On the shortening rate of collections of plane convex curves by the area-preserving mean curvature flow

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Abstract

Area-preserving mean curvature flows can be used to model some phase transitions. Geometrically, in two dimensional case, they describe the shortening of the curves that are interfaces separating the two phases while preserving the areas of each phase, respectively. Scaling arguments suggest that under the area-preserving mean curvature flow, the rescaled total curve length $\tilde{L}(t)$ decreases as a temporal power law $\tilde{L}(t) \sim ct^{-1/2}$, where c is a positive constant. In this paper, we consider the evolution of a collection of non-intersecting smooth convex plane curves and prove a time-averaged lower bound of the decay rate of $\tilde{L}$ which exhibits the aforementioned power law, and get the dependence of the coefficient constant c on the curve shapes.

1 Introduction

For a collection of smooth curves $\{\Gamma_i, i = 1, \cdots, N\}$, where $N = N(t)$ is the number of curves at time t, the total length can be shortened by the mean curvature flow

$$v = -\kappa \text{ on } \Gamma_i. \tag{1.1}$$

Here v is the outer normal velocity of the curve and κ is the curvature, chosen to be positive if Γ_i is convex. The mean curvature flow is defined by local information only and the evolution of each curve is independent of others as long as they are far apart and do not intersect.

Equation (1.1) is the L^2 gradient flow for the curve length $L_i := |\Gamma_i|$ because

$$\frac{d}{dt}|\Gamma_i| = \int_{\Gamma_i} \kappa v \, ds. \tag{1.2}$$

It has been proved that for a smooth planar curve, the mean curvature flow will shrink it to a circle and the area enclosed by each curve decreases with a fixed rate, and consequently the curve disappears in finite time. In the case that Γ_i is a simple curve, $dA_i/dt = -2\pi$ [9, 10], here A_i is the area enclosed by Γ_i.

In some situations we may want to keep the total area enclosed by the curves to be constant. An example is about phase transitions. Suppose we have a two-phase mixture, with a minor phase A dispersed in a background of phase B. These phases are separated by an interface consisting of curves. What happens in general is atoms of phase A detach from the interface, diffuse around and reattach to some other places of the interface. The detachment/attachment process and the diffusion process in general happen in different time scales. When the detachment/attachment process is fast and the diffusion process takes more time to complete, we get diffusion-dominated phase transition models like Mullins-Sekerka ones. While when the detachment/attachment process dominates, we get some model that is similar to mean curvature flow but the areas of each phase are conserved. When these two processes are comparable in time scale, we get a Mullins-Sekerka model with kinetic drag term, see, e.g. [3] for a detailed discussion.

So it is natural to consider the area-preserving mean curvature flow, which is the L^2 gradient flow for the total curve length with the restriction that the total area is conserved. The flow is written as

$$v = -\kappa + \lambda \text{ on } \Gamma_i, \tag{1.3}$$

where $\lambda = \lambda(t)$ is a spatial constant, which is in fact a Lagrange multiplier, chosen to conserve the total area.

A brief discussion about the gradient flow structure will be included in Section 2.

Since $dA_i/dt = \int_{\Gamma_i} v \, ds$, in order to conserve the total area $A := \sum_i A_i$, where the summation goes over all surviving curves at time t, we need

$$0 \;=\; \sum_i \frac{dA_i}{dt} = \sum_i \int_{\Gamma_i} -\kappa + \lambda \, ds \tag{1.4}$$

So

$$\lambda = \frac{\sum_i \int_{\Gamma_i} \kappa \, ds}{\sum_i L_i}, \tag{1.5}$$

When all Γ_i are simple curves, since $\int_{\Gamma_i} \kappa \, ds = 2\pi$, we have

$$\lambda = \frac{2\pi}{L_c}, \tag{1.6}$$

where

$$L_c := \frac{\sum_i L_i}{\sum_i 1} \tag{1.7}$$

is the average length of surviving curves at time t.

Now the system (1.3) is coupled. Even if the curves do not intersect, they still interact with each other through the spatial constant λ. Consequently, some global properties about the whole collection of curves are included in the equations.

