ and } F_2(v) = \|v\|_{L^2}^2,$$

where S_u is the singularity set of u in Ω , $\beta > 0$ and $\mathcal{H}^k(B)$ is the k -dimensional Hausdorff measure of the set B . For an indepth discussion of the Mumford-Shah model, we refer the readers to L. Ambrosio, N. Fusco and D. Pallara [3].

By replacing SBV with BV (the space of functions of Bounded Variation) for X_1 , and keeping $X_2 = L^2$ with $F_1(u) = |Du|(\Omega)$ and $F_2(v) = \|v\|_{L^2}^2$, we obtain the Rudin-Osher-Fatemi model [33], which was originally proposed for image de-noising. See also [2], [31], [9], [8], [45], [44], [1], [18], [28], among others, for the analysis of the variational model with $F_1(u) = |Du|(\Omega)$ and $F_2(v) = \|v\|_{L^1}$, which was proposed for removing impulse noise and texture decomposition.

We recall that $Du = (D_1u, \dots, D_nu)$ is the distributional derivative of u which is defined as a Radon measure on Ω ,

$$\int_{\Omega} \phi dD_iu = - \int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx, \text{ for all } \phi \in C_c^\infty, i = 1, \dots, n,$$

and $|Du|(\Omega)$ defines a seminorm on BV , and $\|u\|_{L^1} + |Du|(\Omega)$ defines a norm, which makes BV a Banach space. From now on, we denote $|Du|(\Omega)$ by $|u|_{BV}$.

In [27], Y. Meyer then proposed to replace the space $X_2 = L^2$ in the Rudin-Osher-Fatemi model with weaker function spaces to model oscillatory components (texture or noise). These spaces of generalized functions are $G = div(L^\infty)$, $F = div(BMO) = B\dot{M}O^{-1}$ ([21], [37]) (BMO = space of functions of Bounded Mean Oscillation [20]), and $E = \dot{B}_{\infty, \infty}^{-1}$, with $F_2(v)$ being the norm of one of these function spaces, while keeping $X_1 = BV$ with $F_1(u) = |u|_{BV}$. D. Mumford and B. Gidas [29] also showed that natural

images are drawn from probability distributions supported by generalized functions. Further analysis and variations of Meyer's models can be found in [43], [4], [5], [17], [26], [22], [13], [14], among others.

A general idea for modeling oscillatory components is, instead of imposing $\|v\|_{L^p}$ be bounded, to impose that $\|K * v\|_{L^p}$ is bounded, for some "averaging" kernel K , which can be realized as a low-pass filter. More specifically, we consider K such that $\widehat{K}(\xi)$ decays rapidly as $|\xi| \rightarrow \infty$. With

$$F_2(v) = \|K * v\|_{L^p} = \left\| \left(\widehat{K}(\xi) \widehat{v}(\xi) \right)^\vee (\cdot) \right\|_{L^p},$$

the model (1) allows $\widehat{v}(\xi)$ to be large when $|\xi|$ is large. This is precisely what allows v to be more oscillatory. With

$$\begin{aligned} \widehat{K}(\xi) &= \widehat{\beta}_\alpha(\xi) = (1 + 4\pi^2|\xi|^2)^{\alpha/2} = e^{\frac{\alpha}{2} \log(1+4\pi^2|\xi|^2)}, \text{ or} \\ \widehat{K}(\xi) &= \widehat{I}_\alpha(\xi) = (2\pi|\xi|)^\alpha = e^{\alpha \log(2\pi|\xi|)}, \text{ for } \alpha < 0, \end{aligned} \quad (2)$$

we obtain that $\|K * v\|_{L^p}$ is an equivalent norm for the potential Sobolev space W_p^α , and its homogeneous version \dot{W}_p^α , respectively. These choices of K were investigated in [32], [5], [23], and [13] for modeling oscillatory components. With K being the Gaussian ($\widehat{K}_t(\xi) = e^{-2\pi t|\xi|^2}$) or the Poisson kernel ($\widehat{P}_t(\xi) = e^{-2\pi t|\xi|}$), we arrive to the homogeneous Besov space $\dot{B}_{p,\infty}^\alpha$, for some $\alpha < 0$. These kernels have faster decay in the Fourier domain compared to (2). Besov spaces can also be characterized by other smooth kernels (see [42]).

In [14], inspired by Y. Meyer [27], the decomposition of an image f into $u + v$ has $X_1 = BV$ and $X_2 = \dot{B}_{p,\infty}^\alpha$, with $F_1(u) = |u|_{BV}$ and $F_2(v) = \|v\|_{\dot{B}_{p,\infty}^\alpha}$. In other words, the variational problem (1) becomes

$$\inf_{(u,v) \in (BV, \dot{B}_{p,\infty}^\alpha)} \left\{ \mathcal{K}(u) = |u|_{BV} + \lambda \|v\|_{\dot{B}_{p,\infty}^\alpha}, v = f - u \right\}, \quad (3)$$

with $\alpha < 0$, $1 \leq p \leq \infty$, and the space $\dot{B}_{p,\infty}^\alpha$ is the homogenous Besov space, which can be characterized as follow.

Let $\Phi(x) = 2^{n/2} e^{-2\pi|x|^2}$, for $x \in \mathbb{R}^n$, and denote $\Phi_t(x) = t^{-n} \Phi(x/t)$, then the heat kernel is defined as $K_t(x) = \Phi_{\sqrt{4t}}(x) = \left(e^{-2\pi t|\xi|^2} \right)^\vee(x)$. We have $\int_{\mathbb{R}^n} K_t(x) dx = 1$, for all $t > 0$. For each $f \in L^p$, let $u(x, t) = K_t * f(x)$. Then u satisfies the heat equation,

$$\left(\frac{\partial}{\partial t} - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \right) u(x, t) = 0, \text{ with } u(x, 0) = f(x). \quad (4)$$

The following definitions can be seen from E. Stein [35] and H. Triebel [41], [42].

Definition 1. For each $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$ such that $m > \alpha/2$, we say a function (or distribution) f belongs to the homogeneous Besov space $\dot{B}_{p,\infty}^\alpha(\mathbb{R}^n)$, $1 \leq p \leq \infty$, if

$$\|f\|_{\dot{B}_{p,\infty}^\alpha(\mathbb{R}^n)} = \sup_{0 < t < \infty} \left\{ t^{m-\alpha/2} \left\| \frac{\partial^m}{\partial t^m} K_t * f \right\|_{L^p(\mathbb{R}^n)} \right\} < \infty, \quad (5)$$

and we say $f \in \dot{B}_{p,\infty}^\alpha(\Omega)$, $1 \leq p \leq \infty$, if

$$\|f\|_{\dot{B}_{p,\infty}^\alpha(\Omega)} = \sup_{0 < t < 1} \left\{ t^{m-\alpha/2} \left\| \frac{\partial^m}{\partial t^m} K_t * f \right\|_{L^p(\Omega)} \right\} < \infty. \quad (6)$$

Equipped with the above norms, $\dot{B}_{p,\infty}^\alpha(\mathbb{R}^n)$ and $\dot{B}_{p,\infty}^\alpha(\Omega)$ become Banach spaces.

From the above definitions, the variational problem (3) becomes

$$\inf_{(u,v) \in (BV, \dot{B}_{p,\infty}^\alpha)} \left\{ \mathcal{K}(u) = |u|_{BV} + \lambda \sup_{t > 0} \left\{ t^{-\alpha/2} \|K_t * (f - u)\|_{L^p} \right\} \right\}. \quad (7)$$

Note that here we pick $m = 0$ for $\alpha < 0$. The minimization problem (7) can be rewritten as

$$\inf_{u \in BV} \left\{ \mathcal{K}(u) = |u|_{BV} + \bar{\lambda} \|K_{\bar{t}} * (f - u)\|_{L^p} \right\}, \quad (8)$$

for some $\bar{t} = \bar{t}(\alpha, f - u) \geq 0$ and $\bar{\lambda} = \lambda \bar{t}^{-\alpha/2}$. The term $\|K_{\bar{t}} * v\|_{L^p}$ from the equation (8) imposes a uniform smoothing of scale \bar{t} on the texture component v . However, texture may have different scales of oscillations locally. A natural extension of (8) is to have \bar{t} as a function of $x \in \Omega$. In other words, the support of the kernel $K_{\bar{t}(x)}$ is locally adapted to the local oscillation. This paper is devoted to studying the local scales of images and using these local scales for multiscale image decompositions.

We note that scale is an important aspect in computer vision, and has been studied rigorously. We mention a few work by [36], [39], [7], [38], [25], [24], [12], and references there in. Our study of local scales is different and is motivated from the function spaces point of view.

2 Local Scales and Nontangential Local Scales in Images

An important function space that measures the local differentiability of functions or distributions is the Triebel-Lizorkin function space [42]. The following definition is the characterization of this function space in term of the Gaussian kernel K_t .

Definition 2. For each $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$ such that $m > \alpha/2$, we say a function (or distribution) f belongs to the homogeneous Triebel-Lizorkin function space $\dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)$, $1 \leq p \leq \infty$, if

$$\|f\|_{\dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)} = \left\| \sup_{0 < t < \infty} \left\{ t^{m-\alpha/2} \left| \frac{\partial^m}{\partial t^m} K_t * f \right| \right\} \right\|_{L^p(\mathbb{R}^n)} < \infty, \quad (9)$$

and we say $f \in \dot{F}_{p,\infty}^\alpha(\Omega)$, $1 \leq p \leq \infty$, if

$$\|f\|_{\dot{F}_{p,\infty}^\alpha(\Omega)} = \left\| \sup_{0 < t < 1} \left\{ t^{m-\alpha/2} \left| \frac{\partial^m}{\partial t^m} K_t * f \right| \right\} \right\|_{L^p(\Omega)} < \infty. \quad (10)$$

Equipped with the above norms, $\dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)$ and $\dot{F}_{p,\infty}^\alpha(\Omega)$ become Banach spaces.

Remark 1. When $p = \infty$, we have $\dot{B}_{\infty,\infty}^\alpha = \dot{F}_{\infty,\infty}^\alpha$ for all $\alpha \in \mathbb{R}$.

Note that in the above definition, any $m \in \mathbb{N}^0$, such that $m > \alpha/2$, would provide an equivalent norm for the space $\dot{F}_{p,\infty}^\alpha$. The following result can be found in E. Stein [35] and H. Triebel [41],[42].

Proposition 1. *For $1 \leq p \leq \infty$, if m_1 and m_2 are two non-negative integers greater than $\alpha/2$, then*

$$\sup_{0 < t < \infty} \left\{ t^{m_1 - \alpha/2} \left\| \frac{\partial^{m_1}}{\partial t^{m_1}} K_t * f \right\|_{L^p(\mathbb{R}^n)} \right\} \text{ and } \sup_{0 < t < \infty} \left\{ t^{m_2 - \alpha/2} \left\| \frac{\partial^{m_2}}{\partial t^{m_2}} K_t * f \right\|_{L^p(\mathbb{R}^n)} \right\} \quad (11)$$

provide equivalent norms for the space $\dot{B}_{p,\infty}^\alpha(\mathbb{R}^n)$. Similarly, if m_1 and m_2 are two non-negative integers greater than $\alpha/2$, then

$$\left\| \sup_{0 < t < \infty} \left\{ t^{m_1 - \alpha/2} \left| \frac{\partial^{m_1}}{\partial t^{m_1}} K_t * f \right| \right\} \right\|_{L^p(\mathbb{R}^n)} \text{ and } \left\| \sup_{0 < t < \infty} \left\{ t^{m_2 - \alpha/2} \left| \frac{\partial^{m_2}}{\partial t^{m_2}} K_t * f \right| \right\} \right\|_{L^p(\mathbb{R}^n)} \quad (12)$$

provide equivalent norms for the space $\dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)$. The same result also holds for the function space $\dot{B}_{p,\infty}^\alpha(\Omega)$ and $\dot{F}_{p,\infty}^\alpha(\Omega)$ for $1 \leq p \leq \infty$.

For each $f \in \dot{F}_{p,\infty}^\alpha$, $\alpha < 2$ and $1 \leq p \leq \infty$, define

$$Sf(x, t) = t^{1-\alpha/2} \frac{\partial}{\partial t} u(x, t) = t^{1-\alpha/2} \frac{\partial}{\partial t} K_t * f(x). \quad (13)$$

Here we implicitly assume that if $f \in \dot{F}_{p,\infty}^\alpha(\Omega)$ then $Sf(x, t)$ is defined for all $0 < t < 1$, and if $f \in \dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)$ then $Sf(x, t)$ is defined for all $t > 0$. Let $\tau = \log_a(t)$. Sometimes we write $Sf(x, \tau)$ to mean that the evaluation in t is in logarithmic scale.

Note that the definition of Sf does not depend on p . However, its smoothness depends on the space $\dot{F}_{p,\infty}^\alpha$ in which f belongs to. We have the following results regarding the regularity for the derivatives of $Sf(x, \tau)$ with respect to $\tau = \log_a(t)$.

Proposition 2. *Let $\tau = \log_a(t)$, and by a change of variable, let*

$$Sf(x, \tau) = t^{1-\alpha/2} \frac{\partial}{\partial t} K_t * f(x).$$

Then for $\alpha < 2$ and $f \in \dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)$, $1 \leq p \leq \infty$,

$$\left\| \sup_{0 < t < \infty, \tau = \log_a(t)} \left| \frac{\partial}{\partial \tau} Sf(\cdot, \tau) \right| \right\|_{L^p(\mathbb{R}^n)} \leq C_1 \|f\|_{\dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)}. \quad (14)$$

In general,

$$\left\| \sup_{0 < t < \infty, \tau = \log_a(t)} \left| \frac{\partial^k}{\partial \tau^k} Sf(\cdot, \tau) \right| \right\|_{L^p(\mathbb{R}^n)} \leq C_k \|f\|_{\dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)}, \quad (15)$$

where C_k are universal constants that depend on n , α and k . The same results also hold for $f \in \dot{F}_{p,\infty}^\alpha(\Omega)$.

