Modeling oscillatory components with the homogeneous spaces $B\dot{M}O^{-\alpha}$ and $\dot{W}^{-\alpha,p}$ *

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Abstract

This paper is devoted to the decomposition of an image f into $u + v$, with u a piecewise-smooth or "cartoon" component, and v an oscillatory component (texture or noise), in a variational approach. The cartoon component u is modeled by a function of bounded variation, while v, usually represented by a square integrable function, is now being modeled by a more refined and weaker texture norm, as a distribution. Generalizing the idea of Y. Meyer [32], where $v \in F = \mathrm{div}(BMO) = B\dot{M}O^{-1}$, we model here the texture component by the action of the Riesz potentials on v that belongs to BMO or to L^p. In an earlier work [26], the authors proposed energy minimization models to approximate (BV, F) decompositions explicitely expressing the texture as divergence of vector fields in BMO. In this paper, we consider an equivalent more isotropic norm of the space F in terms of the Riesz potentials, and study models where the Riesz potentials of oscillatory components belong to BMO or to L^p, $1 \leq p < \infty$ (thus we consider oscillatory components in $B\dot{M}O^{\alpha}$ or in $\dot{W}^{\alpha,p}$, with $\alpha < 0$). Theoretical, experimental results and comparisons to validate the proposed methods are presented.

1 Introduction and motivations

We assume that a given grayscale image f is defined on $\mathbb{R}^n$ or $\Omega = [0,1]^n \subset \mathbb{R}^n$. When f is defined on Ω, we assume that f is periodic and Ω is the fundamental domain. Denote

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the function space defined on $\mathbb{R}^n$ by $X(\mathbb{R}^n)$ and the function space defined on Ω by $X(\Omega)$. When we make no distinction between $X(\mathbb{R}^n)$ or $X(\Omega)$, we write X to mean either $X(\mathbb{R}^n)$ or $X(\Omega)$.

An important problem in image analysis is the decomposition of f into $u+v$, where u is piecewise-smooth containing the geometric components of f and v is oscillatory, typically texture or noise. A general variational method for decomposing $f \in X_1 + X_2$ into $u+v$, with $u \in X_1$ and $v \in X_2$, can be defined by the minimization problem

$$\inf_{(u,v)\in X_1\times X_2} \left\{F_1(u) + \lambda F_2(v) : f = u+v\right\}, \tag{1}$$

where $F_1, F_2 \geq 0$ are functionals and X_1, X_2 are spaces of functions or distributions such that $F_1(u) < \infty$, and $F_2(v) < \infty$, if and only if $(u,v) \in X_1 \times X_2$. The constant $\lambda > 0$ is a tuning parameter. A good model for (1) is given by a choice of X_1 and X_2 so that with the given desired properties of u and v, we obtain $F_1(u) << F_2(u)$ and $F_1(v) >> F_2(v)$.

In more standard or canonical approaches, the space L^2 is used to model v when f denotes the image of a real scene, u is a piecewise-smooth approximation of f (made up of homogeneous regions with sharp boundaries), and v is a residual (additive Gaussian noise or small details). For example, in the Mumford and Shah model [35] for image segmentation, $f \in L^\infty(\Omega) \subset L^2(\Omega)$ is split into $u \in SBV(\Omega)$ [2], a piecewise-smooth function with its discontinuity set J_u composed of a union of curves of total finite length, and $v = f-u \in L^2(\Omega)$ representing noise or texture. The (non-convex) model in the weak formulation is [33]

$$\inf_{(u,v)\in SBV(\Omega)\times L^2(\Omega)} \Big\{ \int_{\Omega\setminus J_u} |\nabla u|^2 dx + \alpha \mathcal{H}^1(J_u) + \beta \|v\|^2_{L^2(\Omega)},\ f = u+v \Big\}, \tag{2}$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure, and $\alpha, \beta > 0$ are tuning parameters. With the above notations, the first two terms in the energy from (2) compose $F_1(u)$, while the third term makes $F_2(v)$. A related decomposition is obtained by the total variation minimization model of Rudin, Osher, and Fatemi [39] for image denoising. The (convex) decomposition model is

$$\inf_{(u,v)\in BV\times L^2} \Big\{ |u|_{BV} + \lambda \|v\|^2_{L^2},\quad f = u+v \Big\}, \tag{3}$$

where $|u|_{BV(\Omega)} = \int |Du|$ (the semi-norm on the space BV) [14], and $\lambda > 0$ is a tuning parameter. This model is strictly convex and is easily solved in practice. However, it has some limitations pointed out by several authors ([44], [45], [32] among others). If $f = \alpha\chi_D$ is a multiple of the characteristic function of a disk D centered at the origin and of radius R, we would like the minimizer u to be f if R is not too small. However, for any $R \geq \frac{1}{\lambda\alpha}$ and any finite $\lambda > 0$, we have [32]

$$u = (\alpha - \frac{1}{\lambda R})\chi_D,\ v = \frac{1}{\lambda R}\chi_D.$$

The model (3) is of the form $|u|_{BV} + \lambda\|f-u\|^q_{L^p}$, $p \geq 1$, $q \geq 1$, and the loss of intensity property is always present when we have $q > 1$ while keeping the total variation. In particular, we no longer have an intensity loss if we substitute $\|\cdot\|_{L^2}^2$ in (3) with $\|\cdot\|_{L^2}$ or $\|\cdot\|_{L^1}$, which was proposed in the continuous case by Cheon, Paranjpye, Vese and Osher [10], and further analysis in the L^1 case was made by Chan and Esedoglu [9], among others).

We are interested in function spaces that give small penalties to oscillations. As noted in [32], oscillatory components do not have small norms in L^2 or L^1. Moreover, Alvarez, Gousseau and Morel [19], [1] argue that BV is not a good choice to model natural images. To overcome these drawbacks, we can relax the condition on $F_1(u) = |u|_{BV}$ or $F_2(v) = \|v\|_{L^p}$, for $p = 1$ or $p = 2$. One way is to use a non-convex regularization in u (like in (2), [17], [7], [48], [29] etc), that is weaker than $|\cdot|_{BV}$. Another way is to use weaker norms than the L^p norm. Here we keep a convex BV regularization, and consider weaker norms than the L^p norm, following [32]. Mumford and Gidas [34] also show that, under some assumptions, natural images are drawn from probability distributions supported by generalized functions, and not by functions.

Y. Meyer [32] questions the model (3) and proposes more refined versions, using weaker norms of generalized functions to model v, instead of the $\|\cdot\|_{L^2}^2$. Among the spaces proposed in [32] to better model the texture component, is the space F and the minimization model

$$\inf_{(u,v)\in BV\times F} \Big\{ |u|_{BV} + \lambda \|v\|_F, \quad f = u + v \Big\}, \tag{4}$$

where F is defined below.

Definition 1. In two dimensions, the space F consists of distributions v which can be written as

$$v = \mathrm{div}(\vec{g}) \text{ in } \mathcal{D}', \quad \vec{g} = (g_1, g_2) \in BMO^2, \text{ with}$$

$$\|v\|_F = \inf \Big\{ \|g_1\|_{BMO} + \|g_2\|_{BMO} : \ v = \mathrm{div}(\vec{g}) \text{ in } \mathcal{D}', \ \vec{g} \in BMO^2 \Big\}.$$

The space BMO is defined below.

Definition 2. We say that $f \in L^1_{loc}$ belongs to BMO [23], [41], if

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| dx < \infty,$$

where Q is a square (it is sufficient to consider squares with sides parallel with the axes), and $f_Q = |Q|^{-1} \int_Q f(x)dx$ denotes the mean value of f over the square Q.

An equivalent norm of BMO can be obtained by taking the supremum over dyadic squares and their 1/3 translations as in [15] by J. Garnett and P. Jones .

In [32], Y. Meyer also proposed two other function spaces to model the oscillatory component v, denoted by G and E, with $u \in BV \subset L^2 \subset G \subset F \subset E$. The space G is defined like F but having $\vec{g}$ in $(L^\infty)^2$ instead of $(BMO)^2$, while $E = \dot{B}^{-1}_{\infty,\infty} = \triangle(\dot{B}^1_{\infty,\infty})$ is a homogeneous Besov space of regularity index -1.

Meyer's G-model is approximated and studied in [49]-[50], [37], [4], [3], [6], [36], [20], [51], [11], [27], [25], among others. Meyer's E model was studied and discussed in [5], [16] and [24].

In [26], the third and fourth authors proposed several methods to numerically compute the BMO-norm of a function defined on a bounded domain Ω, and approximate Meyer's F-model (4) by the convex variational relaxed problem,

$$\inf_{u\in BV(\Omega),\ \vec{g}\in BMO(\Omega)^2}\left\{|u|_{BV(\Omega)}+\mu\|f-u-\mathrm{div}\vec{g}\|^2_{L^2(\Omega)}+\lambda\left[\|g_1\|_{BMO(\Omega)}+\|g_2\|_{BMO(\Omega)}\right]\right\}. \quad (5)$$

As $\mu\to\infty$, this model approximates the model (4). An equivalent model was also proposed in [26], by setting $\vec{g}=\nabla g$, i.e. $v=\Delta g$, and minimizing

$$\begin{aligned}\inf\Big\{&|u|_{BV(\Omega)}+\mu\|f-u-\Delta g\|^2_{L^2(\Omega)}+\lambda\left[\|g_x\|_{BMO(\Omega)}+\|g_y\|_{BMO(\Omega)}\right]\\ &: u\in BV(\Omega), g, \triangle g\in L^2(\Omega), \nabla g\in BMO(\Omega)^2\Big\}.\end{aligned} \quad (6)$$

Formulations (5) and (6) are still approximations to Meyer's F-model. In these models, a given image f is decomposed into $u+v+r$, where $u\in BV(\Omega)$ is piecewise smooth, $v=\mathrm{div}(\vec{g})\in F$ or $v=\Delta g=\mathrm{div}(\nabla g)\in F$ consists of oscillatory components, and $r=f-u-v\in L^2(\Omega)$ is a residual. Numerically, r is negligible. The significance of r is also discussed in [16].

Other related decomposition models using wavelets are by I. Daubechies and G. Teschke [13], R. Coifman and D. Donoho [12], J. L. Starck, M. Elad, and D. Donoho [40], F Malgouyres [31], S. Lintner and F. Malgouyres [30], Haddad and Meyer [20], Haddad, [21], or Gilles [18].

In this paper, we consider an equivalent norm for the space F in terms of the Riesz potentials, and study models where the action of the Riesz potentials with the oscillatory components belong to BMO. In other words, we model the oscillatory component v by imposing that $(-\Delta)^{\alpha/2}v$ belongs to BMO, for some $\alpha<0$, i.e. $v\in B\dot{M}O^{\alpha}$. If $\alpha=-1$, we recover the space F, but now the equivalent norm is defined in an isotropic way and we can obtain exact decompositions (4), and equivalent decompositions as in (5) and (6).

As a byproduct and for comparison, we also consider models when $(-\Delta)^{\alpha/2}v\in L^p$, $1\le p<\infty$, i.e. v belongs to the homogeneous potential Sobolev space $\dot{W}^{\alpha,p}$, for some $\alpha<0$. The case $1\le p<\infty$ and $\alpha=-1$ reduces to the case from [49], [50]. The case $p=2$ and $\alpha=-1$ reduces to the model from [37] in an equivalent PDE formulation, while the more general case with $\alpha<0$ and $p=2$ reduces to the models proposed by L. Lieu [27], [28], and also related with the proposal from [34].

As noted in [32] in more details, the space $F=B\dot{M}O^{-1}$ has also been used in an analysis of the Navier-Stokes equations by Koch and Tataru [22], where $B\dot{M}O^{-1}$ is defined through another isotropic equivalent norm in terms of the Carleson measure.

2 The homogeneous spaces $\dot{BMO}^\alpha$ and $\dot{W}^{\alpha,p}$

In this section, we consider a general form of function spaces, and in the definitions we make no distinction between periodic functions or functions defined on R^n. We recall the definitions of the Riesz potential

$$I_\alpha v(x) = (-\triangle)^{\alpha/2} v(x) = \left((2\pi|\xi|)^\alpha \hat{v}(\xi)\right)^\vee (x) = k_\alpha * v(x),$$

with $k_\alpha(x) = \left((2\pi|\xi|)^\alpha\right)^\vee (x)$, where as usual, $\hat{}$ indicates the Fourier transform and $^\vee$ indicates the inverse Fourier transform.