Recall that in phase transitions of dilute systems, it is widely observed that larger particles grow while smaller ones shrink and disappear. And the morphology of the mixture exhibits some self-similarity and some characteristic length grows as a temporal power law, see, e.g. [1]. The self-similarity can be depicted by, say, the size distribution of particles as in the mean field models (see the review paper [18]). But this can be made rigorous only when particles are spheres and even in this case, the mean field models admit infinitely many classes of self-similar solutions but mathematically there are still situations that self-similarity is impossible [14, 15]. Generically it is unlikely to encounter those non-self-similar cases in experiments.

The growth of a characteristic length obeys a temporal power law according to, heuristically, the self-similarity. Correlation length is a common choice of such a characteristic length but mathematically it is difficult to work with, and hence there is no rigorous mathematical result about it so far as we know.

There is another quantity that is both physically natural and mathematically convenient, namely the surface energy. Due to thermodynamics, phase transitions are driven by the dissipation of surface energy. On the other hand, to avoid the influence of the size of the system, rather than directly working on the total surface energy, we want to work on the "density" of the energy.

Since surface energy is proportional to the area of the surface between the two phases, the density of surface energy is then proportional to the ratio between the surface area and the volume of the system, and hence scales as inverse to length.

The first mathematically rigorous result about the dissipation of the energy density is established by Kohn and Otto [11], for Cahn-Hilliard equations, a diffuse-interface model for phase transitions. For the Cahn -Hilliard equation with constant mobility, the characteristic length grows as $t^{1/3}$ and hence the energy density decays as $t^{-1/3}$. Their main result is a one-sided time-averaged estimate for the energy density which says that, if we look at the dynamics in a large enough time interval, the averaged decay rate of the energy density can not be faster than the corresponding power law.

Another important feature of this estimate is, it does not depend on any assumption about self-similarity. And it is just one-sided because the energy can decay slower since there are unstable equilibria and anyway the system will approach its final stable equilibrium and stop decaying.

The Kohn-Otto framework has since been applied to many problems, including some other models of phase transitions, epitaxial growth, and thin film droplets, etc. [2, 4, 5, 6, 13, 12, 16]. The common feature of these problems is, they are all driven by dissipation of some energy. It is also worth mentioning that it is in [4] that a $t^{-1/2}$ power law was handled for a mean field model of phase transition, which corresponds to an area-preserving mean curvature flow for collections of spheres in $n \geqslant 2$ dimensions.

Back to our area-preseving mean curvature flow. The above mentioned surface energy is now proportional to the total length of the curves and the density will be the ratio between the length and the total area enclosed by the curves. So

$$\tilde{L} := \frac{L}{A} = \frac{\sum L_i}{\sum A_i} \tag{1.8}$$

is the quantity we want to study.

In general, the dynamics can be very complicated since there can be annihilation and coalescence of curves. In this paper, we will only consider a simple case when coalescence is absent. Namely our initial configuration will be a collection of smooth convex curves which are far away from each other. Now the only possible singularity is the disappearance of curves. But we have only finitely many instances that such singularities occur since we have finitely many curves – although the exact number can be very large. The good thing about this simple situation is, we can study the dynamics "piecewisely" in time, namely, in each time interval when no curves disappear.

In terms of phase transition, this situation corresponds to a dilute mixture. In this case, we can still have complicated geometry since we have neither restriction on the number of initial curves nor on the "size" of each curve.

As was discussed, the dissipation rate of $\tilde{L}$ is heuristically determined by the self-similarity of the dynamics. And the self-similarity has an important feature which is the scaling invariance. It is not clear if we have self-similarity, but it is easy to check that the model holds scaling invariance. If we rescale length and time by

$$x = \eta\hat{x}, \quad t = \eta^2\hat{t}, \tag{1.9}$$

in the new variables $\hat{x}$ and $\hat{t}$, the normal velocity v, curvature κ, and spatial constant λ become $v = \hat{v}/\eta, \kappa = \hat{\kappa}/\eta, \lambda = \hat{\lambda}/\eta$. Then equation (1.3) is invariant

$$\hat{v} = -\hat{\kappa} + \hat{\lambda}. \tag{1.10}$$

This scaling-invariance suggests that the only possible temporal power law for the characteristic length should be $t^{1/2}$. Since the averaged curve length $\tilde{L}$ is inverse to length, it then heuristically decays like $t^{-1/2}$.