Proof. Since $\frac{dt}{d\tau} = t \ln a$, we have

$$\begin{aligned}
\left| \frac{\partial}{\partial \tau} S f(x, \tau) \right| &= \left| \frac{\partial}{\partial t} \left(t^{1-\alpha/2} \frac{\partial}{\partial t} K_t * f(x) \right) \frac{\partial t}{\partial \tau} \right| \\
&= \left| \left((1-\alpha/2) t^{1-\alpha/2-1} \frac{\partial}{\partial t} K_t * f(x) + t^{1-\alpha/2} \frac{\partial^2}{\partial t^2} K_t * f(x) \right) t \ln a \right| \\
&\leq \ln a \left((1-\alpha/2) \left| t^{1-\alpha/2} \frac{\partial}{\partial t} K_t * f(x) \right| + \left| t^{2-\alpha/2} \frac{\partial^2}{\partial t^2} K_t * f(x) \right| \right) \\
&\leq \ln a \left((1-\alpha/2) \sup_{0 < t < \infty} \left| t^{1-\alpha/2} \frac{\partial}{\partial t} K_t * f(x) \right| \right. \\
&\quad \left. + \sup_{0 < t < \infty} \left| t^{2-\alpha/2} \frac{\partial^2}{\partial t^2} K_t * f(x) \right| \right).
\end{aligned}$$

By (11), we have

$$\left\| \sup_{0 < t < \infty, \tau = \log_a(t)} \left| \frac{\partial}{\partial \tau} S f(\cdot, \tau) \right| \right\|_{L^p(\mathbb{R}^n)} \leq \ln a (1 - \alpha/2 + A_1) \|f\|_{\dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)},$$

where A_1 is the equivalent norm constant. Therefore,

$$\left\| \sup_{0 < t < \infty, \tau = \log_a(t)} \left| \frac{\partial}{\partial \tau} S f(\cdot, \tau) \right| \right\|_{L^p(\mathbb{R}^n)} \leq C_1 \|f\|_{\dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)}, \quad (16)$$

with $C_1 = \ln a (1 - \alpha/2 + A_1)$. By differentiating $\frac{\partial}{\partial \tau} S f(x, \tau)$ with respect to τ a second time and using similar techniques, we obtain,

$$\begin{aligned}
\frac{\partial^2}{\partial \tau^2} S f(x, \tau) &= \ln a \frac{\partial}{\partial \tau} \left((1-\alpha/2) t^{1-\alpha/2} \frac{\partial}{\partial t} K_t * f(x) + t^{2-\alpha/2} \frac{\partial^2}{\partial t^2} K_t * f(x) \right) \\
&= (\ln a)^2 t \left((1-\alpha/2)^2 t^{1-\alpha/2-1} \frac{\partial}{\partial t} K_t * f(x) \right. \\
&\quad \left. + (1-\alpha/2) t^{1-\alpha/2} \frac{\partial^2}{\partial t^2} K_t * f(x) + (2-\alpha/2) t^{1-\alpha/2} \frac{\partial^2}{\partial t^2} K_t * f(x) \right. \\
&\quad \left. + t^{2-\alpha/2} \frac{\partial^3}{\partial t^3} K_t * f(x) \right) \\
&= (\ln a)^2 \left((1-\alpha/2)^2 t^{1-\alpha/2} \frac{\partial}{\partial t} K_t * f(x) \right. \\
&\quad \left. + (1-\alpha/2) t^{2-\alpha/2} \frac{\partial^2}{\partial t^2} K_t * f(x) + (2-\alpha/2) t^{2-\alpha/2} \frac{\partial^2}{\partial t^2} K_t * f(x) \right. \\
&\quad \left. + t^{3-\alpha/2} \frac{\partial^3}{\partial t^3} K_t * f(x) \right)
\end{aligned}$$

Therefore,

$$\left\| \sup_{0 < t < \infty, \tau = \log_a(t)} \left| \frac{\partial^2}{\partial \tau^2} S f(\cdot, \tau) \right| \right\|_{L^p(\mathbb{R}^n)} \leq C_2 \|f\|_{\dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)}, \quad (17)$$

where $C_2 = A_1(2 - \alpha/2)^2 + A_2$, and A_i 's are the equivalent norm constant. In general, for all $x \in \Omega$,

$$\left\| \sup_{t>0, \tau \log_a(t)} \left| \frac{\partial^k}{\partial \tau^k} Sf(\cdot, \tau) \right| \right\|_{L^p(\mathbb{R}^n)} \leq C_k \|f\|_{\dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)}. \quad (18)$$

□

Note that if we further assume that f is bounded then for each x ,

$$\sup_{0 < t < \infty} \left| \frac{\partial^k}{\partial \tau^k} Sf(x, \tau) \right| \leq C_k \|f\|_{L^\infty(\mathbb{R}^n)},$$

for all $k \geq 0$. In other words, for each x , $Sf(x, \tau)$ is smooth with respect to τ . We have the following result regarding the spatial smoothness of $Sf(x, t)$.

Proposition 3. *Suppose $f \in \dot{F}_{p,\infty}^\alpha$ for some $\alpha < 2$ and $1 \leq p \leq \infty$. For each $t > 0$, we have*

$$\|\Delta Sf(\cdot, t)\|_{L^p} \leq Ct^{-1} \|f\|_{\dot{F}_{p,\infty}^\alpha}, \quad (19)$$

Recall the variational model for decomposing f into $u + v$ via,

$$K(f, \lambda) = \inf_{(u,v) \in (X_1, X_2)} \{ \mathcal{K}(u, v, t) = F_1(u) + \lambda F_2(v), f = u + v \},$$

for $\lambda > 0$. Since the geometric part u is much smoother than the oscillatory part v , we typically have $X_1 \subset X_2$. The parameter $\lambda > 0$ determines the separation of oscillations or scales. We can think of λ as a scale selecting parameter with respect to the Peetre K-functional $K(f, \lambda)$, which provides an equivalent norm for the space $X = X_1 + X_2$. This interpretation of scales was investigated in [40] for multiscale image decompositions with $\mathcal{K}(u, v, \lambda) = |u|_{BV} + \lambda \|v\|_{L^2}^2$. The local scale interpretation of this K-functional is discussed in [39]. See also [8], [45], [44], [1], and [28] for similar interpretations with $\mathcal{K}(u, v, \lambda) = |u|_{BV} + \lambda \|v\|_{L^1}$. In this paper, we consider a different scale interpretation, which will become clearer in the following definitions.

Suppose $f \in \dot{F}_{p,\infty}^\alpha$, for some $\alpha < 2$ and $1 \leq p \leq \infty$. From Proposition 2, for each $x \in \Omega$, $\frac{\partial^k}{\partial \tau^k} Sf(x, \tau)$ is smooth and bounded in $\tau = \log_a(t)$. So as τ increases and if there is a change in the scale of f at x , we also expect the change in $Sf(x, \tau)$ at x . With this interpretation, we have the following definitions of local scales of f at $x \in \Omega$.

Definition 3. Fix an $f \in \dot{F}_{p,\infty}^\alpha$, $\alpha < 2$ and $1 \leq p \leq \infty$.

- The local scales of f at x is defined to be the set $T_f(x)$ consisting of $t \geq 0$, such that $|(Sf(x, t))|$ is a local maximum. Moreover, for each $t \in T_f(x)$, if there exists an interval $[a, b]$, $a < b$, containing t such that for all $t \in [a, b]$, $\frac{\partial}{\partial t}(Sf(x, t)) = 0$. Then we only select $t = a$ to be in $T_f(x)$ and discard all other $t' \in (a, b]$. Therefore, if t and t' belong to $T_f(x)$, then $|t - t'| > 0$.
- We define the first scale of f at x to be $t_{f,1}(x)$, such that

$$t_{f,1}(x) = \inf \{t \in T_f(x)\}.$$

The i^{th} scale of f at x is then defined as

$$t_{f,i}(x) = \inf \{t \in T_f(x) \setminus \{t_{f,1}(x), \dots, t_{f,i-1}(x)\}\}.$$

It is clear that $0 \leq t_{f,1}(x) < t_{f,2}(x) < \dots < t_{f,i}(x) < \dots$

- We define the major scale of f at x up to $s > 0$, as $t_{f,+}(s, x)$ such that

$$t_{f,+}(s, x) = \arg \max_{t \in T_f(x), 0 \leq t \leq s} |Sf(x, t)|. \quad (20)$$

- For each x , we refer to $|Sf(x, t)|$ as the **oscillatory level** of f at scale $t \in T_f(x)$.

In the previous definition, a local scale of f at x is defined as t such that $|Sf(x, t)|$ is a local maximum. To further impose some interaction of scales in the spatial domain, we consider the nontangential control on the local maximum of $|Sf(x, t)|$. In other words, let the following nontangential maximal function be defined as

$$S^*(x, t) := \sup_{|x-y|<t} \left| Sf(y, t) e^{\frac{-|x-y|^2}{2t}} \right|.$$

We then define the nontangential local scales of f at x as the following.

Definition 4. For each $f \in \dot{F}_{p,\infty}^\alpha$, $\alpha < 2$ and $1 \leq p \leq \infty$, we have the following definitions.

- The nontangential local scales of f at x is defined to be the set $T_f^*(x)$ consisting of $t \geq 0$, such that $S^*f(x, t)$ is a local maximum. Moreover, for each $t \in T_f^*(x)$, if there exists an interval $[a, b]$, $a < b$, containing t such that for all $t \in [a, b]$, $\frac{\partial}{\partial t}(S^*f(x, t)) = 0$. Then we only select $t = a$ to be in $T_f^*(x)$ and discard all other $t' \in (a, b]$. Therefore, if t and t' belong to $T_f^*(x)$, then $|t - t'| > 0$.
- We define the first nontangential scale of f at x to be $t_{f,1}^*(x)$, such that

$$t_{f,1}^*(x) = \inf \{t \in T_f^*(x)\}.$$

The i^{th} nontangential scale of f at x is then defined as

$$t_{f,i}^*(x) = \inf \{t \in T_f^*(x) \setminus \{t_{f,1}^*(x), \dots, t_{f,i-1}^*(x)\}\}.$$

It is clear that $0 \leq t_{f,1}^*(x) < t_{f,2}^*(x) < \dots < t_{f,i}^*(x) < \dots$

- We define the major nontangential scale of f at x up to $s > 0$, as $t_{f,+}^*(s, x)$ such that

$$t_{f,+}^*(s, x) = \arg \max_{t \in T_f^*(x), 0 \leq t \leq s} S^*f(x, t). \quad (21)$$

- For each x , we refer to $S^*f(x, t)$ as the **oscillatory level** of f at scale $t \in T_f^*(x)$.

The next proposition shows that the number of local scales at x is invariant under dilation. This is an important property that scales should satisfy. Moreover, the local scales of an image are the local scales of its dilated version times the square of dilating factor.

Proposition 4. Suppose that $f \in \dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)$, $\alpha < 2$, and define the dilating operator $d_\delta f(x) = f(\delta x)$. Then for each $\alpha < 2$ and $\delta > 0$,

$$t_{f,i}(x) = \delta^2 t_{d_\delta f,i}(x). \quad (22)$$

In other words, $\frac{t_{f,i}(x)}{t_{d_\delta f,i}(x)} = \delta^2$, which is quadratic in δ .

Proof. Note that if we let $\Phi(x) = 2^{n/2} e^{-2\pi|x|^2}$, and define $\Phi_t(x) = t^{-n} \Phi(x/t)$, then the heat kernel K_t is just $\Phi_{\sqrt{4t}}(x)$. Without loss of generality, we may assume that $x = 0$. Let

$$t_{f,*}(0) = \arg \max_{0 < t < 1} t^{1-\alpha/2} \left| \frac{\partial K_t}{\partial t} * f(0) \right|.$$

We will show that $t_{f,*}(0) = \delta^2 t_{d_\delta f,*}(0)$. The cases $t_{f,i}(0) = \delta^2 t_{d_\delta f,i}(0)$ follow. We have

$$\begin{aligned} K_t * d_\delta f(0) &= \int K_t(y) f(\delta y) dy = \int (4t)^{-n/2} \Phi\left(\frac{y}{\sqrt{4t}}\right) f(\delta y) dy \\ &= \int (4t)^{-n/2} \Phi\left(\frac{z}{\delta\sqrt{4t}}\right) f(z) \frac{dz}{\delta^n} = \int (4t\delta^2)^{-n/2} \Phi\left(\frac{z}{\sqrt{4t\delta^2}}\right) f(z) dz \\ &= \int K_{t\delta^2}(z) f(z) dz = K_s * f(0) \end{aligned}$$

where $s = \delta^2 t$. This implies that

$$\frac{\partial}{\partial t} (K_t * d_\delta f(0)) = \frac{\partial}{\partial t} (K_s * f(0)) = \delta^2 \frac{\partial}{\partial s} (K_s * f(0))$$

Note that in the last equality, we use $\frac{\partial K_s}{\partial t} = \frac{\partial K_s}{\partial s} \cdot \frac{ds}{dt}$. Therefore,

$$\begin{aligned} \sup_{0 < t < 1} t^{1-\alpha/2} \left| \frac{\partial K_t}{\partial t} * d_\delta f(0) \right| &= \sup_{0 < t < 1} t^{1-\alpha/2} \delta^2 \left| \frac{\partial K_s}{\partial s} * f(0) \right| \\ &= \delta^\alpha \sup_{0 < t < 1} s^{1-\alpha/2} \left| \frac{\partial K_s}{\partial s} * f(0) \right|. \end{aligned} \quad (23)$$

The supremum on the righthand side of equation (23) is attained at $s = t_{f,*}$, which implies that

$$\delta^2 t_{d_\delta f,*}(0) = t_{f,*}(0). \quad (24)$$

Since convolution is invariant under translation, (24) holds for all x . \square

Remark 2. From the previous Proposition, we see that $\frac{t_{f,*}(x)}{t_{d_\delta f,*}(x)} = \delta^2$, which is quadratic in δ . This quadratic behavior comes from the fact that the heat kernel is the rescaling of $\Phi(x) = t^{-n} \Phi(x/t)$ by a multiple of \sqrt{t} . In other words, $K_t(x) = \Phi_{\sqrt{4t}}(x)$. If we were to use the Poisson kernel instead of the heat kernel to study local scales, we then would obtain

$$\frac{t_{f,*}(x)}{t_{d_\delta f,*}(x)} = \delta, \quad (25)$$

which is linear in δ . In other words, for $\alpha < 1$, let $S^\alpha f(x, t) = t^{1-\alpha} \frac{\partial P_t}{\partial t} * f(x)$, where $P_t(x) = \varphi_t(x) = t^{-n} \varphi(x/t)$ with $\varphi(x) = c_n (1 + |x|^2)^{-(n+1)/2}$, $c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}$. Then with the local scales defined as in Definition 3, we obtain (25). We chose to use the heat kernel because it has faster decay and provides a better separation of scales.

By observing $Sf(x, t)$ for each x , we obtain a sequence of scales t_i 's that f exhibits at x , and the value of $|Sf(x, t_i)|$ determines how prominent the oscillation at the scale t_i is at x . For some i , suppose $t_i(x)$ is large at some point x , then we expect that, for some y belonging to a small neighborhood of x , $t_i(y)$ is very close to $t_i(x)$. However, it is no longer true if $t_i(x)$ is small. This statement is precisely the consequence of Proposition 3.

Remark 3. Recall from Proposition 2 that for each $f \in \dot{F}_{p,\infty}^\alpha$,

$$\left| \frac{\partial Sf(x, t)}{\partial t} \right| \leq Ct^{-1} \|f\|_{\dot{F}_{p,\infty}^\alpha}, \text{ and}$$

$$\left| \frac{\partial Sf(x, t)}{\partial \tau} \right| \leq C \ln a \|f\|_{\dot{F}_{p,\infty}^\alpha}.$$

In other words, having Sf as a function of t , we see that based on the definition of scales the transition between scales is of order $O(t)$. However, the transition between scales is of order $O(1)$ in logarithmic scale τ . With the constant a being close to 1, we can take τ to be discrete ($\tau \in \mathbb{Z}$) and local scales computed over the discrete set $\{a^\tau, \tau \in \mathbb{Z}\}$ are still be close to the true local scales of f . Moreover, we also obtain an equivalent norm for the space $\dot{B}_{p,\infty}^\alpha$ and $\dot{F}_{p,\infty}^\alpha$ in terms of this discrete variable. For each $\alpha \in \mathbb{R}$ and $1 \leq p \leq \infty$, let $m \in \mathbb{N}_0$ such that $m > \alpha/2$. Then for any $a > 1$,

$$\|f\|_{\dot{B}_{p,\infty}^\alpha(\mathbb{R}^n)} \approx \sup_{\tau \in \mathbb{Z}} a^{\tau(m-\alpha/2)} \left\| \frac{\partial^m}{\partial t^m} K_{a^\tau} * f(x) \right\|_{L^p(\mathbb{R}^n)},$$

where ' \approx ' means equivalent norm. Similarly,

$$\|f\|_{\dot{F}_{p,\infty}^\alpha(\mathbb{R}^n)} \approx \left\| \sup_{\tau \in \mathbb{Z}} \left| a^{\tau(m-\alpha/2)} \frac{\partial^m}{\partial t^m} K_{a^\tau} * f(x) \right| \right\|_{L^p(\mathbb{R}^n)}.$$

Similar results also hold for the spaces $\dot{B}_{p,\infty}^\alpha(\Omega)$ and $\dot{F}_{p,\infty}^\alpha(\Omega)$, but with $\tau \in \mathbb{Z}$ and $\tau < 0$.

According to definition 3, the local scales of f depend on the choice of α . Figures 1 and 2 show the graph of $|Sf(x, \tau)|$ and $|S^*f(x, \tau)|$, respectively, with respect to the discrete variable τ for different values of $\alpha \in [-0.6, 0.6]$ at various x 's with the assumption that f is periodic and $\Omega = [0, 1]^n$. From this figure, we observe that for $\alpha < 0$ the oscillatory levels $|Sf(x, t_{f,i}(x))|$ and $|S^*f(x, t_{f,i}^*(x))|$ are small for the first few small scales $t_{f,i}(x)$ and $t_{f,i}^*(x)$ respectively, and the local scales are slightly increasing as α decreases. In other words, when $\alpha < 0$ and large in absolute value, the oscillatory level for small scales are low and the oscillatory level for large scales are high. The situation is reversed for large positive α . Also from these figures, α determines which scale is a major local scale at each x . Experimentally for studying the local scales from the test images in figure 3, we find α to be within a small neighborhood of the origin is an appropriate choice.