We also recall the Riesz transforms of a function f in two dimensions:

$$\widehat{(R_j f)}\ (\xi) = \frac{i\xi_j}{|\xi|}\hat{f}(\xi), \quad j = 1, 2,$$

having the property

$$(R_1)^2 + (R_2)^2 = -I,$$

where I is the identity operator. We note that the Riesz operators R_j are bounded in BMO:

$$\|R_j f\|_{BMO} \le C_0 \|f\|_{BMO},$$

for some positive constant C_0.

Our main motivation of this work is the following lemma, which provides an isotropic equivalent norm for F, and easier to use in practice. This will also lead to generalizations.

Lemma 1. *The norm $\|v\|_F$ is equivalent with the norm $\|I_{-1} v\|_{BMO} = \|(-\Delta)^{-1/2} v\|_{BMO}$.*

Proof. Again, we note that the Riesz operators R_j are bounded in BMO [41],

$$\|R_j f\|_{BMO} \le C_0 \|f\|_{BMO},$$

for some positive constant C_0.

We have:

$$\begin{aligned} v &= -((R_1)^2 + (R_2)^2)v = -(-\triangle)^{1/2}(-\triangle)^{-1/2}((R_1)^2 + (R_2)^2)v \\ &= R_1(-\triangle)^{1/2}(-R_1(-\triangle)^{-1/2}v) + R_2(-\triangle)^{1/2}(-R_2(-\triangle)^{-1/2}v) \\ &= R_1(-\triangle)^{1/2}g_1 + R_2(-\triangle)^{1/2}g_2 = div(g_1, g_2), \end{aligned}$$

with $g_j = -R_j((-\triangle)^{-1/2}v)$.

Then $\|g_j\|_{BMO} = \|-R_j((-\triangle)^{-1/2}v)\|_{BMO} \le C_0\|(-\triangle)^{-1/2}v\|_{BMO}$.

Therefore,

$$\|v\|_F := \inf_{\vec{g} \in BMO \times BMO,\ div\vec{g} = v} \left[\|g_1\|_{BMO} + \|g_2\|_{BMO}\right] \le 2C_0\|(-\triangle)^{-1/2}v\|_{BMO}.$$

For the converse inequality, suppose $v = div(g_1, g_2)$, with $g_1, g_2 \in BMO$. Then

$$v = div(g_1, g_2) = (-\triangle)^{1/2}(R_1 g_1 + R_2 g_2),$$

therefore

$$(-\triangle)^{-1/2} v = R_1 g_1 + R_2 g_2,$$

and then

$$\begin{aligned}\|(-\triangle)^{-1/2} v\|_{BMO} &= \|R_1 g_1 + R_2 g_2\|_{BMO} \leq \|R_1 g_1\|_{BMO} + \|R_2 g_2\|_{BMO} \\ &\leq C_0 \|g_1\|_{BMO} + C_0 \|g_2\|_{BMO} = C_0 \Big[\|g_1\|_{BMO} + \|g_2\|_{BMO} \Big].\end{aligned}$$

We conclude that

$$\|(-\triangle)^{-1/2} v\|_{BMO} \leq C_0 \inf_{\vec{g} \in BMO \times BMO,\ div \vec{g} = v} \Big[\|g_1\|_{BMO} + \|g_2\|_{BMO} \Big] = C_0 \|v\|_F,$$

and therefore the two norms are equivalent, since we have obtained

$$\frac{1}{2C_0} \|v\|_F \leq \|(-\triangle)^{-1/2} v\|_{BMO} \leq C_0 \|v\|_F.$$

□

Thus, for $v \in F$, the quantity $\|I_{-1} v\|_{BMO} = \|(-\Delta)^{-1/2} v\|_{BMO}$ provides an equivalent norm for $\|v\|_F$ introduced in Definition 1. This isotropic norm can be used as an alternative way to the models proposed and solved in [26]. Moreover, we are led to consider more general cases, when v is modeled by the space $B\dot{M}O^\alpha$, $\alpha < 0$, defined below.

Definition 3. (See Strichartz [42] and [43]) We say that a function (or distribution) v belongs to the homogeneous space $B\dot{M}O^\alpha = I_\alpha(BMO)$, $\alpha \in \mathbb{R}$, if

$$\|v\|_{B\dot{M}O^\alpha} := \|I_\alpha v\|_{BMO} < \infty.$$

Equipped with $\| \cdot \|_{B\dot{M}O^\alpha}$, $B\dot{M}O^\alpha$ becomes a Banach space.

Elements in BMO or $B\dot{M}O^\alpha$ that are different by a constant are identified. In other words, we can assume that v has zero mean ($\int v(x)dx = 0$) if $v \in BMO$ or $v \in B\dot{M}O^\alpha$.

The space $B\dot{M}O^\alpha$ coincides with the classical Triebel-Lizorkin homogeneous space $\dot{F}^\alpha_{\infty,2}$ [46]. An equivalent norm for $B\dot{M}O^\alpha$ can also be obtained using the Carleson measure, as in [22]: let $\Phi(x) = Ce^{-2\pi|x|^2}$, where C is chosen so that $\int \Phi(x)\, dx = 1$, thus $\Phi(x) = (e^{-2\pi|\xi|^2})^\vee(x)$. Define $\Phi_t(x) = t^{-n}\Phi(\frac{x}{t})$, $x \in \mathbb{R}^n$. For each $v \in L^1_{loc}$, let $w_t(x) = \Phi_{\sqrt{4t}} * v(x)$. We have the following characterization of BMO using Carleson measure [22], [41].

Definition 4. We say that $v \in BMO$ if

$$\begin{aligned}\|v\|_{BMO} &:= \sup_{x,R}\left(\frac{4\pi}{Q(x,R)}\int_{Q(x,R)}\int_0^R t|\nabla(\Phi_t * v)|^2\,dtdy\right)^{1/2}\\ &= \sup_{x,R}\left(\frac{1}{Q(x,R)}\int_{Q(x,R)}\int_0^{R^2} |\nabla w|^2\,dtdy\right)^{1/2} \\ &\approx \sup_{x,R}\left(\frac{1}{Q(x,R)}\int_{Q(x,R)}\int_0^{R^2} \left|(-\Delta)^{1/2}w\right|^2\,dtdy\right)^{1/2} < \infty,\end{aligned} \tag{7}$$

where $Q(x,R)$ denotes a square centered at x with side length R, and "$\approx$" denotes equivalent norm.

Similarly, we have the following Carleson measure characterization of $\dot{BMO}^\alpha$, which could be another alternative approach to the work in [26].

Definition 5. We say that v belongs to $\dot{BMO}^\alpha$, $\alpha \in \mathbb{R}$, if

$$\begin{aligned}\|I^\alpha v\|_{BMO} &= \sup_{x,R}\left(\frac{4\pi}{Q(x,R)}\int_{Q(x,R)}\int_0^R t|\nabla(\Phi_t * (I_\alpha v))|^2\,dtdy\right)^{1/2}\\ &= \sup_{x,R}\left(\frac{4\pi}{Q(x,R)}\int_{Q(x,R)}\int_0^R t|\nabla(I_\alpha(\Phi_t * v))|^2\,dtdy\right)^{1/2}\\ &= \sup_{x,R}\left(\frac{1}{Q(x,R)}\int_{Q(x,R)}\int_0^{R^2} |\nabla(I_\alpha w)|^2\,dtdy\right)^{1/2}\\ &\approx \sup_{x,R}\left(\frac{1}{Q(x,R)}\int_{Q(x,R)}\int_0^{R^2} \left|I_{\alpha+1/2}w\right|^2\,dtdy\right)^{1/2} < \infty.\end{aligned} \tag{8}$$

Again, "$\approx$" denotes equivalent norms.

In the remaining part of this paper, we use Definition 3 for $\dot{BMO}^\alpha$. For comparison, substituting BMO in Definition 3 by L^p, $1 \le p < \infty$, we arrive to the homogeneous potential Sobolev spaces, which we recall here.

Definition 6. We say that a function (or distribution) v belongs to the homogeneous potential Sobolev space $\dot{W}^{\alpha,p}$, for $\alpha \in \mathbb{R}$, $1 \le p \le \infty$, if

$$\|v\|_{\dot{W}^{\alpha,p}} := \|I_\alpha v\|_{L^p} < \infty.$$

Equipped with $\|\cdot\|_{\dot{W}^{\alpha,p}}$, $\dot{W}^{\alpha,p}$ becomes a Banach space.

Note that if $g \in \dot{W}^{\alpha,p}$, $\alpha < 0$, then $\int_\Omega g(x)\,dx = 0$. Some useful properties of $\dot{BMO}^\alpha$ and $\dot{W}^{\alpha,p}$ are recalled below:

- I_s is an isometry from $\dot{BMO}^{\alpha}$ and $\dot{W}^{\alpha,p}$ to $\dot{BMO}^{\alpha-s}$ and $\dot{W}^{\alpha-s,p}$, respectively, for all $s, \alpha \in \mathbb{R}$.

- Let $\tau_\delta f(x) = f(\delta x)$, $\delta > 0$, $x \in \mathbb{R}^n$, be the dilation operator. We have

$$\|\tau_\delta f\|_{L^p(\mathbb{R}^n)} = \delta^{-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)},$$
$$\|\tau_\delta f\|_{\dot{BMO}^{\alpha}(\mathbb{R}^n)} = \delta^{\alpha} \|f\|_{\dot{BMO}^{\alpha}(\mathbb{R}^n)}, \quad \|\tau_\delta f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} = \delta^{-\frac{n}{p}+\alpha} \|f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)}.$$

From this dilation property, we see that, for $\alpha < 0$, $\|\cdot\|_{\dot{W}^{\alpha,p}}$ provides a better separation among different oscillations compared to $\|\cdot\|_{L^p}$, and for the same $\alpha < 0$, $\|\cdot\|_{\dot{W}^{\alpha,p}}$ provides a better separation among different oscillations compared to $\|\cdot\|_{\dot{BMO}^{\alpha}}$. The experimental results in this paper will support these remarks.

In [16], the authors have numerically considered the case when the oscillatory component v belongs to $\dot{B}^{\alpha}_{p,\infty}$, $\alpha < 0$, as a generalization of the space E proposed by Y. Meyer. The following remark shows that $\dot{B}^{\alpha}_{p,q}$ and $\dot{W}^{\alpha,p}$ are in fact close [47].

Remark 1. If $\alpha \in \mathbb{R}$ and $p \geq 1$, then

$$\dot{B}^{\alpha}_{p,1} \subset \dot{W}^{\alpha,p} \subset \dot{B}^{\alpha}_{p,\infty}. \tag{9}$$

3 Modeling oscillations with $\dot{BMO}^{\alpha}$ and $\dot{W}^{\alpha,p}$

Given an image f, we would like to decompose it into $u + v$, where $u \in BV$, and v is an element of $\dot{BMO}^{\alpha}$ or $\dot{W}^{\alpha,p}$, for $\alpha < 0$ and $1 \leq p < \infty$. In other words, we consider modeling oscillatory component v (of zero mean) as Δg, where $g \in \dot{BMO}^{s}$ or $\dot{W}^{s,p}$, for $s < 2$, $1 \leq p < \infty$, in the minimization problems for image decomposition

$$\inf_{u,g} \left\{ |u|_{BV} + \mu \|f - u - \Delta g\|^2_{L^2} + \lambda \|g\|_{\dot{BMO}^{s}} \right\}, \text{ and} \tag{10}$$
$$\inf_{u,g} \left\{ |u|_{BV} + \mu \|f - u - \Delta g\|^2_{L^2} + \lambda \|g\|_{\dot{W}^{s,p}} \right\}. \tag{11}$$

The model (10), when $s = 1$, is equivalent with the model (6). Since v belongs to $\dot{BMO}^{\alpha}$ or $\dot{W}^{\alpha,p}$ with $\alpha = s - 2$, we will also consider the exact decomposition models,

$$\inf_{u} \left\{ |u|_{BV} + \lambda \|f - u\|_{\dot{BMO}^{\alpha}} \right\}, \text{ and} \tag{12}$$
$$\inf_{u} \left\{ |u|_{BV} + \lambda \|f - u\|_{\dot{W}^{\alpha,p}} \right\}. \tag{13}$$

Thus, when $\alpha = -1$ in (12), we recover Meyer's model (4). Theorems 1 and 2 from [16] can be exactly carried out here to show existence of minimizers for the above models (10), (11), (12) and (13).