Again, it is possible for $\tilde{L}$ to decay slower than the power law. It is also possible for $\tilde{L}$ to decay faster, for example, as indicated in [8], for a single convex curve, the area-preserving mean curvature flow will make $\tilde{L}$ decay exponentially to a positive minimum determined by the conserved area. We will include the details in Section 3.

Nevertheless, $\tilde{L}$ still can not decay faster than $t^{-1/2}$ in a time-averaged sense. This is the main result of this paper and we state it as a theorem.

We indicate the shape of a curve Γ_i by its shape indicators $\beta_i := L_i^2/A_i$. Note $\beta_i \geqslant 4\pi$ for any simple curve and equality holds if and only if the curve is a circle.

Theorem 1.1. *Under the area-preserving mean curvature flow, for any $1 < p < 2$, there exist positive constants $\tilde{C}_1, \tilde{C}_2$ depending only on p such that for any collection of smooth convex curves $\{\Gamma_i\}$, the averaged curve length $\tilde{L}$ satisfies*

$$\int_0^T \tilde{L}(t)^p \, dt \geqslant \tilde{C}_1 \left(\frac{8}{9}\sqrt{\frac{\pi}{\beta_0}}\right)^{p/2} \int_0^T (t^{-1/2})^p \, dt \quad \text{for } T \geqslant \tilde{C}_2 \frac{4}{9\sqrt{\beta_0}} M_0^2. \tag{1.11}$$

Here $M_0 = \sum_i A_{i0}^{3/2} / \sum_i A_{i0}$ with A_{i0} being the initial area enclosed by curve Γ_i at time $t = 0$, and $\beta_0 = \max\{\beta_i(0)\}$ is the maximum of all shape indicators for the initial curves.

The structure of this paper is as follows. In section 2 we discuss some properties of the area-preserving mean curvature flow and the corresponding interpretation related to phase transitions. In Section 3 we will include some geometric results of mean-curvature flows for collections of convex curves. In Section 4 we will briefly discuss the Kohn-Otto framework and derive our main result, Theorem 1.1.

2 Some basic properties

In this section we discuss some basic properties of the area-preserving mean curvature flow. First let's briefly discuss its gradient flow structure in a formal way. A gradient flow structure consists of two parts, a manifold and a metric on the tangent space. Formally, we consider the manifold $\mathcal{M}$ of all possible collections of simple closed curves with fixed total area

$$\mathcal{M} := \left\{ \mathbf{\Gamma} = \{\Gamma_i\} : \sum_i A_i = c \right\}, \tag{2.1}$$

where A_i is the area enclosed by Γ_i. The tangent space of $\mathcal{M}$ at $\mathbf{\Gamma}$ will be all possible normal velocities v for curves that do not change the total area. Since the variation of area is determined by the normal velocity

$$< dA, v >= \sum_i \int_{\Gamma_i} v \, ds, \tag{2.2}$$

the tangent space will be written as

$$T_{\mathbf{\Gamma}}\mathcal{M} := \left\{ v : \sum_i \int_{\Gamma_i} v \, ds = 0 \right\}. \tag{2.3}$$