For each $f \in \dot{F}_{p,\infty}^\alpha$, let us for a moment make a change of notation by denoting $Sf(x)$ as $S^\alpha f(x)$, the set of local scales $T_f(x)$ as $T_f^\alpha(x)$, the i^{th} local scale $t_{f,i}(x)$ as $t_{f,i}^\alpha(x)$, and the major scale $t_{f,+}(x)$ as $t_{f,+}^\alpha(x)$. We have the following result concerning the local scales of f for different values of α . The same result also applies to the nontangential case.

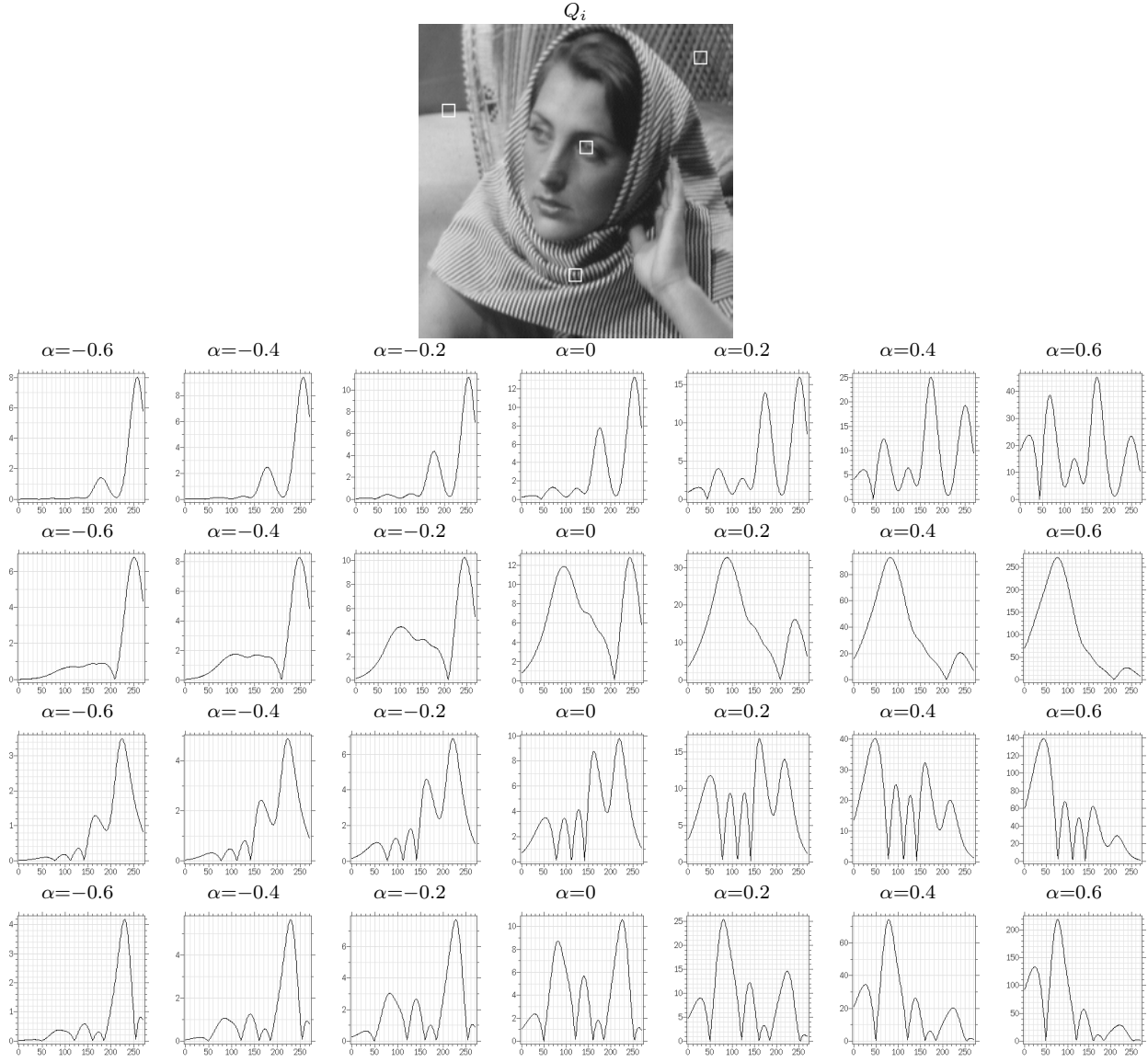


Figure 1: The plots show the graphs of $|Sf(x_i, \log_a(t))|$ for different values of α , and x_i is the center of the square Q_i , in the order from left to right and top to bottom.

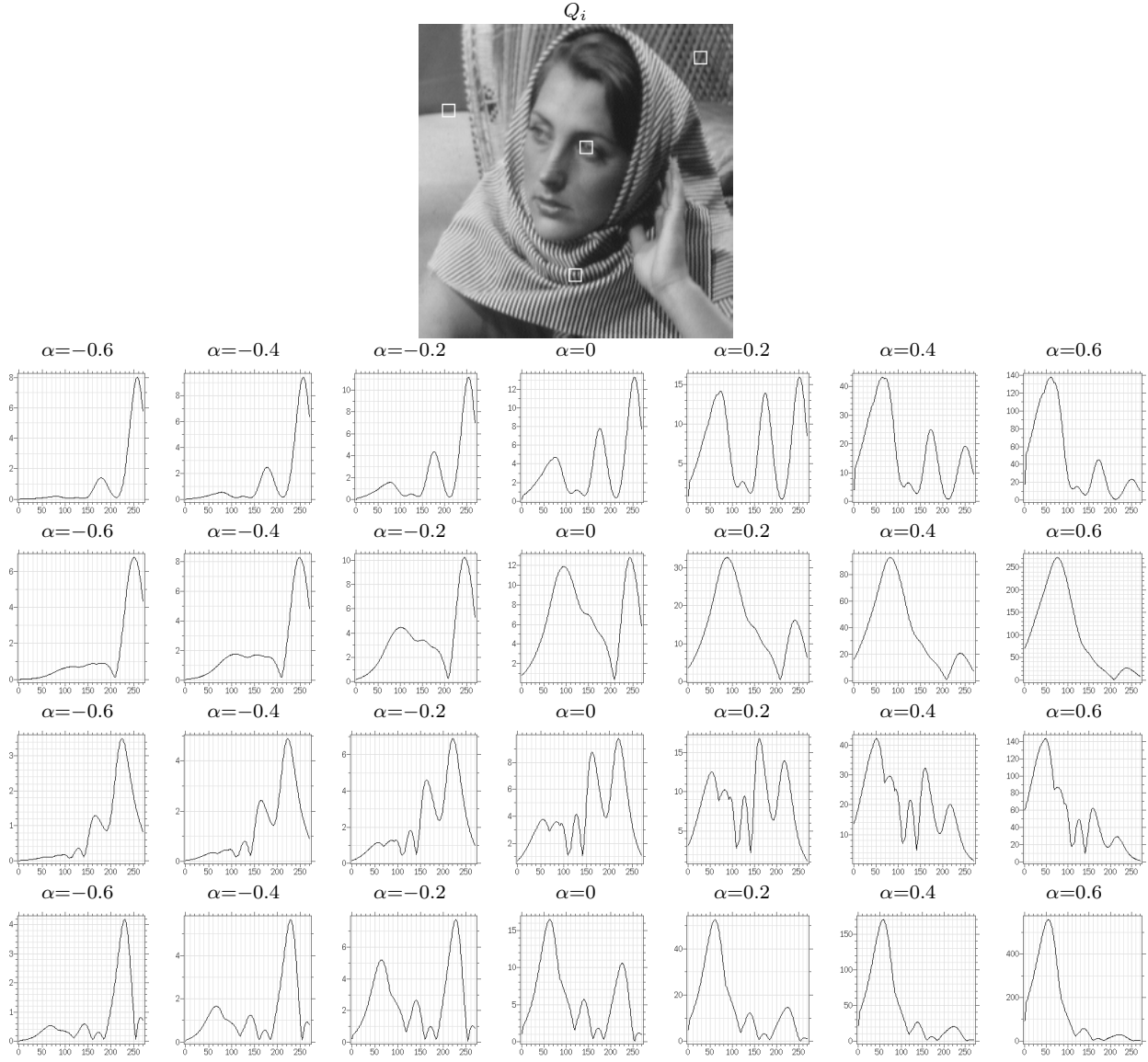


Figure 2: The plots show the graphs of $|S^* f(x_i, \log_a(t))|$ for different values of α , and x_i is the center of the square Q_i , in the order from left to right and top to bottom.

Lemma 1. *Suppose $f \in \dot{F}_{p,\infty}^\alpha$, for all $\alpha \in (a, b)$ and some $p \in [1, \infty]$. If $a < \alpha_1 < \alpha_2 < b < 2$, then for almost every x ,*

$$t_{f,+}^{\alpha_2}(x) \leq t_{f,+}^{\alpha_1}(x).$$

Proof. Let $a < \alpha_1 < \alpha_2 < b$ with $\epsilon = \alpha_2 - \alpha_1 > 0$. We have

$$\begin{aligned} |S^{\alpha_2} f(x, t)| &= t^{1-\alpha_2/2} \left| \frac{\partial K_t}{\partial t} * f(x) \right| = t^{1-(\alpha_1+\epsilon)/2} \left| \frac{\partial K_t}{\partial t} * f(x) \right| \\ &= t^{-\epsilon/2} \left(t^{1-\alpha_1/2} \left| \frac{\partial K_t}{\partial t} * f(x) \right| \right) = t^{-\epsilon/2} |S^{\alpha_1} f(x, t)|. \end{aligned} \quad (26)$$

Since $t_{f,+}^{\alpha_1}(x) = \arg \max_{0 < t < 1} |S^{\alpha_1} f(x, t)|$ and $t^{-\epsilon/2}$ is decreasing,

$$\arg \max_{0 < t < 1} |S^{\alpha_2} f(x, t)| \leq \arg \max_{0 < t < 1} |S^{\alpha_1} f(x, t)|$$

□

As a consequence of the previous lemma, we have

Proposition 5. *Suppose $f \in \dot{F}_{p,\infty}^\alpha$, for all $\alpha \in (a, b)$ and some $p \in [1, \infty]$. Let $a < \alpha_1 < \alpha_2 < b < 2$ and $x \in \Omega$. Then for any $t_1 \in T_f^{\alpha_1}(x)$, there exists $t_2 \in T_f^{\alpha_2}(x)$ such that $t_2 \leq t_1$.*

Recall that *BMO* is the space of functions of Bounded Mean Oscillation. The following John-Nirenberg inequality can be found in E. Stein [34].

Theorem 1. (*John-Nirenberg inequality*) *Suppose $f \in BMO(\Omega)$, and Denote by $f_B = \frac{1}{|B|} \int_B f(x) dx$ the mean of f in B . Then there exists a positive constant C_1 and C_2 so that for every $\alpha > 0$, and every ball $B \subset \Omega$,*

$$|\{x \in B : |f(x) - f_B| > \alpha\}| \leq |B| C_1 e^{-C_2 \alpha / \|f\|_{BMO(\Omega)}}.$$

In particular, Suppose Ω is bounded and by replacing $f(x)$ with $f(x) + f_\Omega$, then

$$|\{x \in \Omega : |f(x)| > \alpha\}| \leq C_1 e^{-C_2 \alpha / \|f\|_{BMO(\Omega)}}.$$

Let us for a moment assume that f belongs to $L^\infty(\Omega)$. Taking $\alpha = 0$ in (13), we have for each $f \in \dot{F}_{p,\infty}^0(\Omega)$,

$$Sf(x, t) = S^0 f(x, t) = t \frac{\partial K_t}{\partial t} * f(x, t).$$

Let $\tau = \log_a(t)$, for some $a > 1$. For each $x \in \Omega$, and $\beta > 0$, define

$$B_\beta(x) = \{t_i \in T f(x) : |Sf(x, t_i)| \geq \beta\}, \quad (27)$$

and

$$B_{\beta,N} = \{x \in \Omega : \#B_\beta(x) \geq N\}, \quad (28)$$

where $\#B$ denotes the discrete measure of the set B .

The following theorem says that the measure of $B_{\beta,N}$ is exponentially decayed with respect to N and the oscillatory level β . In other words, The measure of the set of x 's that are embedded in many prominent scales is exponentially small.

Theorem 2. For each $f \in L^\infty(\Omega)$, let $B_{\beta,N}$ be defined as (28) for some $\beta > 0$, and N a positive integer. Then $B_{\beta,N}$ satisfies the two bounds,

1. $|B_{\beta,N}| \leq C_1 e^{-C_2 N \beta^3 / \|f\|_{L^\infty(\Omega)}^3}$, and
2. $|B_{\beta,N}| \leq C_1 e^{-C_2 (N \beta^3 / \|f\|_{L^\infty(\Omega)}^3)^{1/2}}$,

for some positive constants C_i 's. We also obtain the same result by replacing $|Sf(x, t)|$ with $|S^*f(x, t)|$ in (27). The same result also holds if $\Omega = \mathbb{R}^n$.

Proof. Let the square function of f be defined as

$$[\mathcal{S}f(x)]^2 = \int_0^1 |Sf(x, t)|^2 \frac{dt}{t} = \ln(a) \int_{-\infty}^0 |Sf(x, \tau)|^2 d\tau.$$

We have

$$\|\mathcal{S}f\|_{BMO(\Omega)} \leq C' \|f\|_{BMO(\Omega)} \leq C' \|f\|_{L^\infty(\Omega)}. \quad (29)$$

Since $\left| \frac{\partial^2}{\partial \tau^2} Sf \right|_{L^\infty(\Omega \times \mathbb{R}^+)} \leq C \|f\|_{L^\infty(\Omega)}$, there exists a small $\epsilon > 0$, which depends on β and $\left| \frac{\partial^2}{\partial \tau^2} Sf \right|_{L^\infty(\Omega \times \mathbb{R}^+)}$, such that the intervals $I_i = [\tau_i - \epsilon, \tau_i + \epsilon]$ are disjoint with $|I_i| = 2\epsilon$, and $\epsilon = C\beta \|f\|_{L^\infty(\Omega)}$, for some new constant C . Moreover, In each I_i , $|Sf(x, \tau)|$ is concave. This implies that for $\tau \in I_i$,

$$\begin{aligned} |Sf(x, \tau)| &\geq \beta - \frac{C}{2} \|f\|_{L^\infty(\Omega)} |\tau - \tau_i| \geq \frac{\beta}{2}. \\ \Rightarrow \int_{I_i} |Sf(x, \tau)|^2 d\tau &\geq (\beta/2)^2 |I_i| \geq C\beta^3 / \|f\|_{L^\infty(\Omega)} \\ \Rightarrow [\mathcal{S}f(x)]^2 &= \ln(a) \int_{-\infty}^0 |Sf(x, \tau)|^2 d\tau \geq \ln(a) \sum_{t_i \in B_\beta(x)} \int_{I_i} |Sf(x, \tau)|^2 d\tau \\ &\geq \frac{C\beta^3}{\|f\|_{L^\infty(\Omega)}} \cdot \#\{B_\beta(x)\}. \end{aligned}$$

But

$$B_{\beta,N} \subset \left\{ x \in \Omega : [\mathcal{S}f(x)]^2 \geq \frac{CN\beta^3}{\|f\|_{L^\infty(\Omega)}} \right\}. \quad (30)$$

By John-Nirenberg inequality, we have

$$\left| \left\{ x \in \Omega : [\mathcal{S}f(x)]^2 \geq \frac{CN\beta^3}{\|f\|_{L^\infty(\Omega)}} \right\} \right| \leq C_1 e^{-C_2 N \beta^3 / (\|f\|_{L^\infty(\Omega)} \|\mathcal{S}f\|_{BMO(\Omega)}^2)},$$

for some positive constants C_i 's. Therefore, together with (29) and (30),

$$|B_{\beta,N}| \leq C_1 e^{-C_2 N \beta^3 / \|f\|_{L^\infty(\Omega)}^3},$$

for some new positive constants C_i 's.