Remark 2. We note here the connections between the above decomposition models (12) and (13) with the real and the complex methods of interpolation. These connections could be applied to obtain additional information about the regularity of the data f.

In the real method of interpolation [38], we consider two Banach spaces X_0, X_1 which are continuously embedded in a common Hausdorff topological vector space V. Given any positive number λ, the $K-functional$ is defined by

$$K(\lambda, f) = \inf\{\|u\|_{X_0} + \lambda\|v\|_{X_1} : \ f = u + v, \ u \in X_0, \ v \in X_1\}, \quad f \in X_0 + X_1.$$

Note that for each λ, $K(\lambda, f)$ is a norm on $X_0 + X_1$ equivalent to $\|\cdot\|_{X_0+X_1}$.

Let $1 \le q \le \infty$ and $0 < \theta < 1$. The real interpolation space $(X_0, X_1)_{\theta,q}$ consists of all $f \in X_0 + X_1$ which have finite norm

$$\|f\|_{\theta,q} = \begin{cases} \left(\int_0^\infty (\lambda^{-\theta} K(\lambda, f))^q \frac{d\lambda}{\lambda}\right)^{1/q} & \text{if } 1 \le q < \infty, \\ \sup_{\lambda>0}\left\{\lambda^{-\theta} K(\lambda, f)\right\} & \text{if } q = \infty. \end{cases}$$

Thus, we see that computing optimal decompositions $f = u + v$ is necessary to study the behavior of a given data f.

In the complex method of interpolation, introduced by A. Calderón [8], we consider a pair of complex Banach spaces X_0, X_1, continuously embedded in a complex topological vector space V. We then consider functions $g(\xi)$, $\xi = s + it$ defined on the strip $0 \le s \le 1$ of the $\xi-$plane, with values in $X_0 + X_1$ continuous and bounded with respect to the norm of $X_0 + X_1$ in $0 \le s \le 1$ and analytic in $0 < s < 1$, and such that $g(it) \in X_0$ is X_0-continuous and tends to zero as $|t| \to \infty$, $g(1+it) \in X_1$ is X_1-continuous and tends to zero as $|t| \to \infty$. In this linear space of functions denoted by $\mathcal{G}(X_0, X_1)$, we introduce the norm

$$\|g\|_{\mathcal{G}} = \max[\sup_t \|g(it)\|_{X_0}, \sup_t \|g(1+it)\|_{X_1}].$$

Then $\mathcal{G}$ becomes a Banach space.

Given a real number s, $0 \le s \le 1$, we consider the subspace $X_s = [X_0, X_1]_s$ of $X_0 + X_1$ defined by $X_s = \{f| \ f = g(s), \ g \in \mathcal{G}(X_0, X_1)\}$ endowed with the norm

$$\|f\|_{X_s} = \inf\{\|g\|_{\mathcal{F}}, \ g(s) = f\}.$$

Then X_s becomes a Banach space continuously embedded in $X_0 + X_1$.

We see that the complex method can also be used to analyze the behavior of the data $f \in X_0 + X_1$, but without explicitly computing optimal decompositions $f = u + v$.

We discuss next scaling properties of the proposed minimization models. Recall the dilating operator $\tau_\delta f(x) = f(\delta x)$, $\delta > 0$. We have

$$|\tau_\delta f|_{BV(\mathbb{R}^n)} = \delta^{-n+1}|f|_{BV(\mathbb{R}^n)}, \qquad \|\tau_\delta f\|_{L^p(\mathbb{R}^n)} = \delta^{-n/p}\|f\|_{L^p(\mathbb{R}^n)}, \tag{14}$$

$$\|\tau_\delta f\|_{B\dot{M}O^\alpha(\mathbb{R}^n)} = \delta^{\alpha}\|f\|_{B\dot{M}O^{\alpha,p}(\mathbb{R}^n)}, \qquad \|\tau_\delta f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} = \delta^{-n/p+\alpha}\|f\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)} \tag{15}$$

Following [16], we would like to characterize the parameters μ and λ in the proposed models (10), (11), (12) and (13) when the image f is being dilated by a factor δ (zoom in when $0 < \delta < 1$ and zoom out when $\delta > 1$).

Proposition 1. *Denote*

$$\mathcal{J}_{f,\lambda}(u) = |u|_{BV(\mathbb{R}^n)} + \lambda\|f-u\|_{B\dot{M}O^\alpha(\mathbb{R}^n)}$$

For a fixed f and $\lambda > 0$, let $(u_\lambda, v_\lambda = f - u_\lambda)$ be a minimizer for the energy $\mathcal{J}_{f,\lambda}$. Then for $\lambda' = \lambda\delta^{-n+1-\alpha}$, $(\tau_\delta u_\lambda, \tau_\delta v_\lambda)$ minimizes the energy $\mathcal{J}_{\tau_\delta f,\lambda'}$.

Proof. Since $(u_\lambda, v_\lambda = f - u_\lambda)$ is a minimizer, this implies

$$\mathcal{J}_{f,\lambda}(u_\lambda) = |u_\lambda|_{BV(\mathbb{R}^n)} + \lambda\|v_\lambda\|_{B\dot{M}O^\alpha(\mathbb{R}^n)}$$

is minimal. Applying τ_δ to f, u_λ and v_λ using λ', we have

$$\begin{aligned}\mathcal{J}_{\tau_\delta f,\lambda'}(\tau_\delta u_\lambda) \quad &= |\tau_\delta u_\lambda|_{BV(\mathbb{R}^n)} + \lambda'\|\tau_\delta v_\lambda\|_{B\dot{M}O^\alpha(\mathbb{R}^n)} \\ &= \delta^{-n+1}|u_\lambda|_{BV(\mathbb{R}^n)} + \lambda'\delta^\alpha\|v_\lambda\|_{B\dot{M}O^\alpha(\mathbb{R}^n)}.\end{aligned}$$

We have $\delta^{n-1}\mathcal{J}_{\tau_\delta f,\lambda'}(\tau_\delta u_\lambda)$ is minimized when $\lambda' = \lambda\delta^{-n+1-\alpha}$. Therefore, $(\tau_\delta u_\lambda, \tau_\delta v_\lambda)$ is a minimizer for $\mathcal{J}_{\tau_\delta f,\lambda'}$ with $\lambda' = \lambda\delta^{-n+1-\alpha}$. □

Similarly, when $\|\cdot\|_{B\dot{M}O^\alpha}$ is replaced by $\|\cdot\|_{\dot{W}^{\alpha,p}}$, we have the following result.

Proposition 2. *For a fixed f and $\lambda > 0$, let $(u_\lambda, v_\lambda = f - u_\lambda)$ be a minimizer for the energy,*

$$\mathcal{K}_{f,\lambda}(u) = |u|_{BV(\mathbb{R}^n)} + \lambda\|f-u\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)}$$

Then for $\lambda' = \lambda\delta^{(-n+1)-(-n/p+\alpha)}$, $(\tau_\delta u_\lambda, \tau_\delta v_\lambda)$ minimizes $\mathcal{K}_{\tau_\delta f,\lambda'}$.

Using the same techniques, we obtain the following results for the models (10) and (11).

Proposition 3. *Fix an f, $\mu > 0$, and $\lambda > 0$.*

1. *Let $(u_{\mu,\lambda}, v_{\mu,\lambda})$ be a minimizer for the energy from (10), which can be rewritten as*

$$\mathcal{J}_{f,\mu,\lambda}(u) = |u|_{BV(\mathbb{R}^n)} + \mu\|f-u-v\|^2_{L^2(\mathbb{R}^n)} + \lambda\|v\|_{B\dot{M}O^\alpha(\mathbb{R}^n)}$$

Then for $\mu' = \mu\delta$ and $\lambda' = \lambda\delta^{-n+1-\alpha}$, $(\tau_\delta u_{\mu,\lambda}, \tau_\delta v_{\mu,\lambda})$ minimizes $\mathcal{J}_{\tau_\delta f,\mu',\lambda'}$.

2. *Let $(u_{\mu,\lambda}, v_{\mu,\lambda})$ be a minimizer for the energy from (11), which can be rewritten as*

$$\mathcal{K}_{f,\mu,\lambda}(u) = |u|_{BV(\mathbb{R}^n)} + \mu\|f-u-v\|^2_{L^2(\mathbb{R}^n)} + \lambda\|v\|_{\dot{W}^{\alpha,p}(\mathbb{R}^n)}$$

Then for $\mu' = \mu\delta$ and $\lambda' = \lambda\delta^{(-n+1)-(-n/p+\alpha)}$, $(\tau_\delta u_{\mu,\lambda}, \tau_\delta v_{\mu,\lambda})$ minimizes $\mathcal{K}_{\tau_\delta f,\mu',\lambda'}$.

4 Characterization of minimizers

In this section, we would like to show some results regarding the characterization of minimizers for the exact decompositions (12) and (13) under some assumptions or minor modifications. These can be seen as extensions and generalizations of the results from Lemma 4, Thm. 3 (page 32), Proposition 4 (page 33) and Thm. 4 (page 4) from [32].

4.1 The case $|u|_{BV} + \lambda \|I_\alpha(f-u)\|^2_{BMO}$

We have the following equivalent formulations of BMO for different values of $p \in [1,\infty)$, see [41] for example. For $f \in L^2_{loc}$, we have

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q| dx \le \Big(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^2 dx\Big)^{1/2},$$

thus if $\sup_Q \Big(\frac{1}{|Q|}|f(x) - f_Q|^2 dx\Big)^{1/2} \le C$, then $f \in BMO$. Conversely, if $f \in BMO$ according to Definition 2, then for any $p < \infty$, f is in L^p_{loc} and $\frac{1}{|Q|}\int_Q |f(x)-f_Q|^p dx \le c_p \|f\|^p_{BMO}$, for all squares Q.

Thus consider the problem with $p = 2$ in the definition of the equivalent BMO norm, and we substitute (12) by

$$\inf_u \mathcal{F}(u),$$

where

$$\begin{aligned}\mathcal{F}(u) &= |u|_{BV} + \lambda \sup_Q \frac{1}{|Q|}\int_Q |k_\alpha * (f-u) - (k_\alpha * (f-u))_Q|^2 dx, \text{ or}\\ \mathcal{F}(u) &= |u|_{BV} + \lambda \sup_Q \frac{1}{|Q|}\|k_\alpha * f - (k_\alpha * f)_Q - (k_\alpha * u - (k_\alpha * u)_Q)\|^2_{L^2(Q)}.\end{aligned}$$

Denote $\langle f, g\rangle_{L^2(Q)} := \int_Q fg\ dx$. Consider the quantity $\|\cdot\|_{\alpha,*}$, (possibly attains ∞), defined as

$$\|f\|_{\alpha,*} = \sup_{h\in BV, |h|_{BV}\neq 0} \frac{\frac{1}{|\bar{Q}|}\left\langle k_\alpha * f - (k_\alpha * f)_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}}\right\rangle_{L^2(\bar{Q})}}{|h|_{BV}},$$

where $\bar{Q}$ satisfies

$$\bar{Q} = \arg\max_Q \frac{1}{|Q|}\int_Q |k_\alpha * f - (k_\alpha * f)_Q|^2 dx, \tag{16}$$

Definition 7. Let $\bar{Q}$ satisfies (16). Given an $\alpha \in \mathbb{R}$, we say f satisfies property (**P**) if for any $h \in BV$,

$$\begin{aligned}&\lim_{\epsilon_n\to 0}\frac{1}{|Q_{\epsilon_n}|}\left\langle k_\alpha * f - (k_\alpha * f)_{Q_{\epsilon_n}}, k_\alpha * h - (k_\alpha * h)_{Q_{\epsilon_n}}\right\rangle_{L^2(Q_{\epsilon_n})}\\ &\quad\ge \frac{1}{|\bar{Q}|}\left\langle k_\alpha * f - (k_\alpha * f)_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}}\right\rangle_{L^2(\bar{Q})}\end{aligned} \tag{17}$$

for some sequence of squares Q_{ϵ_n} and of small parameters $\epsilon_n > 0$ converging to zero, such that

$$Q_{\epsilon_n} = \arg\max_Q \frac{1}{|Q|}\int_Q |k_\alpha * (f - \epsilon_n h) - (k_\alpha * (f-\epsilon_n h))_Q|^2 dx.$$

Proposition 4. *Let $\alpha \in \mathbb{R}$ and $\bar{Q}$ be the square depending on α and f, such that*

$$\bar{Q} = \arg\max_{Q} \frac{1}{|Q|} \int_{Q} |k_\alpha * f - (k_\alpha * f)_Q|^2 dx. \tag{18}$$

(i) If $\|f\|_{\alpha,*} \leq \frac{1}{2\lambda}$, then $u = 0$ and $v = f$ is a minimizer.