A metric on the tangent space can be defined. For any two possible normal velocities v^1, v^2,

$$g(v^1, v^2) := \sum_i \int_{\Gamma_i} v^1 v^2 \, ds. \tag{2.4}$$

The differential of curve length L is

$$< dL, \tilde{v} >= \sum_i \int_{\Gamma_i} \kappa \tilde{v} \, ds \quad \text{for all } \tilde{v} \in T_{\mathbf{\Gamma}}\mathcal{M}. \tag{2.5}$$

dL can now be translated by $g(\cdot, \cdot)$ into the corresponding gradient by

$$g(\mathrm{grad} L, \tilde{v}) =< dL, \tilde{v} > \tag{2.6}$$

and we will choose the velocity to be the negative gradient, to dissipate curve length as fast as possible. So we want to find $v = -\mathrm{grad} E$ in the sense that

$$g(v, \tilde{v}) = -g(\mathrm{grad} E, \tilde{v}), \tag{2.7}$$

i.e., we want

$$\sum_i \int_{\Gamma_i} v\tilde{v}\, ds = -\sum_i \int_{\Gamma_i} \kappa\, \tilde{v}\, ds \quad \text{for all } \tilde{v} \in T_{\mathbf{\Gamma}}\mathcal{M}. \tag{2.8}$$

Now because $\sum_i \int_{\Gamma_i} \tilde{v}\, ds = 0$, we conclude that there must be a spatial constant λ such that

$$v = -\kappa + \lambda \quad \text{on } \Gamma_i \text{ for all } i. \tag{2.9}$$

Then λ can be determined by the conservation of total area as was did in Section 1.

It is immediate from the gradient flow structure that the area-preserving mean-curvature flow really decreases the total length L and the dissipation rate is

$$\frac{dL}{dt} = -g(v,v) = -\sum_i \int_{\Gamma_i} (\kappa - \lambda)^2\, ds \leqslant 0, \tag{2.10}$$

and equality holds if and only if $\kappa = \lambda$ for every Γ_i, i.e., when all Γ_i are circles with the same radius. This situation is the only possible unstable equilibrium and the stable equilibrium is when there is only one curve and it is a circle. This has been pointed out in [17]

Now we look at the the evolution of the areas A_i enclosed by these curves. The areas change according to

$$\frac{dA_i}{dt} = \int_{\Gamma_i} v\, ds = \int_{\Gamma_i} -\kappa + \lambda\, ds = 2\pi(-1 + \frac{L_i}{L_c}) \text{ by (1.6)}. \tag{2.11}$$

So the area A_i enclosed by Γ_i decreases when the length of Γ_i is shorter than the average curve length, and increases otherwise.

In terms of phase transitions of dilute systems, domains enclosed by the curves are generally considered as particles, Assuming uniform density, the masses of particles are proportional to the areas. So we can see that particles with short perimeters shrink– in the sense of losing mass–and possibly disappear in finite time, while those with longer perimeters grow.

We mentioned in Section 1 that the generic phenomenon is that larger particles grow while smaller ones shrink and disappear. According to equation (2.11) though, it is not the area – which is a general quantity to measure the size – but the perimeter that determines the growth of particles. However, if the particles are of similar shapes, say if they are all approximately circles, or if they are all ellipses with similar eccentricity, bigger areas still generally indicates bigger perimeters, and heuristically particles with bigger areas grow and those with smaller areas shrink. We need to remind ourselves that the situation is no longer true when particles are of different shapes. For example, an ellipse can have bigger perimeter while enclose smaller area than a circle. Hence the ellipse grows and the circle shrinks.

3 Geometric results related to convex curves

In this section, we will summarize the necessary geometric results for area-preserving mean curvature flow. First, we will see that if the initial collection of curves are convex, then they remain convex until some time when one or more curves shrink to a point and disappear. This is a simple generalization of a result in [8], which only considers the evolution of one single convex curve under equation (1.3).

Lemma 3.1. *A collection of strictly convex curves evolving according to equation (1.3) remain convex.*

The proof is done by considering the evolution of the curvature. As is pointed out in [17], by following the process of [9] or [8], it can be shown that if the curves are convex, then the curvature can be considered a function of t and θ, where θ is the angle between the tangent of this curve and the x axis. And the curvature satisfies a differential equation

$$\frac{\partial \kappa}{\partial t} = \kappa^2 \frac{\partial^2 \kappa}{\partial \theta^2} + \kappa^3 - \lambda \kappa^2 \tag{3.1}$$

with positive initial value. Then it can be proved that there exists a unique smooth solution for this equation upto some finite time when some curves disappear and the solution remains positive by a maximum principle.