It can also be shown that, for some positive constants C_i 's,

$$|B_{\beta,N}| \leq C_1 e^{-C_2 \left(N\beta^3 / \|f\|_{L^\infty(\Omega)}^3 \right)^{1/2}},$$

since

$$B_{\beta,N} \subset \left\{ x \in \Omega : Sf(x) \geq \left(\frac{CN\beta^3}{\|f\|_{L^\infty(\Omega)}} \right)^{1/2} \right\}.$$

□

2.1 Experimental Results for Local Scales and Nontangential Local Scales

For numerical study, we assume that f is periodic and $\Omega = [0, 1]^n$. Therefore, the set of scales of f is contained in $[0, 1]$. From remark 3, we discretize the scale interval $[0, 1]$ in a discrete logarithmic scale $\tau = \log_a t$, for some $a > 1$, and $\tau \in \mathbb{Z}$ with $\tau \leq 0$. The closer the a to 1, the closer we are to capturing the true scales of f . In this paper, all images are of size 256×256 and the local scales are computed using the discrete set

$$S = \{t_0 = a^{\tau_0}, \dots, t_i = a^{\tau_0+i}, \dots, t_N = a^{\tau_0+N}\}, \quad (31)$$

where t_0 and t_N are the smallest and largest scales that we want to detect respectively, $a = 1/0.95$, and $\tau_0 = -290$. Here, τ_0 is chosen so that K_{t_0} approximates the Dirac delta function.

The following figures show the local scales of images which we assume to be in $\dot{F}_{p,\infty}^\alpha(\Omega)$ with $\alpha = 0$. Figures 4-5 show the graphs of $|Sf(x, \tau)|$ and $S^*f(x, \tau)$ with respect to $\tau = \log_a(t)$ for a fixed x centered at the squares which are the significant supports of $K_{t_i(x)}$, where $t_i(x)$ is the i^{th} local scale at x . Recall that a local scale of f at x occurs at t if $|Sf(x, t)|$ is a local maximum. In these two cases, both the local scales and nontangential local scales are equal.

Figures 6-12 show $\log_a(t_1(x))$ where $t_1(x)$ is the first (nontangential) local scale of f at x such that on the first row the oscillatory level $|Sf(x, t_1(x))| > \epsilon$, and on the second row $S^*f(x, t_1(x)) > \epsilon$, for various ϵ 's.

3 Multiscale Image Decompositions

Given an image f , we would like to decompose f into $u+v$, where u is piecewise smooth and v is oscillatory. In [14] the following variational problem was considered,

$$\begin{aligned} \mathcal{K}(u) &= |u|_{BV} + \lambda \|f - u\|_{\dot{B}_{p,\infty}^\alpha} \\ &= |u|_{BV} + \lambda \sup_{0 < t < 1} \left\{ t^{-\alpha/2} \|K_t * (f - u)\|_{L^p} \right\} \end{aligned} \quad (32)$$

for $\alpha < 0$ and $1 \leq p \leq \infty$. Here, u is modeled as a function in BV and v is modeled as a distribution in $\dot{B}_{p,\infty}^\alpha$. The minimization problem (32) above can be rewritten as

$$\inf_u \left\{ \mathcal{K}(u) = |u|_{BV} + \bar{\lambda} \|K_{\bar{t}} * (f - u)\|_{L^p} \right\}, \quad (33)$$

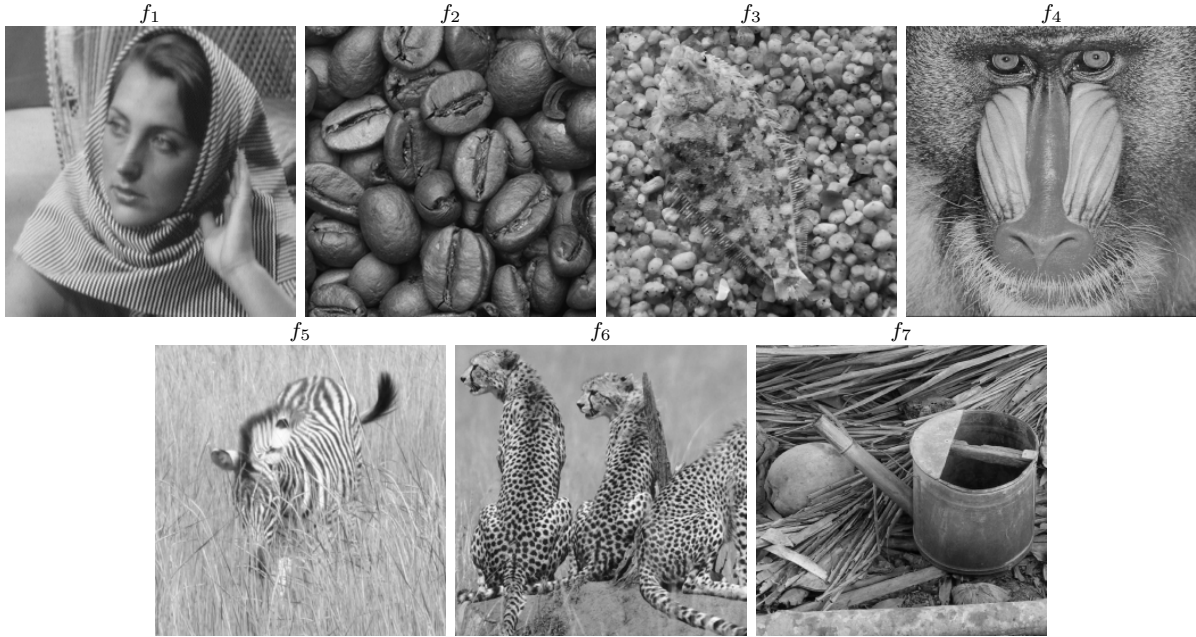


Figure 3: Test images

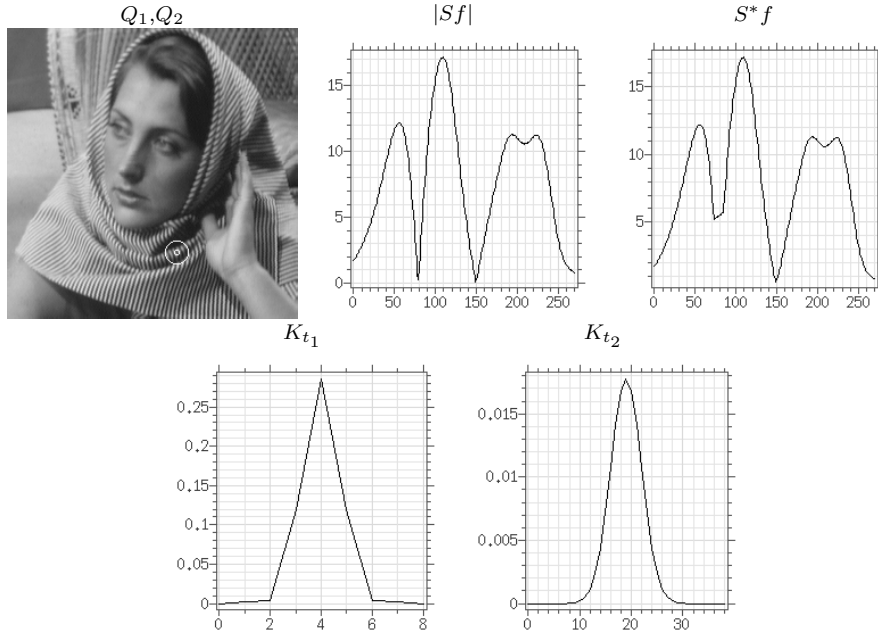


Figure 4: $|Sf|$ and S^*f are the graphs of $|Sf(x, t)|$ and $S^*f(x, t)$, respectively, with respect to $\tau = \log_a(t)$, where $f = f_1$ from figure 3, $\alpha = 0$, x is the center of the squares Q_1 and Q_2 . $K_{t_{f,1}(x)}$ and $K_{t_{f,2}(x)}$ are the heat kernels at time scales $t_{f,1}(x)$ and $t_{f,2}(x)$ with the supports discretely approximated by Q_1 and Q_2 , respectively. Here we see that $t_{f,i}^*(x) \approx t_{f,i}(x)$.

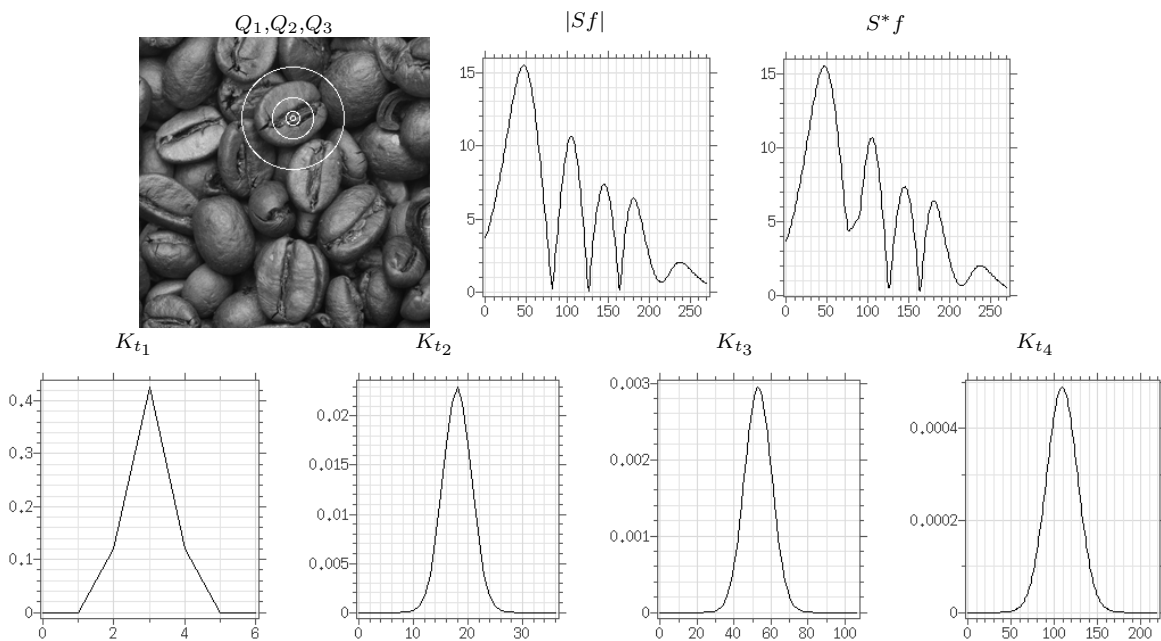


Figure 5: $|Sf|$ and S^*f are the graphs of $|Sf(x, t)|$ and $S^*f(x, t)$, respectively, with respect to $\tau = \log_a(t)$, where $f = f_2$ from figure 3, $\alpha = 0$, x is the center of the squares Q_i , $i = 1, 2, 4$. $K_{t_{f,i}(x)}$ is the heat kernel at time scales $t_{f,i}(x)$ with the support discretely approximated by Q_i , for $i = 1, 2, 3$. Here we also see that $t_{f,i}^*(x) \approx t_{f,i}(x)$.



Figure 6: The first row shows $\log_a(t_1(x))$, where $t_1(x)$ is the smallest local scale at x such that the oscillatory level $|Sf(x, t_1(x))| > \epsilon$. The second row shows $\log_a(t_1^*(x))$, where $t_1^*(x)$ is the smallest nontangential local scale at x such that the oscillatory level $S^*f(x, t_1^*(x)) > \epsilon$ for various ϵ .

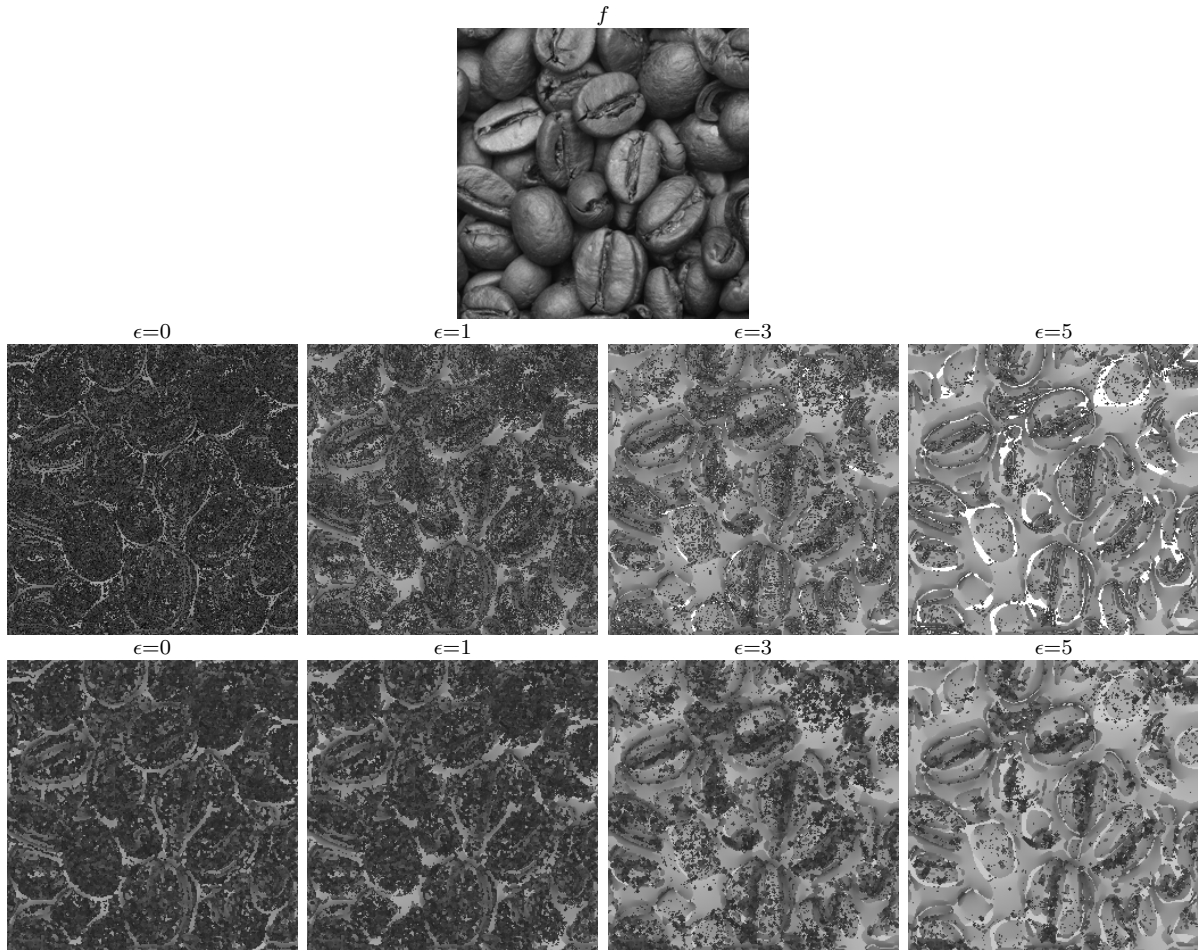


Figure 7: The first row shows $\log_a(t_1(x))$, where $t_1(x)$ is the smallest local scale at x such that the oscillatory level $|Sf(x, t_1(x))| > \epsilon$. The second row shows $\log_a(t_1^*(x))$, where $t_1^*(x)$ is the smallest nontangential local scale at x such that the oscillatory level $S^*f(x, t_1^*(x)) > \epsilon$ for various ϵ .