*(ii) If $u = 0$ and $v = f$ is a minimizer and if, in addition, f satisfies property (**P**) from (17), then $\|f\|_{\alpha,*} \leq \frac{1}{2\lambda}$.*

Proof.

(i) Let $h \in BV$ such that

$$\mathcal{F}(h) = |h|_{BV} + \lambda \sup_{Q} \frac{1}{|Q|} \int_{Q} |k_\alpha * (f-h) - (k_\alpha * (f-h))_Q|^2 dx < +\infty.$$

Since $\|f\|_{\alpha,*} \leq \frac{1}{2\lambda}$, we have for all $h \in BV$,

$$-2\lambda \frac{1}{|\bar{Q}|} \left\langle k_\alpha * f - (k_\alpha * f)_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}} \right\rangle_{L^2(\bar{Q})} \geq -|h|_{BV},$$

Then

$$\begin{aligned}
\mathcal{F}(h) &= |h|_{BV} + \lambda \sup_{Q} \frac{1}{|Q|} \left\| k_\alpha * f - (k_\alpha * f)_Q - (k_\alpha * h - (k_\alpha * h)_Q) \right\|^2_{L^2(Q)} \\
&= |h|_{BV} + \lambda \sup_{Q} \frac{1}{|Q|} \Big[\|k_\alpha * f - (k_\alpha * f)_Q\|^2_{L^2(Q)} \\
&\quad -2 \left\langle k_\alpha * f - (k_\alpha * f)_Q, k_\alpha * h - (k_\alpha * h)_Q \right\rangle_{L^2(Q)} + \|k_\alpha * h - (k_\alpha * h)_Q\|^2_{L^2(Q)} \Big].
\end{aligned}$$

With $\bar{Q}$ defined as in (18), we have

$$\begin{aligned}
\mathcal{F}(h) &\geq |h|_{BV} + \lambda \frac{1}{|\bar{Q}|} \|k_\alpha * f - (k_\alpha * f)_{\bar{Q}}\|^2_{L^2(\bar{Q})} + \frac{1}{|\bar{Q}|} \|k_\alpha * h - (k_\alpha * h)_{\bar{Q}}\|^2_{L^2(\bar{Q})} \\
&\quad - 2\lambda \frac{1}{|\bar{Q}|} \left\langle k_\alpha * f - (k_\alpha * f)_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}} \right\rangle_{L^2(\bar{Q})} \\
&= |h|_{BV} + \lambda \sup_{Q} \frac{1}{|Q|} \|k_\alpha * f - (k_\alpha * f)_Q\|^2_{L^2(Q)} + \frac{1}{|\bar{Q}|} \|k_\alpha * h - (k_\alpha * h)_{\bar{Q}}\|^2_{L^2(\bar{Q})} \\
&\quad - 2\lambda \frac{1}{|\bar{Q}|} \left\langle k_\alpha * f - (k_\alpha * f)_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}} \right\rangle_{L^2(\bar{Q})} \\
&\geq F(0) + \frac{1}{|\bar{Q}|} \|k_\alpha * h - (k_\alpha * h)_{\bar{Q}}\|^2_{L^2(\bar{Q})} \geq F(0).
\end{aligned}$$

Therefore, $u = 0$ is a minimizer.

(ii) Suppose now $u = 0$ and $v = f$ is a minimizer and f satisfies property $(\mathbf{P})$ from (17). We have

$$|h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \int_Q |(k_\alpha * (f-h) - (k_\alpha * (f-h))_Q)|^2 dx \\ \geq \lambda \sup_Q \frac{1}{|Q|} \int_Q |k_\alpha * f - (k_\alpha * f)_Q|^2 dx.$$

Thus

$$\begin{aligned} |h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \Big\{ \int_Q |k_\alpha * k - (k_\alpha * f)_Q|^2 dx \\ - 2 \langle k_\alpha * f - (k_\alpha * f)_Q, k_\alpha * h - (k_\alpha * h)_Q \rangle_{L^2(Q)} \\ + \int_Q |k_\alpha * h - (k_\alpha * h)_Q|^2 dx \Big\} \geq \lambda \sup_Q \frac{1}{|Q|} \int_Q |k_\alpha * f - (k_\alpha * f)_Q|^2 dx. \end{aligned} \tag{19}$$

Let $\hat{Q}$ be defined as the square depending on f and h that achieves the maximum in $\sup_Q \frac{1}{|Q|} \int_Q |k_\alpha * (f-h) - (k_\alpha * (f-h))_Q|^2 dx$.

Then we can rewrite (19) as

$$\begin{aligned} |h|_{BV} + \lambda \frac{1}{|\hat{Q}|} \Big\{ \int_{\hat{Q}} |k_\alpha * f - (k_\alpha * f)_{\hat{Q}}|^2 dx \\ - 2 \left\langle k_\alpha * f - (k_\alpha * f)_{\hat{Q}}, k_\alpha * h - (k_\alpha * h)_{\hat{Q}} \right\rangle_{L^2(\hat{Q})} \\ + \int_{\hat{Q}} |k_\alpha * h - (k_\alpha * h)_{\hat{Q}}|^2 dx \Big\} \geq \lambda \sup_Q \frac{1}{|Q|} \int_Q |k_\alpha * f - (k_\alpha * f)_Q|^2 dx. \end{aligned}$$

This implies

$$\begin{aligned} |h|_{BV} + \lambda \sup_Q \frac{1}{|Q|} \int_Q |k_\alpha * f - (k_\alpha * f)_Q|^2 dx \\ - 2\lambda \frac{1}{|\hat{Q}|} \left\langle k_\alpha * f - (k_\alpha * f)_{\hat{Q}}, k_\alpha * h - (k_\alpha * h)_{\hat{Q}} \right\rangle_{L^2(\hat{Q})} \\ + \lambda \frac{1}{|\hat{Q}|} \int_{\hat{Q}} |k_\alpha * h - (k_\alpha * h)_{\hat{Q}}|^2 dx \geq \lambda \sup_Q \frac{1}{|Q|} \int_Q |k_\alpha * f - (k_\alpha * f)_Q|^2 dx. \end{aligned} \tag{20}$$

Changing h into ϵh in (20), dividing both sides by $\epsilon > 0$, and taking $\epsilon \to 0$, we obtain that for any $h \in BV$,

$$\frac{\frac{1}{|\bar{Q}|} \left\langle k_\alpha * f - (k_\alpha * f)_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}} \right\rangle_{L^2(\bar{Q})}}{|h|_{BV}} \leq \frac{1}{2\lambda}.$$

Therefore, $\|f\|_{\alpha,*} \leq \frac{1}{2\lambda}$. □

Proposition 5. *Assume now* $\|f\|_{\alpha,*} > \frac{1}{2\lambda}$.

(i) Suppose u *is a minimizer and* $f-u$ *satisfies the property* (**P**) *from (17). Then* u *satisfies*

$$\begin{aligned} \frac{1}{2\lambda}|u|_{BV} &= \frac{1}{|\bar{Q}|}\left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\bar{Q}}, k_\alpha * u - (k_\alpha * u)_{\bar{Q}}\right\rangle_{L^2(\bar{Q})} \\ \textit{and } \|k_\alpha * (f-u)\|_* &= \frac{1}{2\lambda}, \end{aligned} \tag{21}$$

where $\bar{Q} = \mathit{argmax}_Q \frac{1}{|Q|}\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)}$.

(ii) If u *satisfies the properties in (21), then* u *is a minimizer.*

Proof.

(i) Assume that u is a minimizer. Then, for any small ϵ and any $h \in BV$, we have

$$\begin{aligned} &|u+\epsilon h|_{BV} + \lambda \sup_Q \frac{1}{|Q|}\|k_\alpha * (f-(u+\epsilon h)) - (k_\alpha * (f-(u+\epsilon h)))_Q\|^2_{L^2(Q)} \\ &\geq |u|_{BV} + \lambda \sup_Q \frac{1}{|Q|}\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)}. \end{aligned} \tag{22}$$

Let $\hat{Q}$ be the square that achieves the maximum in the left-hand-side of the above equation (22), which depends on $k_\alpha * (f-(u+\epsilon h))$. By triangle inequality we obtain

$$\begin{aligned} |u|_{BV} + |\epsilon||h|_{BV} + \lambda\frac{1}{\hat{Q}}\Big[&\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(\hat{Q})} \\ &- 2\epsilon\left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\hat{Q}}, k_\alpha * h - (k_\alpha * h)_{\hat{Q}}\right\rangle_{L^2(\hat{Q})} \\ &+ \epsilon^2\|k_\alpha * h - (k_\alpha * h)_{\hat{Q}}\|^2_{L^2(\hat{Q})}\Big] \\ &\geq |u|_{BV} + \lambda \sup_Q \frac{1}{|Q|}\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)}. \end{aligned}$$

Thus,

$$\begin{aligned} |\epsilon||h|_{BV} + \lambda \sup_Q \frac{1}{|Q|}&\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)} \\ &- 2\lambda\epsilon\frac{1}{|\hat{Q}|}\left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\hat{Q}}, k_\alpha * h - (k_\alpha * h)_{\hat{Q}}\right\rangle_{L^2(\hat{Q})} \\ &+ \lambda\epsilon^2\frac{1}{|\hat{Q}|}\|k_\alpha * h - (k_\alpha * h)_{\hat{Q}}\|^2_{L^2(\hat{Q})} \\ &\geq \lambda \sup_Q \frac{1}{|Q|}\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)}. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&|\epsilon||h|_{BV} + \lambda\epsilon^2 \frac{1}{|\hat{Q}|}\|k_\alpha * h - (k_\alpha * h)_{\hat{Q}}\|^2_{L^2(\hat{Q})} \\
&\qquad \geq 2\lambda\epsilon\frac{1}{|\hat{Q}|}\left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\hat{Q}}, k_\alpha * h - (k_\alpha * h)_{\hat{Q}}\right\rangle_{L^2(\hat{Q})}.
\end{aligned} \tag{23}
$$

Taking in (23) $\epsilon > 0$, dividing by ϵ, and letting $\epsilon \to 0$, we have, for any $h \in BV$,

$$
|h|_{BV} \geq 2\lambda\frac{1}{|\bar{Q}|}\left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\bar{Q}}, k_\alpha * h - (k_\alpha * h)_{\bar{Q}}\right\rangle. \tag{24}
$$

Similarly, with $h = u$ in (23), $\epsilon < 0$, dividing by ϵ, and letting $\epsilon \to 0$, we get

$$
|u|_{BV} \leq 2\lambda\frac{1}{|\bar{Q}|}\left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\bar{Q}}, k_\alpha * u - (k_\alpha * u)_{\bar{Q}}\right\rangle. \tag{25}
$$

Therefore, (24) and (25) imply (21).