Next we quote an isoperimetric inequality for convex curves.

Lemma 3.2. *[7] If Γ is a convex curve and κ is its curvature, then*

$$\int_\Gamma \kappa^2 \, ds \geqslant \pi \frac{L}{A}, \tag{3.2}$$

where L and A are the length of Γ and the area it encloses, respectively.

Now we will consider the evolution of the shapes of the curves.

Lemma 3.3. *For each convex curve $\Gamma_i(t)$, the shape indicator $\beta_i(t) = L_i^2/A_i$ is decreasing in time, with a positive lower bound 4π.*

Proof. The proof is a generalization of the one for Corollary 2.4 in [8]. We include it here for the convenience of the readers.

We know for a closed simple curve γ with fixed length l, the area a it encloses attains the maximum when γ is a circle. It is immediate that

$$\frac{l^2}{a} \geqslant 4\pi. \tag{3.3}$$

So

$$\frac{L_i^2}{A_i} \geqslant 4\pi. \tag{3.4}$$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{L_i^2}{A_i} - 4\pi\right) &= \frac{2}{A_i}\left(L_i\dot{L}_i - \frac{L_i^2}{2A_i}\dot{A}_i\right) \\
&= \frac{2}{A_i}\left(L_i\int_{\Gamma_i}(-\kappa^2+\kappa\lambda)\,ds - \frac{L_i^2}{2A_i}\int_{\Gamma_i}(-\kappa+\lambda)\,ds\right) \\
&= \frac{2}{A_i}\left(-L_i\left(\int_{\Gamma_i}\kappa^2\,ds - \frac{\pi L_i}{A_i}\right) + \frac{\lambda L_i}{2}\left(4\pi - \frac{L_i^2}{A_i}\right)\right) \\
&\leqslant -\frac{\lambda L_i}{A_i}\left(\frac{L_i^2}{A_i} - 4\pi\right) \quad \text{by (3.2)} \qquad (3.5)\\
&\leqslant 0. \qquad (3.6)
\end{aligned}$$

Hence β_i decreases with lower bound 4π. □

Remark 3.1 Corollary 2.4 in [8] considers the evolution of one convex curve under the area-preserving mean-curvature flow. Then its area A is conserved and the length L decreases and $\lambda = 2\pi/L$. Inequality (3.5) becomes

$$\frac{d}{dt}\left(\frac{L^2}{A} - 4\pi\right) \leqslant -\frac{2\pi}{A}\left(\frac{L^2}{A} - 4\pi\right), \tag{3.7}$$

and

$$\frac{L^2}{A} - 4\pi \leqslant Ce^{-2\pi t/A}. \tag{3.8}$$

So in this case L decreases exponentially to $\sqrt{4\pi A}$, although the decay rate is determined by the area A.

Remark 3.2 We mention that if any of the initial curves is a non-circular ellipse, then neither the ordinary mean curvature flow nor the area-preserving mean curvature flow will keep it elliptic, although the symmetries with respect to the original major and minor axes are kept, and so is the symmetry with respect to the original center.

Conservation of symmetry is obvious. The deformation from ellipse into non-ellipse can be proven by contradiction. Assume we have an ellipse γ with semimajor and semiminor axes a and b respectively. If the evolution keeps the curve elliptic, then the shape is determined by the evolution of the semimajor and semiminor axes.