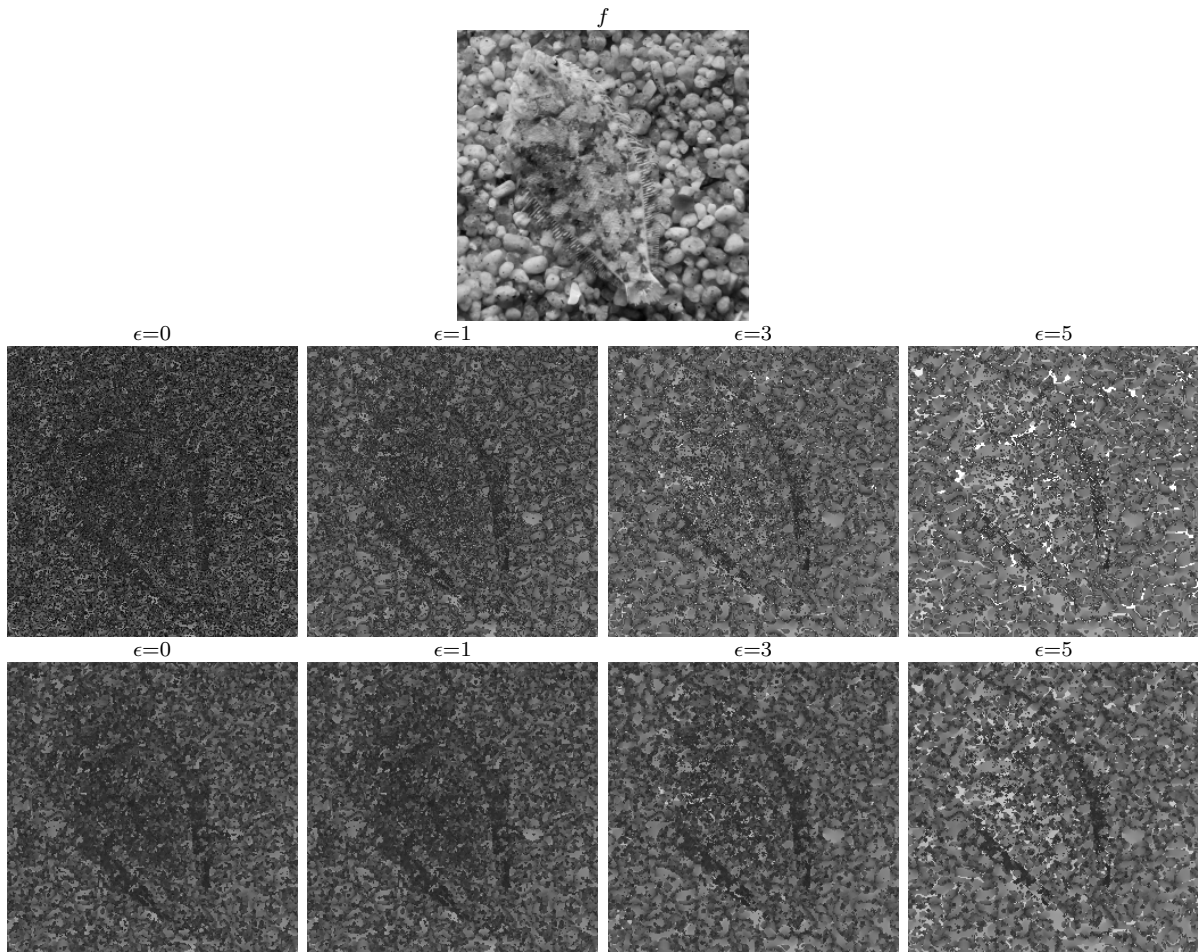


Figure 8: The first row shows $\log_a(t_1(x))$, where $t_1(x)$ is the smallest local scale at x such that the oscillatory level $|Sf(x, t_1(x))| > \epsilon$. The second row shows $\log_a(t_1^*(x))$, where $t_1^*(x)$ is the smallest nontangential local scale at x such that the oscillatory level $S^*f(x, t_1^*(x)) > \epsilon$ for various ϵ .

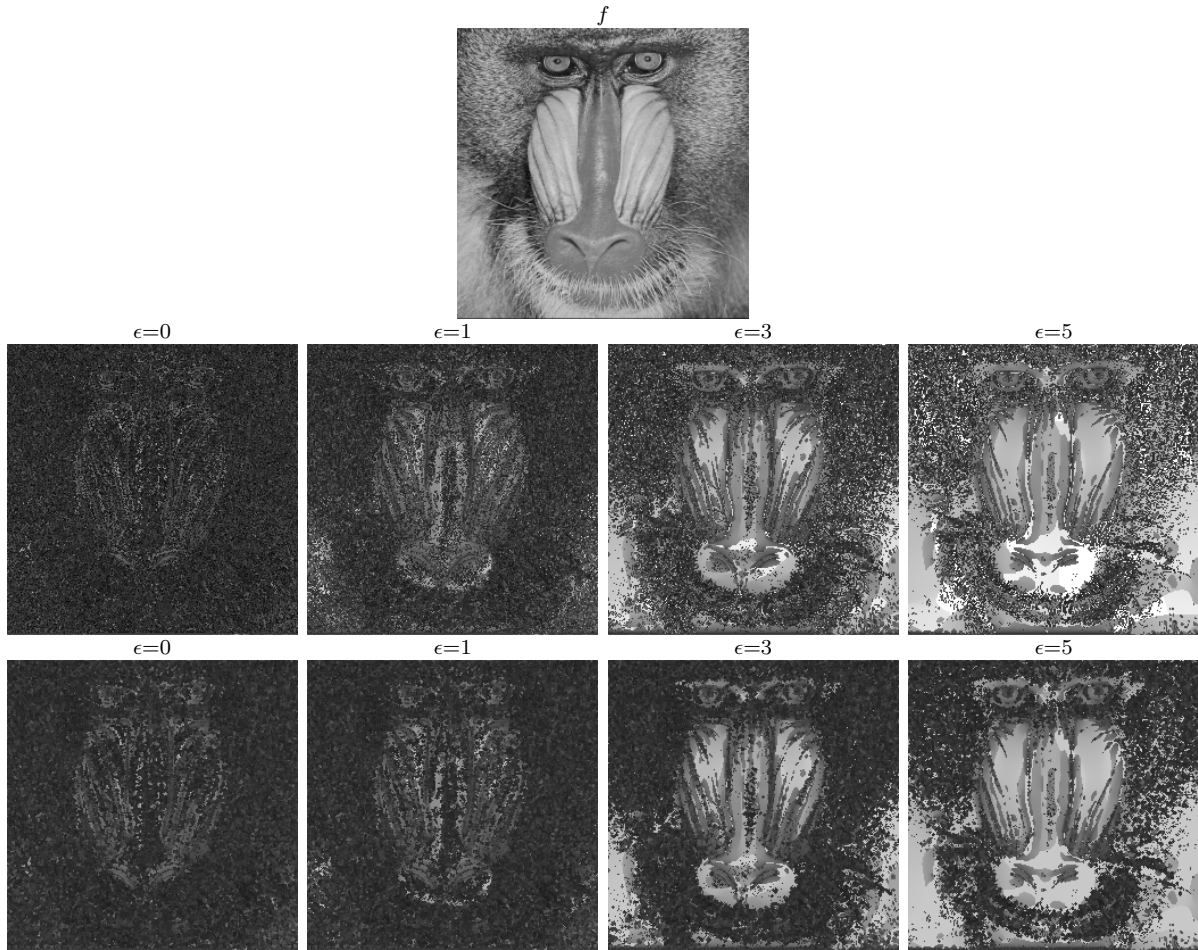


Figure 9: The first row shows $\log_a(t_1(x))$, where $t_1(x)$ is the smallest local scale at x such that the oscillatory level $|Sf(x, t_1(x))| > \epsilon$. The second row shows $\log_a(t_1^*(x))$, where $t_1^*(x)$ is the smallest nontangential local scale at x such that the oscillatory level $S^*f(x, t_1^*(x)) > \epsilon$ for various ϵ .

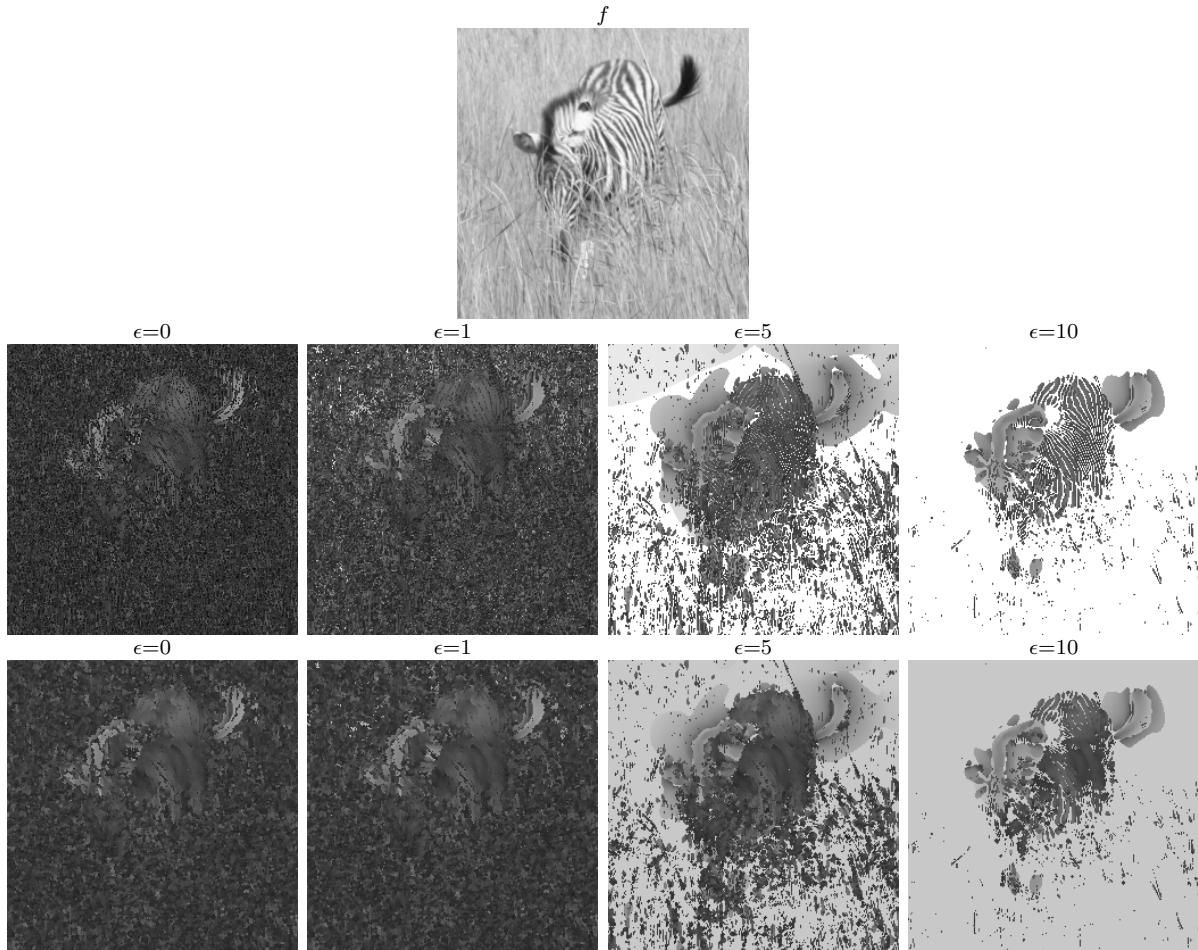


Figure 10: The first row shows $\log_a(t_1(x))$, where $t_1(x)$ is the smallest local scale at x such that the oscillatory level $|Sf(x, t_1(x))| > \epsilon$. The second row shows $\log_a(t_1^*(x))$, where $t_1^*(x)$ is the smallest nontangential local scale at x such that the oscillatory level $S^*f(x, t_1^*(x)) > \epsilon$ for various ϵ .

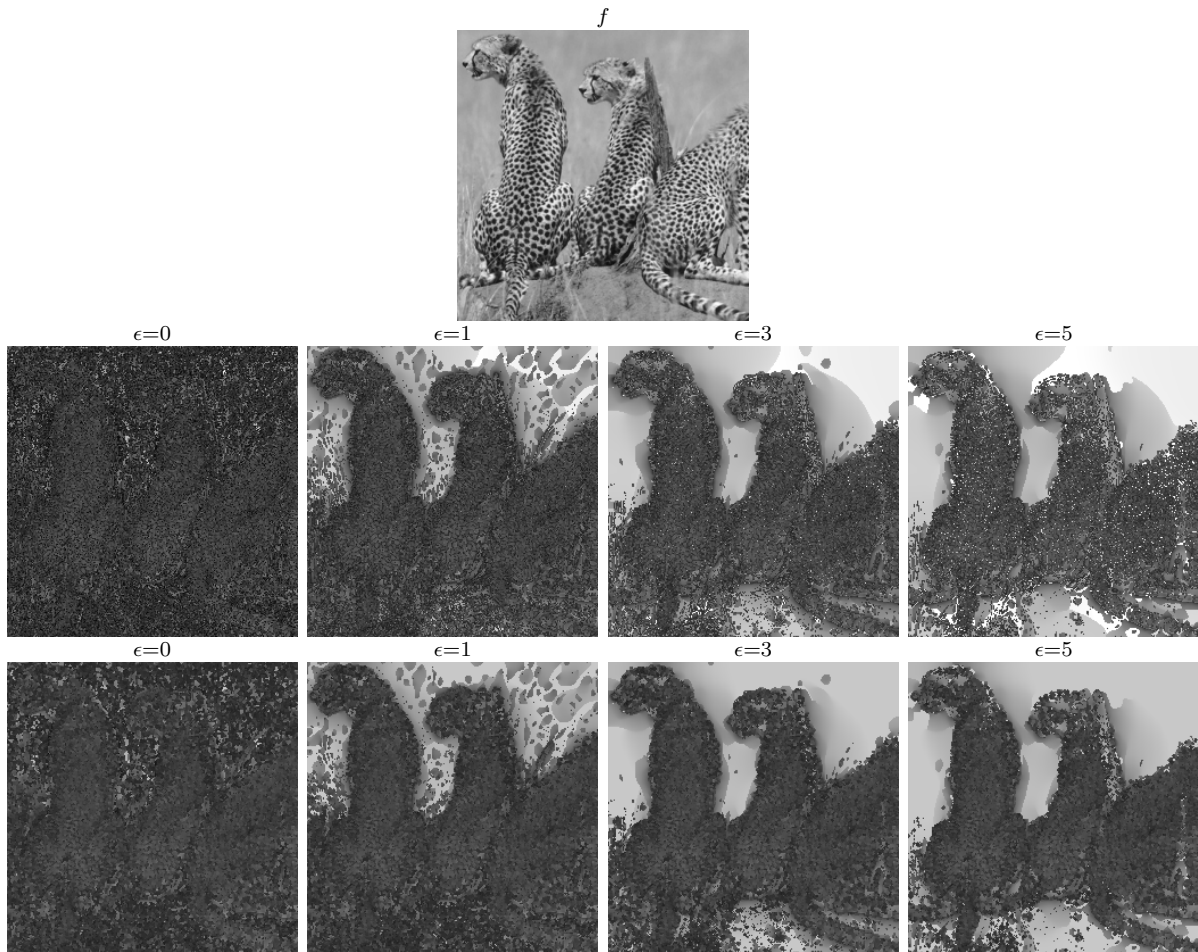


Figure 11: The first row shows $\log_a(t_1(x))$, where $t_1(x)$ is the smallest local scale at x such that the oscillatory level $|Sf(x, t_1(x))| > \epsilon$. The second row shows $\log_a(t_1^*(x))$, where $t_1^*(x)$ is the smallest nontangential local scale at x such that the oscillatory level $S^*f(x, t_1^*(x)) > \epsilon$ for various ϵ .