(ii) Let $w \in BV$ arbitrary, and let $h = w - u \in BV$, or $w = u + h$. We have

$$
\begin{aligned}
\mathcal{F}(w) &= |w|_{BV} + \lambda\sup_Q \frac{1}{|Q|}\|k_\alpha * (f-w) - (k_\alpha * (f-w))_Q\|^2_{L^2(Q)} \\
&= |u+h|_{BV} + \lambda\sup_Q \frac{1}{|Q|}\|k_\alpha * (f-(u+h)) - (k_\alpha * (f-(u+h)))_Q\|^2_{L^2(Q)} \\
&= |u+h|_{BV} + \lambda\sup_Q \Big\{\frac{1}{|Q|}\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)} \\
&- 2\frac{1}{|Q|}\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_Q, k_\alpha * h - (k_\alpha * h)_Q\rangle_{L^2(Q)} \\
&+ \frac{1}{|Q|}\|k_\alpha * h - (k_\alpha * h)_Q\|^2_{L^2(Q)}\Big\}.
\end{aligned} \tag{26}
$$

Let $\bar{Q}$ be the square that achieves the supremum in

$$
\sup_Q \frac{1}{|Q|}\|k_\alpha * (f-u) - (k_\alpha * (f-u))_Q\|^2_{L^2(Q)}.
$$

We have

$$
|u+h|_{BV} \geq 2\lambda\frac{1}{|\bar{Q}|}\left\langle k_\alpha * (f-u) - (k_\alpha * (f-u))_{\bar{Q}}, k_\alpha * (u+h) - (k_\alpha * (u+h))_{\bar{Q}}\right\rangle_{L^2(\bar{Q})}.
$$

This implies

$$\begin{aligned}
\mathcal{F}(w) &\geq 2\lambda\frac{1}{|\bar{Q}|}\left\langle k_\alpha*(f-u)-(k_\alpha*(f-u))_{\bar{Q}}, k_\alpha*u-(k_\alpha*u)_{\bar{Q}}\right\rangle_{L^2(\bar{Q})}\\
&+2\lambda\frac{1}{|\bar{Q}|}\left\langle k_\alpha*(f-u)-(k_\alpha*(f-u))_{\bar{Q}}, k_\alpha*h-(k_\alpha*h)_{\bar{Q}}\right\rangle_{L^2(\bar{Q})}\\
&+\lambda\frac{1}{|\bar{Q}|}\|k_\alpha*(f-u)-(k_\alpha*(f-u))_{\bar{Q}}\|^2_{L^2(\bar{Q})}\\
&-2\lambda\frac{1}{|\bar{Q}|}\left\langle k_\alpha*(f-u)-(k_\alpha*(f-u))_{\bar{Q}}, k_\alpha*h-(k_\alpha*h)_{\bar{Q}}\right\rangle_{L^2(\bar{Q})}\\
&+\lambda\frac{1}{|\bar{Q}|}\|k_\alpha*h-(k_\alpha*h)_{\bar{Q}}\|^2_{L^2(\bar{Q})}\\
&=|u|_{BV}+\lambda\frac{1}{|\bar{Q}|}\|k_\alpha*(f-u)-(k_\alpha*(f-u))_{\bar{Q}}\|^2_{L^2(\bar{Q})}\\
&+\lambda\frac{1}{|\bar{Q}|}\|k_\alpha*h-(k_\alpha*h)_{\bar{Q}}\|^2_{L^2(\bar{Q})}\geq\mathcal{F}(u).
\end{aligned}$$

Therefore, u is a minimizer. □

Property (**P**) from Definition 7 could hold for distributions f, when k_α is a sufficiently smooth kernel. If k_α is not sufficiently smooth, we can introduce a very small amount of smoothing by additional convolution with another kernel, say the Poisson kernel. In other words, the quantity $k_\alpha * f$ could be substituted by $P_\delta * k_\alpha * f$, where P_δ is the Poisson kernel with some small $\delta > 0$.

The following counter-examples in one dimension show that property (**P**) (with equality or inequality) may not hold for instance for discontinuous functions f when $\alpha = 0$ (thus when $k_\alpha * f = f$).

Example 1. Consider on $\mathbb{R}$ the intervals $I_n = [2^{-n-1}, 2^{-n}]$, $n \geq 0$, and let c_n be the midpoint of I_n. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ outside of $[0, 1]$, and

$$f(x) = \begin{cases} +(1-2^{-n}) \text{ if } x \in [2^{-n-1}, c_n] \quad (n \geq 1),\\ -(1-2^{-n}) \text{ if } x \in [c_n, 2^{-n}] \quad (n \geq 1),\\ +1 \text{ if } x \in [\frac{1}{2}, \frac{3}{4}],\\ -1 \text{ if } x \in [\frac{3}{4}, 1]. \end{cases}$$

Then $\|f\|_{BMO} = 1$ and $[\frac{1}{2}, 1]$ is the interval where the norm is attained. Now let $h = -f$ on $[0, \frac{1}{2}]$ and $h \equiv 0$ otherwise. Then if $\epsilon > 0$, $f - \epsilon h$ attains its BMO norm on one of the intervals $[2^{-n-1}, 2^{-n}]$, $n \geq 1$ (actually the norm increases to $1 + \epsilon$ as $n \to \infty$).

But, for $n \geq 1$,

$$\frac{1}{2^{-n}}\int_{I_n}(f-f_{I_n})(h-h_{I_n})dx = 1,$$

while

$$\frac{1}{2}\int_{[\frac{1}{2},1]}(f-f_{I_0})(h-h_{I_0})dx = 0,$$

thus (17) with equality $=$ instead of inequality $\geq$ does not hold. A similar counter-example can be constructed in two dimensions.

The following counter-example shows that inequality $\geq$ also may not hold in (17).

Example 2. Similarly, let $I_n = [2^{-n-1}, 2^{-n})$, $n \geq 0$, and c_n be the midpoint of the interval I_n. Let $J_n = [2^{-n-1}, c_n]$ and $K_n = [c_n, 2^{-n})$. On I_0 let $f = h = \chi_{J_0} - \chi_{K_0}$ and on I_n for $n \geq 1$ let $f = (1 - \frac{1}{n})(\chi_{J_n} - \chi_{K_n})$. Splitting each J_n and each K_n into two half intervals denoted A_n and B_n, let $h = \chi_{A_n} - \chi_{B_n}$. Then f and h have mean zero over all intervals I_n. Again we assume that f and h are zero otherwise. We have $\bar{Q} = I_0$ and

$$\frac{1}{|\bar{Q}|}\int_{\bar{Q}} f(x)h(x)dx = 1,$$

but for $n \geq 1$,

$$\frac{1}{|I_n|}\int_{I_n} |f(x) - \epsilon h(x)|^2 dx = (1 - \frac{1}{n})^2 + \epsilon^2$$

and

$$\frac{1}{|I_n|}\int_{I_n} f(x)h(x)dx = 0.$$

The following example shows that, at least in one dimension, if f and h are sufficiently smooth (for example polynomials or analytic functions), then property (**P**) from Definition 7 holds with equality in (17).

Example 3. Let f and h be polynomials or analytic functions on a bounded interval I in $\mathbb{R}$. Let $Q = [x_0 - r, x_0 + r]$, be an arbitrary interval included in I. Then the quantities $\frac{1}{2r}\int_Q |f - f_Q|^2 dx$, $\frac{1}{2r}\langle f - f_Q, h - h_Q\rangle_{L^2(Q)}$, and $\frac{1}{2r}\int_Q |h - h_Q|^2 dx$ remain polynomials or analytic functions of (x_0, r). Let $P(\epsilon, x_0, r) = \frac{1}{2r}\int_Q |(f - \epsilon h) - (f - \epsilon h)_Q|^2 dx$, polynomial or analytic function in (x_0, r) and quadratic polynomial in ϵ. If (x_0^0, r^0) achieves the maximum of $P(0, x_0, r)$, and if $(x_0^\epsilon, r^\epsilon)$, a bounded sequence, achieves the maximum of $P(\epsilon, x_0, r)$, then there is a subsequence $(x_0^{\epsilon_n}, r^\epsilon)$ and $\epsilon_n \to 0$ such that $\lim_{\epsilon_n \to 0} P(\epsilon_n, x_0^{\epsilon_n}, r^{\epsilon_n}) = P(0, x_0^0, r^0)$, thus property (**P**) is satisfied in this case.

4.2 The case of $\dot{W}^{\alpha,p}$

Consider here the minimization

$$\inf_u \left\{\mathcal{F}(u) = |u|_{BV} + \lambda\|f - u\|_{\dot{W}^{\alpha,p}}\right\}, \tag{27}$$

for some $\alpha < 0$, and $1 \leq p < \infty$. Thus $\mathcal{F}(u)$ is the sum of two non-differentiable functionals at the origin. Assume that we "regularize" the second term (this is often done in practice, for valid computational calculations) by smoothing at the origin the L^p norm; thus substituting $\|f - u\|_{L^p}$ by $R_\delta(f - u) = \left\{\int \sqrt{\delta^2 + |f - u|^{2p}}dx\right\}^{1/p}$, for small $\delta > 0$.

Therefore, substitute the problem (27) by the regularized functional

$$\inf_u \left\{ \mathcal{F}_\delta(u) = |u|_{BV} + \lambda R_\delta(I_\alpha(f-u)) \right\}. \tag{28}$$

Let $f \in V = \dot{W}^{\alpha,p}$, and let V' be the topological dual of V. We have $V' = \dot{W}^{-\alpha,p'}$, where p' is the conjugate of p. Denote by $\langle \cdot, \cdot \rangle$ the duality pairing for V and V'.

Problem (28) can be seen as a particular case of a more general case, where R_δ is a Gateaux-differentiable functional on the Banach space V, with continuous Gateaux derivative. For (any) fixed $f \in V$, we have $R'_\delta(f) \in V'$ and $\langle R'_\delta(f), -v\rangle = \lim_{\epsilon\to 0} \frac{R_\delta(f-\epsilon v) - R_\delta(f)}{\epsilon}$, for any $v \in V$. For any $f \in V$, define now the quantity $\|\cdot\|_{\alpha,*}$ (in $[0,+\infty]$) as

$$\|f\|_{\alpha,*} = \sup_{h\in BV,\ |h|_{BV}\neq 0} \frac{\langle R'_\delta(k_\alpha * f), k_\alpha * h\rangle}{|h|_{BV}}.$$

We also assume that for any $f, h \in V$,

$$R_\delta(f-\epsilon h) = R_\delta(f) + \epsilon \langle R'_\delta(f), -h\rangle + O(\epsilon^2)$$

in a neighborhood of the origin. Using the notation $g(\epsilon) = R_\delta(f-\epsilon h)$ for fixed f and h, this is equivalent with

$$g(\epsilon) = g(0) + \epsilon g'(0) + O(\epsilon^2),$$

where $g'(0) = \langle R'_\delta(f), -h\rangle$.

We have the following characterizations of minimizers for (28), a more general case than (27). Note that these are more general than the quadratic case considered in [32]. For the converse implications below, to show that some u is a minimizer, we need more conditions on R_δ related to convexity. The functional $R_\delta = \left\{ \int \sqrt{\delta^2 + |f-u|^{2p}}dx \right\}^{1/p}$, defined in the particular case of interest to us, satisfies the assumptions mentioned above and the additional ones that are given below.

Proposition 6.

(i) Assume that $u = 0$ is a minimizer of (28). Then $\|f\|_{\alpha,*} \leq \frac{1}{\lambda}$.

(ii) Assume that $\|f\|_{\alpha,*} \leq \frac{1}{\lambda}$, and assume that R''_δ exists and it is a continuous bilinear form on V, satisfying $R''_\delta(v)(h,h) \geq 0$, for any $v, h \in V$. Moreover, we assume that in a neighborhood of the interval $[-1,1]$ we have

$$g(\epsilon) = g(0) + \epsilon g'(0) + \frac{\epsilon^2}{2} g''(\xi_\epsilon),$$

with ξ_ϵ between 0 and ϵ, and $g''(\xi_\epsilon) = R''_\delta(f - \xi_\epsilon h)(-h,-h) \geq 0$.

Then $u = 0$ is a minimizer of (28).

Proof.