The curvatures at the endpoints of the major and minor axes are a/b^2 and b/a^2, respectively. So for the ordinary mean curvature flow,

$$\dot{a} = -\frac{a}{b^2}, \quad \dot{b} = -\frac{b}{a^2}; \tag{3.9}$$

and for the area-preserving mean curvature flow,

$$\dot{a} = -\frac{a}{b^2} + \lambda, \quad \dot{b} = -\frac{b}{a^2} + \lambda. \tag{3.10}$$

Now the area of this ellipse is $A = \pi ab$. If the curves are kept ellipses, then for the ordinary mean curvature flow, the change of A would be

$$\dot{A} = \pi(\dot{a}b + a\dot{b}) = -\pi(\frac{a}{b} + \frac{b}{a}); \tag{3.11}$$

and for the area-preserving mean curvature flow it would be

$$\dot{A} = \pi(\dot{a}b + a\dot{b}) = -\pi(\frac{a}{b} + \frac{b}{a} - \lambda(a+b)). \tag{3.12}$$

On the other hand, under the ordinary mean curvature flow,

$$\dot{A} = \int_\gamma v\, ds = \int_\gamma -\kappa\, ds = -2\pi; \tag{3.13}$$

and under the area-preserving mean curvature flow,

$$\dot{A} = \int_\gamma (-\kappa + \lambda)\, ds = -2\pi + \lambda|\gamma|. \tag{3.14}$$

Equations (3.13) and (3.14) contradict (3.11) and (3.12), respectively, as long as $a \neq b$.

4 Proof of the main theorem

In this section we will prove Theorem 1.1. Even though the theorem is an estimate only about $\tilde{L}(t)$, it is done by considering the relations between $\tilde{L}$ and an auxiliary length $M(t)$. This method was introduced in [11].

The choice of $M(t)$ is not obvious and there is no standard method to define one, so far as we know. In our case, we define

$$M := \frac{\sum_i A_i^{3/2}}{\sum_i A_i}. \tag{4.1}$$

Recall that the summation is over all surviving curves at time t.

The first relation between $\tilde{L}$ and M is an isoperimetric inequality.

Lemma 4.1. (isoperimetric inequality)
For all t, we have

$$\tilde{L}M \geqslant 2\sqrt{\pi}. \tag{4.2}$$

Proof. Since $L_i^2/A_i \geqslant 4\pi$, we have $L_i \geqslant 2\sqrt{\pi}A_i^{1/2}$. By applying the Cauchy-Schwarz inequality, we have

$$\tilde{L}M = \frac{\sum A_i^{3/2} \sum L_i}{(\sum A_i)^2} \geqslant 2\sqrt{\pi}\frac{\sum A_i^{3/2} \sum A_i^{1/2}}{(\sum A_i)^2} \geqslant 2\sqrt{\pi}. \tag{4.3}$$

□

The second relation about $\tilde{L}$ and M is an inequality about their dissipation rates.

Lemma 4.2. (dissipation inequality)
Let

$$\beta_0 = \max\left\{\frac{L_i(0)^2}{A_i(0)}\right\}$$

be the upper bound of the shaper indicators of the initial curves. Then for all times with no curves disappearing, we have

$$\left(\frac{dM}{dt}\right)^2 \leqslant \frac{9}{4}\sqrt{\beta_0}M\left(-\frac{d\tilde{L}}{dt}\right). \tag{4.4}$$

Proof. As is discussed in Section 2, the dissipation rate of $\tilde{L}$ is a consequence of the gradient flow structure. By (2.10), we have

$$\frac{d\tilde{L}}{dt} = -\frac{1}{A}\sum_i \int_{\Gamma_i} (\kappa - \lambda)^2 \, ds. \tag{4.5}$$

On the other hand, by (2.11), we have

$$\left|\frac{dA_i}{dt}\right| = \left|\int_{\Gamma_i} -\kappa + \lambda \, ds\right| \leqslant L_i^{1/2}\left(\int_{\Gamma_i} (\kappa - \lambda)^2 \ ds\right)^{1/2}, \tag{4.6}$$

$$\left|\frac{dM}{dt}\right| = \frac{3}{2A}\left|\sum_i A_i^{1/2}\frac{dA_i}{dt}\right| \tag{4.7}$$

$$\leqslant \frac{3}{2A}\left|\sum_i A_i^{1/2}L_i^{1/2}\left(\int_{\Gamma_i} (\kappa - \lambda)^2 \ ds\right)^{1/2}\right| \tag{4.8}$$

$$\leqslant \frac{3}{2A}\left(\sum_i A_i L_i\right)^{1/2}\left(\sum_i \int_{\Gamma_i} (\kappa - \lambda)^2 \ ds\right)^{1/2}. \tag{4.9}$$