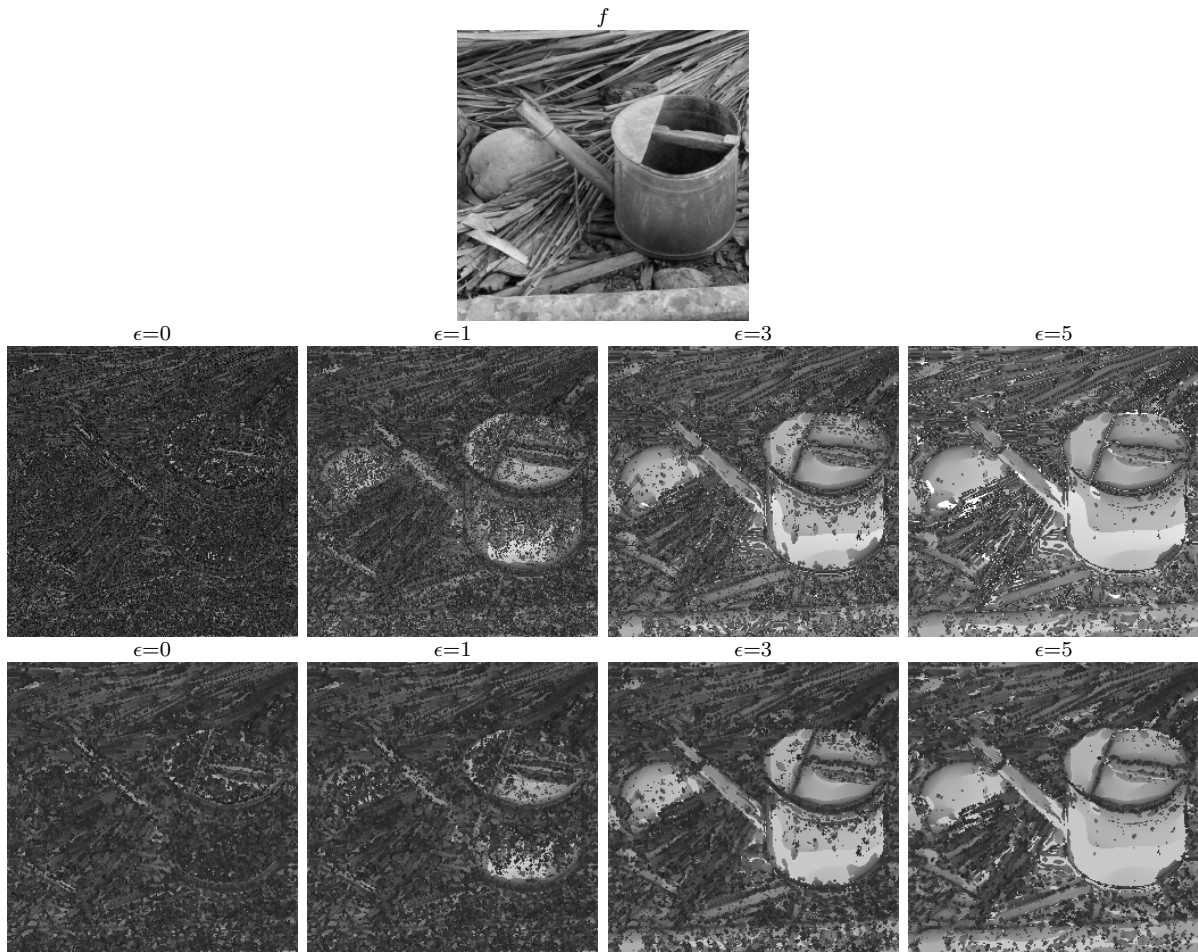


Figure 12: The first row shows $\log_a(t_1(x))$, where $t_1(x)$ is the smallest local scale at x such that the oscillatory level $|Sf(x, t_1(x))| > \epsilon$. The second row shows $\log_a(t_1^*(x))$, where $t_1^*(x)$ is the smallest nontangential local scale at x such that the oscillatory level $S^*f(x, t_1^*(x)) > \epsilon$ for various ϵ .

for some $\bar{t} = \bar{t}(\alpha, f - u) \geq 0$ and $\bar{\lambda} = \lambda \bar{t}^{-\alpha/2}$. The term $\|K_{\bar{t}} * v\|_{L^p}$ from the equation (33) imposes a uniform smoothing of scale \bar{t} on the oscillatory component v . However, oscillatory components may have different scales of oscillations locally. Therefore, a natural extension of (33) is to consider the following minimization problem,

$$\inf_u \{ \mathcal{K}(u) = |u|_{BV} + \lambda \|K_{\bar{t}(\cdot)} * (f - u)(\cdot)\|_{L^p} \}, \quad (34)$$

where now $\bar{t}(x)$ is a preferred scale at x , and $\lambda > 0$ is a constant.

Recall that each point x in the image f may exhibit multiple scales, and each choice of $\bar{t}(x)$ arises to a different decomposition. One obvious choice for $\bar{t}(x)$ is the first local scale $t_{f,1}(x)$ or nontangential local scale $t_{f,1}^*(x)$, but other choices can be used depending on applications. We will consider two cases for the choice of $\bar{t}(x)$:

1. $\bar{t}(x)$ depends on f .
2. $\bar{t}(x)$ depends on the unknown $v = f - u$.

Remark 4. For each $\bar{t}(x) \geq 0$, denote $Kf = K_{\bar{t}} * f$. Then K is linear and bounded on $L^p(\mathbb{R}^n)$, $1 < p \leq \infty$. We note that $|Kf(x)| \leq Mf(x)$, where Mf is the Hardy-Littlewood maximal function defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

and Theorem 1.3.1 of A. Stein [35] shows that Mf is bounded on $L^p(\mathbb{R}^n)$, $1 < p \leq \infty$. In other words, $\|Mf\|_{L^p(\mathbb{R}^n)} \leq A_p \|f\|_{L^p(\mathbb{R}^n)}$, for $1 < p \leq \infty$. Therefore, $\|Kf\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}$, for $1 < p \leq \infty$.

Next we would like to show existence for (34) with $p = 1$, $\bar{t}(x)$ is chosen a priori and $\Omega \subset \mathbb{R}^2$ is bounded.

Theorem 3. *Let $f \in L^\infty(\Omega)$, $\Omega = [0, 1]^2 \subset \mathbb{R}^2$. Note that $L^\infty(\Omega) \subset L^p(\Omega)$, for all $p \geq 1$. Fixing $\lambda > 0$ and $\bar{t}(x) : \Omega \rightarrow [0, 1]$, and denote $Kf(x) = K_{\bar{t}(x)} * f(x)$. Then there exists a minimizer for the variational problem,*

$$\inf_{u \in BV(\Omega)} \{ \mathcal{K}(u) = |u|_{BV(\Omega)} + \lambda \|K(f - u)\|_{L^1(\Omega)} \}. \quad (35)$$

Proof. We note that this is a standard proof of existence in variational calculus. Let $\{u_n\}$ be a minimizing sequence for (35). We have,

$$|u_n|_{BV(\Omega)} \leq C \quad (36)$$

$$\|K(f - u_n)\|_{L^1(\Omega)} \leq C. \quad (37)$$

First, we make a claim that conditions (36) and (37) imply $u_{n,\Omega} = \frac{1}{|\Omega|} \int_\Omega u_n dx$ is uniformly bounded. Now, assume this is true, by the Poincaré inequality,

$$\|u_n - u_{n,\Omega}\|_{L^p(\Omega)} \leq C_p |u_n|_{BV(\Omega)}, \text{ for all } 1 \leq p \leq 2, \quad (38)$$

where C_p only depends on Ω . By (36) and the uniform boundedness of $u_{n,\Omega}$, we obtain that u_n is uniformly bounded in $L^1(\Omega)$. Therefore u_n is uniformly bounded in $BV(\Omega)$.

This implies that there exists a $u \in BV(\Omega)$ such that, up to a subsequence, u_n converges weak* to u in $BV(\Omega)$ and strongly in $L^1(\Omega)$. We have

$$|u|_{BV(\Omega)} \leq \liminf_{n \rightarrow \infty} |u_n|_{BV(\Omega)}, \text{ and} \quad (39)$$

$$\|u\|_{L^1(\Omega)} \leq \liminf_{n \rightarrow \infty} \|u_n\|_{L^1(\Omega)}. \quad (40)$$

We also have $u_n \rightarrow u$ in $L^p(\Omega)$ since $BV(\Omega)$ is compactly embedded in $L^p(\Omega)$ for $1 \leq p < 2$. The boundedness of K on $L^p(\Omega)$, $p > 1$, implies that upto a subsequence, for a p such that $1 < p < 2$,

$$\|Ku_n - Ku\|_{L^1(\Omega)} \leq |\Omega|^{1-1/p} \|Ku_n - Ku\|_{L^p(\Omega)} \leq C_p \|u_n - u\|_{L^p(\Omega)} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Therefore, $\|Ku\|_{L^1(\Omega)} \leq \liminf_{n \rightarrow \infty} \|Ku_n\|_{L^1(\Omega)}$. Similarly,

$$\|K(f - u)\|_{L^1(\Omega)} \leq \liminf_{n \rightarrow \infty} \|K(f - u_n)\|_{L^1(\Omega)}. \quad (41)$$

By (39) and (41), $\mathcal{K}(u) \leq \mathcal{K}(u_n)$. Therefore, $u \in BV(\Omega)$ is a minimizer, and $u_\Omega = f_\Omega$.

To prove the claim: Let $w_n = u_n - u_{n,\Omega}$, then $|w_n|_{BV(\Omega)} = |u_n|_{BV(\Omega)}$, and $w_{n,\Omega} = 0$, for all n . By Poincare's inequality,

$$\|w_n\|_{L^p(\Omega)} = \|w_n - w_{n,\Omega}\|_{L^p(\Omega)} \leq C_p |w_n|_{BV(\Omega)} \leq C, \text{ for } 1 \leq p \leq 2.$$

We have,

$$\begin{aligned} C &\geq \|K(f - u_n)\|_{L^1(\Omega)} = \|K(f - w_n) - Ku_{n,\Omega}\|_{L^1(\Omega)} \\ &\geq \left| \|K(f - w_n)\|_{L^1(\Omega)} - \|Ku_{n,\Omega}\|_{L^1(\Omega)} \right| \end{aligned} \quad (42)$$

But $\|K(f - w_n)\|_{L^1(\Omega)}$ is uniformly bounded since, for any $1 < p \leq 2$,

$$\|K(f - w_n)\|_{L^1(\Omega)} \leq |\Omega|^{1-1/p} \|K(f - w_n)\|_{L^p(\Omega)} \leq A_p \|f - w_n\|_{L^p(\Omega)} \leq C.$$

Therefore, $|u_{n,\Omega}|$ is also uniformly bounded. \square

In general, when $1 < p \leq \frac{n}{n-1}$, $f \in L^p$ and K defined as above, we still obtain existence of a minimizer for the following variational problem,

$$\inf_{u \in BV} \{ \mathcal{K}(u) = |u|_{BV} + \lambda \|K(f - u)\|_{L^p} \}. \quad (43)$$

Next we would like to consider the variational model where the local scales depend on the oscillatory component v . For each $f \in L^\infty(\Omega)$, $0 < s \leq 1$, and $\epsilon > 0$, consider the variational problem,

$$\inf_{u \in BV(\Omega)} \left\{ \mathcal{K} = |u|_{BV(\Omega)} + \lambda \|K_{\bar{t}(\cdot)} * v(\cdot)\|_{L^1(\Omega)}, v = f - u \right\}, \quad (44)$$

where $\tau = \log_a(t)$, $\alpha = 0$, and

$$\bar{t}(x) = \begin{cases} \arg \max \{ |Sv(x, t)|, 0 \leq t \leq s, t \in T_v(x), |Sv(x, \tau)| > \epsilon \}, & \text{if exists} \\ 0, & \text{otherwise.} \end{cases} \quad (45)$$

or we may consider

$$\bar{t}(x) = \begin{cases} \arg \max \{S^*v(x, t), 0 \leq t \leq s, t \in T_v^*(x), S^*v(x, \tau) > \epsilon\}, & \text{if exists} \\ 0, & \text{otherwise.} \end{cases} \quad (46)$$

For short notation, we write conditions (45) and (46) as

$$\bar{t}(x) = \arg \max S(v, s, \epsilon), \text{ and } \bar{t}(x) = \arg \max S^*(v, s, \epsilon), \text{ respectively.} \quad (47)$$

3.1 Numerical Computations and Results

In this section, we show numerical results for texture decompositions. In [16] and [19], the authors statistically showed that texture-like natural images when being convolved with compact averaging kernels of zero mean have Laplacian probability distributions. For image decomposition, the texture part v on a suitable compact set has zero mean. Therefore, we consider the model (35) with $p = 1$. Recall the functional in (35),

$$\mathcal{K}(u) = |u|_{BV(\Omega)} + \lambda \|K(f - u)\|_{L^1(\Omega)} = \int_{\Omega} |\nabla u| \, dx + \int_{\Omega} |K(f - u)| \, dx,$$

where $Kf = K_{\bar{t}(x)} * f$ and $\bar{t}(x)$ is fixed. Minimizing $\mathcal{K}(u)$ with respect to u , we obtain the following differential equation

$$\frac{\partial \mathcal{K}}{\partial u} = -\operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) - \lambda K^* \left[\frac{K(f - u)}{|K(f - u)|} \right], \quad (48)$$

where K^* is the adjoint operator of K . Using gradient descend method, we then solve the following PDE,

$$\frac{\partial u}{\partial \tau} = -\frac{\partial \mathcal{K}}{\partial u} = \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) + \lambda K^* \left[\frac{K(f - u)}{|K(f - u)|} \right]. \quad (49)$$

Note that $Kf(x) = K_{\bar{t}(x)} * f(x)$ is the convolution with the scale t locally depends on x . However, $K^*f(x)$ is not the pointwise convolution. We have

$$\begin{aligned} (K^*f, u) &= (f, Ku) = \int f(y)Ku(y) \, dy = \int f(y) \int K_{\bar{t}(y)}(y - z)u(z) \, dz \, dy \\ &= \int \left(\int f(y)K_{\bar{t}(y)}(y - z) \, dy \right) u(z) \, dz. \end{aligned}$$

Therefore, K^* is defined as $K^*f(z) = \int f(y)K_{\bar{t}(y)}(y - z) \, dy$. The problem (44) is computed in a similar fashion, but K is updated at each iteration. Denote $K_n f(x) = K_{\bar{t}_n(x)} * f(x)$. The algorithm is as follows:

1. Fix a $\lambda > 0$ and $\alpha < 0$. Let $u_0 = f$ and the local scales \bar{t}_0 for K_0 is initialized to be the first local scales of f . I.e. $\bar{t}_0(x) = \arg \max S(f, s, \epsilon)$ (or $\bar{t}_0(x) = \arg \max S^*(v, s, \epsilon)$), (see (47)).

2. Assume that we have obtained u_n and K_n , u_{n+1} is computed as follow

$$u_{n+1} = u_n + \Delta\tau \left(\operatorname{div} \left(\frac{\nabla u_{n+1}}{|\nabla u_n|} \right) + \lambda K_n^* \left[\frac{K_n(f - u_n)}{|K_n(f - u_n)|} \right] \right),$$

where $\Delta\tau$ is an artificial time stepping for the gradient descend method. K_{n+1} is then computed by (45), over the discrete set of scales S as defined in (31), using $v_{n+1} = f - u_{n+1}$. I.e. $\bar{t}_{n+1}(x) = \arg \max S(v_{n+1}, s, \epsilon)$ (or $\bar{t}_{n+1}(x) = \arg \max S^*(v_{n+1}, s, \epsilon)$), (see 47). Here we use $\Delta\tau = 1$, and the discrete spatial grid size $\Delta x = 1$.

Remark 5. We do not claim that a minimizer for the variational problem (44)-(46) exists. Our scheme to solve this problem is an iterative scheme (as described above). Numerically, with the above initial values for u_0 and \bar{t}_0 , we obtain that the energy $\mathcal{K}(u_n) = |u_n|_{BV} + \lambda \|K_n * (f - u_n)\|_{L^1}$ decreases (as shown in figure 15) and converges to a local minimum.

Figures 13-14 show the decompositions using the variational model (34) with $\lambda = 2$ and $\alpha = 0$ for different choices of $\bar{t}(x)$. For computing the local scales, we use the set S as defined in (31).

Figures 15-19 show the decompositions using the variational model (44) with $\lambda = 2$ and $\alpha = 0$. The choices for $\bar{t}(x)$, which is updated at each iteration, are discussed in each figure.