(i) For any $\epsilon \in \mathbb{R}$ and any $h \in BV$, we have

$$
\begin{aligned}
|\epsilon h|_{BV} + \lambda R_\delta(k_\alpha * (f - \epsilon h)) &\geq \lambda R_\delta(k_\alpha * f), \\
|\epsilon||h|_{BV} + \lambda\Big[R_\delta(f) + \epsilon \left\langle R'_\delta(k_\alpha * f), -k_\alpha * h\right\rangle + O(\epsilon^2)\Big] &\geq \lambda R_\delta(f), \\
|\epsilon||h|_{BV} + \lambda\epsilon \left\langle R'_\delta(k_\alpha * f), -k_\alpha * h\right\rangle + \lambda O(\epsilon^2) &\geq 0.
\end{aligned}
$$

Taking $\epsilon > 0$, dividing by ϵ and letting $\epsilon \to 0$, we obtain

$$
|h|_{BV} \geq \lambda \left\langle R'_\delta(k_\alpha * f), k_\alpha * h\right\rangle, \quad \text{thus } \frac{1}{\lambda} \geq \|f\|_{\alpha,*}.
$$

(ii) Conversely, take any $h \in BV$. Then using the assumptions, we have

$$
\begin{aligned}
|h|_{BV} + \lambda R_\delta(k_\alpha * (f - h)) &\geq \lambda \left\langle R'_\delta(k_\alpha * f), k_\alpha * h\right\rangle + \lambda R_\delta(k_\alpha * f) \\
&\quad + \lambda \left\langle R'_\delta(k_\alpha * f), -k_\alpha * h\right\rangle + \frac{\lambda}{2} g''(\xi_1) \\
&= \lambda R_\delta(k_\alpha * f) + \frac{\lambda}{2} g''(\xi_1) \geq \lambda R_\delta(k_\alpha * f).
\end{aligned}
$$

Therefore, $u = 0$ is a minimizer. □

Proposition 7. *Assume that $\|f\|_{\alpha,*} > \frac{1}{\lambda}$.*

(i) If u is a minimizer, then

$$
\frac{1}{\lambda} = \|f - u\|_{\alpha,*} \text{ and } \frac{1}{\lambda}|u|_{BV} = \left\langle R'_\delta(k_\alpha * (f - u)), k_\alpha * u\right\rangle.
$$

(ii) Suppose that $u \in BV$ satisfies

$$
\frac{1}{\lambda} = \|f - u\|_{\alpha,*} \text{ and } \frac{1}{\lambda}|u|_{BV} = \left\langle R'_\delta(k_\alpha * (f - u)), k_\alpha * u\right\rangle,
$$

and assume in addition the same conditions from Proposition 6 (ii) on the regularity and convexity of R_δ. Then u is a minimizer.

Proof. By the assumption and the previous result, $u = 0$ cannot be a minimizer.

(i) If $u \in BV$ is a minimizer, then

$$
|u + \epsilon h|_{BV} + \lambda R_\delta(k_\alpha * (f - (u + \epsilon h))) \geq |u|_{BV} + \lambda R_\delta(k_\alpha * (f - u)).
$$

Thus

$$
\begin{aligned}
|u + \epsilon h|_{BV} + \lambda R_\delta(k_\alpha * (f - u)) + \lambda\epsilon \left\langle R'_\delta(k_\alpha * (f - u), -k_\alpha * h\right\rangle + O(\epsilon^2) \\
\geq |u|_{BV} + \lambda R_\delta(k_\alpha * (f - u)).
\end{aligned} \tag{29}
$$

By triangle inequality, we also obtain

$$
\begin{aligned}
|u| + |\epsilon||h|_{BV} + \lambda R_\delta(k_\alpha * (f - u)) + \lambda\epsilon \left\langle R'_\delta(k_\alpha * (f - u), -k_\alpha * h\right\rangle + O(\epsilon^2) \\
\geq |u|_{BV} + \lambda R_\delta(k_\alpha * (f - u)).
\end{aligned}
$$

After terms cancellation and division by $\epsilon > 0$, taking $\epsilon \to 0$, we obtain that for any $h \in BV$,

$$|h|_{BV} \geq \lambda \left\langle R'_\delta(k_\alpha * (f-u), k_\alpha * h\right\rangle,$$

$$\text{therefore } \frac{1}{\lambda} \geq \|f-u\|_{\alpha,*}. \tag{30}$$

Taking now $h = u$ in (29), with $-1 < \epsilon < 0$, after cancellations and division by $\epsilon < 0$ and letting $\epsilon \to 0$, we obtain

$$|u|_{BV} \leq \lambda \left\langle R'_\delta(k_\alpha * (f-u), k_\alpha * u\right\rangle. \tag{31}$$

Combining (30) and (31), we obtain the desired results,

$$\frac{1}{\lambda} = \|f-u\|_{\alpha,*}, \quad \frac{1}{\lambda}|u|_{BV} = \left\langle R'_\delta(k_\alpha * (f-u), k_\alpha * u\right\rangle.$$

(ii) Conversely, by the assumptions and taking $\epsilon = 1$, we have

$$\begin{aligned}|u+h|_{BV} + \lambda R_\delta(k_\alpha * (f-(u+h))) &= |u+h|_{BV} + \lambda R_\delta(k_\alpha * (f-u)) \\ &+ \lambda \left\langle R'_\delta(k_\alpha * (f-u), -k_\alpha * h\right\rangle + \frac{\lambda}{2} g''(\xi_1) \\ &\geq \lambda \left\langle R'_\delta(k_\alpha * (f-u)), k_\alpha * (u+h)\right\rangle + \lambda R_\delta(k_\alpha * (f-u)) \\ &+ \lambda \left\langle R'_\delta(k_\alpha * (f-u), -k_\alpha * h\right\rangle + \frac{\lambda}{2} g''(\xi_1) \\ &= |u|_{BV} + \lambda R_\delta(k_\alpha * (f-u)) + \frac{\lambda}{2} g''(\xi_1) \\ &\geq |u|_{BV} + \lambda R_\delta(k_\alpha * (f-u)),\end{aligned}$$

thus u is a minimizer. □

5 Numerical minimization algorithms

For numerical studies, we consider spaces of functions or distributions that are periodic and $\Omega = [0,1]^2$ is the fundamental domain in $\mathbb{R}^2$. We give in this section the ingredients for minimizing in practice the proposed decomposition models from Section 3, in a gradient descent and purely PDE approach. In other words, we formally compute the associated Euler-Lagrange equations, which are then discretized and solved by finite differences. In a future work, related minimization models will be described and solved in a non-PDE framework, using a multiscale bottom-up approach.

5.1 Algorithms for the decompositions using BV and $\dot{BMO}^\alpha$

For $\alpha < 0$, recall the minimization problem (12) for exact decompositions

$$\inf_u \left\{ \mathcal{E}(u) = |u|_{BV(\Omega)} + \lambda \|I_\alpha(f-u)\|_{BMO(\Omega)} = |u|_{BV(\Omega)} + \lambda \|k_\alpha * (f-u)\|_{BMO(\Omega)} \right\}, \tag{32}$$

where we recall that $k_\alpha(x) = \left((2\pi|\xi|)^\alpha\right)^\vee(x)$, and here the dimension is $n = 2$.

We show the steps to solve (32). Using the classical definition of the BMO norm, we re-write (32) as

$$\inf_{u\in BV(\Omega)}\Big\{\mathcal{E}(u) = \int_\Omega |\nabla u|dx + \lambda \sup_Q \frac{1}{|Q|}\int_Q |k_\alpha * (f-u) - c_Q|dx\Big\},$$

where $\alpha < 0$, Q is a square with sides parallel with the axes, and c_Q denotes a constant which depends on $k_\alpha * (f-u)$ in Q. Here we take c_Q to be the median of $k_\alpha * (f-u)$ in Q.

The main steps of the algorithm are as follows (following [26]):

1. Start with an initial guess u^0.

2. If u^n is computed, $n \geq 0$, evaluate $k_\alpha * (f-u^n)$ using the Fast Fourier Transform and find a square $Q = Q^n$ that achieves the BMO norm of $k_\alpha * (f-u)$ in Ω (by one of the methods proposed in [26]; here, we use the dyadic squares and their 1/3 translations, as explained in [15]).

3. Fix Q the square obtained at the previous step, denote by χ_Q the characteristic function of this square Q, and minimize with respect to $u = u^{n+1}$ the energy

$$\mathcal{E}(u) = \int_\Omega |\nabla u|dx + \lambda\frac{1}{|Q|}\int_\Omega \Big|k_\alpha * (f-u) - c_Q\Big|\chi_Q dx, \tag{33}$$

and the associated Euler-Lagrange equation in $u = u^{n+1}$ can be computed, and we obtain using gradient descent

$$\frac{\partial u}{\partial t} = \frac{\lambda}{|Q|}k_\alpha * \text{sign}\left[\left(k_\alpha * (f-u) - c_Q\right)\chi_Q\right] + \text{div}\Big(\frac{\nabla u}{|\nabla u|}\Big), \tag{34}$$

with $Q = Q^n$ and $u = u^{n+1}$. Note that c_Q is the median of $k_\alpha * (f-u)$ in Q.

4. Repeat steps 2, and 3 using equation (34), until convergence (update u^{n+1} and Q^{n+1} each time and repeat).

Similarly, for the minimization problem (10), again with $s < 2$ ($\alpha = s-2$),

$$\begin{aligned}\inf_{u,g}\Big\{\mathcal{A}(u,g) &= |u|_{BV(\Omega)} + \mu\|f-u-\Delta g\|^2_{L^2(\Omega)} + \lambda\|I_s g\|_{BMO(\Omega)} \\ &= |u|_{BV(\Omega)} + \mu\|f-u-\Delta g\|^2_{L^2} + \lambda\|k_s * g\|_{BMO(Omega)}\Big\},\end{aligned} \tag{35}$$

re-written as

$$\inf_{u,g}\Big\{\mathcal{A}(u,g) = \int_\Omega |\nabla u|dx + \mu\int_\Omega |f-u-\triangle g|^2dx + \lambda\sup_Q\frac{1}{|Q|}\int_Q |k_s * g - c_Q|dx\Big\},$$

where c_Q is the median of $k_s * g$ over the square Q, the main steps are as follows.

1. Start with initial guess u^0, g^0.

2. If u^n and g^n are computed, $n \geq 0$, evaluate $k_s * g^n$ using the Fast Fourier Transform and find a square $Q = Q^n$ that achieves the BMO norm of $k_s * g$ in Ω (by one of the methods proposed in [26]).

3. Fix Q the square obtained at the previous step, denote by χ_Q the characteristic function of this square Q, and minimize with respect to $u = u^{n+1}$ and $g = g^{n+1}$ the energy

$$\mathcal{A}(u,g) = \int_\Omega |\nabla u| dx + \mu \int_\Omega |f - u - \triangle g|^2 dx + \lambda \frac{1}{|Q|} \int_\Omega |k_s * g - c_Q| \chi_Q dx,$$

by solving the associated Euler-Lagrange equations using gradient descent

$$\frac{\partial u}{\partial t} = 2\mu(f - u - \triangle g) + \operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big),$$

$$\frac{\partial g}{\partial t} = -\frac{\lambda}{|Q|} k_s * \operatorname{sign}\left[(k_s * g - c_Q)\,\chi_Q\right] + 2\mu \triangle (f - u - \triangle g)$$

with $Q = Q^n$ and $u = u^{n+1}$, $g = g^{n+1}$. Note that c_Q is the median of $k_s * (f - u)$ in Q.

4. Repeat steps 2, and 3 using equation (34), until convergence (update u^{n+1}, g^{n+1} and Q^{n+1} each time and repeat).

5.2 Algorithms for the decompositions using BV and $\dot{W}^{\alpha,p}$

For $\alpha < 0$, recall the minimization problem (13)

$$\inf_u \Big\{ \mathcal{E}(u) = |u|_{BV(\Omega)} + \lambda \|I_\alpha (f-u)\|_{L^p(\Omega)} = |u|_{BV(\Omega)} + \lambda \|k_\alpha * (f-u)\|_{L^p(\Omega)} \Big\}, \tag{36}$$

which is again minimized using Euler-Lagrange equation and gradient descent, as follows. Solve to steady state

$$\frac{\partial u}{\partial t} = \lambda \|k_\alpha * (f-u)\|^{1-p}_{L^p(\Omega)} k_\alpha * \Big[|k_\alpha * (f-u)|^{p-2} k_\alpha * (f-u)\Big] + \operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big),$$

computing the convolutions using the Fast Fourier Transform.