We have proved that the shape indicators $\beta_i(t) = \frac{L_i^2}{A_i}$ are decreasing in time with positive minimum 4π. Taking $\beta_0 = \max\{\beta_i(0)\}$ to be the upper bound of the shape indicators for the initial curves, we have

$$L_i = \sqrt{\beta_i A_i} \leqslant \sqrt{\beta_0 A_i}. \tag{4.10}$$

Hence

$$\left(\frac{dM}{dt}\right)^2 \leqslant \frac{9}{4}\sqrt{\beta_0}\frac{\sum_i A_i^{3/2}}{A}\left(\frac{\sum_i \int_{\Gamma_i}(\kappa-\lambda)^2\ ds}{A}\right) = \frac{9}{4}\sqrt{\beta_0}M\left(-\frac{d\tilde{L}}{dt}\right). \tag{4.11}$$

□

Now we are ready to prove Theorem 1.1. Define

$$\tilde{M} = \frac{1}{2\sqrt{\pi}}M, \quad \tilde{t} = \frac{9\sqrt{\beta_0}}{8\sqrt{\pi}}t. \tag{4.12}$$

Then the isoperimetric inequality (4.2) and the dissipation inequality (4.4) indicate that

$$\tilde{L}\tilde{M} \geqslant 1, \quad \left(\frac{d\tilde{M}}{d\tilde{t}}\right)^2 \leqslant \tilde{M}\left(-\frac{d\tilde{L}}{d\tilde{t}}\right). \tag{4.13}$$

Applying Lemma 4.2 in [4], for any $1 < p < 2$, we can find positive constants $\tilde{C}_1, \tilde{C}_2$ depending only on p such that

$$\int_0^{\tilde{T}} \tilde{L}(\tilde{t})^p\ d\tilde{t} \geqslant \tilde{C}_1 \int_0^{\tilde{T}} (\tilde{t}^{-1/2})^p\ d\tilde{t} \quad \text{if } \tilde{T} \geqslant \tilde{C}_2\tilde{M}(0)^2. \tag{4.14}$$

Changing variables back into t and M, the estimate reads

$$\int_0^{T} \tilde{L}(t)^p\ dt \geqslant C_1 \int_0^{T} (t^{-1/2})^p\ dt \quad \text{if } T \geqslant C_2 M(0)^2, \tag{4.15}$$

where

$$C_1 = \tilde{C}_1\left(\frac{8}{9}\sqrt{\frac{\pi}{\beta_0}}\right)^{p/2}, \qquad C_2 = \tilde{C}_2\frac{4}{9\sqrt{\beta_0}}. \tag{4.16}$$

This completes the proof of Theorem 1.1. □

Remark. The estimate depends on the shape of the initial curves through β_0, which is the maximum of all initial shape indicators. When the initial data corresponds to a dilute mixture, the initial curves are not far away from circles and hence β_0 is not too big compared to 4π. For example, for an ellipse with eccentricity e, the shape indicator β satisfies

$$4\pi \leqslant \beta = \frac{16\alpha}{\pi} E(e)^2, \tag{4.17}$$

where $\alpha = (1 - e^2)^{-1/2}$ is the long-short axes ratio and $E(e)$ is the complete elliptic integral of the second kind with elliptic modulus e, i.e.,

$$E(e) = \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta. \tag{4.18}$$

Since $E(e)$ decreases from $\pi/2$ to 1 as e increases from 0 to 1, β is approximately $4\pi\alpha$ when e is near 0 and β is approximately $16\alpha/\pi$ when e is about 1. Either way β increases at most linearly in α and the constants C_1, C_2 decreases sublinearly.

Since the constants C_1 and C_2 do not depend on the number of curves or any other information, we say this kind of estimate exhibits some universality.

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