4 Discussions

4.1 Distinguishing Different Orientations of Oscillations

Definition 3 and 4 define the local scales in terms of the heat kernel K_t which is isotropic. We remark that the knowledge of the local scales and their oscillatory levels at each point can be used for segmenting different scales of texture as can be seen from figures 6-12. However, isotropic kernels do not distinguish different orientations of texture. To further capture the orientations, one can consider nonisotropic kernels or the following idea suggested by R. Coifman: Suppose $\alpha < 2$ and $f \in \dot{F}_{p,\infty}^\alpha$. Recall that $Sf(x, t)$ is defined as

$$Sf(x, t) = t^{1-\alpha/2} \frac{\partial K_t}{\partial t} * f(x) = t^{1-\alpha/2} [\Delta K_t * f(x)],$$

where $x = (x_1, \dots, x_n)$ and $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$. Suppose $t_i(x) = t_{f,i}(x)$ is an i^{th} local scale of f at x . The orientation of the oscillation of f at the local scale $t_i(x)$ can be obtained from the principal components of the n-vector

$$(t_i(x))^{1-\alpha/2} \left(\frac{\partial^2}{\partial x_1^2} K_{t_i(x)} * f(x), \dots, \frac{\partial^2}{\partial x_n^2} K_{t_i(x)} * f(x) \right).$$

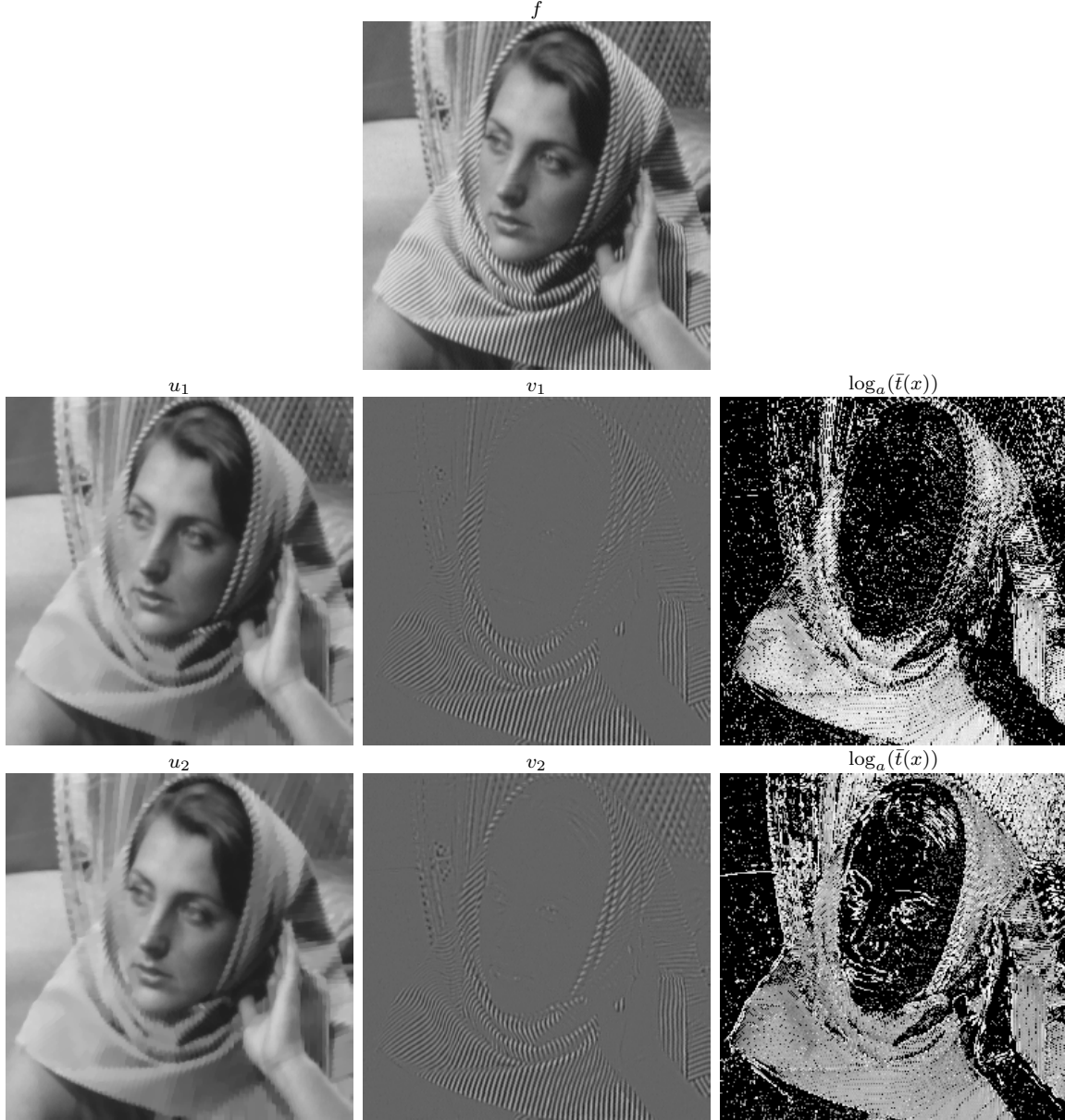


Figure 13: The decompositions, using (34), of f into: 1) $u_1 + v_1$ with the local scales $\bar{t}(x) = t_1(x)$ if $t_1(x) < a^{-220}$ and $\bar{t}(x) = 0$ otherwise, 2) $u_2 + v_2$ with the local scales $\bar{t}(x) = t_1(x)$ if $t_1(x) < a^{-200}$ and $\bar{t}(x) = 0$ otherwise. In both cases $t_1(x)$ is the first local scale of f at x such that the oscillatory level $|Sf(x, t_1)| > 1$.

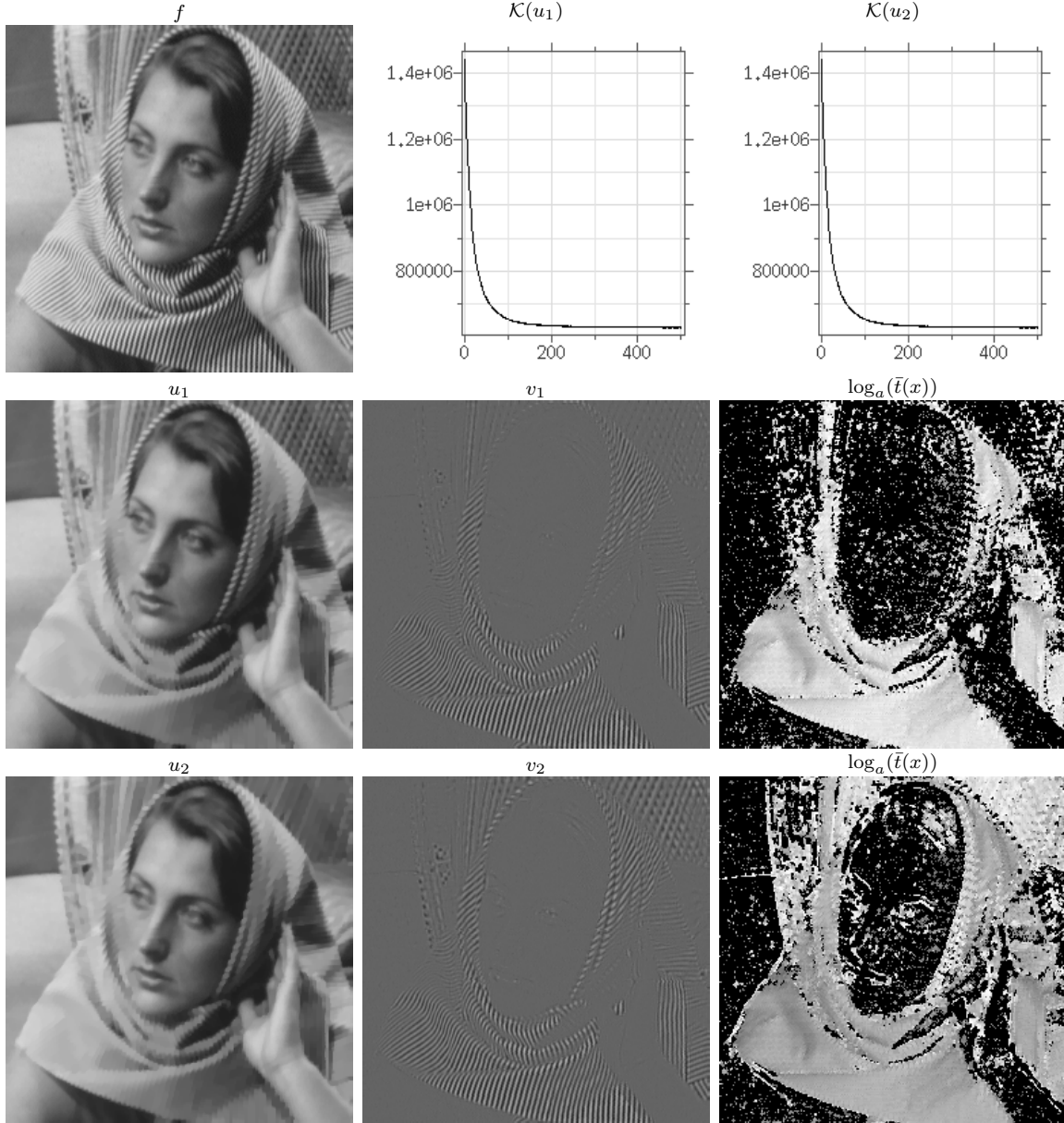


Figure 14: The decompositions, using (34), of f into: 1) $u_1 + v_1$ with the local scales $\bar{t}(x) = t_1(x)$ if $t_1(x) < a^{-220}$ and $\bar{t}(x) = 0$ otherwise, 2) $u_2 + v_2$ with the local scales $\bar{t}(x) = t_1(x)$ if $t_1(x) < a^{-200}$ and $\bar{t}(x) = 0$ otherwise. In both cases $t_1(x)$ is the first local scale of f at x such that the oscillatory level $S^*f(x, t_1) > 1$. $\mathcal{K}(u_i)$ shows the evolution of the energy corresponds to the decomposition $f = u_i + v_i$, $i = 1, 2$.

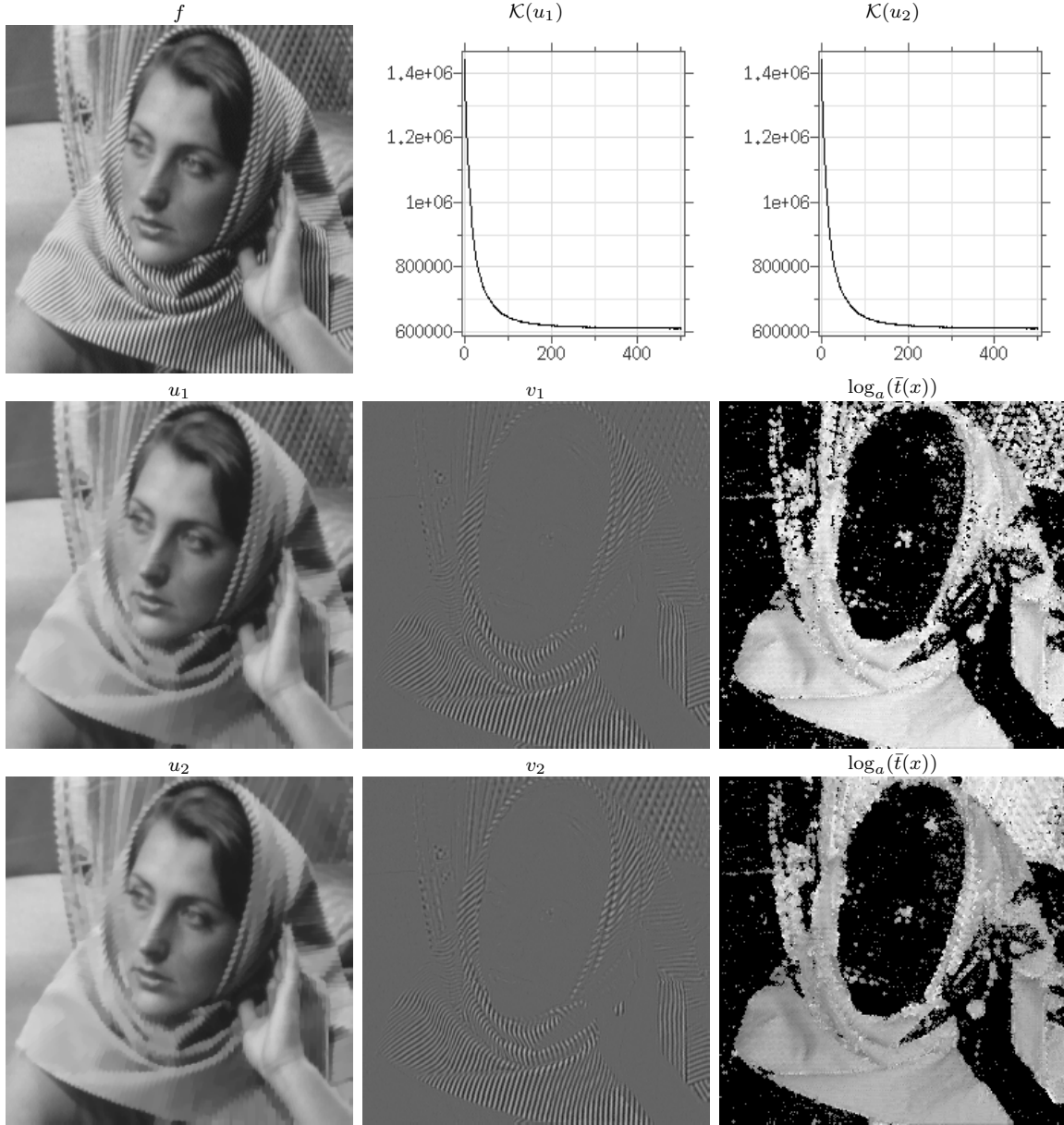


Figure 15: The decomposition, using (44), of f into: 1) $u_1 + v_1$ with $\bar{t}(x)$ defined as (46), with $s = a^{-220}$ and $\epsilon = 1$; 2) $u_2 + v_2$ with $\bar{t}(x)$ defined as (46), with $s = a^{-200}$ and $\epsilon = 1$. $\mathcal{K}(u_i)$ shows the evolution of the energy corresponds to the decomposition $f = u_i + v_i$, $i = 1, 2$.

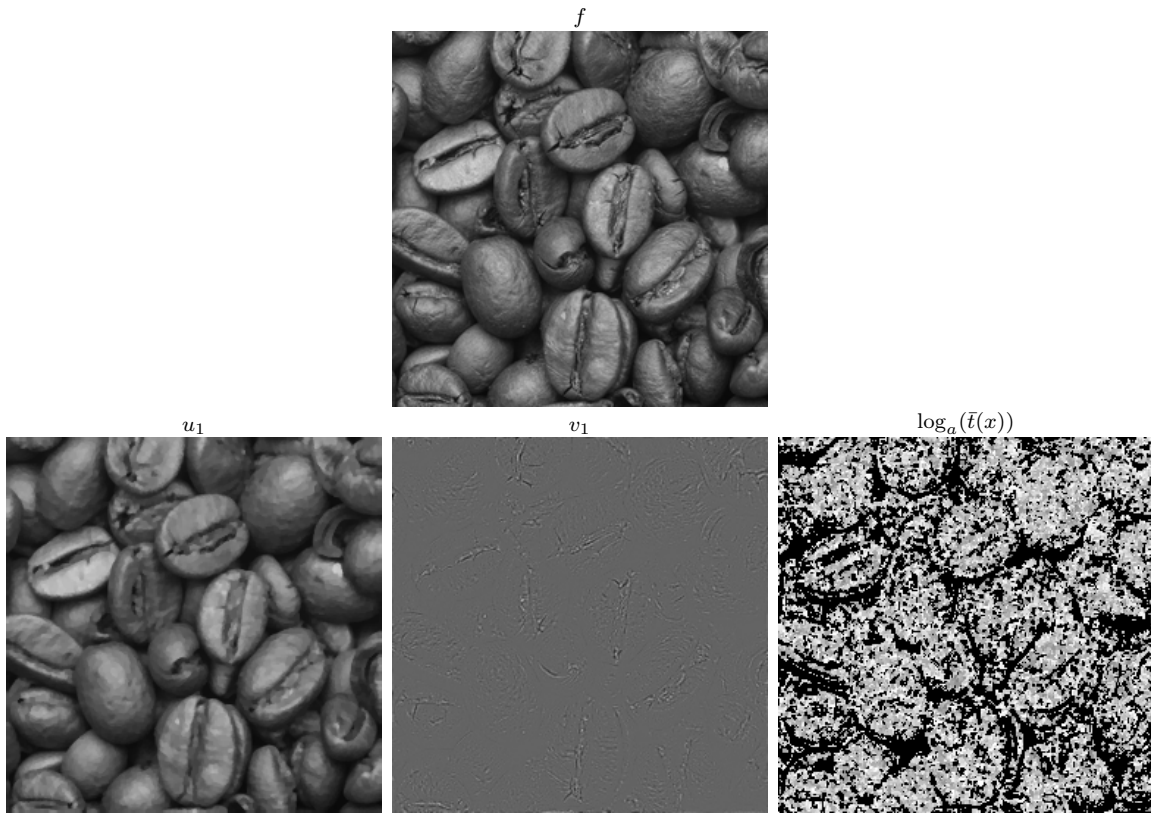


Figure 16: The decomposition, using (34), of f into $u_1 + v_1$ with the local scales $\bar{t}(x) = t_1(x)$ if $t_1(x) < a^{-220}$ and $\bar{t}(x) = 0$ otherwise. Here, $t_1(x)$ is the first local scale of f at x such that the oscillatory level $S^*f(x, t_1) > 1$.