Finally, for the minimization problem (11), recalled here with $s < 2$ ($\alpha = s - 2$),

$$\begin{aligned} \inf_{u,g} \Big\{ \mathcal{A}(u,g) &= |u|_{BV(\Omega)} + \mu \|f - u - \Delta g\|^2_{L^2(\Omega)} + \lambda \|I_s g\|_{L^p(\Omega)} \\ &= |u|_{BV(\Omega)} + \mu \|f - u - \Delta g\|^2_{L^2(\Omega)} + \lambda \|k_s * g\|_{L^p(\Omega)} \Big\}, \end{aligned} \tag{37}$$

we use again the associated Euler-Lagrange equations and gradient descent, formally written as

$$\frac{\partial u}{\partial t} = 2\mu(f - u - \triangle g) + \operatorname{div}\Big(\frac{\nabla u}{|\nabla u|}\Big),$$

$$\frac{\partial g}{\partial t} = -\lambda \|k_s * g\|_{L^p(\Omega)}^{1-p} k_s * \left[|k_s * g|^{p-2} k_s * g \right] + 2\mu \triangle (f - u - \triangle g).$$

In practice, the above Euler-Lagrange equations are discretized using finite differences. The calculations are stable and the numerical energy decreases versus iterations.

6 Numerical results and comparisons

Figure 1 shows three test Barbara images, to be used in our experimental calculations.

Figure 2 shows a decomposition of f_1 from Figure 1 using the Rudin-Osher-Fatemi model (3). Note the loss of intensity on the face area.

Figure 3 shows a decomposition of f_1 from Figure 1 using the model (5) from [26]. Here oscillatory component is modeled as $v = div(\vec{g})$, $\vec{g} \in (BMO)^2$. We a obtain an improvement in the loss of intensity, however vertical and horizontal textures are still kept in u.

Figure 4 shows a decomposition of f_1 from Figure 1 using the model (6) from [26]. Here the oscillatory component is modeled as $v = \Delta g$, $\nabla g \in (BMO)^2$. The decomposition is now more isotropic, textures are well captured in v including non-repeated patterns. This comes from the property of BMO.

Figure 5 shows a decomposition of f_1 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$, $s = 0.2$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$. Now, mostly repeated patterns are captured in v.

Figure 6 shows a decomposition of f_2 from Figure 1 using the model (10). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{BMO}^s$ with $s = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.0011$. As remarked earlier, non-repeated patterns are also captured in v.

Figures 7-8 show decompositions of f_2 and f_3 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 0$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$.

Figures 9-10 show decompositions of f_2 and f_3 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.0005$.

Figure 11 shows a decomposition of f_2 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1.5$, and $p = 1$. The parameters used are: $\mu = 10$, and $\lambda = 5e\neg 05$.

Figure 12 shows a decomposition of f_2 from Figure 1 using the model (12). Here the oscillatory component $v \in \dot{BMO}^{-0.5}$, $\lambda = 200$.

Figure 13 shows a decomposition of f_2 from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.1$, $p = 1$, $\lambda = 1.25$.

Figure 14 shows a decomposition of f_2 from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.5$, $p = 1$, $\lambda = 15$.

Figure 15 shows a decomposition of f_2 from Figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.6$, $p = 1$, $\lambda = 30$.

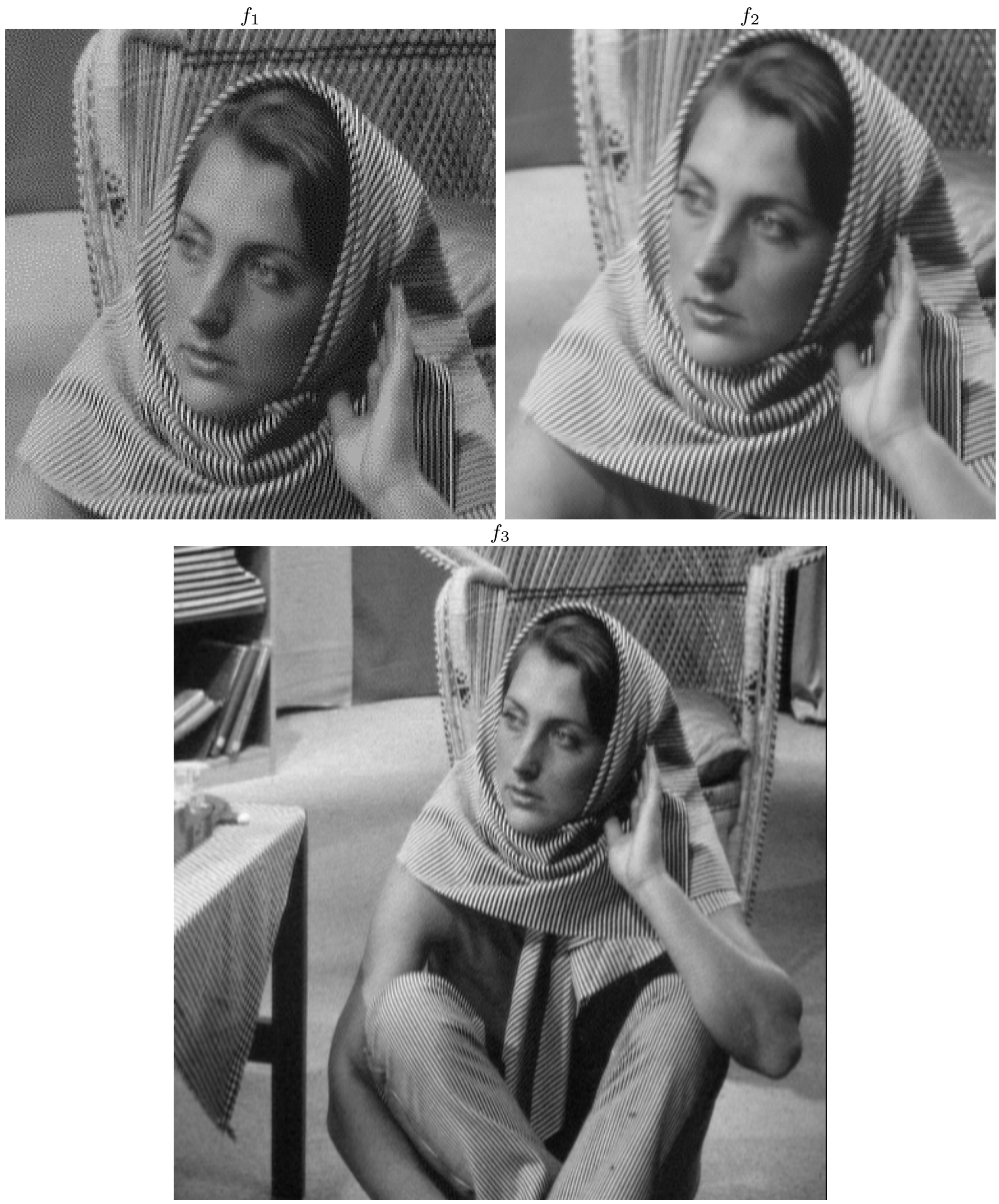

Figure 1: Test images to be decomposed.

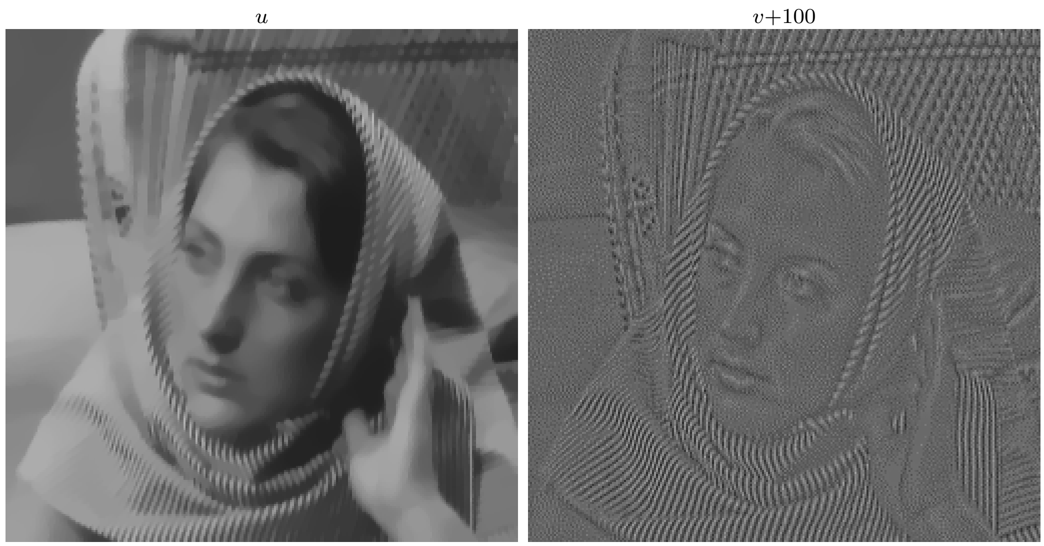

Figure 2: A decomposition of f_1 from Figure 1 using the Rudin-Osher-Fatemi model (3). (3)

Figure 3: A decomposition of f_1 from Figure 1 using the model (5) from [26]. Here oscillatory component is modeled as $v = div(\vec{g}),\ \vec{g} \in (BMO)^2$.

Figure 4: A decomposition of f_1 from Figure 1 using the model (6) from [26]. Here the oscillatory component is modeled as $v = \Delta g$, $\nabla g \in (BMO)^2$.

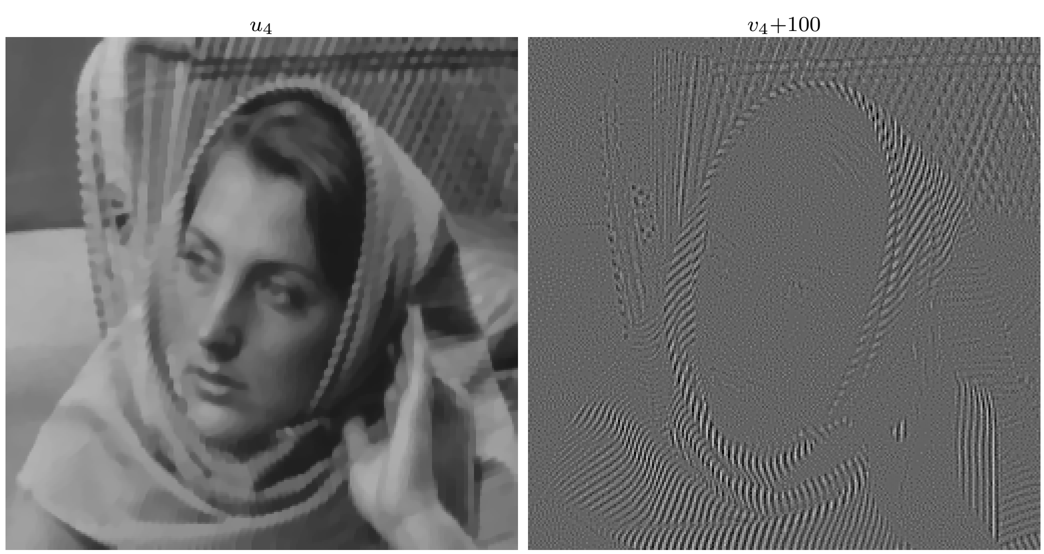

Figure 5: A decomposition of f_1 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$, $s = 0.2$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$.

Figure 6: A decomposition of f_2 from Figure 1 using the model (10). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{BMO}^s$ with $s = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.0011$.