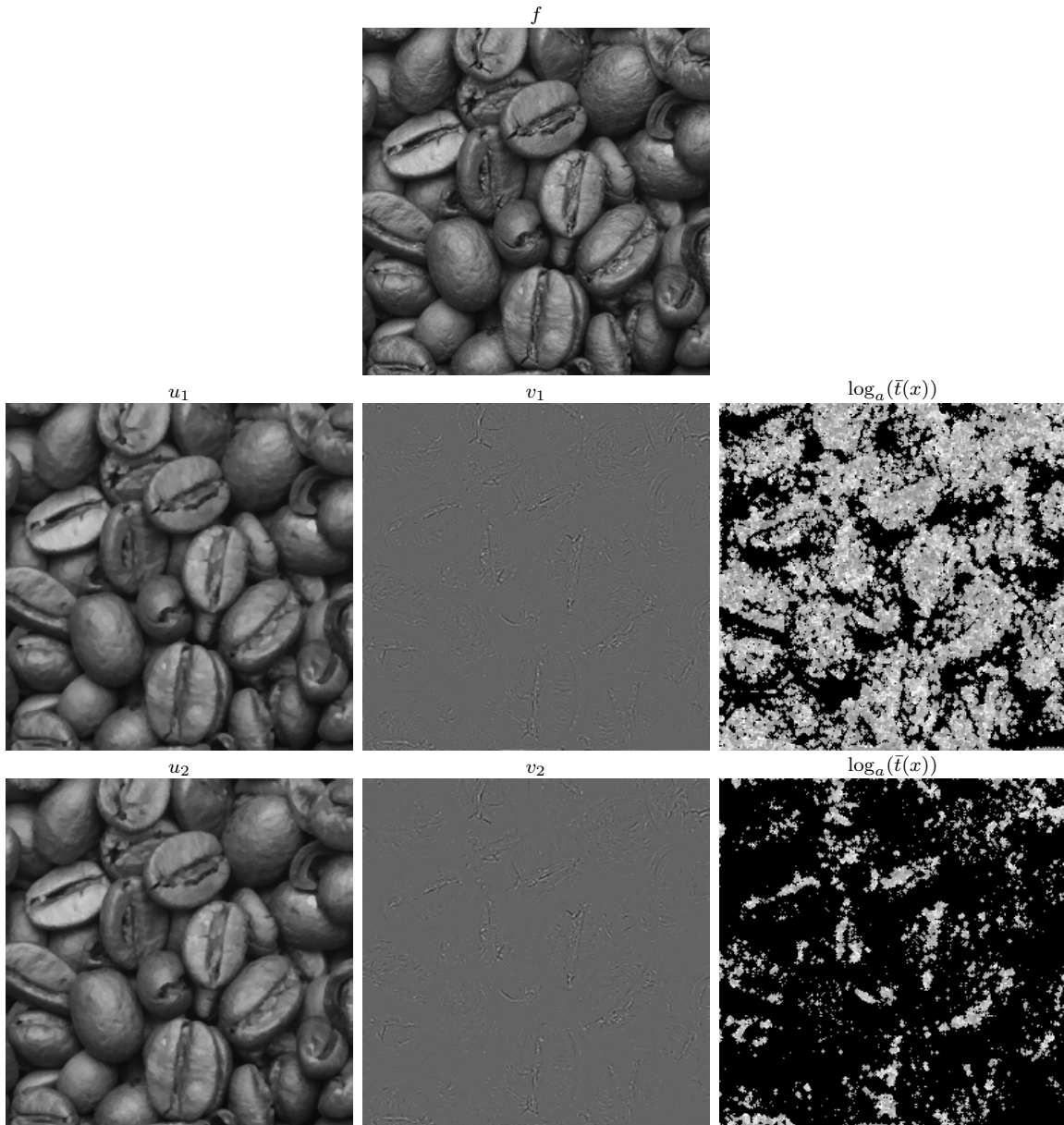


Figure 17: The decomposition, using (44), of f into: 1) $u_1 + v_1$ with $\tilde{t}(x)$ defined as (46), $s = a^{-220}$ and $\epsilon = 1$; 2) $u_2 + v_2$ with $\tilde{t}(x)$ defined as (46), $s = a^{-220}$ and $\epsilon = 3$.

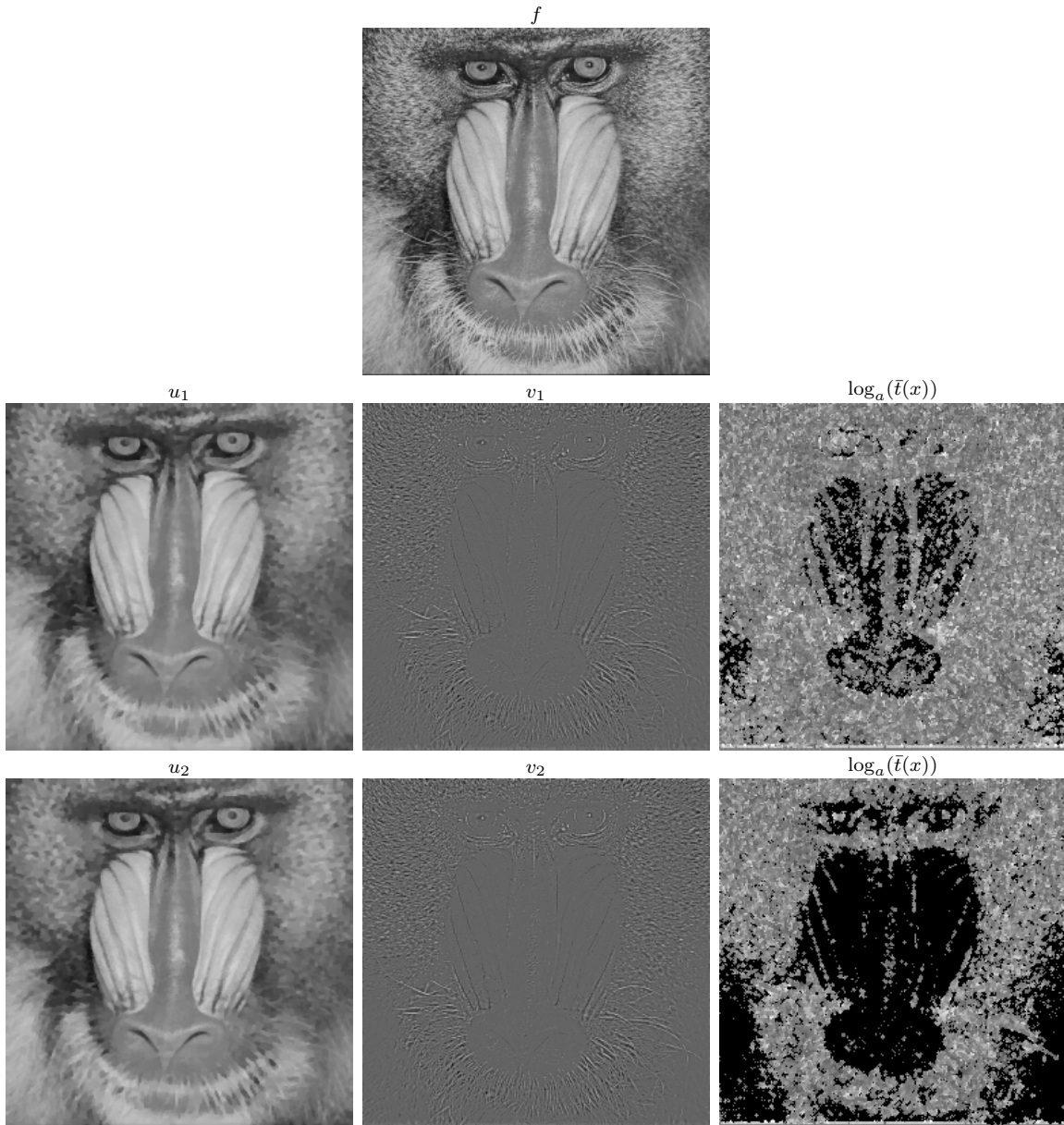


Figure 18: The decomposition, using (44), of f into: 1) $u_1 + v_1$ with $\bar{t}(x)$ defined as (46), $s = a^{-200}$ and $\epsilon = 1$; 2) $u_2 + v_2$ with $\bar{t}(x)$ defined as (46), $s = a^{-200}$ and $\epsilon = 3$.

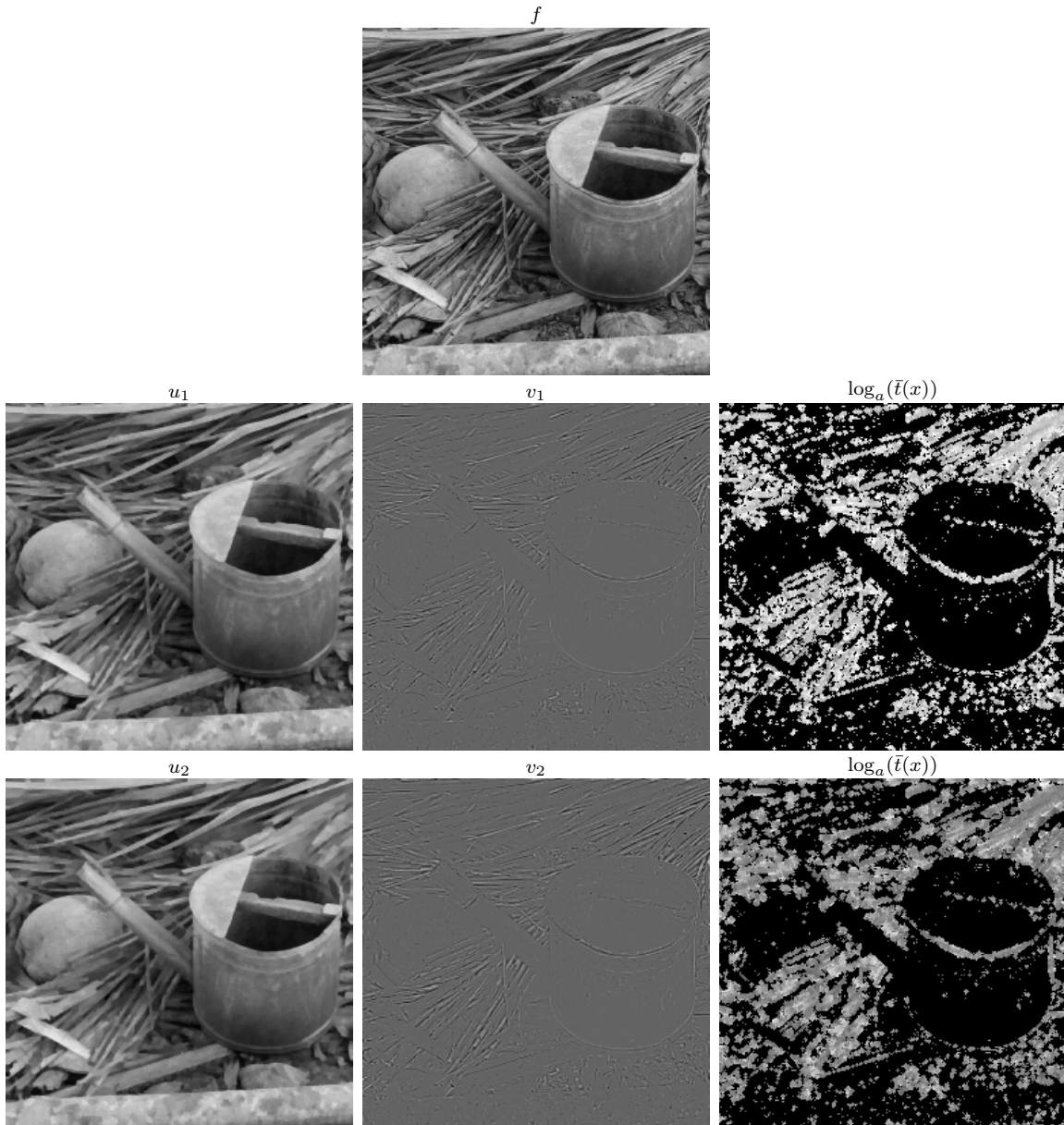


Figure 19: The decomposition, using (44), of f into: 1) $u_1 + v_1$ with $\tilde{t}(x)$ defined as (46), $s = a^{-230}$ and $\epsilon = 3$; 2) $u_2 + v_2$ with $\tilde{t}(x)$ defined as (46), $s = a^{-200}$ and $\epsilon = 3$.

4.2 Other Interpretations of Local Scales

In this paper, we study the local scales which are governed by the linear isotropic equation,

$$\frac{\partial u}{\partial t} = \Delta u, u(x, 0) = f(x),$$

which has the solution $u(x, t) = K_t * f(x)$ for $t > 0$, where K_t is the heat kernel. For $\alpha < 2$, let $s = 1 - \alpha/2$. Rewrite $Sf(x, t)$ as $S_s f(x, t)$, we have

$$S_s f(x, t) = t^s \frac{\partial u}{\partial t}(x, t) = t^s \Delta u(x, t).$$

One can consider different evolution equations to study local scales. One example is using the p-Laplacian. More specifically, for $p > 0$, consider the evolution equation

$$\frac{\partial u}{\partial t} = \Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u), u(x, 0) = f(x), \quad (50)$$

Now, for $p > 0$, define $S_{s,p}f(x, t)$ and its nontangential version $S_{s,p}^*f(x, t)$ as

$$\begin{aligned} S_{s,p}f(x, t) &= t^s \frac{\partial u}{\partial t}(x, t) = t^s \Delta_p u(x, t), \\ S_{s,p}^*f(x, t) &= \sup_{|x-y|<t} \left| S_{s,p}f(y, t) e^{-\frac{|x-y|^2}{2\sigma}} \right|. \end{aligned}$$

The local scales corresponding to the evolution equation (50) can be defined as the local maximums of $|S_{s,p}f(x, t)|$ or $S_{s,p}^*f(x, t)$ for different values of s and p .

In [7], T. Brox and J. Weickert considered a different scale interpretation for the case $p = 1$, where the local scale $m(x)$ is defined as, for some choice of T ,

$$\frac{1}{m(x)} = \frac{1}{2} \frac{\int_0^T |\partial_t u(x, t)| dt}{\int_0^T (1 - \delta_{\partial_t u(x, t), 0}) dt}, \quad (51)$$

where $\partial_t u(x, t) = \Delta_1 = \operatorname{div} \left(\frac{\nabla u(x, t)}{|\nabla u(x, t)|} \right)$ with $u(x, 0) = f(x)$, and $\delta_{a,b} = 1$ if $a = b$ and 0 otherwise. From the above equation, we see that the average of $\partial_t u(x, t)$ upto time T is used to define the local scale at x . In other words,

$$\frac{1}{m(x)} = \frac{1}{2} \frac{\int_0^T |S_{0,1}f(x, t)| dt}{\int_0^T (1 - \delta_{\partial_t u(x, t), 0}) dt}.$$

We refer the readers to [7] and [38] and references there in for further analysis of scales in images.

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