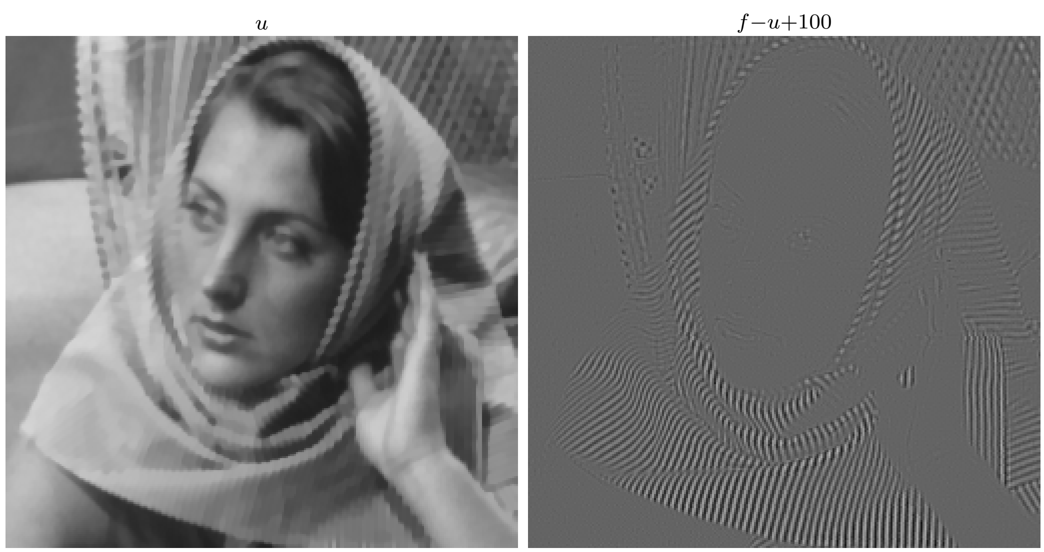

Figure 7: A decomposition of f_2 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 0$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$.

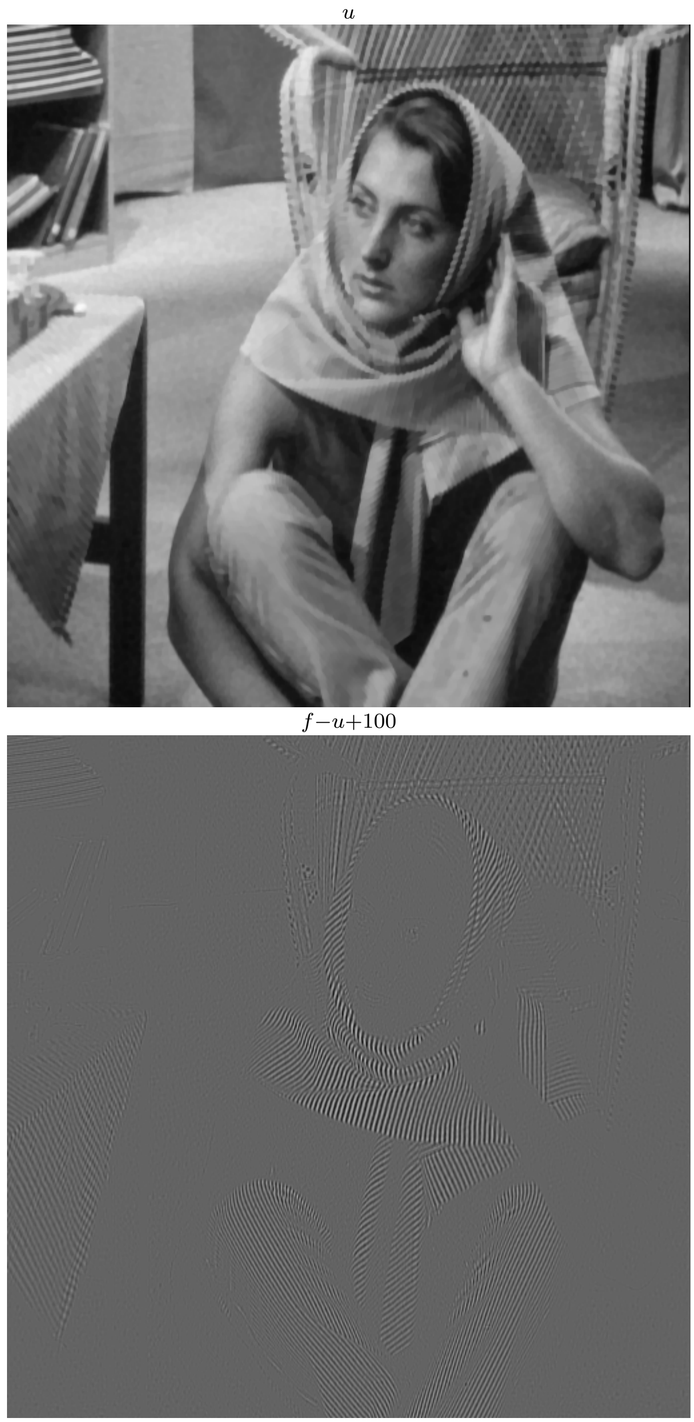

Figure 8: A decomposition of f_3 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 0$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 1$.

Figure 9: A decomposition of f_2 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.0005$.

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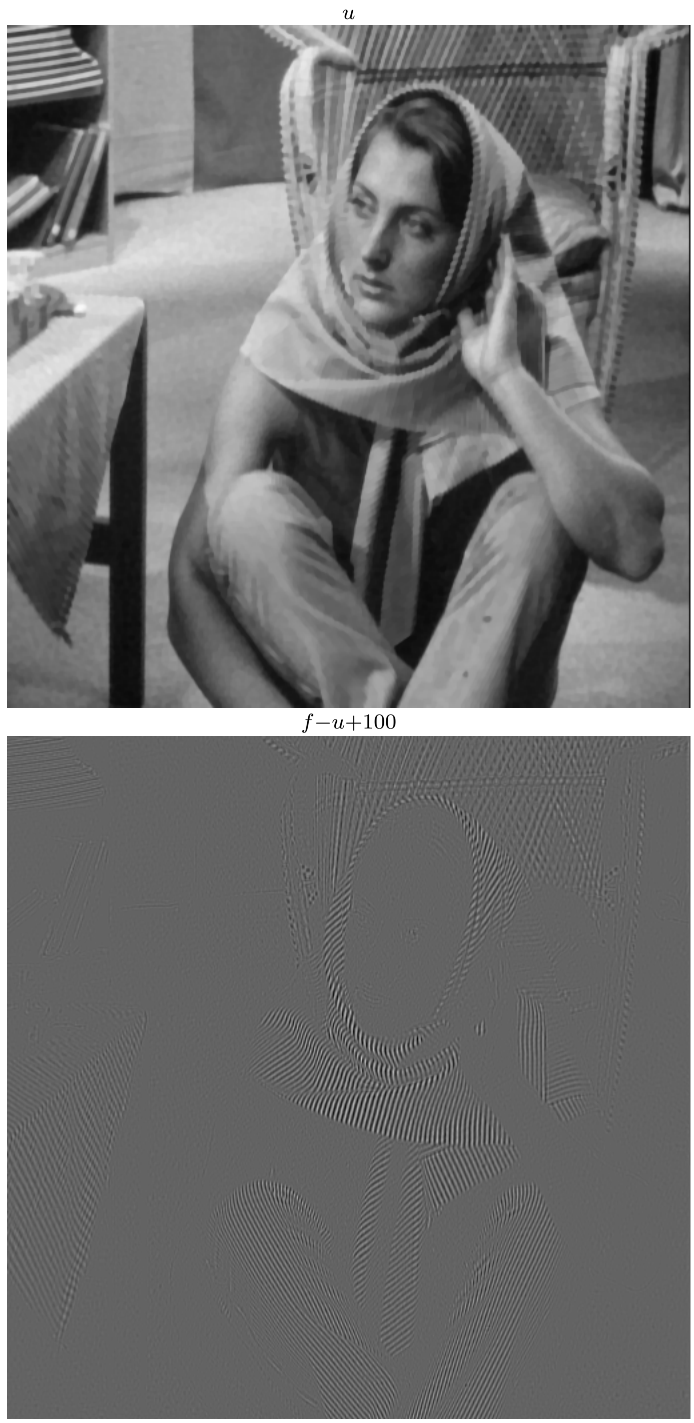

Figure 10: A decomposition of f_3 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1$, and $p = 1$. The parameters used are: $\mu = 1$, and $\lambda = 0.00025$.

Figure 11: A decomposition of f_2 from Figure 1 using the model (11). Here the oscillatory component is modeled as $v = \Delta g$, $g \in \dot{W}^{s,p}$ with $s = 1.5$, and $p = 1$. The parameters used are: $\mu = 10$, and $\lambda = 5e\neg 05$.

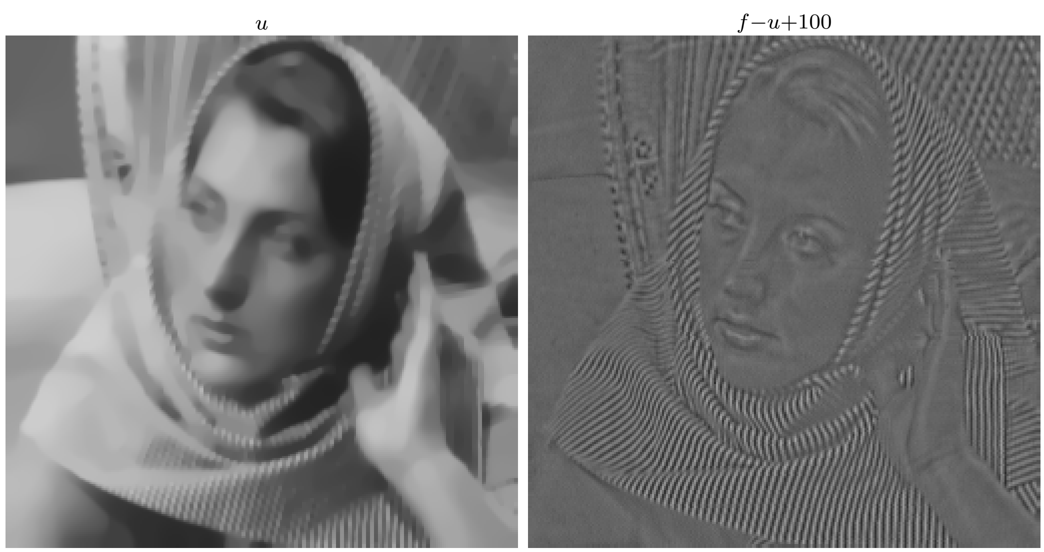

Figure 12: A decomposition of f_2 using (12) with $p = 1$. Here the oscillatory component $v \in B\dot{M}O^{\alpha}$, $\alpha = -0.5$, $\lambda = 200$.

Figure 13: A decomposition of f_2 from figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.1$, $p = 1$, $\lambda = 1.25$.

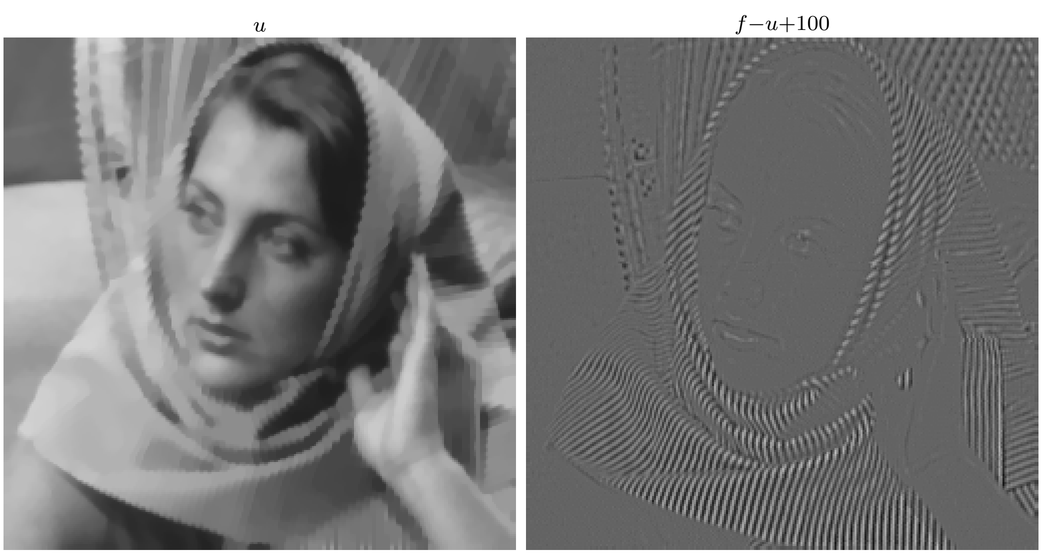

Figure 14: A decomposition of f_2 from figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.5$, $p = 1$, $\lambda = 15$.

Figure 15: A decomposition of f_2 from figure 1 using the model (13). Here the oscillatory component $v \in \dot{W}^{\alpha,p}$, $\alpha = -0.5$, $p = 1$, $\lambda = 30$.

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